Interpolation by Lipschitz holomorphic functions*

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Introduction

Let $\mathbf{C}^d$ be d-dimensional complex space ($d>1$) with norm $|z|=(|z_1|^2+\ldots+|z_d|^2)^{1/2}$ and unit ball $B=\{z\in\mathbf{C}^d\colon\ |z|<1\}$. By μ we shall denote the rotation-invariant, normalized Borel measure on $S=\partial B$ and by $C(S)$ — the space of continuous functions on S. If $f\in C(S)$ has a continuous extension $\tilde{f}\colon \bar{B}\to\mathbf{C}$, holomorphic on B, then we shall write $f\in A(B)$. We shall denote $CA=S-A$ for $A\subset S$ and by $[z_1, z_2]$ — any shortest path on S joining z_1 with z_2 ($z_1, z_2\in S$). Let $\varrho(z_1, z_2)$ be the length of a path $[z_1, z_2]$, let $q(z_1, z_2)=|1-\langle z_1, z_2\rangle|$ and let $K(z, r)=\{\xi\in S\colon\ q(z, \xi)<r\}$ ($\langle z_1, z_2\rangle$ be the scalar product of the vectors z_1 and z_2). We say that $f\in\operatorname{Lip}\alpha$, where $0<\alpha\leqq 1$, if $f\in C(S)$ and there exists a constant C such that

$$|f(z)-f(\xi)|\leqq C\varrho(z, \xi)$$

for $z, \xi\in S$.

Aleksandrov proved [2] that for every real function $g\in C(S)$ and for every $\varepsilon>0$ there exist functions $f\in A(B)$ such that $\operatorname{Re} f\leqq g$ and $\mu(\{z\in S\colon\ \operatorname{Re} f(z)=g(z)\})\geqq 1-\varepsilon$. Sibony proved [4] that if $f\in A(B)\cap\operatorname{Lip}\alpha$ is a nonconstant function with norm $\|f\|_\infty\leqq 1$, then $\mu(\{z\in S\colon\ |f(z)|=1\})=0$. This theorem was strengthened by Henkin (see [3] sect. 11.4), who obtained the following result: If $f\in A(B)\cap\operatorname{Lip}\alpha$ is a nonconstant function such that $\operatorname{Re} f\leqq 0$ and $1\geqq\alpha>1/2$, then $\mu(\{z\in S\colon \operatorname{Re} f(z)=0\})=0$. It is still an open problem, if the assumption $1\geqq\alpha>1/2$ can be replaced by a weaker condition $1\geqq\alpha>b$, where $b<1/2$. We shall show that b has to be positive:

Theorem. *For every* $\varepsilon>0$ *there exists* $\alpha>0$ *such that for every real function* $g\in\operatorname{Lip} 1$ *it is possible to find nonconstant functions* $f\in A(B)\cap\operatorname{Lip}\alpha$ *such that* $\operatorname{Re} f\leqq g$ *on* S, *and*

$$\mu(\{z\in S\colon\ \operatorname{Re} f(z) = g(z)\})\geqq 1-\varepsilon.$$

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Corollary 1. *For every* $\varepsilon>0$ *there exists* $\alpha>0$ *such that for every function* $g\in\mathrm{Lip}\,1$, $g>0$ *there exist nonconstant functions* $f\in A(B)\cap\mathrm{Lip}\,\alpha$ *such that* $|f(z)|\leqq g(z)$ *for* $z\in S$, *and*

$$\mu(\{z\in S\colon |f(z)| = g(z)\}) \geqq 1-\varepsilon.$$

Proof. Define $\tilde{g}=\log(g)$ and apply the Theorem to the function $\tilde{g}$ instead of g. We shall get some functions $\tilde{f}\in A(B)$. The functions $e^{\tilde{f}}$ will satisfy the assertion of Corollary 1.

Corollary 2. *There exists* $\alpha>0$ *such that for every* $\varepsilon>0$ *it is possible to find nonconstant functions* $f\in A(B)\cap\mathrm{Lip}\,\alpha$ *such that* $\|f\|_\infty\leqq 1$ *and*

$$\mu(\{z\in S\colon |f(z)| = 1\}) \geqq 1-\varepsilon.$$

Proof. Let us apply the assertion of Corollary 1 for $\varepsilon_0=1/2$ and $g\equiv 1$. We shall get functions $f\in A(B)\cap\mathrm{Lip}\,\alpha$, for some $\alpha>0$, such that $|f(z)|\leqq 1$ for $z\in S$, and $\mu(E)\geqq 1/2$, where $E=\{z\in S\colon |f(z)|=1\}$. Let $u=P[\chi_E]$ be the Poisson integral of the characteristic function of the set E. Let us fix $\varepsilon>0$. Then $u(a)>1-\varepsilon$ for some point $a\in B$. Let $\psi\in\mathrm{Aut}\,(B)$ be an automorphism of the ball B such that $\psi(0)=a$ and let $F=f\circ\psi$. Then $\chi_E\circ\psi=\chi_K$, where $K=\{z\in S\colon |F(z)|=1\}$. Moreover

$$\mu(K) = \int_S \chi_K\, d\mu = \int_S \chi_E\circ\psi\, d\mu = P[\chi_E\circ\psi](0) = P(\chi_E)(\psi(0)) = u(a) \geqq 1-\varepsilon.$$

Also $F\in A(B)\cap\mathrm{Lip}\,\alpha$ and $|F|\leqq 1$ on S. This ends the proof of Corollary 2.

To prove the assertion of the Theorem, we shall need the following lemmas:

Lemma 1 *(Aleksandrov). Let* $a, N>0$, $0<p<1$. *There exists a number* $r_0>0$ *and* $\sigma=\sigma(a,N,p,d)>0$ *such that for every number* $r<r_0$ *and* $K(\xi,r)$ $(\xi\in S)$, *it is possible to find a function* $h\in A(B)$ *satisfying the following conditions:*

(1) $$\mathrm{Re}\, h(0) = 0$$

(2) $$|h(z)| \leqq a \qquad \text{for } z\in K(\xi,r)$$
$$\leqq a\left(\frac{r}{q(\xi,r)}\right)^N \quad \text{for } z\in S-K(\xi,r).$$

(3) $$\int_{K(\xi,r)} |\mathrm{Re}\, h-1|^p\, d\mu + \int_{CK(\xi,r)} |h|^p\, d\mu \leqq (1-\sigma)\mu(K(\xi,r)).$$

This Lemma was proved by Aleksandrov [1]. The example of the function h, given by Aleksandrov, is $h(z)=g_1\left(\frac{R'}{r}-\frac{R'}{r}\langle z,\xi\rangle\right)$, where R' is some number independent of r and $g_1(z)=ai(1+z)^{-N}$. Hence h is a function defined on some neighborhood of S and it is constant in the directions $w\in\mathbf{C}^n$ such that $\langle w,\xi\rangle=0$.

It follows that

(4) $$h'_w(z) = 0 \quad \text{for} \quad z \in S \quad \text{and} \quad w \in \mathbf{C}^n \quad \text{such that} \quad \langle w, \xi \rangle = 0,$$

where h'_w is a directional derivative of the function f at the (complex) direction w.

Since the directional derivative of the function $\frac{R'}{r} - \frac{R'}{r}\langle z, \xi \rangle$ at the direction ξ is $-\frac{R'}{r}$, we have

$$h'_\xi(z) = g'_1\left(\frac{R'}{r} - \frac{R}{r}\langle z, \xi \rangle\right)\left(-\frac{R'}{r}\right) = \frac{R'}{r} Nia\left(1 + \frac{R'}{r} - \frac{R'}{r}\langle z, \xi \rangle\right)^{-N-1}.$$

But $\left|1 + \frac{R'}{r} - \frac{R'}{r}\langle z, \xi \rangle\right| \geqq \max\left(1, \frac{R'}{r} q(z, \xi)\right)$, hence

(5) $$h'_\xi(z) \leqq \frac{E}{r} \qquad \text{for} \quad z \in K(\xi, r)$$

$$\leqq \frac{E}{r}\left(\frac{r}{q(z, \xi)}\right)^{-N-1} \quad \text{for} \quad z \in S - K(\xi, r),$$

where E is some constant independent of r.

Lemma 2. *Let, for* $z \in [z_1, z_2]$, $v = v(z)$ *be a unit vector tangent to the path* $[z_1, z_2]$ *at the point* z. *Then*

$$b_1 q(z_1, z_2) \geqq \int_{[z_1, z_2]} |\langle v, z \rangle| \, da(z),$$

where a *is the "length measure", i.e.* $da(z) = d\varrho(z_1, z)$, *and* b_1 *is some constant.*

Proof. Let $z_0 \in S$ be a vector such that $\langle z_1, z_0 \rangle = 0$ and the (complex) linear space generated by z_1 and z_0 contains z_2. Hence, there are numbers $\alpha_1, \alpha_2 \in \mathbf{C}$ such that α_1 is real, $|\alpha_1|^2 + |\alpha_2|^2 = 1$ and

$$z_2 = \cos t_0 z_1 + \sin t_0 (\alpha_1 i z_1 + \alpha_2 z_0),$$

where $0 \leqq t_0 \leqq \pi$. It follows that the function

$$\Gamma(t) = (\cos t + i\alpha_1 \sin t) z_1 + \alpha_2 \sin t z_0,$$

where $0 \leqq t \leqq t_0$, is a parametrization of the path $[z_1, z_2]$. If $z = \Gamma(t)$ for some $0 \leqq t \leqq t_0$, then $da(z) = |\Gamma'(t)| \, dt = dt$ and

$$v = v(z) = \Gamma'(t) = (-\sin t + i\alpha_1 \cos t) z_1 + \alpha_2 \cos t z_0.$$

Hence, $\langle v, z \rangle = -i\alpha_1$ and $\int_{[z_1, z_2]} |\langle v, z \rangle| \, d\lambda(z) = |\alpha_1| t_0$. On the other hand, $q(z_1, z_2) =$

$|1-\langle z, \Gamma(t_0)\rangle| = |1-\cos t_0 + i\cdot\alpha_1\cdot\sin t_0| \geqq \max(1-\cos t_0, \sin t_0)\cdot|\alpha_1|$. Hence

$$\int_{[z_1, z_2]} |\langle v, z\rangle|\, d\lambda(z) \geqq \frac{t_0}{\max(1-\cos t_0, \sin t_0)} q(z_1, z_2) \geqq \frac{\pi}{2} q(z_1, z_2).$$

This ends the proof of Lemma 2.

For $g\in C(S)$, $g>0$, let $\beta(g)=\sup\left(\frac{g(z_1)}{g(z_2)} q(z_1, z_2)^{-1}\right)$, where the supremum is taken over all points $z_1, z_2\in S$ such that $\frac{g(z_1)}{g(z_2)}\geqq 2$. If there are not such points, we define $\beta(g)=1$.

For $\xi, z\in S$, let

$$\gamma_\xi(z) = \limsup_{\eta\to 0} \left|\frac{g\left(\frac{z+\eta\xi}{|z+\eta\xi|}\right)-g(z)}{\eta}\right|.$$

We shall say that $\gamma(g)\leqq R$, if $\frac{\gamma_\xi(g)(z)}{g(z)}\leqq R|\langle z, \xi\rangle|+\sqrt{R}$ for all $z, \xi\in S$.

For $g\in C(S)$ (not necessarily positive) we define

$$T(g) = \sup_{z, \xi\in S, z\neq\xi} \left|\frac{g(z)-g(\xi)}{z-\xi}\right|.$$

There exists a constant C_1 such that, for every $r\geqq 0$, $k\geqq 1$, $z\in S$, the inequality $\mu(K(z, kr))\leqq C_1 k^d \mu(K(z, r))$ holds (see [3] sect. 5.1.4).

Lemma 3. *Assume that* $g\in C(S)$, $g>0$, $\gamma(g)\leqq R$, $s>1$, $z_1, z_2\in S$ *and* $\frac{g(z_1)}{g(z_2)}\geqq s$. *Then* $q(z_1, z_2)\geqq C_2\left(\frac{s-1}{s}\right)^2 R^{-1}$, *where* C_2 *is some constant.*

Proof. Let g, s, z_1, z_2 satisfy the assumption of Lemma 3. Let us take $z\in[z_1, z_2]$ such that $g(z)=g(z_1)$ and $g(\xi)\leqq g(z_1)$ for every $\xi\in[z, z_2]$. Let $v=v(\xi)$ be a unit vector tangent to $[z_1, z_2]$ at the point $\xi\in[z, z_2]$. Then

$$g(z)-g(z_2) \leqq \int_{[z_1, z_2]} \gamma_v(g)(\xi)\, d\lambda(\xi) \leqq \int_{[z_1, z_2]} g(\xi)(R|\langle v, \xi\rangle|+\sqrt{R})\, da(\xi)$$

$$\leqq [b_1 R q(z, z_2)+R\varrho(z, z_2)]\, g(z_1) \leqq [b_1 R q(z_1, z_2)+\sqrt{R}\varrho(z_1, z_2)]\, g(z_1),$$

because of Lemma 2 and the inequalities $q(z, z_2)\leqq q(z_1, z_2)$, $\varrho(z, z_2)\leqq\varrho(z_1, z_2)$. Dividing by $g(z_1)$, we get

$$q(z_1, z_2) \geqq 1/2\,\frac{g(z_1)-g(z_2)}{Rbg(z_1)} \geqq 1/2\, b_1^{-1} R^{-1}\frac{s-1}{s}$$

or

$$\varrho(z_1, z_2) \geqq 1/2 \frac{g(z_1)-g(z_2)}{g(z_1)} R^{-1/2} \geqq 1/2 \frac{s-1}{s} R^{-1/2}.$$

Since there exists a constant b_2 such that, for $z', z'' \in S$, $\varrho(z', z'')^2 \leqq b_2 q(z', z'')$, the assertion of Lemma 3 follows. Let us fix some number $0<p<1$.

Lemma 4. *There exist constants* $C, \tau>0$, *with the following properties: If* $g \in C(S)$, $g>0$, $R \geqq 1$ *and* $\max(\beta(g), \gamma(g)) \leqq R$, *then there exists a function* $h \in A(B)$ *such that*

(i) $|h| \leqq \frac{1}{10} g$ *on* S,

(ii) $\|g-\operatorname{Re} h\|_p^p \leqq (1-\tau)\|g\|_p^p$,

(iii) $\max(\beta(g-\operatorname{Re} h), \gamma(g-\operatorname{Re} h)) \leqq CR$, *and*

(iv) $T(h) \leqq CR\|g\|_\infty$.

Proof. Let $N=d+4$, $P=\sum_{n=0}^{\infty}\sum_{k=2^n}^{\infty} 2^d(k+2)^{d+1}2^{n+1}k^{-N}$ and $a=(20PC_1)^{-1}$. Let $\sigma=\sigma(a, N, p, d)$ and r_0 be numbers given by Lemma 1 and let η be a constant such that $0 \leqq \eta \leqq r_0$ and

$$\left(1-\left(4\frac{\eta}{C_1}\right)^{1/2}\right)^{-1} \leqq \min\left[2, 1+\left(\frac{1}{2}\sigma\right)^{1/p}, \left(1-\frac{1}{2}\sigma\right)^{-1/2p}\right].$$

From Lemma 3 it follows that if one of the inequalities

(6)
$$|g(z_1)-g(z_2)|^p \geqq g(z_2)^p 1/2\sigma,$$
$$(1-1/2\sigma)^{1/2p} g(z_2) \geqq g(z_1),$$
$$1/2 g(z_1) \geqq g(z_2) \quad \text{or} \quad 1/2 g(z_2) \geqq g(z_1)$$

holds with g satisfying the assumptions of Lemma 4, then $q(z_1, z_2) \geqq 4r$, where $r=\eta R^{-1}$.

Let $\mathfrak{F}=\{K(\xi_j, r)\}_{j=1}^M$ be a maximal family of disjoint balls and let $D=\cup\mathfrak{F}$. Since (6) fails for $z_1=\xi_j$ and $z_2 \in K(\xi_j, 4r)$, we have

$$\int_{K(\xi_j, 4r)} g^p \, d\mu \leqq 2^p g(\xi_j)^p \mu(K(\xi_j, 4r))$$
$$\leqq 2^p C_1 4^d g(\xi_j)^p \mu(K(\xi_j, r)) \leqq F\int_{K(\xi_j, r)} g^p \, d\mu,$$

where $F=2^p C_1 4^d 2^p$.

Summing over all $j=1, 2, \ldots, M$ and applying the equality $S=\bigcup_{j=1}^M K(\xi_j, 4r)$ (because $q^{1/2}$ is a metric), we get

(7)
$$\int_S g^p \, d\mu \leqq F\int_D g^p \, d\mu.$$

Lemma 1 yields functions h_j $(j=1, 2, \ldots, M)$ associated to $K(\xi_j, r)$ with a and

N defined above. We claim that the function $h=\sum_{j=1}^{M} g(\xi_j)h_j$ satisfies the conclusion of Lemma 4. Let us denote $H_j=g(\xi_j)\operatorname{Re} h_j$ for $j=1,2,\dots,M$. We have

$$\int_{K(\xi_j,r)} |g-H_j|^p\,d\mu+\int_{CK(\xi_j,r)} |H_j|^p\,d\mu \tag{8}$$
$$\leqq \int_{K(\xi_j,r)} |g-g(\xi_j)|^p\,d\mu+g(\xi_j)^p\int_{K(\xi_j,r)} |1-h_j|^p\,d\mu+g(\xi_j)^p\int_{CK(\xi_j,r)} |h_j|^p\,d\mu.$$

Since (b) fails for $z_1=\xi_j$ and $z_2\in K(\xi_j,r)$, we get

$$\int_{K(\xi_j,r)} |g-g(\xi_j)|^p\,d\mu \leqq 1/2\sigma\mu(K(\xi_j,r))g(\xi_j)^p. \tag{9}$$

Using the same argument, we show that

$$[(1-1/2\sigma)^{1/2p}g(\xi_j)]^p\mu(K(\xi_j,r)) \leqq \int_{K(\xi_j,r)} g^p\,d\mu. \tag{10}$$

Combining (8), (9), (3) and (10) we obtain

$$\int_{K(\xi_j,r)} |g-H_j|^p\,d\mu+\int_{CK(\xi_j,r)} |H_j|^p\,d\mu \leqq (1-1/2\sigma)\mu(K(\xi_j,r))g(\xi_j)^p \tag{11}$$
$$\leqq (1-1/2\sigma)^{1/2}\int_{K(\xi_j,r)} g^p\,d\mu = (1-\tau^*)\int_{K(\xi_j,r)} g^p\,d\mu,$$

where $\tau^*=1-(1-1/2\sigma)^{1/2}$. Let $D=\bigcup_{j=1}^{M} K(\xi_j,r)$. On $K(\xi_j,r)$ the following inequality

$$|g-\operatorname{Re} h|^p \leqq |g-H_j|^p+\sum_{i=j} |H_i|^p$$

holds, and on CD,

$$|g-\operatorname{Re} h|^p \leqq g^p+\sum_{i=1}^{M} |H_i|^p.$$

Hence

$$\int_S |g-\operatorname{Re} h|^p\,d\mu \leqq \sum_{j=1}^{M}\Big[\int_{K(\xi_j,r)} |g-H_j|^p\,d\mu+\sum_{i\neq j}\int_{K(\xi_j,r)} |H_i|^p\,d\mu\Big]$$
$$+\int_{CD} g^p\,d\mu+\int_{CD}\sum_{i=1}^{M} |H_i|^p\,d\mu.$$

Each function $|H_i|^p$ is integrated over CD and over all $K(\xi_j,r)$ with $j\neq i$, hence over $CK(\xi_i,r)$. Thus

$$\int_S |g-\operatorname{Re} h|^p\,d\mu \leqq \sum_{j=1}^{M}\int_{K(\xi_j,r)} |g-H_j|^p\,d\mu \tag{12}$$
$$+\sum_{j=1}^{M}\int_{CK(\xi_j,r)} |H_j|^p\,d\mu+\int_{CD} g^p\,d\mu.$$

Summing (11) over $j=1,2,\dots,M$ and applying to (12), we get

$$\int_S |g-\operatorname{Re} h|^p\,d\mu \leqq (1-\tau^*)\int_D g^p\,d\mu+\int_{CD} g^p\,d\mu = \int_S g^p\,d\mu-\tau^*\int_D g^p\,d\mu,$$

and because of (7)

$$\int_S |g-\operatorname{Re} h|^p\, d\mu \leqq \int_S g^p\, d\mu - \frac{\tau^*}{F}\int_S g^p\, d\mu = (1-\tau)\int_S g^p\, d\mu, \tag{13}$$

for $\tau = \dfrac{\tau^*}{F}$. This proves (ii).

Next, *we prove* (i). Let us fix a point $z \in S$. Define

$$A_n = A_n(z) = \{\xi_j\colon\, 2^n g(z) \leqq g(\xi_j) < 2^{n+1} g(z)\}$$

for $n=1, 2, \ldots$, $A_0 = A_0(z) = \{\xi_j\colon\, g(\xi_j) < 2g(z)\}$. Let us assume that $\xi_j \in A_n$ for some $n>0$. Then $R \geqq \beta(g) \geqq \dfrac{g(\xi_j)}{g(z)} q(z, \xi_j)^{-1} \geqq 2^n q(z, \xi_j)^{-1}$, hence

$$q(z, \xi_j) \geqq 2^n R^{-1} \geqq 2^n r.$$

Hence, if $A_n^k = \{\xi_j \in A_n\colon\, kr \leqq q(z, \xi_j) < (k+1)r\}$, then $A_n^k = \emptyset$ for $k = 1, 2, \ldots, 2^n - 1$. Since $K(\xi_j, r) \subset K(z, 2(k+2)r)$ for $\xi_j \in A_n^k$ (this inclusion follows from the fact that $q^{1/2}$ is a metric on S), we have

$$|A_n^k|\, \mu(K(\xi_j, r)) \leqq \mu(K(z, 2(k+2)r)) \leqq C_1 2^d (k+2)^d \mu(K(z, r))$$

and since $\mu(K(\xi_j, r)) = \mu(K(z, r))$, it follows that $|A_n^k| \leqq C_1 2^d (k+2)^d$. Because of (2) and the definitions of A_n^k and a, we have

$$|h(z)| \leqq \sum_{i=1}^M |h_j(z)| g(\xi_j) \leqq \sum_{n=0}^\infty \sum_{k=0}^\infty \sum_{\xi_j \in A_n^k} (k^{-N} a) 2^{n+1} g(z) \tag{14}$$

$$\leqq \sum_{n=0}^\infty \sum_{k=2^n}^\infty C_2 2^d (k+2)^d 2^{n+1} k^{-N} a g(z) \leqq \frac{1}{10} g(z),$$

which proves (i).

We turn to (iv). Let $\xi, z \in S$. For $\xi_j \in A_n^k(z)$, we have

$$|\langle \xi, \xi_j\rangle| \leqq |\langle \xi, z\rangle| + |\langle \xi, \xi_j - z\rangle|$$

$$\leqq |\langle \xi, z\rangle| + |\xi_j - z| \leqq |\langle \xi, z\rangle| + \sqrt{2}\, q(\xi_j, z)^{1/2} \leqq |\langle \xi, z\rangle| + \sqrt{2}\, [(k+1)r]^{1/2}.$$

Now, applying (4) and (5), we obtain

$$|g(\xi_j)(h_j)'_\xi|(z) \leqq E \frac{a}{r} k^{-N-1} g(\xi_j) |\langle \xi, \xi_j\rangle|$$

$$\leqq E \frac{a}{r} k^{-N-1} 2^{n+1} g(z) \big(|\langle \xi, z\rangle| + \sqrt{2}\, [(k+1)r]^{1/2}\big).$$

Summing over $j=1, 2, \ldots, M$ and applying the same argument as before, we get

$$(15)\qquad |h'_\xi(z)| \leqq \sum_{j=1}^{M} |g(\xi_j)(h_j)'_\xi(z)| \leqq Ear^{-1}C_2 2^d g(z)$$

$$\cdot \sum_{n=0}^{\infty} \sum_{k=2^n}^{\infty} (k+2)^d k^{-N-1} 2^{n+1}\big(|\langle \xi, z\rangle| + \sqrt{2}[(k+1)r]^{1/2}\big)$$

$$\leqq C_3 g(z)\big(|\langle \xi, z\rangle| R + \sqrt{R}\big),$$

where C_3 is some constant. This inequality shows that

$$(16)\qquad T(h) \leqq 2C_3 R \|g\|_\infty.$$

Proof of (iii): Because of (15) and (14), we have

$$\gamma_\xi(g-\mathrm{Re}\, h)(z) \leqq [\gamma_\xi(g)+|h'_\xi|](z) \leqq (1+C_3)\big(|\langle \xi, z\rangle| R+\sqrt{R}\big) g(z)$$

$$\leqq \frac{10}{9}(1+C_3)\big(|\langle \xi, z\rangle| R+\sqrt{R}\big)(g-\mathrm{Re}\, h)(z).$$

This proves that

$$(17)\qquad \gamma(g-\mathrm{Re}\, h) \leqq \left(\frac{10}{9}(1+C_3)\right)^2 R.$$

Let us assume that $\dfrac{(g-\mathrm{Re}\, h)(z_1)}{(g-\mathrm{Re}\, h)(z_2)} = s \geqq 2$. Because of (14), we have $\dfrac{g(z_1)}{g(z_2)} \geqq \dfrac{9}{11} s$. If $\dfrac{9}{11} s \geqq 2$, then (because $\beta(g) \leqq R$)

$$\left(\frac{9}{11} s\right) q(z_1, z_2)^{-1} \leqq R,$$

hence

$$\frac{(g-\mathrm{Re}\, h)(z_1)}{(g-\mathrm{Re}\, h)(z_2)} q(z_1, z_2)^{-1} \leqq \frac{11}{9} R.$$

If $\dfrac{9}{11} s \leqq 2$, then applying Lemma 3 and the inequality $s_0 \overset{\text{def}}{=} \dfrac{9}{11} s \leqq \dfrac{18}{11}$, we have $q(z_1, z_2) \geqq C_2 \left(\dfrac{s_0-1}{s_0}\right)^2 R^{-1} \geqq C_2 \dfrac{49}{324} R^{-1}$, hence

$$\frac{(g-\mathrm{Re}\, h)(z_1)}{(g-\mathrm{Re}\, h)(z_2)} q(z_1, z_2)^{-1} \leqq s\frac{324}{49C_2} R \leqq \frac{792}{49C_2} R,$$

because $s \leqq \dfrac{22}{9}$. This shows that $\beta(g-\mathrm{Re}\, h) \leqq \max\left(\dfrac{11}{9}, \dfrac{792}{49C_2}\right) R$ and together with (17) concludes the proof of (iii).

Lemma 5. *To every* $\psi>0$ *corresponds a number* $W=W(\psi)>0$ *with the following property: If* $g\in C(S)$, $g>0$, $\max(\beta(g),\gamma(g))\leqq R$, *then there is an* $h\in A(B)$, *with* $\operatorname{Re} h(0)=0$, *so that*

(i) $\|h\|_\infty\leqq W\|g\|_\infty$,

(ii) $\operatorname{Re} h<g$ on S,

(iii) $\|g-\operatorname{Re} h\|_p^p\leqq\psi\|g\|_p^p$,

(iv) $\max(\beta(g-\operatorname{Re} h),\gamma(g-\operatorname{Re} h))\leqq WR$,

(v) $T(h)\leqq WR\|g\|_\infty$.

Proof. Let us take a function g satisfying the assumptions of Lemma 5 and a number $n=\left[\frac{\log\psi}{\log(1-\tau)}\right]+1$. We shall construct two sequences of functions: $\{g_0,g_1,\dots,g_n\}$ and $\{h_1,h_2,\dots,h_n\}$. Let us put $g_0=g$. Now let us assume that for $0\leqq i<n$ we constructed a sequence $\{g_1,g_2,\dots,g_i\}$ of positive and continuous functions on S such that

(a) $\max(\beta(g_i),\gamma(g_i))\leqq C^iR$,

where C is a constant as in Lemma 4. Of course, this condition is satisfied for $i=0$. Lemma 4, applied to g_i and C^iR in place of g and R, yields a function $h_{i+1}\in A(B)$, $h_{i+1}(0)=0$, satisfying the following conditions:

(b) $|h_{i+1}(z)|\leqq\frac{1}{10}g_i(z)$ for $z\in S$,

(c) $\|g_i-\operatorname{Re} h_{i+1}\|_p^p\leqq(1-\tau)\|g_i\|_p^p$,

(d) $\max(\beta(g_i-\operatorname{Re} h_{i+1}),\gamma(g_i-\operatorname{Re} h_{i+1}))\leqq C^{i+1}R$,

(e) $T(h_{i+1})\leqq C^{i+1}R\|g_i\|_\infty$.

Let us define $g_{i+1}=g_i-\operatorname{Re} h_{i+1}$. From (b) it follows that $g_{i+1}\geqq 0$. The condition (d) is the condition (a) for $i-1$.

We shall prove that the function $h=\sum_{i=1}^n h_i$ satisfies the conditions (i)—(v) of Lemma 4 with $W=(2\tilde{C})^{n+1}$, where $\tilde{C}=\max(1,C)$. We claim that $\|g_i\|_\infty\leqq 2^i\|g_0\|_\infty$ for $i=0,1,\dots,n$. The equality holds for $i=0$. Let us assume that this is true for some $0\leqq i<n$. Then

$$\|g_{i+1}\|_\infty=\|g_i-\operatorname{Re} h_{i+1}\|_\infty\leqq\|g_i\|_\infty+\|h_{i+1}\|_\infty\leqq\|g_i\|_\infty+\frac{1}{10}\|g_i\|_\infty$$

$$\leqq\left(1+\frac{1}{10}\right)2^i\|g_0\|_\infty\leqq 2^{i+1}\|g_0\|_\infty,$$

because of (b) and our assumption. Hence,

$$\|h\|_\infty \leqq \sum_{i=1}^n \|h_i\|_\infty \leqq \frac{1}{10}\sum_{i=1}^n \|g_{i-1}\| \leqq \sum_{i=1}^n 2^{i-1}\|g_0\|_\infty \leqq 2^n \|g\|_\infty,$$

because of (b) and the definition of g_0. This proves (i), since $2^n \leqq W$.

We have $0<g_n=g_0-(\mathrm{Re}\,h_1+\mathrm{Re}\,h_2+\ldots+\mathrm{Re}\,h_n)=g-\mathrm{Re}\,h$ and (ii) follows.

We shall show that $\|g_i\|_p^p \leqq (1-\tau)^i \|g\|_p^p$. The equality holds for $i=0$. If it is true for some $0\leqq i<n$, then $\|g_{i+1}\|_p^p \leqq (1-\tau)\|g_i\|_p^p \leqq (1-\tau)^{i+1}\|g\|_\infty$, because of (c) and the definition of g_{i+1}. It follows that $\|g-\mathrm{Re}\,h\|_p^p=\|g_n\|_p^p \leqq (1-\tau)^n \|g\|_p^p \leqq \psi\|g\|_p^p$, because of our choice of n. This proves (iii).

The condition (iv) follows from (d) for $i=n-1$, since $g_{n-1}-\mathrm{Re}\,h_n=g-\mathrm{Re}\,h$.

Finally, because of (e) and the inequality $\|g_i\|_\infty \leqq 2^i \|g\|_\infty$, we have

$$T(h) \leqq \sum_{i=1}^n T(h_i) \leqq \sum_{i=1}^n C^i R \|g_{i-1}\|_\infty \leqq \sum_{i=1}^n C^i 2^{i-1} R \|g\|_\infty,$$

which proves (v).

Lemma 6. *Let us assume that* $h_i \in C(S)$, $\|h_i\|_\infty \leqq w_1 2^{-i}$, $T(h_i) \leqq w_2 W^i$, *where* w_1, w_2, W *are some constants,* $W \geqq 2$, $i=1,2,\ldots$. *Then* $h=\sum_{i=1}^\infty h_i \in \mathrm{Lip}\,\alpha$ *for* $\alpha \leqq \dfrac{1}{2}\dfrac{\log 2}{\log W}$.

Proof. Let us take any number $0<\varkappa\leqq 1$ and an integer n such that $W^{-(2n+2)} \leqq \varkappa \leqq W^{-2n}$. Define $f_1=\sum_{i=1}^n h_i$, $f_2=\sum_{i=n+1}^\infty h_i$. Then $T(f_1) \leqq \sum_{i=1}^n T(h_i) \leqq w_2 W^{n+1}$. Hence, if $\varrho(z_1, z_2)=\varkappa$, then $|f_1(z_1)-f_1(z_2)| \leqq w_2 W^{n+1}\varkappa \leqq w_2 W^{-n+1}$. For $V=2\max(w_2 W^2, 4w_1)$ and $\alpha \leqq \dfrac{1}{2}\dfrac{\log 2}{\log W}$, we have

$$\begin{aligned} |h(z_1)-h(z_2)| &\leqq |f_1(z_1)-f_1(z_2)|+|f_2(z_1)-f_2(z_2)| \\ &\leqq w_2 W^{-n+1}+\sum_{i=n+1}^\infty \big(|h_i(z_1)|+|h_i(z_2)|\big) \leqq w_2 W^{-n+1}+2w_1 2^{-n} \\ &\leqq 2\max(w_2 W^2, 4w_1) 2^{-(n+1)} \leqq V\varkappa^\alpha = V\varrho(z_1, z_2). \end{aligned}$$

This ends the proof of Lemma 6.

To prove the assertion of the Theorem, let us assume at first that $g\in \mathrm{Lip}\,1$ and $g>0$. Let $1/2 \geqq \varepsilon > 0$, $\psi=1/4\varepsilon$ and let $W=W(\psi)$ be a corresponding number from Lemma 5. It follows that $\max(\beta(g), \gamma(g)) \leqq R$ for some number $R\geqq 1$. We shall construct two sequences of functions: $\{g_i\}_{i=0}^\infty$ and $\{h_i\}_{i=1}^\infty$ such that $g_i \in C(S)$, $g_i>0$ and

$$(*) \qquad \max(\beta(g_i), \gamma(g_i)) \leqq RW^i$$

for $i=0, 1, \ldots$.

Let $g_0=g$ and let us assume that, for some $i\geqq 0$, we constructed $g_i \in C(S)$, $g_i>0$, satisfying the condition $(*)$. Lemma 5, applied to g_i and RW^i in place of g and R, yields a function $h_{i+1}\in A(B)$, $h_{i+1}(0)=0$, satisfying the following conditions:

(i)′ $\|h_{i+1}\|_\infty \leqq W\|g_i\|_\infty$,

(ii)′ $\operatorname{Re} h_{i+1} < g_i$ on S,

(iii)′ $\|g_i-\operatorname{Re} h_{i+1}\|_p^p \leqq \psi\|g_i\|$,

(iv)′ $\max(\beta(g_i-\operatorname{Re} h_{i+1}), \gamma(g_i-\operatorname{Re} h_{i+1})) \leqq RW^{i+1}$.

(v)′ $T(h_{i+1}) \leqq W^{i+1}R\|g_i\|_\infty$.

We define

(vi)′ $g_{i+1} = \min(g_i-\operatorname{Re} h_{i+1}, 2^{-i-1}\|g\|_\infty)$.

The definition (vi)′ and the condition (iv)′ show that $\gamma(g_{i+1})\leqq RW^{i+1}$. Let $z_1, z_2\in S$ be points such that $\dfrac{g_{i+1}(z_1)}{g_{i+1}(z_2)}\geqq 2$. Then

$$g_{i+1}(z_2) = (g_i-\operatorname{Re} h_{i+1})(z_2)$$

and

$$\frac{g_{i+1}(z_1)}{g_{i+1}(z_2)}\varrho(z_1, z_2)^{-1} \leqq \frac{(g_i-\operatorname{Re} h_{i+1})(z_1)}{(g_i-\operatorname{Re} h_{i+1})(z_2)}\varrho(z_1, z_2)^{-1} \leqq RW^{i+1},$$

because of (iv)′ and the inequality $\dfrac{(g_i-\operatorname{Re} h_{i+1})(z_1)}{(g_i-\operatorname{Re} h_{i+1})(z_2)}\geqq 2$. Hence, $\beta(g_{i+1})\leqq RW^{i+1}$ and this ends the proof of $(*)$ for $i+1$ instead of i.

Moreover, from (vi)′ it follows that $\|g_i\|_\infty\leqq 2^{-i}\|g\|_\infty$. Hence, because of (i)′,

(a)′ $\|h_{i+1}\|_\infty \leqq W2^{-i}\|g\|_\infty$.

Since $0<g_{i+1}\leqq g_i-\operatorname{Re} h_{i+1}$, from (iii)′ and by easy induction, it follows that

(b)′ $\|g_i\|_p^p \leqq \psi^i\|g\|_\infty$.

The condition (v)′, applied for $i-1$ instead of i, together with the inequality $\|g_{i-1}\|_\infty\leqq 2^{-i+1}\|g\|_\infty\leqq\|g\|_\infty$, gives us

(c)′ $T(h_i) \leqq W^iR\|g\|_\infty$.

Because of Lemma 6, (a)′ and (c)′, it follows that $f=\sum_{i=1}^\infty h_i\in\operatorname{Lip}\alpha$, for $\alpha\leqq \dfrac{1}{2}\dfrac{\log 2}{\log W}$. Moreover $\operatorname{Re} f\leqq g$. Let

$$A_i = \{g_i-\operatorname{Re} h_{i+1} \leqq 2^{-i-1}\|g\|_\infty\} = \{g_i-\operatorname{Re} h_{i+1} = g_{i+1}\}.$$

Then, because of (iii)′ and (b)′,

$$\mu(S-A_i) \leqq \frac{\|g_i - \operatorname{Re} h_{i+1}\|_p^p}{[2^{-i-1}\|g\|_\infty]^p} \leqq \psi^{i+1} 2^{i+1}.$$

Hence, $\mu(\bigcap_{i=1}^\infty A_i) = 1 - \mu(\bigcup_{i=1}^\infty (S-A_i)) \geqq 1 - \sum_{i=1}^\infty \mu(S-A_i) \geqq 1 - \sum_{i=1}^\infty (2\psi)^i \geqq 1-\varepsilon$.

For $z \in \bigcap_{i=1}^\infty A_i$, we have $\operatorname{Re} f(z) = g(z)$. This ends the proof of the assertion of the Theorem in the case $g > 0$. The general case follows by replacing g by $g+c$, if necessary, where c is some positive constant.

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