Adjoint boundary value problems for the biharmonic equation on C^1 domains in the plane

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This is the first of a two-part paper on adjoint problems for the biharmonic equation in a C^1 domain in the plane. The study of problems which arise as adjoints to Dirichlet problems was suggested by the paper of Fabes and Kenig on H^1 spaces of C^1 domains [7]. Their paper shows that the Neumann problem on bounded C^1 domains with h^1 boundary data (see § 1.1 of [7] for details) is solvable as a single layer potential. In addition, the gradient of the single layer potential is used to establish a connection between the h^1 data space and the H^1 space of vectors of harmonic functions in Ω satisfying a generalized Cauchy—Riemann system.

In our papers we show how analogous adjoint problems for the biharmonic equation arise from the potentials and Green's formulae used to study the Dirichlet problem. We give solutions in the form of lower order potentials, a device we introduced in § 5 of [3] to solve the Dirichlet problem.

In the first paper we use the lower order potential to solve the adjoint problems with data in the dual of the space of Dirichlet data. We further show that by considering biharmonic functions as the real parts of solutions of the equation $\bar{\partial}^2 f=0$, $(\bar{\partial}=\partial_x+i\partial_y)$, we are able to solve a fundamental problem in two dimensional elasticity with a modified form of the lower order potential.

The boundary data considered in the first paper is a space of cosets of linear functionals acting on the Dirichlet data. To obtain convergence for the potentials at the boundary we must extend the meaning of the coset space to include functions defined on a system of local parallel translates of the boundary. The trace of the potentials on the boundary is shown to have an inverse in the coset space and a solution of the adjoint problem is obtained.

In the second paper [4] we extend our results to show that for the elasticity problem the potential has non-tangential point-wise limits almost everywhere char-

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acterized by singular integrals. Furthermore, we show that as an operator defined point-wise almost everywhere, the trace of the potentials is invertible. This yields a solution to the elasticity problem with non-tangential pointwise convergence at the boundary. We point out that the invertibility of the pointwise defined trace depends crucially on its invertibility as an operator acting on the coset space.

The potentials in these papers were developed to take advantage of recent developments in the study of singular integrals, most notably the work of A. P. Calderón [2] and Coifman, McIntosh and Meyer [5] on the Cauchy integral on Lipschitz curves. The results we obtain extend previous results to domains with C^1 boundaries and data in $L^p(\partial\Omega)$ for $1<p<\infty$.

These papers are the first step in the development of an H^p theory for biharmonic functions. One application of such a theory will be the extension of results for the adjoint problems to data spaces of L^p functions where $p\leqq 1$.

The appearance of the $\bar{\partial}^2$ equation in a study of biharmonic functions is not surprising. A basic step in the application of complex methods in elasticity is writing a biharmonic function v as $v(x,y)=\operatorname{Re}\{\bar{z}f(z)+g(z)\}$ where f and g are analytic. (See Chapter 5 of Muskhelishvili [9] for details). We point out that $\bar{\partial}^2(\bar{z}f(z)+g(z))=0$ so it is clear that the connection between the $\bar{\partial}^2$ equation and biharmonic functions plays an important role in the theory of elasticity.

0. Introduction

In this paper we study adjoint boundary value problems for the biharmonic equation, $\Delta^2 u=0$, in a bounded C^1 domain in $\mathbf{R}^2$. The solutions of the problems posed here are applications of the methods and estimates developed by Cohen and Gosselin in [3] in obtaining multiple layer potential solutions to the interior Dirichlet problem for the biharmonic equation.

We show how the adjoint problem is determined by the form of the multiple layer potential and solve it in both the interior and exterior of Ω. In addition we solve the exterior Dirichlet problem.

We next show that the fundamental solution F for the biharmonic operator Δ^2 is the real part of a complex valued function satisfying $\partial^2(F+i\tilde{F})=0$. The real and imaginary parts of the solutions of this equation satisfy a system of partial differential equations which we apply to $F+i\tilde{F}$ to obtain a new form for the multiple layer potential. This in turn enables us to solve the adjoint problem:

$$\Delta^2 v = 0 \quad \text{in} \quad \Omega,$$

$$(0.1) \qquad (v_{xx}x_s+v_{xy}y_s,\ v_{xy}x_s+v_{yy}y_s) = (\varphi, \psi) \quad \text{on} \quad \partial\Omega.$$

Finally we point out that this last problem is equivalent to finding the stress function from the x and y components of normal stress on the boundary of a thin plate, a fundamental problem in the theory of two dimensional elasticity.

In the first section we outline briefly the main results and methods used in [3] to solve the interior Dirichlet problem for the biharmonic equation. We introduce Agmon's multiple layer potential $u(\dot{f}; X)$ where $\dot{f}=(f, g, h)\in\mathscr{B}_p$, a space of compatible triples defined on the boundary $\partial\Omega$. We recall that $\dot{u}=(u, u_x, u_y)$ has boundary values of the form $(\dot{I}+\dot{\mathscr{K}})\dot{f}$ where $\dot{\mathscr{K}}$ is compact from $\mathscr{B}_p$ to itself. We show that $(\dot{I}+\dot{\mathscr{K}})$ is invertible by considering the dual space $\mathscr{B}_p^*$, introducing a lower order potential, and applying the relevant Green's formula. This argument shows that the adjoint $(\dot{I}+\dot{\mathscr{K}})^*$ is one to one on the space $\mathscr{B}_p^*$.

In the second section we look at triples of functions defined in a neighborhood of $\partial\Omega$, and make precise what we mean by convergence at the boundary to elements in $\mathscr{B}_p^*$. We introduce the adjoint boundary value problem and show how the exterior version is solved by the lower order potential.

We solve the exterior Dirichlet problem with data in $\mathscr{B}_p$ and the interior adjoint problem with data in a subspace of $\mathscr{B}_p^*$ satisfying certain moment conditions.

In the third section we consider biharmonic functions as the real parts of complex valued functions satisfying $\bar{\partial}^2 f=0$. We show that the real and imaginary parts of such functions satisfy necessary conditions analogous to the Cauchy—Riemann equations. By applying these conditions to the components of the kernel of the multiple layer potential we obtain a different form of the multiple layer potential and a solution to the simpler form of the Dirichlet problem

$$\Delta^2 u = 0 \quad \text{in} \quad \Omega,$$

$$(u_x, u_y) \to (g, h)\in L^p(\partial\Omega)\times L^p(\partial\Omega). \tag{0.2}$$

We show that this form of the Dirichlet problem has the natural adjoint problem in (0.1) which we solve in an appropriate distributional sense.

In the fourth section we make some concluding remarks. We explain the connection of the adjoint problem (0.1) with a fundamental problem in two dimensional elasticity, and we give a physical interpretation to the moment conditions satisfied by the solution of the interior adjoint problem. We raise some questions about pointwise solutions and uniqueness and suggest possible extensions of these results to L^1 data and to Lipschitz domains.

1. Review of the solution to interior Dirichlet Problem

In this section we review the multiple layer potential solution of the Dirichlet problem for the biharmonic equation in a bounded, simply connected C^1 domain in $\mathbf{R}^2$. We introduce Agmon's multiple layer potential $u(\dot{f}; X)$ with density $\dot{f}$ in a space of compatible triples. We recall the estimates for $\dot{u}=(u, u_x, u_y)$ on and near the boundary $\partial\Omega$ which give the interior and exterior limits of the form $(\pm\dot{I}+\dot{\mathcal{K}})\dot{f}$ where $\dot{\mathcal{K}}$ is compact. The lower order potential is introduced and we show how it is used to show $(\dot{I}+\dot{\mathcal{K}})^*$ is one to one.

1.1. The multiple layer potential

We start with the Green's formula:

For $u, v\in C^4(\bar{\Omega})$

(1.1.1)
$$\iint_\Omega \operatorname{Re}(\partial_x+i\partial_y)^2 u(\partial_x-i\partial_y)^2 v\,dx\,dy$$
$$= \int_{\partial\Omega} uK_1v+u_xK_2v+u_yK_3v\,ds(Q)+\iint_\Omega u\Delta^2 v\,dx\,dy,$$

where

(1.1.2)
$$K_1v = \frac{\partial\Delta v}{\partial N}+2\frac{\partial v_{xy}}{\partial s},$$
$$K_2v = (v_{xx}-v_{yy})y_s,$$
$$K_3v = (v_{xx}-v_{yy})x_s+(4v_{xy})y_s$$

and $x(s)\vec{\imath}+y(s)\vec{\jmath}$ represents the boundary sketched counterclockwise and parametrized by arclength. x_s and y_s denote the arclength derivatives of x and y and $\frac{\partial v}{\partial N}$ and $\frac{\partial v}{\partial s}$ denote the inner normal and tangential derivatives of v on the boundary.

We let $\dot{u}=(u, u_x, u_y)$, $\vec{K}v=(K_1v, K_2v, K_3v)$, $\bar{\partial}=(\partial_x+i\partial_y)$ and $\partial=(\partial_x-i\partial_y)$. With this notation the Green's formula (1.1.1) has the simpler form

(1.1.3)
$$\iint_\Omega \operatorname{Re}\bar{\partial}^2 u\partial^2 v\,dx\,dy = \int_{\partial\Omega} \dot{u}\vec{K}v^T\,ds(Q)+\iint_\Omega u\Delta^2\,dx\,dy,$$

where $\vec{K}v^T$ denotes the transpose (a column vector in this case) of $\vec{K}v$.

We next set $v=F(X-Q)$ where for $X=(x, y)$

(1.1.4)
$$F(X) = \frac{-1}{4\pi}\left[(x^2+y^2)\log\sqrt{x^2+y^2}+y^2\right]$$

is the fundamental solution for the biharmonic operator used by Agmon in [1]. Substituting for v in (1.1.3) we get:

(1.1.5)

$$\iint_\Omega \operatorname{Re} \bar{\partial}^2 u \partial^2 (F(X-\cdot))\,dx\,dy = \int_{\partial\Omega} \dot{u}(Q) \vec{K} F(X-Q)^T\,ds(Q) + \begin{cases} -2u(X), & X\in\Omega \\ 0, & X\notin\bar{\Omega}. \end{cases}$$

We introduce the following space of polynomials. For $\Omega\subset\mathbf{R}^2$, $\mathscr{I}(\Omega)=\{f\colon f=\alpha x+\beta y+\gamma(x^2+y^2)+\delta$ where $\alpha,\beta,\gamma,\delta\in\mathbf{R}$, $(x,y)\in\Omega\}$. $\mathscr{I}$ is the four dimensional space spanned by 1, x, y and x^2+y^2. We let $\mathscr{I}(\partial\Omega)=\{\dot{f}\colon f\in\mathscr{I}$, and $\dot{f}$ is the restriction of (f, f_x, f_y) to $\partial\Omega\}$. A simple computation shows that $\bar{\partial}^2 f=\partial^2 f=0$ for all $f\in\mathscr{I}$. Setting $u=f\in\mathscr{I}$ in (1.1.5) we have the reproducing property

$$\int_{\partial\Omega} \dot{f}(Q) \vec{K}^Q F(X-Q)^T\,ds(Q) = \begin{cases} 2f(X), & X\in\Omega \\ 0, & X\notin\overline{\partial\Omega} \end{cases} \tag{1.1.6}$$

where $\vec{K}^Q F(X-Q)^T$, the superscript Q denotes the variable with respect to which $\vec{K}$ acts.

We next consider the space of compatible triples

$$\mathscr{B}_p = \left\{\dot{f} = (f, g, h)\colon \dot{f}\in L_1^p(\partial\Omega)\times L^p(\partial\Omega)\times L^p(\partial\Omega) \quad\text{and}\quad \frac{df}{ds} = g x_s + h y_s \ \text{ a.e.}\right\}.$$

Definition (1.1.7). *The multiple layer potential* $u(\dot{f}; X)$ *with density* $\dot{f}\in\mathscr{B}_p$ *is defined:*

$$u(\dot{f}; X) = \int_{\partial\Omega} \dot{f}(Q) \vec{K}^Q F(X-Q)^T\,ds(Q).$$

1.2. The basic estimates

For $u=u(\dot{f}; X)$, the multiple layer potential with density $\dot{f}\in\mathscr{B}_p$, we let $\dot{u}$ denote the triple (u, u_x, u_y). For $X\notin\partial\Omega$

$$\dot{u}(X) = \int_{\partial\Omega} \dot{f}(Q) k(X, Q)\,ds(Q) \tag{1.2.1}$$

where $k(X, Q)$ is the 3×3 matrix given by the matrix product

$$k(X, Q) = (\vec{K}^Q)^T (I, \partial_x^X, \partial_y^X) F(X-Q) \tag{1.2.2}$$

where the superscripts indicate the variable on which the differential operators act.

For any $P\neq Q$ the matrix $k(P, Q)$ can be defined by (1.2.2). For $P\in\partial\Omega$ we tentatively define the operator $\mathscr{K}\colon \mathscr{B}_p\to\mathscr{B}_p$ by

$$\mathscr{K}\dot{f}(P) = \text{p.v.} \int_{\partial\Omega} \dot{f}(Q) k(P, Q)\,ds(Q). \tag{1.2.3}$$

Theorem (1.2.4). *For* $\dot{f} \in \mathscr{B}_p$,

(i) $\dot{\mathscr{K}}\dot{f}(P)$ *exists a.e. and is compact and bounded from* $\mathscr{B}_p$ *to* $\mathscr{B}_p$ *in the* $L_1^p(\partial\Omega)\times L^p(\partial\Omega)\times L^p(\partial\Omega)$ *norm,*

(ii) *non-tangential* $\lim_{X\to P\in\partial\Omega} \dot{u}(\dot{f}, X) = \begin{cases} (\dot{I}+\dot{\mathscr{K}})\dot{f}(P) & a.e., \quad X\in\Omega \\ (-\dot{I}+\dot{\mathscr{K}})\dot{f}(P) & a.e., \quad X\notin\bar{\Omega}. \end{cases}$

(iii) *For* $\dot{f}\in\mathscr{I}(\partial\Omega)$, $\dot{\mathscr{K}}\dot{f}(P)=\dot{f}(P)$.

Proof. The proof of theorem (1.2.4) is the contents of chapters three and four of Cohen and Gosselin [3]. It consists of applying Calderón's theorem on the Cauchy integral along Lipschitz curves [2] to the multiple layer potential and its derivatives. The boundedness arguments are straightforward applications of Calderón's theorem but the compactness arguments require considerable care. They involve utilizing all the representations of the multiple layer potential employed by Agmon in his paper on multiple layer potentials [1].

1.3. The adjoint and the lower order potential

The existence of a solution to the Dirichlet problem, $\Delta^2 u=0$ in Ω, $\dot{u}=\dot{f}\in\mathscr{B}_p$ on $\partial\Omega$, is established if we can invert the boundary operator $(\dot{I}+\dot{\mathscr{K}})$. Because the boundary is C^1, it is necessary to consider the adjoint operator $(\dot{I}+\dot{\mathscr{K}})^*$. To do this we must introduce a lower order potential with density in $\mathscr{B}_p^*$ the dual of $\mathscr{B}_p$.

Briefly, the dual space of $\mathscr{B}_p$ is the triple of cosets of L^q functions $\mathscr{B}_p^*=\{\dot{\theta}=(\theta,\varphi,\psi)\colon \dot{\theta}\in L^q(\partial\Omega)\times L^q(\partial\Omega)\times L^q(\partial\Omega)/\mathscr{B}_p^\perp\}$ where

$$\mathscr{B}_p^\perp = \left\{\dot{\theta}\colon \int_{\partial\Omega} \dot{f}(Q)\dot{\theta}(Q)^T\,ds(Q) = 0 \ \text{ for all } \dot{f}\in\mathscr{B}_p\right\}.$$

For $\dot{f}\in\mathscr{B}_p$ and $\dot{\theta}\in\mathscr{B}_p^*$ we define the dual pairing by $\langle \dot{f},\dot{\theta}\rangle=\int_{\partial\Omega}\dot{f}(Q)\dot{\theta}(Q)^T\,ds(Q)$.

To find $\dot{\mathscr{K}}^*\dot{\theta}$ we proceed formally. For $\dot{f}\in\mathscr{B}_p$, $\dot{\theta}\in\mathscr{B}_p^*$,

$$\text{(1.3.1)} \qquad \langle \dot{f}, \dot{\mathscr{K}}^*\dot{\theta}\rangle = \langle \dot{\mathscr{K}}\dot{f},\dot{\theta}\rangle = \int_{\partial\Omega}\int_{\partial\Omega} \dot{f}(Q)\dot{k}(P,Q)\,ds(Q)\,\dot{\theta}(P)^T\,ds(P)$$

$$= \int_{\partial\Omega} \dot{f}(Q)\int_{\partial\Omega} \dot{k}(P,Q)\dot{\theta}(P)^T\,ds(P)\,ds(Q)$$

$$= \int_{\partial\Omega} \dot{f}(Q)\left\{\vec{K}^Q\int_{\partial\Omega} F(P-Q)\theta(P)+\partial_x^P F(P-Q)\varphi(P)+\partial_y^P F(P-Q)\psi(P)\,ds(P)\right\}^T ds(Q).$$

This calculation is formal since the integral $\int_{\partial\Omega}\dot{k}(P,Q)\dot{\theta}(P)^T\,ds(P)$ need not exist even in the principal value sense. Nonetheless, the calculation suggests introducing the following potential:

Definition (1.3.2). *The lower order potential* $v=v(\dot\theta; X)$ *with density* $\dot\theta\in\mathscr{B}_p^*$ *is defined for* $X\neq P$ *by*

$$v(\dot\theta; X) = \int_{\partial\Omega} F(P-X)\theta(P)+\partial_x^P F(P-X)\varphi(P)+\partial_y^P F(P-X)\psi(P)\,ds(P)$$

where F *is the fundamental solution given in* (1.1.4).

Theorem (1.3.3). *The lower order potential is well defined on the quotient space* $\mathscr{B}_p^*$ *and is globally* C^1. *Furthermore, if* $\dot\theta\in\ker(\dot I+\dot{\mathscr{K}})^*$, $\langle\dot f, \dot\theta\rangle=0$ *for* $\dot f\in\mathscr{I}(\partial\Omega)$ *and* $v(\dot\theta; X)$ *satisfies the estimates:*

(1.3.4) $$v(X) = O(\log|X|) \quad as \quad |X|\to\infty,$$

$$\frac{\partial^{\alpha+\beta}}{\partial x^\alpha \partial y^\beta} v(X) = O(|X|^{-\alpha-\beta}) \quad as \quad |X|\to\infty \quad for \quad 1\leqq\alpha+\beta\leqq 3.$$

The proof of theorem (1.3.3) is in theorem (5.1.3) of [3].

1.4. Weak identities and the invertibility of $(\dot I+\dot{\mathscr{K}})^*$

In this section we apply the Green's formula (1.1.3) to the form $\mathrm{Re}\,\bar\partial^2 v\partial^2 v$. Since the lower order potential may have non-integrable singularities we must introduce a partition of unity to obtain the relevant weak identities.

There exist balls $B_j=B(P_j; \delta_j)$, $j=1, \dots, N$, $P_j\in\partial\Omega$, such that $\partial\Omega\subset\bigcup_{j=1}^N B_j$ and $\mathrm{dist}(\partial\Omega, [\bigcup_{j=1}^N B_j]^c)=\delta_0>0$. Since $\partial\Omega$ is C^1 there are local coordinates $\alpha_j\in C_0^1(\mathbf{R})$, $j=1, \dots, N$, centered at P_j, satisfying

(i) $\alpha_j(0) = \alpha_j'(0)$,

(ii) $B(P_j, 4\delta_j)\cap\Omega = \{(z, w)\colon w > \alpha_j(z)\}\cap\Omega$,

(iii) $B(P_j, 4\delta_j)\cap\bar\Omega^c = \{(z, w)\colon w < \alpha_j(z)\}\cap\bar\Omega^c$.

We let $\mathscr{O}_0$ be an open set such that $\mathscr{O}_0\subset\Omega$, $(\bigcup_{j=1}^N B_j)\cup\mathscr{O}_0\supset\Omega$, $\mathscr{O}_0\supset\left\{X\in\Omega\colon \mathrm{dist}(\partial\Omega, X)>\frac{\delta_0}{2}\right\}$ and $\mathscr{O}_0\cap\left\{X\in\Omega\colon \mathrm{dist}(\partial\Omega, X)\leqq\frac{\delta_0}{4}\right\}=\emptyset$.

Let B_R be a ball centered at the origin such that $B_{R/2}\supset\Omega$. Let $\Omega_R=B_R\setminus\bar\Omega$. Let $\mathscr{O}_R$ be an open set such that $\mathscr{O}_R\cup(\bigcup_{j=1}^N B_j)\supset\Omega_R$, $\mathscr{O}_R\subset\bar\Omega^c$, $\mathscr{O}_R\supset\left\{X\in\Omega_R\colon \mathrm{dist}(\partial\Omega, X)>\frac{\delta_0}{2}\right\}$ and $\mathscr{O}_R\cap\left\{X\in\Omega_R\colon \mathrm{dist}(\partial\Omega, X)\leqq\frac{\delta_0}{4}\right\}=\emptyset$.

We next define the following subsets and subarcs. For $0<t<\delta_0/8$,

$$D_{j,t} = \{(z, w)\colon w > \alpha_j(z)+t\} \cap B(P_j, 4\delta_j),$$
$$\gamma_{j,t} = \{(z, \alpha_j(z)+t)\} \cap B(P_j;\ 4\delta_j)$$

for $j=1, 2, \dots, N$ and $D_{n+1,t}=\mathcal{O}_0$.

Then let

$$E_{j,t} = \{(z, w)\colon z < \alpha_j(x)-t\} \cap B(P_j;\ 4\delta_j),$$
$$\varepsilon_{j,t} = \{(z, \alpha_j(z)-t)\} \cap B(P_j; 4\delta_j)$$

for $j=1, \dots, N$ and

$$E_{N+1,t} = \mathcal{O}_R.$$

Finally we assume that $\{\eta_j\}_{j=1}^{N+1}$ is a smooth partition of unity subordinate to the cover $B_1, \dots, B_N$, $\mathcal{O}_0$ and $\{\zeta_j\}_{j=1}^{N+1}$ is a smooth partition of unity subordinate to the cover $B_1, \dots, B_N$, $\mathcal{O}_R$. We then have the following theorem:

Theorem (1.4.1). *Let* $v(\dot\theta; X)$ *denote the lower order potential with density* $\dot\theta \in \mathscr{B}_p^*$. *Let* $\vec N_j$ *denote the unit inner normal at* P_j. *Then,*

(i) *If* $f \in C^2(\Omega) \cap C^1(\bar\Omega)$, $\dot f$ *denotes* (f, f_x, f_y) *at* $\partial\Omega$ *and* $t>0$, *then*

$$\lim_{t\to 0} \sum_{j=1}^{N+1} \int_{D_{j,t}} \operatorname{Re} \eta_j \bar\partial^2 f \partial^2 v\, dx\, dy$$
$$= \lim_{t\to 0} \sum_{j=1}^{N} \int_{\partial\Omega} \int_{\partial\Omega} \eta_j(Q) \dot f(Q) \not{k}(P - t\vec N_j, Q)\, ds(Q) \dot\theta(P)^T\, ds(P)$$
$$= \langle(-\dot I + \dot{\mathscr{K}})\dot f, \dot\theta\rangle = \langle \dot f, (-\dot I + \dot{\mathscr{K}})^* \dot\theta\rangle.$$

(ii) *If* $\langle \dot f, \dot\theta\rangle = 0$ *for every* $\dot f \in \mathscr{I}(\partial\Omega)$ *and* $v = v(\dot\theta; X)$ *is the lower order potential with density* $\dot\theta \in \mathscr{B}_p^*$ *and* $f \in C^2(\bar\Omega^c) \cap C^1(\Omega^c)$ *with the growth properties* $f = O(\log |X|)$ *as* $|X| \to \infty$, $f_x = O(|X|^{-1})$ *as* $|X| \to \infty$ *and* $f_y = O(|X|^{-1})$ *as* $|X| \to \infty$, *then*

$$\lim_{R\to\infty} \lim_{t\to 0} \sum_{j=1}^{N+1} \iint_{E_{j,t}} \operatorname{Re} \zeta_j \bar\partial^2 f \partial^2 v\, dx\, dy$$
$$= \lim_{t\to 0} -\sum_{j=1}^{N} \int_{\partial\Omega} \int_{\partial\Omega} \zeta_j(Q) f(Q) \not{k}(P + t\vec N_j, Q)\, ds(Q) \dot\theta(P)^T\, ds(P)$$
$$= -\langle(\dot I + \dot{\mathscr{K}})\dot f, \dot\theta\rangle = -\langle \dot f, (\dot I + \dot{\mathscr{K}})^* \dot\theta\rangle.$$

Proof. (i) is (5.2.25) of [3]. Part (ii) is essentially proved in lemma (5.2.2) of [3], the one difference being that in [3] it was just done for $f = v(\dot\theta; X)$. A close examination of the argument shows that the theorem still holds with the growth conditions on f given in this theorem.

We next have the following theorem which is a corollary of theorem (1.4.1) and which shows that $\operatorname{Ker}(\dot I + \dot{\mathscr{K}})^* \subset \mathscr{B}_p^\perp$.

Theorem (1.4.2). *If* $\dot{\theta}\in\operatorname{Ker}(\dot{I}+\dot{\mathscr{K}})^*$ *and* $v=v(\dot{\theta};X)$ *is the lower order potential with density* $\dot{\theta}\in\mathscr{B}_p^*$, *then*

(i) $\iint_{\bar{\Omega}^c}\operatorname{Re}\bar{\partial}^2 v\partial^2 v\,dx\,dy=-\langle\dot{f},(\dot{I}+\dot{\mathscr{K}})^*\dot{\theta}\rangle=0,$

(ii) $v\in\mathscr{I}(\bar{\Omega}^c)$, $\dot{v}\in\mathscr{I}(\partial\Omega)$ *and* $\dot{\mathscr{K}}\dot{v}=\dot{v}$,

(iii) $\iint_\Omega\operatorname{Re}\bar{\partial}^2 v\partial^2 v\,dx\,dy=\langle(-\dot{I}+\dot{\mathscr{K}})\dot{v},\dot{\theta}\rangle=0,$

(iv) $v\in\mathscr{I}$ *in* Ω *and* $\partial^2 v=0$ *in* Ω.

(v) *For* $f\in C^2(\Omega)\cap C^1(\bar{\Omega})$, $t>0$

$$\langle\dot{f},(-\dot{I}+\dot{\mathscr{K}})^*\dot{\theta}\rangle=\lim_{t\to 0}\sum_{j=1}^{N+1}\iint_{D_{j,t}}\operatorname{Re}\zeta_j\bar{\partial}^2 f\partial^2 v\,dx\,dy=0.$$

Finally we have the existence of a solution to the interior Dirichlet problem:

Theorem (1.4.3). *For* $\dot{f}\in\mathscr{B}_p$, *there exists* $\dot{\mathring{f}}\in\mathscr{B}_p$ *such that* $u=u(\dot{\mathring{f}};X)$ *satisfies*

$$\Delta^2 u=0 \quad in \quad \Omega,$$

$$\text{non-tangential}\lim_{\substack{X\to P\in\partial\Omega\\ X\in\Omega}}\dot{u}(X)=\dot{f}(P)\quad a.e.$$

Proof. Since $\dot{\theta}\in\operatorname{Ker}(\dot{I}+\dot{\mathscr{K}})^*\Rightarrow\langle\dot{f},(\dot{I}+\dot{\mathscr{K}})^*\dot{\theta}\rangle=0$ and $\langle\dot{f},(-\dot{I}+\dot{\mathscr{K}})^*\dot{\theta}\rangle=0$, we have $\langle\dot{f},\dot{\theta}\rangle=0$ for all $\dot{f}\in\mathscr{B}_p$. Hence $\dot{\theta}\in\mathscr{B}_p^\perp$. This shows that $(\dot{I}+\dot{\mathscr{K}})^*$ is one to one. By the Fredholm theory we know that $(\dot{I}+\dot{\mathscr{K}})$ is one to one and onto. Hence for $\dot{f}\in\mathscr{B}_p$, there exists $\dot{\mathring{f}}\in\mathscr{B}_p$ such that $(\dot{I}+\dot{\mathscr{K}})\dot{\mathring{f}}=\dot{f}$. Setting $u=u(\dot{\mathring{f}};X)$ we meet the conditions of the theorem.

2. The adjoint problem

The calculations in (1.3.1), though purely formal, suggest that in some sense the operator $\dot{\mathscr{K}}^*\dot{\theta}=\vec{K}v(P)$ where v is the lower order potential with density $\dot{\theta}\in\mathscr{B}_p^*$ and $\vec{K}$ is the triple of differential operators defined in (1.1.2). The adjoint boundary value problem is then to find a function v which is biharmonic in Ω and which satisfies $\vec{K}v=\dot{\theta}\in\mathscr{B}_p^*$ on the boundary of Ω.

In this section we make precise the notion of convergence at the boundary in the space $\mathscr{B}_p^*$. Then we solve the exterior and interior adjoint problems, obtaining a solution to the exterior Dirichlet problem as well.

2.1. Convergence in $\mathscr{B}_p^*$ and the exterior adjoint problem

We let $B_1, \ldots, B_N$ and $\{\zeta_j\}_{j=1}^N$, $\{\eta_j\}_{j=1}^N$ be as in §1. Let $\Omega_0 = \{X\colon \operatorname{dist}(X, \partial\Omega) < \delta_0/8\}$. For $\dot\theta = (\theta, \varphi, \psi)$, a triple of functions defined on Ω_0, $\vec N_j$ the unit inner normal at P_j, and $0 < t < \delta_0/8$, we define

(2.1.1) $$\dot\theta_t^-(P) = \sum_{j=1}^N \eta_j(P)\dot\theta(P + t\vec N_j)$$

and

(2.1.2) $$\dot\theta_t^+(P) = \sum_{j=1}^N \zeta_j(P)\dot\theta(P - t\vec N_j).$$

We say $\dot\theta_t^- \in L^q(\partial\Omega)\times L^q(\partial\Omega)\times L^q(\partial\Omega)$ if $\int_{\partial\Omega}\eta_j(P)\dot f(P)\dot\theta(P+t\vec N_j)^T\,ds(P) < \infty$ for $j = 1, \ldots, N$ and all $\dot f \in \mathscr{B}_p$ and similarly we say $\dot\theta_t^+ \in L^q(\partial\Omega)\times L^q(\partial\Omega)\times L^q(\partial\Omega)$ if $\int_{\partial\Omega}\zeta_j(P)\dot f(P)\dot\theta(P-t\vec N_j)^T\,ds(P) < \infty$ for $j = 1, \ldots, N$ and all $\dot f \in \mathscr{B}_p$. We define the dual pairings

$$\langle \dot f, \dot\theta_t^-\rangle = \sum_{j=1}^N \int_{\partial\Omega}\eta_j(P)\dot f(P)\dot\theta(P+t\vec N_j)^T\,ds(P)$$

and

$$\langle \dot f, \dot\theta_t^+\rangle = \sum_{j=1}^N \int_{\partial\Omega}\zeta_j(P)\dot f(P)\dot\theta(P-t\vec N_j)^T\,ds(P).$$

If we identify elements $(\dot\theta_1)_t^+$ and $(\dot\theta_2)_t^+$ for which $\langle \dot f, (\dot\theta_1)_t^+\rangle = \langle \dot f, (\dot\theta_2)_t^+\rangle$ for all $\dot f \in \mathscr{B}_p$, then we consider them as elements in $\mathscr{B}_p^*$. For $\dot\alpha \in \mathscr{B}_p^*$ we define

(2.1.3) $$\dot\alpha = \dot\theta^+ = \lim_{t\to 0}\theta_t^+ \quad \text{if} \quad \lim_{t\to 0}\langle \dot f, \dot\theta_t^+\rangle = \langle \dot f, \dot\alpha\rangle$$

for all $\dot f \in \mathscr{B}_p$. We also define

(2.1.4) $$\dot\alpha = \dot\theta^- = \lim_{t\to 0}\dot\theta_t^- \quad \text{if} \quad \lim_{t\to 0}\langle \dot f, \dot\theta_t^-\rangle = \langle \dot f, \dot\alpha\rangle$$

for all $\dot f \in \mathscr{B}_p$.

We know from theorem (1.4.3) that $(\dot I + \dot{\mathscr{K}})^*$ is one to one and onto on $\mathscr{B}_p^*$. Hence, there exists a triple $\dot{\ddot\theta} \in \mathscr{B}_p^*$ such that $(\dot I + \dot{\mathscr{K}})^*\dot{\ddot\theta} = \dot\theta$. Let $v = v(\dot{\ddot\theta}, X)$ be the lower order potential with density $\dot{\ddot\theta}$. If $\vec K$ is the set of differential operators defined by (1.1.2), then define

$$(\vec K v)_t^+ = \sum_{j=1}^N \zeta_j(P)\vec K v(P - t\vec N_j).$$

Note that $\vec K$ is an operator on the boundary and in defining $(\vec K v)_t^+$ we are extending $\vec K v$ to the point $P - t\vec N_j$ as follows: for $\vec T_P = x_s(P)\vec\imath + y_s(P)\vec\jmath$, the unit tangent vector at $P \in \partial\Omega$, and $\vec N_P$ the unit inner normal at P, we define $\vec K v(P - t\vec N_j) = (K_1, K_2, K_3)(P - t\vec N_j)$ where $K_1 v(P - t\vec N_j) = \langle \vec\nabla \Delta v(P - tN_j), \vec N_P\rangle + 2\langle \vec\nabla v_{xy}(P - t\vec N_j), \vec T_P\rangle$, $K_2 v(P - t\vec N_j) = \big(v_{xx}(P - t\vec N_j) - v_{yy}(P - t\vec N_j)\big)\, y_s(P)$, and $K_3 v(P - t\vec N_j) = \big(v_{xx}(P - t\vec N_j) - v_{yy}(P - t\vec N_j)\big)\, x_s(P) + 4v_{xy}(P - t\vec N_j) y_s(P)$.

From the proof of theorem (1.4.1), (see lemma 5.2.2 of [3] for details), we know that for any $\dot{f}\in\mathscr{B}_p$,

$$(2.1.5)\qquad \lim_{t\to 0}\langle \dot{f}, (\vec{K}v)_t^+\rangle = \lim_{t\to 0}\sum_{j=1}^N\int_{\partial\Omega}\eta_j(P)f(P)\vec{K}v(P-t\vec{N}_j)^T\,ds(P)$$

$$= \langle(\dot{I}+\dot{\mathscr{K}})\dot{f},\dot{\dot{\theta}}\rangle = \langle \dot{f}, (\dot{I}+\dot{\mathscr{K}})^*\dot{\dot{\theta}}\rangle = \langle \dot{f}, \dot{\theta}\rangle.$$

We thus have the following existence theorem for the adjoint problem.

Theorem (2.1.6). *For $\dot{\theta}\in\mathscr{B}_p^*$, there exists a function v defined in $\bar{\Omega}^c$ such that*

$$\Delta^2 v = 0 \quad in \quad \bar{\Omega}^c,$$

$$\lim_{t\to 0}\langle \dot{f}, (\vec{K}v)_t^+\rangle = \langle \dot{f}, \dot{\theta}\rangle,$$

for all $\dot{f}\in\mathscr{B}_p$ and in fact the function v is a lower order potential with density in $\mathscr{B}_p^*$.

2.2. The interior adjoint and exterior Dirichlet problems

In this section we consider the exterior Dirichlet and interior adjoint problems for the biharmonic equation. Throughout this section we assume Ω is a bounded simply connected C^1 domain in $\mathbf{R}^2$. We start by noting that for $\dot{f}\in\mathscr{I}(\partial\Omega)$, $(\dot{I}-\dot{\mathscr{K}})\dot{f}=0$. This tells us that for some elements in $\mathscr{B}_p$ the exterior Dirichlet problem cannot be solved with a multiple layer potential. We begin this section by defining

$$(2.2.1)\qquad V_+ = \{\dot{f}\in\mathscr{B}_p\colon (\dot{I}-\dot{\mathscr{K}})\dot{f}=0\},$$

$$W_+ = \{\dot{\theta}\in\mathscr{B}_p^*\colon (\dot{I}-\dot{\mathscr{K}})^*\dot{\theta}=0\}.$$

The main element in solving the boundary value problems is the following:

Theorem (2.2.2). *For $V_+^\perp=\{\dot{\theta}\in\mathscr{B}_p^*\colon \langle \dot{f},\dot{\theta}\rangle=0$ for all $\dot{f}\in V_+\}$ and $W_+^\perp=\{\dot{f}\in\mathscr{B}_p\colon \langle \dot{f},\dot{\theta}\rangle=0$ for all $\dot{\theta}\in W_+\}$*

(i) $\mathscr{B}_p^*=V_+^\perp\oplus W_+=\text{Range}\,(\dot{I}-\dot{\mathscr{K}})^*\oplus W_+$,

(ii) $\mathscr{B}_p=\text{Range}\,(\dot{I}-\dot{\mathscr{K}})\oplus V_+$.

Proof. The proof follows from two lemmas.

Lemma (2.2.3). *If $\dot{\theta}\in W_+$ and $v=v(\dot{\theta};X)$ is the lower order potential with density $\dot{\theta}$, then $v\in\mathscr{I}(\Omega)$ and $\dot{v}\in\mathscr{I}(\partial\Omega)$.*

Proof. Using theorem (1.4.2) part (v) and the monotone convergence theorem,

$$\iint_\Omega \operatorname{Re}\bar{\partial}^2 v\partial^2 v\,dx\,dy = \langle \dot{v}, (-\dot{I}+\dot{\mathscr{K}}^*)\dot{\theta}\rangle = 0,$$

since $\dot{\theta}\in W_+$. This implies $v_{xx}-v_{yy}=v_{xy}=0$ since $\operatorname{Re}\bar{\partial}^2 v\partial^2 v=(v_{xx}-v_{yy})^2+4v_{xy}^2$. By lemma 13.1 of Agmon [1], $v\in\mathcal{I}(\Omega)$ and by the continuity of $\dot{v}$, we have $\dot{v}\in\mathcal{I}(\partial\Omega)$.

Lemma (2.2.4). *Let* $\mathcal{I}(\partial\Omega)^\perp=\{\dot{\theta}\in\mathcal{B}_p^*\colon\ \langle\dot{f},\dot{\theta}\rangle=0$ *for all* $\dot{f}\in\mathcal{I}(\partial\Omega)\}$. *Then* $\mathcal{I}(\partial\Omega)^\perp\cap W_+=\{0\}$.

Proof. Assume $\dot{\theta}\in\mathcal{I}(\partial\Omega)^\perp\cap W_+$. We know from lemma (2.2.3) that for $\dot{\theta}\in W_+$, $v(\dot{\theta};X)\in\mathcal{I}(\Omega)$ and since v is globally C^1 we have $\dot{v}\in\mathcal{I}(\partial\Omega)$. Since $\dot{\theta}\in\mathcal{I}(\partial\Omega)^\perp$ we know from (5.1.6) and (5.1.7) of [3] that $v(\dot{\theta};X)$ satisfies the growth condition of theorem (1.4.1) part (ii). Then because $\dot{v}\in\mathcal{I}(\partial\Omega)$ and $\dot{\theta}\in\mathcal{I}(\partial\Omega)^\perp$, we can apply (1.4.1) to get

$$(2.2.5)\qquad \iint_{\bar{\Omega}^c}\operatorname{Re}\bar{\partial}^2 v\partial^2 v\,dx\,dy=-\langle(\dot{I}+\dot{\mathcal{K}})\dot{v},\dot{\theta}\rangle=-2\langle\dot{v},\dot{\theta}\rangle=0.$$

From Agmon's lemma (lemma 13.1 of [1]) we know that $v\in\mathcal{I}(\bar{\Omega}^c)$.

If f satisfies the hypothesis of theorem (1.4.1) part (ii) and $\dot{f}=(f,f_x,f_y)$ restricted to $\partial\Omega$ then we have

$$(2.2.6)\qquad \begin{aligned}\langle\dot{f},\dot{\theta}\rangle&=-\tfrac{1}{2}\{\langle\dot{f},(-\dot{I}+\dot{\mathcal{K}})^*\dot{\theta}\rangle-\langle\dot{f},(\dot{I}+\dot{\mathcal{K}})^*\dot{\theta}\rangle\}\\&=-\tfrac{1}{2}\Big\{\lim_{t\to0}\sum_{j=1}^{N+1}\iint_{D_{j,t}}\operatorname{Re}\eta_j\bar{\partial}^2 f\partial^2 v\,dx\,dy\\&\quad-\lim_{R\to\infty}\lim_{t\to0}\sum_{j=1}^{N+1}\iint_{E_{j,t}}\operatorname{Re}\zeta_j\bar{\partial}^2 f\partial^2 v\,dx\,dy\Big\}=-\tfrac{1}{2}\{0-0\},\end{aligned}$$

since $v\in\mathcal{I}(\Omega\cup\bar{\Omega}^c)\Rightarrow\partial^2 v\equiv 0$.

The restriction of C^2 functions and their gradients to the boundary $\partial\Omega$ is dense in $\mathcal{B}_p$ so we conclude that $\langle\dot{f},\dot{\theta}\rangle=0$ for all $\dot{f}\in\mathcal{B}_p$. In other words $\mathcal{I}(\partial\Omega)^\perp\cap W_+=\{0\}$.

Proof of theorem (2.2.2) part (i). $\mathcal{I}(\partial\Omega)\subset V_+$, by theorem (1.2.4), and $\dim\mathcal{I}(\partial\Omega)=4$. By the Fredholm theory V_+ is finite dimensional so $\dim W_+=\dim V_+=m\geqq 4$. Furthermore $\mathcal{I}(\partial\Omega)^\perp\supset V_+^\perp$ and by lemma (2.2.4) $\mathcal{I}(\partial\Omega)^\perp\cap W_+=\{0\}$ so $V_+^\perp\cap W_+=\{0\}$. Since $\dot{I}-\dot{\mathcal{K}}$ has Fredholm index zero, $V_+^\perp$ is a subspace of $\mathcal{B}_p^*$ of codimension m. Thus we have $\mathcal{B}_p^*=V_+^\perp\oplus W_+$. Finally, $V_+^\perp=\operatorname{Range}(\dot{I}-\dot{\mathcal{K}})^*$ so we also have $\mathcal{B}_p^*=\operatorname{Range}(\dot{I}-\dot{\mathcal{K}})^*\oplus W_+$.

To show part (ii) we need only show that $\operatorname{Range}(\dot{I}-\dot{\mathcal{K}})\cap V_+=\{0\}$. Let $\dot{f}_0\in\operatorname{Range}(\dot{I}-\dot{\mathcal{K}})\cap V_+$. Then for $\dot{\theta}\in\mathcal{B}_p^*$, we know from part (i) that $\dot{\theta}=\dot{\theta}_1+\dot{\theta}_2$ where $\dot{\theta}_1\in V_+^\perp$ and $\dot{\theta}_2\in W_+$. It follows that $\langle\dot{f}_0,\dot{\theta}\rangle=\langle\dot{f}_0,\dot{\theta}_1+\dot{\theta}_2\rangle=\langle\dot{f}_0,\dot{\theta}_1\rangle+\langle\dot{f}_0,\dot{\theta}_2\rangle=0$ since $\langle\dot{f}_0,\dot{\theta}_1\rangle=0$ as $\dot{f}_0\in V_+$ and $\dot{\theta}_1\in V_+^\perp$, and $\langle\dot{f}_0,\dot{\theta}_2\rangle$ since $\dot{f}_0\in\operatorname{Range}(\dot{I}-\dot{\mathcal{K}})=W_+^\perp$ and $\dot{\theta}_2\in W_+$. Thus $\dot{f}_0=0$ so $\mathcal{B}_p=\operatorname{Range}(\dot{I}-\dot{\mathcal{K}})\oplus V_+$.

Corollary (2.2.7). $\mathcal{I}(\partial\Omega)=V_+$ *and* $\dim V_+=\dim W_+=4$.

Proof. $\mathscr{B}_p^* = V_+^\perp \oplus W_+ = \mathscr{I}(\partial\Omega)^\perp \oplus W_+$ implies that $V_+^\perp = \mathscr{I}(\partial\Omega)^\perp$. Hence $V_+ = \mathscr{I}(\partial\Omega)$ since V_+ and $\mathscr{I}(\partial\Omega)$ are subspaces. Finally $\dim V_+ = \dim \mathscr{I}(\partial\Omega) = 4$.

Theorem (2.2.8). *If $\dot{f} \in \mathscr{B}_p$, there exists a function w such that*

$$\Delta^2 w = 0 \quad in \quad \bar{\Omega}^c,$$

$$\text{non-tangential} \lim_{\substack{X \to P \in \partial\Omega \\ X \in \Omega}} \dot{w}(X) = \dot{f}(P) \quad a.e.$$

where Ω is a bounded, simply connected C^1 domain in $\mathbf{R}^2$.

Proof. Since $\mathscr{B}_p = \operatorname{Range}(\dot{I} - \dot{\mathscr{K}}) \oplus V_+$, for any $\dot{f} \in \mathscr{B}_p$ we have $\dot{f} = \dot{f}_0 + \dot{f}_1$ where $\dot{f}_0 = (-\dot{I} + \dot{\mathscr{K}})\mathring{f}_0$ and $\dot{f}_1 \in V_+ = \mathscr{I}(\partial\Omega)$.

Since $\dot{f}_1 \in \mathscr{I}(\partial\Omega)$, it is the restriction to $\partial\Omega$ of a triple $\dot{g} = (g, g_x, g_y)$ where $g \in \mathscr{I}$. Set $w = u(\mathring{f}_0; X) + g(X)$ where $u(\mathring{f}_0; X)$ is the multiple layer potential with density $\mathring{f}_0$. By theorem (1.2.4) and the definition of g, $\Delta^2 w = 0$ in $\bar{\Omega}^c$ and

(2.2.9) $$\text{non-tangential} \lim_{\substack{X \to P \in \partial\Omega \\ X \in \Omega}} \dot{w}(X) = (-\dot{I} + \dot{\mathscr{K}})\mathring{f}_0(P) + \dot{g}(P)$$

$$= \dot{f}_0(P) + \dot{f}_1(P) = \dot{f}(P).$$

Remark (2.2.10). If $\dot{\theta} \in W_+$, we know from lemma (2.2.3) that the lower order potential $v(\dot{\theta}; X)$ has boundary values $\dot{v} \in \mathscr{I}(\partial\Omega)$. For $X \in \Omega$ we know $v(X) \in \mathscr{I}(\Omega)$, but for $X \in \bar{\Omega}^c$, $v(X)$ need not be a polynomial. In fact for $\dot{\theta} \in W_+$ and $\dot{\theta} \neq 0$, we know from lemma (2.2.3) that $\dot{\theta} \notin V_+^\perp$. This suggests that $|v(\dot{\theta}; X)| \sim (|X|^2 \log |X|)$ as $|X| \to \infty$ since the moment condition of theorem (1.3.3) is not satisfied. This means that $v(\dot{\theta}; X)$ is not a polynomial in $\bar{\Omega}^c$ but agrees with the polynomial $\dot{v}$ on the boundary $\partial\Omega$. In other words, the exterior Dirichlet problem with data in $\mathscr{I}(\partial\Omega)$ can have two distinct solutions.

Theorem (2.2.11). *For $\dot{\theta} \in V_+^\perp$, there exists a function v such that:*

$$\Delta^2 v = 0 \quad in \quad \Omega$$

$$\lim_{t \to 0} \langle \dot{f}, (\vec{K}v)_t^- \rangle = \langle \dot{f}, \dot{\theta} \rangle$$

for all $\dot{f} \in \mathscr{B}_p$.

Proof. From theorem (2.2.2) part (i), $\mathscr{B}_p^* = V_+^\perp \oplus W_+$. Since $\operatorname{Range}(\dot{I} - \dot{\mathscr{K}})^* = V_+^\perp$ we know that for $\dot{\theta} \in V_+^\perp$ there is a $\mathring{\theta} \in \mathscr{B}_p^*$ such that $(-\dot{I} + \dot{\mathscr{K}})^* \mathring{\theta} = \dot{\theta}$.

Let $v = v(\mathring{\theta}; X)$ be the lower order potential with density $\mathring{\theta}$. Then v is biharmonic in Ω and for $\dot{f} \in \mathscr{B}_p$

(2.2.12) $$\lim_{t \to 0} \langle \dot{f}, (\vec{K}v)_t^- \rangle = \langle (-\dot{I} + \dot{\mathscr{K}})\dot{f}, \mathring{\theta} \rangle = \langle \dot{f}, (-\dot{I} + \dot{\mathscr{K}})^* \mathring{\theta} \rangle = \langle \dot{f}, \dot{\theta} \rangle.$$

Remark (2.2.13). The restriction that $\dot{\theta}\in V_{+}^{\perp}$ is to be expected. From the Green's formula (1.1.3) we know that for $f\in\mathcal{I}(\Omega)$ and $w\in C^4(\bar{\Omega})$ biharmonic, $\bar{\partial}^2 f=0$ so that

$$0=\iint_{\Omega}\operatorname{Re}\bar{\partial}^2 f\partial^2 w\,dx\,dy=\int_{\partial\Omega}\dot{f}(Q)\vec{K}w(Q)^T\,ds(Q). \tag{2.2.14}$$

In other words, if w is biharmonic and smooth in $\bar{\Omega}$, $\vec{K}w\in V_{+}^{\perp}$.

3. The biharmonic Cauchy—Riemann equations

In this section we consider biharmonic functions as the real parts of complex valued functions f satisfying $\bar{\partial}^2 f=0$. The real and imaginary parts of solutions of $\bar{\partial}^2(U+iV)=0$ satisfy a system of partial differential equations which we shall refer to as the biharmonic Cauchy—Riemann equations.

The fundamental solution F for the biharmonic operator (1.1.4) is the real part of the complex valued function $F+i\tilde{F}=\frac{-1}{4\pi}\left[\bar{z}z\log z+\frac{1}{2}\bar{z}z-\frac{1}{2}z^2\right]$ which is a solution of the equation $\bar{\partial}^2(F+i\tilde{F})=0$. The components of the kernel of the multiple layer potential are formed by applying a vector of differential operators to the fundamental solution F. The form of these differential operators enables us to apply the biharmonic Cauchy—Riemann equations and transform the kernel to be written as a new vector of differential operators applied to the "biharmonic conjugate" $\tilde{F}$.

Using the compatibility condition $\left(\frac{df}{ds}=gx_s+hy_s\right)$ and the "conjugate" form of the kernel, the multiple layer potential can be integrated by parts to obtain a multiple layer potential which essentially depends only on g and h. This multiple layer potential solves the version of the Dirichlet problem which calls for a biharmonic function in Ω whose gradient is specified on the boundary.

The most important result of changing the form of the multiple layer potential is that it also changes the adjoint problem. We introduce a new dual space, a new set of boundary conditions and a modified lower order potential to solve the new adjoint boundary value problem. In our concluding remarks we point out how the new adjoint boundary value problem is one of the fundamental problems in the theory of two dimensional elasticity.

Throughout this chapter we assume that Ω is a bounded simply connected C^1 domain in $\mathbf{R}^2$, bounded by a simple closed contour $\partial\Omega$. If this is not explicitly stated in the following definitions, lemmas and theorems, it is to be assumed.

3.1. The biharmonic Cauchy—Riemann equations and the modified multiple layer potential

Let $f=U+iV$ and assume $\bar{\partial}^2 f=0$ where we recall $\bar{\partial}=\partial_x+i\partial_y$. Equating real and imaginary parts we see that U and V satisfy the coupled system of second order partial differential equations:

(3.1.1)
$$U_{xx}-U_{yy}=2V_{xy},$$
$$-2U_{xy}=V_{xx}-V_{yy}.$$

Since $\Delta f=\partial\bar{\partial}f$ and $\bar{\partial}^2 f=0$ we see that $\bar{\partial}\Delta f=\partial\bar{\partial}^2 f=0$ so that Δf is analytic. This guarantees that the usual Cauchy—Riemann equations apply to ΔU and ΔV. That is,

(3.1.2)
$$(\Delta U)_x=(\Delta V)_y,$$
$$(\Delta U)_y=-(\Delta V)_x.$$

The systems of equations (3.1.1) and (3.1.2) will be referred to as the biharmonic Cauchy—Riemann equations (B.C.R.).

The fundamental solution for the biharmonic operator introduced by Agmon in [1] and used by Cohen and Gosselin in [3] is $F(X)=\frac{-1}{4\pi}[(x^2+y^2)\log(x^2+y^2)^{1/2}+y^2]$ where $X=(x,y)$. A simple computation shows:

(3.1.3)
$$F=\operatorname{Re}\frac{-1}{4\pi}\left\{\bar{z}z\log z+\frac{1}{2}\bar{z}z-\frac{1}{2}z^2\right\}=\operatorname{Re}(F+i\tilde{F})$$

where

(3.1.4)
$$\tilde{F}=\frac{-1}{4\pi}\{(x^2+y^2)\arg(x+iy)-xy\}.$$

$\tilde{F}$ is not single valued since $\arg(x+iy)$ is not. However, when necessary we will be able to specify a specific branch of the argument.

It follows from the B.C.R. equations that

(3.1.5)
$$F_{xx}-F_{yy}=2\tilde{F}_{xy},$$
$$-2F_{xy}=\tilde{F}_{xx}-\tilde{F}_{yy}$$

and

$$\frac{\partial\Delta F}{\partial N}=-\frac{\partial\Delta\tilde{F}}{\partial s},$$

where $\frac{\partial}{\partial N}$ represents differentiation with respect to the inner normal and $\frac{\partial}{\partial s}$ represents tangential differentiation where the arc is traversed counterclockwise.

(The equation $\frac{\partial \Delta F}{\partial N} = -\frac{\partial \Delta \tilde{F}}{\partial s}$ is simply the usual Cauchy—Riemann equation in rotated coordinates.)

We next consider the kernel of the multiple layer potential. By applying (3.1.1) and (3.1.2) we get

$$\text{(3.1.6)} \qquad \vec{K}^Q F(X-Q)^T = \begin{bmatrix} \frac{\partial \Delta F}{\partial N}(X-Q) + \frac{\partial}{\partial s} 2F_{xy}(X-Q) \\ (F_{xx}(X-Q) - F_{yy}(X-Q))\, y_s(Q) \\ (F_{xx} - F_{yy})(X-Q)\, x_s(Q) + 4F_{xy}(X-Q)\, y_s(Q) \end{bmatrix}$$

$$= \begin{bmatrix} -2\frac{\partial}{\partial s}\tilde{F}_{xx}(X-Q) \\ 2\tilde{F}_{xy}(X-Q)\, y_s(Q) \\ 2\tilde{F}_{xy}(X-Q)\, x_s(Q) + 2(\tilde{F}_{yy}(X-Q) - \tilde{F}_{xx}(X-Q))\, y_s(Q) \end{bmatrix}$$

where all the differentiations are taken with respect to Q.

We would like to substitute $-2\frac{\partial}{\partial s}\tilde{F}_{xx}$ for $\frac{\partial \Delta F}{\partial N} + \frac{\partial}{\partial s} 2F_{xy}$ in the first component of the kernel of the multiple layer potential and apply an integration by parts to $\int_{\partial\Omega} f(Q)\left(-2\frac{\partial}{\partial s}\tilde{F}_{xx}(X-Q)\right) ds(Q)$. Some care must be taken since $\tilde{F}_{xx} = \frac{1}{2\pi}\left[\arg(x+iy) - \frac{xy}{x^2+y^2}\right]$ which is not single-valued. We fix $P_0 \in \partial\Omega$ and since Ω is simply connected we can define a branch of $\arg(X-P_0)$ for $X \in \Omega$. To perform the integration we fix a path $\partial\Omega_0$ beginning at $P_0 \in \partial\Omega$ and proceed counterclockwise around the boundary. For each fixed $X \in \Omega$ we continue $\arg(X-Q)$ as Q traverses $\partial\Omega_0$. Using the compatability condition $(f' = f x_s + h y_s)$ we get

$$\text{(3.1.7)} \qquad \int_{\partial\Omega} f(Q)\left(-2\frac{\partial}{\partial s}\tilde{F}_{xx}(X-Q)\, ds(Q) = -2f(Q)\tilde{F}_{xx}(X-Q)\Big|_{P_0^+}^{P_0^-}\right.$$

$$+2\int_{\partial\Omega_0} g(Q)\, x_s(Q)\tilde{F}_{xx}(X-Q) + h(Q)\, y_s(Q)\tilde{F}_{xx}(X-Q)\, ds(Q).$$

This computation gives us a new expression for the multiple layer potential.

$$\text{(3.1.8)} \qquad u(\dot{f};\, X) = -2f(Q)\tilde{F}_{xx}(X-Q)\Big|_{P_0^+}^{P_0^-} + 2\int_{\partial\Omega_0} \vec{g}(Q)\vec{L}\tilde{F}(X-Q)^T\, ds(Q)$$

where $\vec{L}v = (L_1 v, L_2 v)$ with $L_1 v = v_{xx} x_s + v_{xy} y_s$ and $L_2 v = v_{xy} x_s + v_{yy} y_s$ and $\vec{g}(Q) = (g(Q), h(Q))$.

For $X \notin \bar{\Omega}$ fixed we can define a branch of $\arg(X-Q)$ for $Q \in \bar{\Omega}$ and integrate by parts to obtain the same expression as in (3.1.7). Since $f \in L_1^p(\partial\Omega)$ we may assume

f is continuous at P_0 and so we can evaluate the boundary term in (3.1.7) to get

(3.1.9) $$-2f(Q)\tilde{F}_{xx}(X-Q)\Big|_{P_0^+}^{P_0^-} = \begin{cases} -4\pi f(P_0), & X\in\Omega \\ 0, & X\notin\bar{\Omega} \end{cases}$$

which is independent of the choice of X in each of the two regions Ω and $\bar{\Omega}^c$. We know that the multiple layer potential $u(\vec{f}; X)$ is single-valued so that by (3.1.9) we know that $\int_{\partial\Omega_0} \bar{g}(Q)\vec{L}\tilde{F}(X-Q)^T\, ds(Q)$ is also single-valued where $\bar{g}$ and $\vec{L}\tilde{F}$ are defined by (3.1.8).

It also follows from (3.1.9) that if $u(\vec{f}; X)$ is the multiple layer potential with density $\vec{f}$ and $\bar{u}=(u_x, u_y)$ then from (3.1.7) we have the representation

(3.1.10) $$\bar{u}(X) = \int_{\partial\Omega_0} \bar{g}(Q)\ell(X, Q)\, ds(Q)$$

where

$$\ell(X, Q) = 2\begin{bmatrix} \partial_x^X L_1^Q \tilde{F}(X-Q) & \partial_y^X L_1^Q \tilde{F}(X-Q) \\ \partial_x^X L_2^Q \tilde{F}(X-Q) & \partial_y^X L_2^Q \tilde{F}(X-Q) \end{bmatrix}.$$

It is useful to note that all the derivatives of the function $\tilde{F}$ in the components of $\ell(X, Q)$ are single valued. This can be verified by a simple computation.

3.2. The Dirichlet problem and the modified multiple layer potential

If we want to study the Dirichlet problem $\Delta^2 u=0$ in Ω, $\bar{u}=(u_x, u_y)=(g, h)$ on $\partial\Omega$ we see from § 3.1 that we need only study the simpler form $\bar{u}$ given by (3.1.10). The kernel matrix ℓ is only 2×2 and the components of ℓ have singularities of the order $|X|^{-1}$. Furthermore, the Dirichlet data involves only pairs (g, h) rather than compatible triples (f, g, h).

Definition (3.2.1). *Let* $\mathscr{C}_p=\{(g, h)\colon g\in L^p(\partial\Omega), h\in L^p(\partial\Omega)$ *and* $\int_{\partial\Omega} g\,dx+h\,dy=0\}$.

Lemma (3.2.2). *For* $\bar{g}=(g, h)\in\mathscr{C}_p$ *and* $P_0\in\partial\Omega$, *define* $A(P)=\int_{P_0}^P g\,dx+h\,dy$. *where the path of integration is counterclockwise along the boundary. Let* $\dot{A}=(A, g, h)$. *Then* $\dot{A}\in\mathscr{B}_p$ *and* $\|A\|_{p,1}\leq c(\|g\|_p+\|h\|_p)$ *where* $\|A\|_{p,1}=\|A\|_p+\left\|\frac{dA}{ds}\right\|_p$.

Definition (3.2.3). *The modified multiple layer potential* $u_m=u_m(\bar{g}; X)$ *with density* $\bar{g}\in\mathscr{C}_p$ *is defined for* $X\notin\partial\Omega$ *by*

(3.2.4) $$u_m(\bar{g};\, X) = \int_{\partial\Omega_0} \bar{g}(Q)L(X, Q)^T\, ds(Q)$$

where $L(X, Q)=2\vec{L}^Q\tilde{F}(X-Q)$, *a particular branch of the argument is chosen to define* $\tilde{F}$ *and the integration begins at the point* $P_0\in\partial\Omega$.

Lemma (3.2.5). *For* $\dot{A}=(A, g, h)$ *where* $\dot{g}=(g, h)\in\mathscr{C}_p$ *and* $A=\int_{P_0}^{P} g\,dx+h\,dy$, *let* $u=u(\dot{A}; X)$ *be the multiple layer potential with density* A. *Then*

(3.2.6) $$\text{non-tangential} \lim_{X\to P\in\partial\Omega} \dot{u}(\dot{A}; X)=\begin{cases}(\dot{I}+\dot{\mathscr{K}})\dot{A}(P), & X\in\Omega\\ (-\dot{I}+\dot{\mathscr{K}})\dot{A}(P), & X\in\bar{\Omega}^c.\end{cases}$$

and

(3.2.7) $$\|\dot{\mathscr{K}}\dot{A}\|_{L_1^p x L^p x L^p} \leqq c(\|g\|_p+\|h\|_p).$$

Proof. The limits and estimates are a consequence of applying lemma (3.2.2) to theorem (1.2.4).

For $\dot{f}\in\mathscr{B}_p$ let $\Pi\dot{f}$ be the projection of $\dot{f}$ onto the last two components ($\Pi\dot{f}=(g, h)$). Define $\bar{\mathscr{K}}\dot{f}=\Pi\dot{\mathscr{K}}\dot{f}$ and $\bar{I}\dot{f}=\Pi\dot{f}$. Clearly, Π: $\mathscr{B}_p\to\mathscr{C}_p$. For $\bar{g}\in\mathscr{C}_p$, $A=\int_{P_0}^{P} g\,dx+h\,dy$ and $\dot{A}=(A, g, h)$, define $\bar{\mathscr{L}}\bar{g}=\bar{\mathscr{K}}\dot{A}=\Pi\dot{\mathscr{K}}\dot{A}$. We will also use $\bar{I}$ to denote the identity on $\mathscr{C}_p$.

Proposition (3.2.8). *For* A, $\dot{A}$ *and* $\bar{\mathscr{L}}$ *defined as above*

(i) $\bar{\mathscr{L}}$: $\mathscr{C}_p\to\mathscr{C}_p$ *is bounded in the* $L^p(\partial\Omega)\times L^p(\partial\Omega)$ *norm and compact from* $\mathscr{C}_p$ *to* $\mathscr{C}_p$,

(ii) non-tangential $\lim_{X\to P\in\partial\Omega} \bar{u}(\dot{A}; X)$

$$=\text{non-tangential} \lim_{X\to P\in\partial\Omega} \int_{\partial\Omega} \bar{g}(Q)\ell(X, Q)\,ds(Q)=\begin{cases}(\bar{I}+\bar{\mathscr{L}})(\bar{g})(P), & X\in\Omega\\ (-\bar{I}+\bar{\mathscr{L}})(\bar{g})(P), & X\in\bar{\Omega}^c,\end{cases}$$

(iii) $\bar{I}+\bar{\mathscr{L}}$ *is invertible on* $\mathscr{C}_p$.

Proof. The boundedness of $\bar{\mathscr{L}}$ comes from theorem (1.2.4). Compactness follows from the compactness of $\dot{\mathscr{K}}$ on $\mathscr{B}_p$.

To show (ii), let $\dot{A}=(A, g, h)$ and let $u(\dot{A}; X)$ be the multiple layer potential with density $\dot{A}$. From theorem (1.2.4), if $\bar{u}=(u_x, u_y)$

(3.2.9) $$\text{non-tangential} \lim_{X\to P} \bar{u}(X)=\begin{cases}\bar{g}(P)+\bar{\mathscr{K}}\dot{A}(P), & X\in\Omega\\ -\bar{g}(P)+\bar{\mathscr{K}}\dot{A}(P), & X\in\bar{\Omega}^c.\end{cases}$$

On the other hand, from (3.1.8), $\bar{u}(X)=\int_{\partial\Omega}\bar{g}(Q)\ell(X, Q)\,ds(Q)$. Thus, using the definition of $\bar{\mathscr{L}}$ and (3.2.9) we get (ii).

(iii) Let $\bar{g}\in\mathscr{C}_p$. Define $\dot{A}$ as before. From theorem (1.4.3) $\dot{I}+\dot{\mathscr{K}}$ is invertible. Let $\dot{\tilde{A}}=(\dot{I}+\dot{\mathscr{K}})^{-1}\dot{A}$. Then $\dot{\tilde{A}}=(\tilde{A}, \tilde{g}, \tilde{h})\in\mathscr{B}_p$ and so $(\tilde{g}, \tilde{h})\in\mathscr{C}_p$. Next let $u=u(\dot{\tilde{A}}; X)$ be the multiple layer potential with density $\dot{\tilde{A}}$. The interior non-tangential $\lim_{X\to P}\dot{u}(X)=(\dot{I}+\dot{\mathscr{K}})\dot{\tilde{A}}=\dot{A}=(A, g, h)$. Furthermore, the interior non-tangential $\lim_{X\to P}(u_x, u_y)=\Pi(\dot{I}+\dot{\mathscr{K}})\dot{\tilde{A}}=(g, h)=\bar{g}$. We also have $\Pi(\dot{I}+\dot{\mathscr{K}})\dot{\tilde{A}}=(\bar{I}+\bar{\mathscr{L}})(\bar{\tilde{g}})$ where $\bar{\tilde{g}}=(\tilde{g}, \tilde{h})$. Hence $(\bar{I}+\bar{\mathscr{L}})(\bar{\tilde{g}})=\bar{g}$. This implies that $\bar{I}+\bar{\mathscr{L}}$ maps $\mathscr{C}_p$ onto

$\mathscr{C}_p$. Since $\bar{\mathscr{L}}$ is compact, we know from the Fredholm theory that $\bar{I}+\bar{\mathscr{L}}$ is one to one and thus invertible.

We have shown the following:

Theorem (3.2.10). *Let Ω be a simply connected bounded C^1 domain in $\mathbf{R}^2$ with boundary $\partial\Omega$. Let $\bar{g}\in\mathscr{C}_p$. Then there exists a function u such that*

$$\Delta^2 u = 0 \quad in \quad \Omega$$

$$\text{non-tangential} \quad \lim_{\substack{X\to P\in\partial\Omega \\ X\in\Omega}} (u_x, u_y) = (g, h) \quad a.e..$$

Furthermore we can write u as the modified multiple layer potential $u=u_m(\bar{\bar{g}}; X)$ where u_m is defined in (3.2.5) and $\bar{\bar{g}}=(\bar{I}+\bar{\mathscr{L}})^{-1}(\bar{g})$.

3.3. The exterior adjoint problem

As in §2 we now consider the adjoint problem for the modified multiple layer potential. We want to find a function v such that $\Delta^2 v=0$ in $\bar{\Omega}^c$ and the exterior $\lim_{X\to P\in\partial\Omega}(L_1 v, L_2 v)=(\varphi,\psi)$ where φ and ψ are functions defined on $\partial\Omega$.

To be more precise we start by considering the dual space $\mathscr{C}_p^*=L^q(\partial\Omega)\times L^q(\partial\Omega)/\mathscr{C}_p^\perp$ where $\mathscr{C}_p^\perp=\{(\varphi,\psi)\colon \varphi\in L^q(\partial\Omega),\ \psi\in L^q(\partial\Omega)$ and $\int_{\partial\Omega} g\varphi+h\psi\,ds=0\ \forall\bar{g}=(g,h)\in\mathscr{C}_p\}$. From the definition of $\mathscr{C}_p$ it is clear that (x_s, y_s) is a basis for $\mathscr{C}_p^\perp$.

We must also define a notion of convergence in $\mathscr{C}_p^*$ for pairs of functions defined in a neighborhood of $\partial\Omega$. Let $\{\eta_j\}_{j=1}^n$ and $\{\zeta_j\}_{j=1}^n$ be as in §1.4. For a pair $\bar{\varphi}=(\varphi,\psi)$ defined for all X with $\operatorname{dist}(X,\partial\Omega)\leqq\delta_0$ and $0<t<\delta_0/8$, we define $\bar{\varphi}_t^+(P)=\sum_{j=1}^N \zeta_j(P)\bar{\varphi}(P-t\vec{N}_j)$ where $\vec{N}_j$ is the unit inner normal to $\partial\Omega$ at P_j. We say that $\lim_{t\to 0}\bar{\varphi}_t^+=\bar{\alpha}\in\mathscr{C}_p^*$ if

$$\text{(3.3.1)} \qquad \lim_{t\to 0}\sum_{j=1}^N\int_{\partial\Omega}\zeta_j(Q)\bar{g}(Q)\bar{\varphi}(Q-tN_j)^T\,ds(Q) = \int_{\partial\Omega}\bar{g}(Q)\bar{\alpha}(Q)^T\,ds(Q)$$

for all $\bar{g}\in\mathscr{C}_p$.

In other words, if $\langle\bar{g},\bar{\varphi}_t^+\rangle=\int_{\partial\Omega}\bar{g}\bar{\varphi}\,ds$ denotes the dual pairing of $\mathscr{C}_p$ and $\mathscr{C}_p^*$, then

$$\lim_{t\to 0}\langle\bar{g},\bar{\varphi}_t^+\rangle = \langle\bar{g},\bar{\alpha}\rangle$$

for all $\bar{g}\in\mathscr{C}_p$.

Definition (3.3.2). *The modified lower order potential $V_m=V_m(\vec{\varphi}; X)$ with density $\vec{\varphi}=(\varphi,\psi)$ is defined by:*

$$\text{(3.3.3)} \qquad V_m(\vec{\varphi};\,X) = 2\int_{\partial\Omega_0}\partial_x^P\tilde{F}(P-X)\varphi(P)+\partial_y^P\tilde{F}(P-X)\psi(P)\,ds(P)$$

where $\vec{\varphi}\in L^q(\partial\Omega)\times L^q(\partial\Omega)$, the integration begins and ends at $P_0\in\partial\Omega$ and a particular *branch of the argument is chosen to define* $\tilde{F}(P-X)$. *(The branch may be chosen as in the development of* (3.1.7)*).*

Remarks. The modified lower order potential is defined for densities $\vec{\varphi}\in L^q(\partial\Omega)\times L^q(\partial\Omega)$. If $\vec{\varphi}\in\mathscr{C}_p^*$ and $\vec{\varphi}=(\varphi,\psi)$ and $\vec{\varphi}_1=(\varphi_1,\psi_1)$ are representatives of the coset $\bar{\varphi}$, then $\vec{\varphi}-\vec{\varphi}_1=\lambda(x_s,y_s)$. Letting $\vec{T}=(x_s,y_s)$, a simple computation shows that $V_m(\vec{T};X)=1/2|P_0-X|^2$. Applying the operator $\vec{L}_p=(x_s(P)\partial^X_{xx}+y_s(P)\partial^X_{xy}, x_s(P)\partial^X_{xy}+y_s(P)\partial^X_{yy})$ we see that $\vec{L}_pV_m(\vec{\varphi};X)-\vec{L}_pV_m(\vec{\varphi}_1;X)=\frac{\lambda}{2}(x_s(P),y_s(P))\in\mathscr{C}_p^\perp$. This calculation shows that although different representatives of $\mathscr{C}_p^*$ give rise to different lower order potentials, the function $\vec{L}_pV_m(\vec{\varphi};X)$ agrees on representatives of the coset $\bar{\varphi}$ and so is well defined on the coset space $\mathscr{C}_p^*$.

From the definition of the modified lower order potential it can be seen that for $X\in\Omega$, V_m is single-valued. However for $X\in\bar{\Omega}^c$, V_m may be multiple-valued. A simple computation shows $(\tilde{F}_x,\tilde{F}_y)=\frac{1}{2\pi}((x\arg(x+iy)-y), y\arg(x+iy))$ which is multiple-valued. It is interesting to note that since the argument changes by 2π with a complete circuit around the origin, it follows from the definition of V_m that it is single-valued if the density (φ,ψ) satisfies the moment condition $\int_{\partial\Omega}(x+\alpha)\varphi+(y+\beta)\psi\,ds=0$ for $\alpha,\beta\in\mathbf{R}$.

Finally, a direct calculation shows that even if V_m is multiple-valued, second order derivatives of V_m are single-valued. For an important application in elasticity, the computation of the stress tensor, only second order derivatives of the lower order potential are needed.

Theorem (3.3.4). *If Ω is a simply connected, bounded C^1 domain in $\mathbf{R}^2$ and $\bar{\varphi}\in\mathscr{C}_p^*$, then there exists a function v, possibly multiple-valued, such that*

$$\Delta^2 v = 0 \quad in \quad \bar{\Omega}^c,$$
$$\lim_{t\to 0}\langle\bar{g},(\vec{L}v)_t^+\rangle = \langle\bar{g},\bar{\varphi}\rangle$$

for all $\bar{g}\in\mathscr{C}_p$.

Proof. Since $\bar{I}+\bar{\mathscr{L}}$ is invertible by (3.2.8) part (iii), we know from the Fredholm theory that $(\bar{I}+\bar{\mathscr{L}})^*$ is also invertible. For $\bar{\varphi}\in\mathscr{C}_p^*$ we choose $\bar{\bar{\varphi}}$ such that $\bar{\varphi}=(\bar{I}+\bar{\mathscr{L}})^*\bar{\bar{\varphi}}$. Then let $V_m=V_m(\bar{\bar{\varphi}};X)$ be the modified lower order potential with density $\bar{\bar{\varphi}}$. We have for $\bar{g}\in\mathscr{C}_p$

$$(3.3.5)\quad \lim_{t\to 0}\langle\bar{g},(\vec{L}V_m)_t^+\rangle = \lim_{t\to 0}\sum_{j=1}^N\int_{\partial\Omega}\zeta_j(Q)\bar{g}(Q)\vec{L}V_m(P,Q-tN_j)^T\,ds(Q)$$
$$= \lim_{t\to 0}\sum_{j=1}^N\int_{\partial\Omega}\int_{\partial\Omega}\zeta_j(Q)\bar{g}(Q)\ell(P+t\vec{N}_j,Q)\,ds(Q)\bar{\bar{\varphi}}(P)^T\,ds(P)$$
$$= \int_{\partial\Omega}(\bar{I}+\bar{\mathscr{L}})\bar{g}(P)\bar{\bar{\varphi}}(P)^T\,ds(P) = \langle\bar{g},(\bar{I}+\bar{\mathscr{L}})^*\bar{\bar{\varphi}}\rangle = \langle\bar{g},\bar{\varphi}\rangle.$$

3.4. The exterior Dirichlet and interior adjoint problems

The solvability of the exterior Dirichlet and interior adjoint problems follow from the results for the similar problems in §2. If we let $\bar{V}_+ = \text{Ker}\,(\bar{I}-\bar{\mathscr{L}})$ and $\bar{W}_+ = \text{Ker}\,(\bar{I}-\bar{\mathscr{L}})^*$ we have the following:

Proposition (3.4.1). (i) $\mathscr{C}_p = \bar{V}_+ \oplus \text{Range}\,(\bar{I}-\bar{\mathscr{L}})$,
(ii) $\mathscr{C}_p^* = \bar{W}_+ \oplus \bar{V}_+^\perp = \bar{W}_+ \oplus \text{Range}\,(\bar{I}-\bar{\mathscr{L}})^*$.

Proof. (i) If $\bar{g}\in\mathscr{C}_p$, let $A(P)=\int_{P_0}^P g\,dx+h\,dy$ and let $\dot{A}=(A,g,h)$. Then $\dot{A}\in\mathscr{B}_p$ so by theorem (2.2.2) $\dot{A}=\dot{A}_1+\dot{A}_2$ where $\dot{A}_1\in\text{Ker}\,(\dot{I}-\dot{\mathscr{K}})$, $\dot{A}_2\in\text{Range}\,(\dot{I}-\dot{\mathscr{K}})$. But $\dot{A}=(A_1,g_1,h_1)\in\mathscr{B}_p$ and $\dot{A}_2=(A_2,g_2,h_2)\in\mathscr{B}_p$. So $\bar{g}_1=(g_1,h_1)=\Pi\dot{A}_1$ and $(\bar{I}-\bar{\mathscr{L}})\bar{g}_1=\Pi(\dot{I}-\dot{\mathscr{K}})\dot{A}_1=0$ which implies that $\bar{g}_1\in\bar{V}_+$. Since $\dot{A}_2\in\text{Range}\,(\dot{I}-\dot{\mathscr{K}})$, there exists $\dot{\bar{A}}_2$ such that $\dot{A}_2=(\dot{I}-\dot{\mathscr{K}})\dot{\bar{A}}_2$. Hence $\bar{g}_2=\Pi\dot{A}_2=\Pi(\dot{I}-\dot{\mathscr{K}})\dot{\bar{A}}_2=(\bar{I}-\bar{\mathscr{L}})(\dot{\bar{g}}_2)$, where $\bar{g}_2=\Pi\dot{\bar{A}}_2$. Finally, $\bar{V}_+\cap\text{Range}\,(\bar{I}-\bar{\mathscr{L}})=\{0\}$ because $\bar{V}_+\cap\text{Range}\,(\dot{I}-\dot{\mathscr{K}})=\{0\}$.

For (ii) use an argument similar to the proof of theorem (2.2.2) part (ii).

We now have the solvability of the exterior Dirichlet problem and interior adjoint problems.

Theorem (3.4.2). *For Ω a bounded simply connected C^1 domain in $\mathbf{R}^2$ with $\bar{g}\in\mathscr{C}_p$ and $\bar{\varphi}\in(\text{Ker}\,(\bar{I}-\bar{\mathscr{L}}))^\perp$ there exist functions u and w such that*

$$\Delta^2 u = 0 \quad in \quad \bar{\Omega}^c,$$

(3.4.3)
$$\text{non-tangential} \lim_{\substack{X\to P\in\partial\Omega \\ X\in\Omega}} \nabla u(X) = \bar{g}(P) \quad a.e.$$

and

$$\Delta^2 w = 0 \quad in \quad \Omega,$$

(3.4.4)
$$\lim_{t\to 0}\langle\bar{g}, (\bar{L}w)_t\rangle = \langle\bar{g}, \bar{\varphi}\rangle$$

for all $\bar{g}\in\mathscr{C}_p$.

Proof. For (3.4.3) we write $\bar{g}=\bar{g}_1+\bar{g}_2$ where $\bar{g}_1\in\text{Ker}\,(\bar{I}-\bar{\mathscr{L}})$ and $\bar{g}_2=(-\bar{I}+\bar{\mathscr{L}})(\bar{\bar{g}}_2)$. Then set $u=u_m(\bar{\bar{g}}_2;\,X)+b(X)$ where $b\in\mathscr{I}$ and $\nabla b=\bar{g}_1$ on $\partial\Omega$.

For (3.4.4) we assume $\bar{\varphi}\in\text{Range}\,(\bar{I}-\bar{\mathscr{L}})$. Therefore there exists a $\bar{\bar{\varphi}}$ such that $\bar{\varphi}=(-\bar{I}+\bar{\mathscr{L}})^*(\bar{\bar{\varphi}})$ and we set $w=V_m(\bar{\bar{\varphi}};\,X)$.

4. Concluding remarks

In this section we make some additional remarks about the problems studied in this paper. We point out their connection with the theory of plane elasticity and mention some questions which remain to be done.

4.1. Application to plane elasticity

In a thin elastic plate the x and y components of stress per unit length (X_n, Y_n) o a small line segment ds through a point is given by the matrix product $\begin{bmatrix} X_x & X_y \\ Y_x & Y_y \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \end{bmatrix}=\begin{bmatrix} X_n \\ Y_n \end{bmatrix}$ where the 2×2 matrix is the stress tensor and $\vec{n}=n_1\vec{i}+n_2\vec{j}$ is a unit vector perpendicular to ds. The stress function v is a biharmonic function such that $v_{xx}=X_y$, $v_{xy}=-X_y=-Y_x$, $v_{yy}=X_x$. Under conditions of elastic equilibrium the stress function can be determined from the external stresses at the boundary by solving the boundary value problem

(4.1.1)
$$\Delta^2 v = 0 \quad \text{in} \quad \Omega,$$
$$v_{xx}x_s+v_{xy}y_s = -Y_n \quad \text{on} \quad \partial\Omega,$$
$$v_{xy}x_s+v_{yy}y_s = X_n \quad \text{on} \quad \partial\Omega$$

where X_n and Y_n are the x and y components of normal stress at the boundary $\partial\Omega$. (See Muskhelishvili [9], pages 105 and 113 for more details.)

The adjoint problem studied in § 3 is then the problem of determining the stress function from the boundary stresses. For the interior problem we know we can solve (4.1.1) provided the data $\bar{\varphi}\in \bar{V}_+^{\perp}$. From the discussion in § 3.4, $\bar{V}_+$ consists of the restriction to $\partial\Omega$ of the gradient of functions in $\mathscr{I}(\Omega)$. That is, $\bar{V}_+$ is the subspace of $\mathscr{C}_p$ spanned by the pairs of functions $\{(x, y), (1, 0), (0, 1)\}$. If V is biharmonic and $L_2v=-Y_n$, $L_1v=X_n$ at the boundary we have as a necessary condition that $(-Y_n, X_n)\in\bar{V}_+^{\perp}$. This means that (X_n, Y_n) satisfies the moment conditions

(4.1.2)
$$\int_{\partial\Omega} X_n\, ds = \int_{\partial\Omega} Y_n\, ds = 0$$

and

(4.1.3)
$$\int_{\partial\Omega} yX_n - xY_n\, ds = 0$$

which means that the resultant of external stresses is zero and the moment about the origin is also zero.

4.2. Uniqueness

In this paper and in [3], existence results have been established for a variety of boundary value problems involving biharmonic functions on C^1 domains in $\mathbf{R}^2$. The question of uniqueness has not been addressed and it would be very useful to establish conditions on the data and solutions which will guarantee it. In particular, one would like to know if there is a growth condition at ∞ that will guarantee uniqueness for the exterior Dirichlet problems. The growth properties of the lower order potential suggest that the condition $|u(X)|=o(|X|^2 \log |X|)$ as $|X|\to\infty$ might be a necessary condition.

4.3. Pointwise estimates

The solution of either form of the adjoint problem is a weak solution in the sense that the convergence of $(K_1 v, K_2 v, K_3 v)$ or $(L_1 v_m, L_2 v_m)$ takes place in the appropriate dual space. Furthermore, the limits are cosets of functions rather than functions. On the other hand the lower order potentials are functions. So one would like to know whether the potentials $\vec{K}v$ or $\vec{L}v_m$ have pointwise limits.

The singularities in $\vec{K}v$ are too large to expect pointwise limits but the singularities in $\vec{L}v_m$ suggest that it should have non-tangential limits almost everywhere characterized by singular integrals. In the second paper [4] we show that Calderón's theorem on the Cauchy integral [2] can be utilized to establish pointwise limits for the interior adjoint potential $\vec{L}v_m$ of the form $(-I+\mathscr{L})^*\vec{\varphi}$. This, together with the invertibility of the operator $(-\bar{I}+\bar{\mathscr{L}})^*$ on a subspace of the coset space $\mathscr{C}_p^*$, enables us to solve the elasticity problem in a pointwise sense. In that paper we give a more detailed explanation of the connection with elasticity.

4.4. H^1 theory

The solutions of the adjoint problem suggest a Hardy space theory for biharmonic equations. The "size" of the biharmonic functions here is measured by a kind of integrability of adjoint potentials along local parallel translates of the boundary. What differentiates Hardy space theory from Sobelev theory is that the integrability of the potentials is on spaces of dimension equal to that of the boundary rather than integrability in the domain itself.

Following the paper of Fabes and Kenig [7] one would like to know if there is an H^1 theory for adjoint potentials. More specifically, can we construct an atomic h^1 theory which extends the solvability of the adjoint problems to appropriate subspaces of L^1. Is there an analog of the space BMO which is preserved by the operator $\dot{\mathscr{K}}$. Finally, what is the connection between the solutions of the complex equation $\bar{\partial}^2 f=0$ and the space of biharmonic functions.

4.5. Lipschitz domains

The question arises as to whether the results in this paper extend to Lipschitz domains. The main problem of extension is that for Lipschitz domains the operator $\dot{\mathscr{K}}$ ceases to be compact. For L^2 type data it might be possible to show that $(\dot{I}+\dot{\mathscr{K}})^*$ or $(\bar{I}+\bar{\mathscr{L}})^*$ has closed range. Verchota [10] has done this for related problems involving Laplace's equation. If one could find analogs of his methods for the potentials of this paper, the results for both the Dirichlet and adjoint problems could be extended to Lipschitz domains. It is not clear, however, that Verchota's methods can be applied to the potentials in this paper.

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