On the primary ideal structure at infinity for analytic Beurling algebras

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0. Introduction

An algebra B is a Banach algebra if there is a norm defined on it such that B is a Banach space and the multiplication is continuous. As far as we are concerned, all Banach algebras are assumed commutative. For standard facts in the elementary theory of commutative Banach algebras we refer to [10] and [22].

Suppose w is a locally bounded measurable (weight) function on $\mathbf{R}$, which satisfies

$$\text{(0.1)} \qquad \begin{cases} w(x) \geqq 1, & x\in\mathbf{R}, \\ w(x+y) \leqq w(x)w(y), & x,\, y\in\mathbf{R}. \end{cases}$$

Then the space $L^1_w(\mathbf{R})$ of (equivalence classes of) functions f, Lebesgue measurable on $\mathbf{R}$ and satisfying

$$\|f\|_w = \int_{-\infty}^{\infty} |f(x)|w(x)\,dx < \infty$$

is a Banach algebra under convolution multiplication, which we denote by $*$:

$$(f*g)(x) = \int_{-\infty}^{\infty} f(x-t)\,g(t)\,dt \quad \text{for arbitrary} \quad f,\, g\in L^1_w(\mathbf{R}).$$

Since they were introduced by Beurling in [3], these Banach algebras are called Beurling algebras.

One can show that the limits

$$\alpha = \lim_{x\to+\infty} x^{-1}\log w(x)$$

$$\beta = \lim_{x\to-\infty} x^{-1}\log w(x)$$

are finite, and that $\beta\leqq 0\leqq\alpha$.

In Beurling's classification, one speaks of the analytic case when $\beta<\alpha$. When $\alpha=\beta=0$, one distinguishes between the non-quasianalytic case when

$$\int_{-\infty}^{\infty}(1+x^2)^{-1}\log w(x)\,dx<\infty$$

and the quasianalytic case when this integral is infinite.

Let S_w be the strip or real line $\{z\in\mathbf{C}\colon \beta\leqq \operatorname{Im} z\leqq\alpha\}$. If we extend $L^1_w(\mathbf{R})$ by adding a unit, the corresponding maximal ideal space can be identified with $S_w\cup\{\infty\}$, the one-point compactification of S_w, with the Gelfand transform on $L^1_w(\mathbf{R})$ identified with the Fourier transform

$$\hat{f}(z)=\int_{-\infty}^{\infty}f(t)e^{-itz}\,dt,\quad z\in S_w.$$

Since $L^1_w(\mathbf{R})\subset L^1(\mathbf{R})$, $L^1_w(\mathbf{R})$ is semisimple, that is, the Fourier transform $f\mapsto\hat{f}$ is injective. Hence $L^1_w(\mathbf{R})$ can be viewed as a subalgebra of $A_0(S_w)$, the Banach algebra of functions, holomorphic in S_w^0 and continuous on $S_w\cup\{\infty\}$, having the value 0 at ∞, under the supremum norm on S_w. One can easily show that $L^1_w(\mathbf{R})$ is dense in $A_0(S_w)$. Let us sketch the argument: One shows that functions with rational Fourier transform with poles outside S_w belong to $L^1_w(\mathbf{R})$ and that they are dense in $A_0(S_w)$.

An ideal I in $L^1_w(\mathbf{R})$ (or $A_0(S_w)$; then the Gelfand transform is the identity) is said to be primary at ∞ if it is closed and

$$Z(I)=\bigcap_{f\in I}\{z\in S_w\colon \hat{f}(z)=0\}=\emptyset.$$

If there are no non-trivial primary ideals at ∞, spectral analysis is said to hold in the algebra.

If $\alpha=\beta=0$, so that S_w is a line, it is evident that $A_0(S_w)$ is regular, which implies that the only primary ideal at ∞ is the trivial one: the algebra $A_0(S_w)$ itself.

N. Wiener [21] showed in 1932 that the same result holds for the subalgebra $L^1(\mathbf{R})$. This is often referred to as the General Tauberian Theorem.

A few years later, A. Beurling [3] extended Wiener's theorem to the algebra $L^1_w(\mathbf{R})$, where the weight w has some slight regularity properties and satisfies the non-quasianalyticity condition

$$\int_{-\infty}^{\infty}(1+x^2)^{-1}\log w(x)\,dx<\infty.$$

In his thesis [16] 1950, B. Nyman managed to construct counterexamples in some quasianalytic cases, which is to say that he found certain non-trivial ideals, primary at ∞.

Recently, Y. Domar [9] showed by extending the technique developed by Vretblad in [19] that counterexamples can be found in all quasianalytic cases. His result will be used to give a relation between two conditions on subadditive functions; it is presented in Lemma 2.6.

We shall discuss here the analytic case $\alpha > \beta$, that is, when the strip S_w has positive width.

Let D be the open unit disc in the complex plane, and let the disc algebra $A(D)$ be the Banach algebra of holomorphic functions in D which extend continuously to the boundary, with the supremum norm.

Mapping S_w^0 conformally onto D, we can identify $A_0(S_w)$ with the closed ideal

$$\{f \in A(D)\colon\ f(-1) = f(1) = 0\} \quad \text{in} \quad A(D).$$

The Beurling—Rudin theorem (see [17]) can be used to give a description of all closed ideals in $A_0(S_w)$, especially those primary at ∞. However, this result turns out to be non-trivial. It is proved in Theorem 3.1.

One obtains a doubly indexed chain $\{I_{\xi\eta}\}$, which contains all ideals that are primary at ∞.

B. Nyman [16] showed that the corresponding ideals appear in $L^1_w(\mathbf{R})$, too, and that every primary ideal at ∞ is contained in one of these ideals; if w is the weight

$$w(x) = e^{\alpha|x|} \quad (\alpha > 0),$$

which clearly satisfies (0.1).

In 1958, B. I. Korenblum [14] showed, independently of Nyman's thesis, that all primary ideals at ∞ are indeed of the form $I_{\xi\eta} \cap L^1_w(\mathbf{R})$, where w is the weight considered by Nyman.

In 1973, A. Vretblad [19] extended Nyman's investigations to very general weights, in a way which connects the evasive ideals in the analytic and quasianalytic cases.

The object of this paper is to investigate for which analytic Beurling algebras the primary ideal structure at infinity is tha same as in $A_0(S_w)$, in the sense that every ideal, primary at ∞, is of the form

$$I_{\xi\eta} \cap L^1_w(\mathbf{R}).$$

The conclusion, formulated in Theorems 3.3, 3.5 and Remark 3.6, is that this is the case if the weight w satisfies certain rather weak conditions on w:s deviation from the exponential case.

This is a generalization of Korenblum's result which cannot be obtained without major modifications of his method. In a sense, one could say that the method is simpler because it is more clearly seen why certain arguments work. For example, we use Domar's [8] method to obtain the analytic continuation of the Carleman transform, the rather strong theorem of Levinson and Sjöberg, and Ahlfors's delicate distortion inequalities.

The question if the description of the evasive ideals in the quasianalytic case given by Vretblad in [19] is complete is still open, although there might be reason to suspect that his description is incomplete.

1. Assumptions and notation

Now and onwards, we let w be a real-valued, continuous function satisfying the following set of conditions, (1.1—1.2).

$$w(x) = \exp\left(\frac{\pi}{2}|x|+\psi(x)\right), \quad x\in\mathbf{R}, \tag{1.1}$$

where $\psi(x) = o(|x|)$ as $|x| \to\infty$,

$$w(x+y) \leqq w(x)w(y),\ x,\ y\in\mathbf{R}. \tag{1.2}$$

From (1.1) and (1.2) it follows that $\psi|\mathbf{R}_+$ and $\psi|\mathbf{R}_-$ are subadditive, and in a second step that $\psi(x)\geqq 0$, $x\in\mathbf{R}$. For all weights w satisfying (1.1-2), $S_w=\left\{z\in\mathbf{C}: |\mathrm{Im}\, z|\leqq\frac{\pi}{2}\right\}$, so to simplify our notation, we write S instead of S_w.

The condition that

$$\lim_{x\to+\infty} x^{-1}\log w(x) = \frac{\pi}{2}$$

and

$$\lim_{x\to-\infty} x^{-1}\log w(x) = -\frac{\pi}{2}$$

is only a normalization and the results can be transferred easily to all other analytical cases, that is, when

$$\lim_{x\to+\infty} x^{-1}\log w(x) > \lim_{x\to-\infty} x^{-1}\log w(x).$$

Define

$$\tilde{\psi}(x) = \begin{cases} \sup\limits_{t\geqq 0}\left(\psi(t)-\psi(t+x)\right), & x\geqq 0, \\ \sup\limits_{t\leqq 0}\left(\psi(t)-\psi(t+x)\right), & x<0. \end{cases}$$

Note that if ψ is subadditive, which is a slightly stronger condition than (1.2), $\tilde{\psi}(x)\leqq\psi(-x)$ for every real x. Define for $\varepsilon>0$

$$\begin{cases} M_\psi(\varepsilon) = \int_{-\infty}^{\infty} e^{-\varepsilon|x|+\psi(x)}\,dx \\ M_{\tilde{\psi}}(\varepsilon) = \int_{-\infty}^{\infty} e^{-\varepsilon|x|+\tilde{\psi}(x)}\,dx. \end{cases} \tag{1.3}$$

Sometimes we shall need the following additional assumptions:

(1.4) $$\int_0^\infty \log^+ \log^+ M_\psi(\varepsilon)\, d\varepsilon < \infty,$$

(1.5) $$\int_0^\infty \log^+ \log^+ M_{\tilde\psi}(\varepsilon)\, d\varepsilon < \infty.$$

The dual space $(L_w^1(\mathbf{R}))^*$ can be identified, in an obvious way, with the space $L_w^\infty(\mathbf{R})$ of (equivalence classes of) measurable functions f such that $f/w \in L^\infty(\mathbf{R})$. In bracket notation, $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)g(x)\,dx$, for $f \in L_w^1(\mathbf{R})$ and $g \in L_w^\infty(\mathbf{R})$. This identification could, of course, be made for continuous weight functions w not satisfying (1.1—2), too. Put

$$\delta_+(f) = -\limsup_{\xi\to+\infty} e^{-\xi} \log|\hat f(\xi)| \geqq 0$$

and

$$\delta_-(f) = -\limsup_{\xi\to-\infty} e^{\xi} \log|\hat f(\xi)| \geqq 0.$$

Define, for $\alpha, \beta \geqq 0$, the ideals

$$I_\alpha^+ = \{f \in L_w^1(\mathbf{R}):\ \delta_+(f) \geqq \alpha\}$$

and

$$I_\beta^- = \{f \in L_w^1(\mathbf{R}):\ \delta_-(f) \geqq \beta\}$$

which satisfy $I_{\alpha_1}^+ \subset I_{\alpha_2}^+$ for $\alpha_1 \geqq \alpha_2$ and $I_{\beta_1}^- \subset I_{\beta_2}^-$ for $\beta_1 \geqq \beta_2$. For $\alpha = \beta = 0$ this means that

$$I_0^+ = I_0^- = L_w^1(\mathbf{R}).$$

According to the classification of A. Vretblad [19], it follows that I_α^+ and I_β^- are closed primary ideals corresponding to the point at infinity and that $I_{\alpha_1}^+ \neq I_{\alpha_2}^+$ if $\alpha_1 \neq \alpha_2$ and $I_{\beta_1}^- \neq I_{\beta_2}^-$ if $\beta_1 \neq \beta_2$, under the growth condition

(1.6) $$\int_{-\infty}^{\infty} \frac{\psi(x)}{1+x^2}\, dx < \infty.$$

He adds the condition that w be even, but his results remain valid without it. However, in our restricted case, the fact that these ideals are closed is most easily demonstrated using the Beurling—Rudin theorem in the way suggested in the introduction. This will be done in the proof of Theorem 3.5.

Lemma 2.6 shows that in case ψ is subadditive, (1.4) is equivalent to (1.6); since then $\tilde\psi(x) \leqq \psi(-x)$ for $x \in \mathbf{R}$, (1.5) is a simple consequence of (1.4).

2. Preparatory lemmas

Our first lemma relies heavily on a technique developed by Y. Domar in [8]. $\vee$ denotes the transform $\check{f}(x)=f(-x)$, as is standard in Fourier analysis.

Lemma 2.1. *Let* $f\in L^1_w(\mathbf{R})\setminus\{0\}$ *and* $g\in L^\infty_w(\mathbf{R})$ *satisfy*

$$g*\check{f}=0 \tag{2.1}$$

or, in other words, let g annihilate the non-zero ideal generated by f. Denote by $Z(f)$ *the set* $\{z\in S\colon \hat{f}(z)=0\}$. *Then the function*

$$\mathcal{G}(z)=\begin{cases}\int_0^\infty g(t)e^{itz}\,dt, & \operatorname{Im} z>\frac{\pi}{2},\\ -\int_{-\infty}^0 g(t)e^{itz}\,dt, & \operatorname{Im} z<-\frac{\pi}{2},\end{cases} \tag{2.2}$$

which determines g uniquely, can be continued analytically to a function holomorphic in $\mathbf{C}\setminus Z(f)$.

By (2.1), *we can define*

$$h(x)=\int_0^\infty g(t)f(t+x)\,dt=-\int_{-\infty}^0 g(t)f(t+x)\,dt,\quad x\in\mathbf{R}. \tag{2.3}$$

If $M_{\tilde{\psi}}(\varepsilon)$, *defined by* (1.3), *is finite for every* $\varepsilon>0$, *so that*

$$\int_{-\infty}^\infty |h(x)|\,e^{\left(\frac{\pi}{2}-\varepsilon\right)|x|}\,dx<\infty$$

for all $\varepsilon>0$, *which implies that*

$$\hat{h}(z)=\int_{-\infty}^\infty h(t)e^{-itz}\,dt$$

is holomorphic in S^0, *the extension of* $\mathcal{G}$ *is given by*

$$\frac{\hat{h}(z)}{\hat{f}(z)},\quad \text{for } z\in S^0\setminus Z(f). \tag{2.4}$$

Proof. Let B be the algebra $L^1_w(\mathbf{R})$ extended with a unit. The unit can be identified with the Dirac measure δ.

Let φ be the Möbius mapping

$$z\mapsto (z-2i)^{-1}, \tag{2.5}$$

and define the function a by

$$a(t)=\begin{cases} ie^{-2t}, & t\geqq 0\\ 0, & t<0\end{cases}$$

which obviously belongs to $L^1_w(\mathbf{R})$. Then the Fourier transform $\hat{a}$ of a, defined as 0 at ∞, coincides with φ on S.

Let us introduce the functions

$$b_+(t) = \begin{cases} e^{-\alpha t}, & t \geqq 0 \\ 0, & t < 0 \end{cases}$$

and

$$b_-(t) = \begin{cases} e^{\alpha t}, & t \leqq 0 \\ 0, & t > 0, \end{cases}$$

where $\alpha \in \mathbf{C}$ is arbitrary, and call them cut-off exponentials. To ensure that $b_+, b_- \in L^1_w(\mathbf{R})$, $Re\alpha$ has to be larger than $\frac{\pi}{2}$.

We claim that rational functions of a with poles in $\mathbf{C} \diagdown \hat{a}(S \cup \{\infty\})$ form a dense subspace of B. To see this, observe that the Fourier transforms of cut-off exponential functions belonging to $L^1_w(\mathbf{R})$ are rational functions of $\hat{a}$, and that these cut-off exponentials span a dense subspace of $L^1_w(\mathbf{R})$. The maximal ideal space of B can be identified with $S \cup \{\infty\}$, or, in the sense of [8], with $\hat{a}(S \cup \{\infty\})$.

Now regard B as a function algebra on $\hat{a}(S \cup \{\infty\})$, which is a compact subset of $\mathbf{C}$, bounded by the two circles $\left|z - \frac{i}{4-\pi}\right| = \frac{1}{4-\pi}$ and $\left|z - \frac{i}{4+\pi}\right| = \frac{1}{4+\pi}$ that touch at the origin. This can be done since B is semi-simple.

Extend the functional g on $L^1_w(\mathbf{R})$ to B by defining $\langle \delta, g \rangle = 0$.

By assumption $g * \check{f} = 0$, which implies that $\langle f, g \rangle = 0$, so that the extended g annihilates $(\bar{f})$, the closure of the ideal (f) generated by f in B.

Therefore g defines a corresponding continuous functional $g_{(\bar{f})}$ on $B/(\bar{f})$ by the relation

$$\langle b + (\bar{f}), g_{(\bar{f})} \rangle = \langle b, g \rangle$$

for all $b \in B$.

In [8] Domar defines the analytic transform G of g:

$$G(\zeta) = \langle (a - \zeta\delta + (\bar{f}))^{-1}, g_{(\bar{f})} \rangle,$$
$$\zeta \in \mathbf{C} \diagdown \hat{a}(Z(f) \cup \{\infty\}),$$

which is shown to be holomorphic in $\mathbf{C} \diagdown \hat{a}(Z(f) \cup \{\infty\})$ (Theorem 2.4 [8]).

For $\zeta \in \mathbf{C} \diagdown \hat{a}(S \cup \{\infty\})$, $G(\zeta) = \langle (a - \zeta\delta)^{-1}, g \rangle$.

Substituting $\zeta = \varphi(z)$, with φ defined by (2.5) and recalling that $\langle \delta, g \rangle = 0$, we obtain the relation

$$G \circ \varphi(z) = -i(z-2i)^2 \mathscr{G}(z), \quad z \in \mathbf{C} \diagdown S, \tag{2.6}$$

where $\mathscr{G}$ is the analytic function defined by (2.2). Since G is holomorphic in $\mathbf{C} \diagdown \hat{a}(Z(f) \cup \{\infty\})$, we conclude that $\mathscr{G}$ can be extended analytically to $\mathbf{C} \diagdown Z(f)$.

We will keep the symbol $\mathcal{G}$ for the extension. By Theorem 5.1 in [8], there exists a one-to-one correspondence between the functional g and the holomorphic function G; hence $\mathcal{G}$ determines g uniquely. The first part of the lemma is now proved.

It should be observed that $\mathbf{C}\setminus Z(f)$ is a connected open subset of $\mathbf{C}$ since f is non-zero and $\hat{f}$ is analytic in S^0, so that $\mathcal{G}$ is determined by its behaviour on one of the connected components of $\mathbf{C}\setminus S$. This means that g is determined uniquely by its values on, for example, $\mathbf{R}_+$.

Now we assume that $M_{\tilde{\psi}}(\varepsilon)$ is finite for all $\varepsilon>0$. For $z\in S^0$, a simple estimate gives that the function

$$c_z(t) = \begin{cases} \int_0^\infty f(x+t)e^{-ixz}\,dx, & t \geqq 0 \\ -\int_{-\infty}^0 f(x+t)e^{-ixz}\,dx, & t<0 \end{cases}$$

belongs to $L^1_w(\mathbf{R})$ and satisfies

$$\|c_z\|_w \leqq \|f\|_w M_{\tilde{\psi}}(d(z,\partial S)) \tag{2.7}$$

(d is the Euclidean metric in $\mathbf{C}$). By (2.3) and the definition of c_z,

$$\hat{h}(z) = \langle c_z, g\rangle \tag{2.8}$$

for $z\in S^0$. A straightforward calculation shows that

(2.9)

$$\hat{c}_{z_0}(z) = \int_0^\infty\int_0^\infty f(t+x)e^{-itz-ixz_0}\,dx\,dt - \int_{-\infty}^0\int_{-\infty}^0 f(t+x)e^{-itz-ixz_0}\,dx\,dt$$

$$= \frac{i}{z-z_0}\int_{-\infty}^\infty (e^{-iz\tau}-e^{-iz_0\tau})f(\tau)\,d\tau = \frac{i}{z-z_0}(\hat{f}(z)-\hat{f}(z_0))$$

for $z_0\in S^0$ and $z\in S$.

We are going to show that

$$\mathcal{G}(z) = \frac{\hat{h}(z)}{\hat{f}(z)} \quad \text{for} \quad z\in S^0\setminus Z(f).$$

By (2.6), this amounts to proving that

$$G(\zeta) = -i\zeta^{-2}\frac{\hat{h}\circ\varphi^{-1}(\zeta)}{\hat{f}\circ\varphi^{-1}(\zeta)}$$

for $\zeta\in\hat{a}(S^0\setminus Z(f))$. Substituting $\zeta=\varphi(z)$ in (2.8), we obtain

$$-i\zeta^{-2}\frac{\hat{h}\circ\varphi^{-1}(\zeta)}{\hat{f}\circ\varphi^{-1}(\zeta)} = -\frac{i\zeta^{-2}}{\hat{f}\circ\varphi^{-1}(\zeta)}\langle c_{\varphi^{-1}(\zeta)}, g\rangle.$$

Using (2.9), we find that

$$\frac{-i\zeta^{-2}}{\hat{f}\circ\varphi^{-1}(\zeta)}\hat{c}_{\varphi^{-1}(\zeta)}(\varphi^{-1}(z))-\zeta^{-1}=\frac{\zeta^{-2}}{\hat{f}\circ\varphi^{-1}(\zeta)}\frac{\hat{f}\circ\varphi^{-1}(z)-\hat{f}\circ\varphi^{-1}(\zeta)}{\varphi^{-1}(z)-\varphi^{-1}(\zeta)}-\zeta^{-1}$$

$$=\frac{1}{z-\zeta}\left(-\frac{z\zeta^{-1}}{\hat{f}\circ\varphi^{-1}(\zeta)}\hat{f}\circ\varphi^{-1}(z)+1\right).$$

Multiplying this expression by $z-\zeta$, we see that

$$-\frac{i\zeta^{-2}}{\hat{f}\circ\varphi^{-1}(\zeta)}c_{\varphi^{-1}(\zeta)}-\frac{1}{\zeta}\delta$$

is the inverse of $a-\zeta\delta$ modulo $(\bar{f})$ for $\zeta\in\hat{a}(S^0\setminus Z(f))$. Recalling that $\langle\delta, g\rangle=0$, this proves the lemma.

Remark 2.2. The function $\mathcal{G}$ defined by (2.2—4) is frequently called the Carleman transform of g. There has been some doubt about who invented this device; maybe the analytic transform is a better term.

We shall need the following estimate:

Lemma 2.3. *Assume* (1.4—5). *Suppose* $f\in L^1_w(\mathbf{R})\setminus\{0\}$ *and* $g\in L^\infty_w(\mathbf{R})$ *satisfy* $g*\check{f}=0$, *and that the function* $\mathcal{G}$ *defined by* (2.2—4) *is entire. Then there exists a positive constant* c *such that*

$$|\mathcal{G}(z)|\leqq\exp(ce^{2|\mathrm{Re}\,z|}),\quad z\in\mathbf{C}.$$

Proof. Note that (1.5) implies that $M_{\tilde{\psi}}(\varepsilon)$ is bounded for every $\varepsilon>0$. This is due to the fact that $M_{\tilde{\psi}}$ is a decreasing function.

In the following we use the function h as defined in (2.3). Let us write $z=x+iy$.

A simple calculation shows that the first of the following estimates is valid.

$$|\mathcal{G}(z)|\leqq\|g\|_{L^\infty_w}M_\psi(d(z,\partial S)),\quad z\in\mathbf{C}\setminus S, \tag{2.10}$$

$$|\hat{h}(z)|\leqq\|g\|_{L^\infty_w}\|f\|_w M_{\tilde{\psi}}(d(z,\partial S)),\quad z\in S^0, \tag{2.11}$$

where d is, as in Lemma 2.1, the Euclidean metric in $\mathbf{C}$, and M_ψ and $M_{\tilde{\psi}}$ were defined in (1.3). (2.11) is a combination of (2.7) and (2.8).

In the upper half-plane, we have an explicit formula for the Poisson kernel. To use this, let φ_ε be the conformal mapping $z\mapsto i\exp\left(\frac{z}{1-\varepsilon}\right)$, $z\in S^0$, for $0\leqq\varepsilon\leqq\frac{1}{2}$ (see Figure 2.4).

Let S_ε be the set $\{z\in\mathbf{C}\colon |\mathrm{Im}\,z|<\frac{\pi}{2}(1-\varepsilon)\}$, which φ_ε maps onto the open upper half-plane.

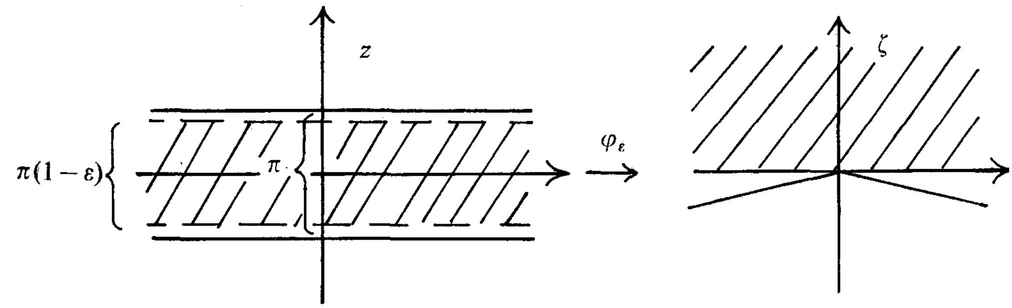

Figure 2.4

Define

$$F_\varepsilon(\zeta) = \hat{f}\circ\varphi_\varepsilon^{-1}(\zeta), \quad \mathscr{G}_\varepsilon(\zeta) = \mathscr{G}\circ\varphi_\varepsilon^{-1}(\zeta),$$

and

$$H_\varepsilon(\zeta) = \hat{h}\circ\varphi_\varepsilon^{-1}(\zeta) \quad \text{for} \quad \zeta\in\varphi_\varepsilon(S).$$

By (2.4) and (2.11), $\mathscr{G}_\varepsilon$ is a quotient of bounded holomorphic functions in the upper half-plane for every $\varepsilon>0$.

Without loss of generality, we can assume that $\mathscr{G}_\varepsilon(z)\not\equiv 0$. Hence, according to a factorization theorem for Hardy spaces (see [13], pages 160—161) we have for every $\varepsilon>0$ and ζ in the open upper half-plane Π that

(2.12)
$$F_\varepsilon(\zeta) = e^{i\gamma_{F_\varepsilon}}\cdot B_{F_\varepsilon}(\zeta)\, e^{i\alpha_{F_\varepsilon}\zeta}$$
$$\cdot\exp\left[\frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\zeta-t}+\frac{t}{t^2+1}\right)\left(\log|F_\varepsilon(t)|\,dt-d\mu_{F_\varepsilon}(t)\right)\right]$$

and

(2.13)
$$H_\varepsilon(\zeta) = e^{i\gamma_{H_\varepsilon}}\cdot B_{H_\varepsilon}(\zeta)\, e^{i\alpha_{H_\varepsilon}\zeta}$$
$$\cdot\exp\left[\frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\zeta-t}+\frac{t}{t^2+1}\right)\left(\log|H_\varepsilon(t)|\,dt-d\mu_{H_\varepsilon}(t)\right)\right],$$

where γ_{F_ε} and γ_{H_ε} are real,

(2.14)
$$\alpha_{F_\varepsilon} = -\limsup_{\xi\to+\infty}\xi^{-1}\log|F_\varepsilon(i\xi)| \geqq 0,$$

(2.15)
$$\alpha_{H_\varepsilon} = -\limsup_{\xi\to+\infty}\xi^{-1}\log|H_\varepsilon(i\xi)| \geqq 0,$$

and the positive singular measures μ_{F_ε} and μ_{H_ε} satisfy

$$\int_{-\infty}^{\infty} \frac{d\mu_{F_\varepsilon}(t)}{1+t^2} < \infty$$

and

$$\int_{-\infty}^{\infty} \frac{d\mu_{H_\varepsilon}(t)}{1+t^2} < \infty.$$

B_{F_ε} and B_{H_ε} denote the respective Blaschke products.

Since α_{F_ε} and α_{H_ε} are finite for every $\varepsilon>0$, we conclude, by the definition of φ_ε, (2.14) and (2.15) that

(2.16) $$\alpha_{F_\varepsilon} = \alpha_{H_\varepsilon} = 0$$

for all $\varepsilon>0$.

To see this, choose $\varepsilon'>\varepsilon>0$. We will show that $\alpha_{F_{\varepsilon'}}=0$ provided that $\alpha_{F_\varepsilon}<\infty$. Now we have that

$$\alpha_{F_{\varepsilon'}} = -\limsup_{\xi\to+\infty} \xi^{-1}\log|F_{\varepsilon'}(i\xi)|$$

$$= -\limsup_{\xi\to+\infty} \xi^{-1}\log|F\circ\varphi_{\varepsilon'}^{-1}(\xi)| = -\limsup_{\xi\to+\infty} \xi^{-1}\log|F\circ\varphi_\varepsilon^{-1}\circ\varphi_\varepsilon\circ\varphi_{\varepsilon'}^{-1}(i\xi)|$$

$$= -\limsup_{\xi\to+\infty} \xi^{-1}\log\left|F_\varepsilon\left(i\xi^{\frac{1-\varepsilon'}{1-\varepsilon}}\right)\right| = -\limsup_{\xi\to+\infty} \xi^{-1-\frac{\varepsilon'-\varepsilon}{1-\varepsilon}}\log|F_\varepsilon(i\xi)| = 0,$$

since $\alpha_{F_\varepsilon}<\infty$, which is the desired conclusion. The corresponding calculation for α_{H_ε} is similar.

Since F_ε and H_ε are analytic in $\varphi_\varepsilon(S^0)$, which contains $\bar{\Pi}\setminus\{0\}$, we conclude that

$$\mu_{F_\varepsilon} = \beta_{F_\varepsilon}\delta_0$$

and

$$\mu_{H_\varepsilon} = \beta_{H_\varepsilon}\delta_0,$$

where δ_0 is the Dirac measure at 0, and β_{F_ε} and β_{H_ε} are non-negative.

β_{F_ε} and β_{H_ε} are given explicitly by the relations

$$\beta_{F_\varepsilon} = -\pi\cdot\limsup_{\xi\to0^+} \xi\cdot\log|F_\varepsilon(i\xi)|,$$

$$\beta_{H_\varepsilon} = -\pi\cdot\limsup_{\xi\to0^+} \xi\cdot\log|H_\varepsilon(i\xi)|.$$

In the same fashion as we derived (2.16) we obtain

(2.17) $$\beta_{F_\varepsilon} = \beta_{H_\varepsilon} = 0$$

for every $\varepsilon>0$.

Inserting (2.16) and (2.17) into (2.12) and (2.13, resp.) we get for $\varepsilon>0$ and $\zeta\in\Pi$ that

$$(2.18)\qquad F_\varepsilon(\zeta)=e^{i\gamma_{F_\varepsilon}}\cdot B_{F_\varepsilon}(\zeta)\cdot\exp\left(\frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\zeta-t}+\frac{t}{t^2+1}\right)\log|F_\varepsilon(t)|\,dt\right),$$

$$(2.19)\qquad H_\varepsilon(\zeta)=e^{i\gamma_{H_\varepsilon}}\cdot B_{H_\varepsilon}(\zeta)\cdot\exp\left(\frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{\zeta-t}+\frac{t}{t^2+1}\right)\log|H_\varepsilon(t)|\,dt\right).$$

By (2.4), we have the equality

$$\mathscr{G}_\varepsilon(\zeta)=\frac{H_\varepsilon(\zeta)}{F_\varepsilon(\zeta)}.$$

Inserting (2.18) and (2.19), we obtain the following estimate in Π:

$$(2.20)\qquad \log|\mathscr{G}_\varepsilon(\zeta)|\leqq\frac{\operatorname{Im}\zeta}{\pi}\int_{-\infty}^{\infty}\frac{\log|H_\varepsilon(t)|-\log|F_\varepsilon(t)|}{|\zeta-t|^2}\,dt.$$

The reason why $\left|\frac{B_{H_\varepsilon}}{B_{F_\varepsilon}}\right|\leqq 1$ in Π is that $\mathscr{G}$ is entire. By (2.11) and the definition of φ_ε,

$$(2.21)\qquad |H_\varepsilon(t)|\leqq\|g\|_{L_w^\infty}\|f\|_w\cdot M_{\tilde\psi}\left(\frac{\pi}{2}\varepsilon\right),\quad t\in\mathbf{R}.$$

Choose a point ζ_0 in the open upper half-plane Π such that

$$F_0(\zeta_0)\neq 0.$$

Then, due to the boundedness above and the subharmonicity of $\zeta\mapsto\log|F_\varepsilon(\zeta)|$, we have the following inequality:

$$\log|F_\varepsilon(\zeta_0)|\leqq\frac{\operatorname{Im}\zeta_0}{\pi}\int_{-\infty}^{\infty}\frac{\log|F_\varepsilon(t)|}{|\zeta_0-t|^2}\,dt,\quad 0\leqq\varepsilon\leqq\frac{1}{2}.$$

Since $F_0(\zeta_0)\neq 0$, the left hand side of this inequality is, by continuity, uniformly bounded from below for ε less than some $\varepsilon_0>0$ and we obtain the estimate

$$(2.22)\qquad \int_{-\infty}^{\infty}\frac{\log|F_\varepsilon(t)|}{1+t^2}\,dt\geqq -C_0\quad(>-\infty),\quad 0\leqq\varepsilon\leqq\varepsilon_0.$$

Letters C with indices stand, here and in the following, for positive quantities independent of ζ and ε.

Similarly, if we vary ζ_0, we get an estimate (2.22) on a lot of open intervals, the union of which covers $[\varepsilon_0,\frac{1}{2}]$. By a compactness argument, a finite number of open intervals suffice to cover $[\varepsilon_0,\frac{1}{2}]$, so that the estimate (2.22) extends to the whole interval $[0,\frac{1}{2}]$, possibly with a different constant C_0.

We get from (2.20), (2.21), (2.22) and the observation

$$\|F_\varepsilon\|_{H^\infty(\Pi)} \leqq \|\hat{f}\|_\infty \leqq \|f\|_w,$$

so that F_ε has a bound from above independent of ε,

$$\log|\mathcal{G}_\varepsilon(\zeta)| \leqq \log M_\Psi\left(\frac{\pi}{2}\varepsilon\right) + C_1\frac{1+|\zeta|^2}{\operatorname{Im}\zeta}$$

for $\varepsilon>0$ and $\zeta\in\Pi$.

If $z\in S_\varepsilon$, this means that

$$\log^+|\mathcal{G}(z)| \leqq \log^+ M_\Psi\left(\frac{\pi}{2}\varepsilon\right) + C_2\frac{e^{|x|/(1-\varepsilon)}}{\cos(y/(1-\varepsilon))} \tag{2.23}$$

for $0<\varepsilon\leqq\frac{1}{2}$.

Thus we have the desired estimate in every strip $S_{2\varepsilon}$, $\varepsilon>0$.

In particular, this is the case for $\varepsilon=\frac{1}{4}$, giving

$$\log^+|\mathcal{G}(z)| \leqq C_3 e^{\frac{4}{3}|x|}, \quad z\in S_{1/2}. \tag{2.24}$$

To extend the estimate to all of $\mathbf{C}$, we proceed as follows.

Choose $0<\varepsilon\leqq\frac{1}{4}$. On the lines $\partial S_{2\varepsilon}$, (2.23) becomes

$$\log^+|\mathcal{G}(z)| \leqq \log^+ M_\Psi\left(\frac{\pi}{4}\cos y\right) + C_4\frac{e^{2|x|}}{\cos y}. \tag{2.25}$$

Combining (2.24) and (2.25), we find that the inequality (2.25) holds for every $z\in S^0$.

Let n be an integer and $z\in[n, n+2]\times i[-\pi, \pi]$. Then

$$\log^+\log^+\left(|\mathcal{G}(z)|^{\exp(-2|n|)}\right) \leqq \max(0, -2|n|+\log^+\log^+|\mathcal{G}(z)|),$$

which by (2.25) can be estimated from above by

$$\max\left(0,\, C_5+\log^+\log^+ M_\Psi\left(\frac{\pi}{4}\cos y\right)+\log^+\frac{1}{\cos y}\right)$$

for $|y|<\frac{\pi}{2}$, and by

$$\max(0, 2+\log^+\log^+(\|g\|_{L_w^\infty}\cdot M_\Psi(d(z,\partial S))))$$

for $\frac{\pi}{2}<|y|\leqq\pi$, due to (2.10).

We have thus established the estimate

$$\log^+\log^+\left(|\mathscr{G}(z)|^{\exp(-2|n|)}\right) \leqq$$

$$\leqq \begin{cases} \max\left(0,\ C_5+\log^+\log^+ M_{\tilde{\psi}}\left(\frac{\pi}{4}\cos y\right)+\log^+\frac{1}{\cos y}\right), & |y| < \frac{\pi}{2} \\ \max(0,\ 2+\log^+\log^+(\|g\|_{L^\infty_w}\cdot M_\psi(d(z,\partial S)))), & \frac{\pi}{2} < |y| \leqq \pi. \end{cases}$$

Observe that $z\mapsto\log^+\log^+\left(|\mathscr{G}(z)|^{\exp(-2|n|)}\right)$ is a subharmonic function.

Since the above estimate of $z\mapsto\log^+\log^+\left(|\mathscr{G}(z)|^{\exp(-2|n|)}\right)$ does not depend on x and n and is integrable in the y-interval $[-\pi, \pi]$, due to (1.4—5), there exists, according to the log-log theorem of Levinson and Sjöberg (see [15], Theorem XLII, Theorème III in [18] or Theorem 4.1 in [11], and, for some further developments, [7]) a (finite) constant M, independent of n, such that

$$|\mathscr{G}(z)|^{\exp(-2|n|)} \leqq e^M \quad \text{if} \quad |y| \leqq \frac{3}{4}\pi.$$

or, equivalently

$$|\mathscr{G}(z)| \leqq \exp(Me^{2|n|})$$

for $z\in[n, n+2]\times i\left[-\frac{3}{4}\pi, \frac{3}{4}\pi\right]$.

Thus we have the estimate

$$|\mathscr{G}(z)| \leqq \exp(C_6 e^{2|x|})$$

for $|y|\leqq\frac{3}{4}\pi$.

Since $\mathscr{G}(z)$ is bounded for $|y|>\frac{3}{4}\pi$ by (2.10), this inequality is valid in the whole complex plane, which proves the lemma.

Remark 2.5. When one tries to obtain a Tauberian theorem like Theorem 3.3 for the ideal chains in [19] in cases when (1.4—5) are not satisfied, the extension of Lemma 2.3 is the major crucial point.

Here follows a lemma about subadditive functions. Recall that M_ψ was defined by (1.3).

Lemma 2.6. *Assume ψ is subadditive. Then the conditions*

(1.4) $$\int_0^\infty \log^+\log^+ M_\psi(x)\,dx < \infty$$

and

(1.6) $$\int_{-\infty}^\infty \frac{\psi(x)}{1+x^2}\,dx < \infty$$

are equivalent.

Proof. That (1.4) is a consequence of (1.6) for subadditive ψ follows from Gurariĭ [11], Lemma 3.5. At least he shows that $\log^+ \log^+ M_\psi$ is integrable over every finite interval $(0, \alpha)$, but since $M_\psi(x) \to 0$ as $x \to \infty$ by (1.1), this amounts to the same thing.

To get the opposite implication, assume (1.6) does not hold, that is

$$\int_{-\infty}^{\infty} \frac{\psi(x)}{1+x^2}\, dx = \infty,$$

which makes $L^1_{\exp\psi}(\mathbf{R})$ a quasianalytic Beurling algebra. Domar shows in [9] that there exists a proper closed ideal I in $L^1_{\exp\psi}(\mathbf{R})$ which is primary at ∞. For any arbitrary functional $g \in L^1_{\exp\psi}(\mathbf{R})^* = L^\infty_{\exp\psi}(\mathbf{R})$ which annihilates I, we define its analytic (Carleman) transform

$$G(z) = \begin{cases} \int_0^\infty g(t) e^{itz}\, dt, & \operatorname{Im} z > 0 \\ -\int_{-\infty}^0 g(t) e^{itz}\, dt, & \operatorname{Im} z < 0. \end{cases}$$

Domar [8] shows that G extends to an entire function (see [8], Theorem 2.4 and Example 3.2), and that G determines g uniquely (see [8], Theorem 5.1).

A simple calculation shows that the following estimate is valid:

$$|G(z)| \leqq \|g\|_{L^\infty_{\exp\psi}} \cdot M_\psi\big(d(z, \mathbf{R})\big), \quad z \in \mathbf{C} \setminus \mathbf{R},$$

where d is, as in Lemma 2.1, the Euclidean metric in $\mathbf{C}$.

If (1.4) holds we can apply the log-log theorem of Levinson and Sjöberg (see [15], Theorem XLII) to deduce that G is bounded in the whole complex plane. By Liouville's theorem G is a constant, which has to be 0, since by the definition of G, $G(z) \to 0$ as $|\operatorname{Im} z| \to \infty$. Hence $g = 0$, and by Hahn—Banach's theorem, $I = L^1_w(\mathbf{R})$. This gives us a contradiction; hence (1.4) cannot hold. That does it.

Remark 2.7. Observe that the previous proof implicitly proves the Tauberian theorem for Beurling algebras, without using the fact that $L^1_{\exp\psi}(\mathbf{R})$ is regular if ψ satisfies (1.6). This should be compared to Dales and Hayman's article [6].

Lemma 2.6 could probably be deduced from Lemma 1 in [4].

3. Main results

As in the introduction, $A_0(S)$ denotes the Banach algebra of functions analytic on S^0 and continuous on $S \cup \{\infty\}$, having the value 0 at ∞, under the uniform norm. Mapping S^0 conformally onto the unit disc D, we identify $A_0(S)$ with the closed ideal

$$I_0 = \{f \in A(D):\ f(-1) = f(1) = 0\}$$

in the disc algebra $A(D)$.

For each closed ideal I in $A(D)$ we can form the ideal $I \cap I_0$, which is a closed ideal in I_0.

Conversely, is it true that each closed ideal J in I_0 is an ideal in $A(D)$? We shall see that the answer is affirmative. Hence the Beurling—Rudin theorem (see [17]) can be used to give a complete description of all closed ideals in $A_0(S)$.

Theorem 3.1. *Each closed ideal J in I_0 is a (closed) ideal in $A(D)$.*

Proof. Define, for each positive integer n,

$$e_n(z) = n\frac{z-1}{n(z-1)-1}\, n\frac{z+1}{n(z+1)+1}, \quad z \in D.$$

One readily sees that $e_n \in I_0$ for each n, and that

$$e_n(z) \to 1, \quad \text{as} \quad n \to \infty \tag{3.1}$$

uniformly on each compact subset of $\bar{D} \setminus \{-1, 1\}$. Putting $\zeta = i\frac{z+1}{z-1}$, which maps D onto the lower half plane, we see that the first factor $n\frac{z-1}{n(z-1)-1}$ is transformed into $\frac{2ni}{2ni-\zeta+i}$, the modulus of which is bounded by 1 in the lower half-plane. A similar argument for the second factor yields that

$$\|e_n\|_\infty \leqq 1, \quad n = 1, 2, 3, \ldots.$$

This, together with (3.1), implies that

$$\|e_n x - x\|_\infty \to 0, \quad \text{as} \quad n \to \infty, \tag{3.2}$$

for each $x \in I_0$.

This is to say that $\{e_n\}$ form an approximate identity in I_0 (see [22]). Select two arbitrary elements, $x \in J$ and $y \in A(D)$. It is sufficient to prove that $xy \in J$. Since $J \subset I_0$ and I_0 is an ideal, $xy \in I_0$. Hence, by (3.2)

$$xye_n \to xy, \quad \text{as} \quad n \to \infty.$$

Now $ye_n \in I_0$ since $e_n \in I_0$. Thus $xye_n \in J$, so that

$$xy \in \bar{J} = J,$$

which establishes the theorem.

Remark 3.2. The main ingredient in the proof of Theorem 3.1 is (essentially) the fact that $A(D)$ meets the strong analytic Ditkin condition, which for instance is satisfied for the Wiener algebra $F(l^1(\mathbf{N}))$, too.

Here the two main results follow. $\delta_+(f)$ and $\delta_-(f)$ were introduced in Section 1.

Theorem 3.3. *Assume* (1.4—5). *Let M be a family of functions in $L^1_w(\mathbf{R})$ whose Fourier transforms have no common zeros in S and suppose that*

$$\inf_{f \in M} \delta_+(f) = \inf_{f \in M} \delta_-(f) = 0.$$

Let I_M be the smallest closed ideal in $L^1_w(\mathbf{R})$ containing M. Then

$$I_M = L^1_w(\mathbf{R}).$$

Proof. Write $z = x + iy$. Let the functional $g \in L^\infty_w(\mathbf{R})$ annihilate the closed ideal I_M.

Due to the theorem of Hahn—Banach it is sufficient to show that $g = 0$. $g \in I_M^\perp$ is equivalent to

$$g * \check{f} = 0$$

for all f in M.

Let $\mathscr{G}$ be the corresponding analytic (Carleman) transform, that is, the function defined by (2.2—4). By Lemma 2.1 $\mathscr{G}$ is entire. Applying Lemma 2.3, we find a constant c such that

$$|\mathscr{G}(z)| \leqq \exp(ce^{2|x|}), \quad z \in \mathbf{C}. \tag{3.3}$$

We will first investigate the behaviour of $\mathscr{G}$ in the right half-plane. Let f be an arbitrary non-zero function in M. By the Ahlfors—Heins theorem (see [2], p. 341, or [5], Theorem 7.2.6),

$$\lim_{x \to +\infty} e^{-x} \log |\hat{f}(x+iy)| = -\delta_+(f) \cos y,$$

for almost all y in the interval $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

For this dense set of y in the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, $\mathscr{G}(x+iy)$ does not grow faster than $\exp((\delta_+(f)+\varepsilon)e^x)$, as $x \to +\infty$, where $\varepsilon > 0$ is arbitrary, since $\hat{h}(x+iy)$ is bounded for each fixed y in the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, by (2.11). Taking into account

(3.3) and applying the Phragmén—Lindelöf principle, we get

$$\mathcal{G}(z) = 0(\exp((\delta_+(f)+\varepsilon)e^x)), \quad \text{as} \quad x \to +\infty,$$

for all $\varepsilon>0$, uniformly in $-\frac{3}{8}\pi \leqq y \leqq \frac{3}{8}\pi$. A second application of the Phragmén—Lindelöf principle, using (2.10) and (3.3), easily shows that

$$\mathcal{G}(z) = 0(\exp((\delta_+(f)+\varepsilon)e^x)), \quad \text{as} \quad x \to +\infty,$$

for all $\varepsilon>0$, uniformly in $y \in \mathbf{R}$. Since, by assumption,

$$\inf_{f \in M} \delta_+(f) = 0,$$

this becomes

$$\mathcal{G}(z) = 0(\exp(\varepsilon e^x)) \quad \text{as} \quad x \to +\infty \tag{3.4}$$

for every $\varepsilon>0$, uniformly in $y \in \mathbf{R}$.

Put

$$\mathcal{G}_0(z) = \exp(-2e^{z/2})\mathcal{G}(z)$$

and

$$\lambda(x) = \log^+ \log^+ M_\psi(x), \quad x > 0.$$

Then λ is a continuous decreasing function. In our terminology, decreasing does not demand strict decrease. Due to the definition of M_ψ, $\lambda(x)$ is unbounded as $x \to 0$, and $\lambda(x)=0$ for x sufficiently large.

In $\{x \in \mathbf{R}_+ : \lambda(x)>0\}$, which is a connected neighbourhood of 0, λ is a strictly decreasing C^∞ function, and therefore it has an inverse λ^{-1}.

Let θ: $\mathbf{R}_+ \to \mathbf{R}_+$ be the bounded continuous decreasing function

$$x \mapsto \lambda^{-1}\left(\frac{x}{2}\right).$$

$\theta(x)$ tends to 0 as $x \to +\infty$. By (1.4)

$$\int_0^\infty \lambda(x)\, dx < \infty,$$

so that

$$\int_0^\infty \theta(x)\, dx < \infty. \tag{3.5}$$

Define the domains

$$\Omega_+ = \left\{z = x+iy \in \mathbf{C}: \ x > 0 \quad \text{and} \quad |y| < \frac{\pi}{2}+\theta(x)\right\}$$

and

$$\Omega_- = \left\{z = x+iy \in \mathbf{C}: \ x < 0 \quad \text{and} \quad |y| < \frac{\pi}{2}+\theta(-x)\right\}.$$

For $x>0$ we have, by the definition of θ, that

$$M_\psi(\theta(x)) = \exp(e^{\lambda\circ\theta(x)}) = \exp(e^{x/2}). \tag{3.6}$$

Combining (3.6) with (2.10), we get

$$|\mathscr{G}(z)| \leqq \|g\|_{L_w^\infty} \exp(e^{x/2}) \quad \text{for} \quad z \in \partial\Omega_+,$$

with $\operatorname{Re} z>0$, which makes $\mathscr{G}_0$ bounded on $\partial\Omega_+$, say by the constant M.

Put

$$\mathscr{G}_1(z) = \mathscr{G}_0(z)/M.$$

Then either $|\mathscr{G}_1(z)| \leqq 1$ in Ω_+, or there exists a point $z_0 = x_0 + iy_0$ in Ω_+ such that $|\mathscr{G}_1(z_0)|>1$.

We assume the latter and will try to obtain a contradiction.

Let $\omega(z, \xi)$ be the harmonic measure having the value 1 on the set $\{z \in \partial\Omega_+ \colon x=\xi$ and $|y| \leqq \frac{\pi}{2} + \theta(x)\}$, and the value 0 on the parts of the boundary $\partial\Omega_+$ with $x<\xi$ (see Figure 3.4).

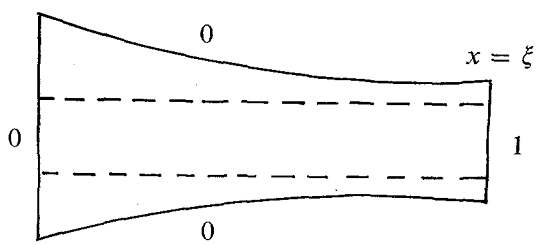

Figure 3.4

Define

$$M(\xi) = \sup_{\substack{z \in \partial\Omega_+ \\ \operatorname{Re} z = \xi}} |\mathscr{G}_1(z)|.$$

We are now about to use a generalized Phragmén—Lindelöf argument.

According to the first distortion inequality of Ahlfors (see [1] and K. Haliste [12], p. 3 or Theorem 3.2) we have the estimate

$$\omega(z_0, \xi) \leqq \frac{4}{\pi} \exp\left(4\pi - \pi \int_{x_0}^{\xi} \frac{dt}{\pi + 2\theta(t)}\right) \quad \text{if} \quad \int_{x_0}^{\xi} \frac{dt}{\pi + 2\theta(t)} > 2.$$

In the following, letters C with indices stand for positive (finite) quantities that do not depend on ξ.

Since $\log|\mathscr{G}_1|$ is a subharmonic function, we have

$$0 < \log|\mathscr{G}_1(z_0)| \leqq \omega(z_0, \xi)\log M(\xi) \leqq C_1 \log M(\xi)\exp\left(-\int_{x_0}^{\xi}\frac{dt}{1+\frac{2}{\pi}\theta(t)}\right)$$

$$\leqq C_2 \log M(\xi)\exp\left(-\int_{x_0}^{\xi}(1-\theta(t))\,dt\right) \leqq C_3 e^{-\xi}\log M(\xi)$$

due to (3.5), so that

$$M(\xi) \geqq \exp\left(C_3^{-1}e^{\xi}\log|\mathscr{G}_1(z_0)|\right),$$

contradicting the estimate (3.4). Thus

$$|\mathscr{G}_0(z)| \leqq M \text{ in } \Omega_+,$$

so that

$$|\mathscr{G}(z)| \leqq M\exp(2e^{x/2}) \text{ in } \Omega_+.$$

By (2.10), this inequality (possibly with a different constant M) is valid in the whole right half-plane, since $|\mathscr{G}(z)| \leqq \|g\| M_\psi(\theta(x)) \leqq \|g\| \exp(e^{x/2})$ for all $z \notin \Omega_+$ in the right half-plane. However, since $\mathscr{G}(z)$ is bounded for $|y| > \frac{3}{4}\pi$, again by (2.10), the principle of Phragmén—Lindelöf yields that it is bounded in the right half-plane.

The same kind of argument works in the left half-plane, too, so that $\mathscr{G}$ is bounded in the whole complex plane.

Applying Liouville's theorem, we find that $\mathscr{G}$ is a constant which has to be 0 by (2.2). Thus — by Lemma 2.1 — $g=0$, and the theorem is established.

The proof of the following theorem was inspired by ideas from Korenblum and Nyman. This time (1.4—5) do not suffice; observe that by Lemma 2.6, the condition on ψ in Theorem 3.5 actually implies (1.4—5). The ideals I_α^+ and I_β^- were introduced in Section 1.

Theorem 3.5. *Suppose ψ is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$ and satisfies the non-quasianalyticity condition*

$$\int_{-\infty}^{\infty}\frac{\psi(x)}{1+x^2}\,dx < \infty. \tag{1.6}$$

Let M be a family of functions in $L_w^1(\mathbf{R})$ whose Fourier transforms have no common zeros in S; let I_M be the smallest closed ideal containing M. If

$$\delta_1 = \inf_{f\in M}\delta_+(f) \quad\text{and}\quad \delta_2 = \inf_{f\in M}\delta_-(f),$$

then

$$I_M = I_{\delta_1}^+ \cap I_{\delta_2}^-.$$

Proof. It is evident that $I_M \subset I_{\delta_1}^+ \cap I_{\delta_2}^-$, since $I_{\delta_1}^+$ and $I_{\delta_2}^-$ are closed ideals, a fact which was mentioned in Section 1. We will now prove this. By the Beurling—Rudin theorem (see [17])

$$J_\alpha = \{f \in A(D)\colon f(-1) = f(1) = 0 \quad \text{and} \quad \limsup_{x \to 1^-} (1-x) \log |f(x)| \leqq -\alpha\}$$

is a closed ideal in $A(D)$ for $\alpha \geqq 0$.

Map S^0 conformally onto D by $z \mapsto \frac{e^z - 1}{e^z + 1}$. Identifying $A_0(S)$ and

$$\{f \in A(D)\colon f(-1) = f(1) = 0\},$$

J_α corresponds to

$$\left\{f \in A_0(S)\colon \limsup_{x \to +\infty} e^x \log |f(x)| \leqq -\frac{\alpha}{2}\right\}.$$

$I^+_{\alpha/2}$ is the intersection of this closed ideal in $A_0(S)$ with $L^1_w(\mathbf{R})$; hence by the simple norm inequality $\|\hat{f}\|_\infty \leqq \|f\|_w$, $I^+_{\alpha/2}$ is closed. This shows that $I^+_{\delta_1}$ is closed.

A similar argument for $I^-_{\delta_2}$ shows that this, too, is a closed ideal. Vretblad [19] showed this result with some extra regularity on the weight w, but his technique was quite different from ours.

To show the opposite inclusion, that is, that $I_M \supset I^+_{\delta_1} \cap I^-_{\delta_2}$, it is sufficient to prove that every functional $g \in L^\infty_w(\mathbf{R})$ satisfying

$$g * \check{f} = 0$$

for all functions f in M also satisfies this equality for all $f \in I^+_{\delta_1} \cap I^-_{\delta_2}$. Roughly speaking, our technique will be to multiply the functions in I_M by a suitable growth function and then to apply the Tauberian Theorem 3.3.

We write $z = x + iy$ and $\zeta = \xi + i\eta$. Let us keep the notation

$$\lambda(x) = \log^+ \log^+ M_\psi(x)$$

and

$$\theta(x) = \lambda^{-1}\left(\frac{x}{2}\right)$$

from the proof of Theorem 3.3.

According to the assumptions on ψ (one of which is (1.2)), ψ is subadditive, so by Lemma 2.6, (1.6) implies (1.4—5). Hence the conditions of Theorem 3.3 are satisfied.

Put $\Omega = \overline{\Omega_+ \cup \Omega_-^0}^{\,0}$, or explicitly,

$$\Omega = \left\{z = x + iy \in \mathbf{C}\colon |y| < \frac{\pi}{2} + \theta(|x|)\right\}.$$

Let φ denote a conformal mapping $\Omega \to S^0$, which can be extended continuously so that $+\infty$ and $-\infty$ are mapped onto themselves. Such a function φ exists and extends to a homeomorphism $\bar{\Omega} \cup \{-\infty, +\infty\} \to S \cup \{-\infty, +\infty\}$ — due to a famous theorem by Carathéodory — since the boundary $\partial\Omega \cup \{-\infty, +\infty\}$ is a Jordan curve.

By a refined version of Ahlfors's distortion inequalities, due to Warschawski (see [20], pp. 290—296, Theorems III(a) and IV(a)), we find, using (3.5), that

$$\text{(3.7)} \qquad \operatorname{Re} \varphi(z) = x + 0(1),$$

as $|x| \to \infty$, uniformly in y.

We have to check that Warschawski's conditions are satisfied. To this end, we will first show that $\log M_\psi$ is a convex (and decreasing) function. Choose $0<t<1$ arbitrarily. Then, by Hölder's inequality,

$$M_\psi(tx+(1-t)y) = \int_{-\infty}^{\infty} e^{-(tx+(1-t)y)|\xi|+\psi(\xi)}\, d\xi$$

$$= \int_{-\infty}^{\infty} e^{-tx|\xi|+t\psi(\xi)-(1-t)y|\xi|+(1-t)\psi(\xi)}\, d\xi$$

$$\leqq \Big(\int_{-\infty}^{\infty} e^{-x|\xi|+\psi(\xi)}\, d\xi\Big)^t \Big(\int_{-\infty}^{\infty} e^{-y|\xi|+\psi(\xi)}\, d\xi\Big)^{1-t} = (M_\psi(x))^t (M_\psi(y))^{1-t}$$

for $x, y>0$, which shows that $\log M_\psi$ is convex.

Since $\log M_\psi$ is strictly decreasing, unbounded, and convex, a simple argument, which we are about to present, shows that

$$-(\log M_\psi)' \geqq c \log M_\psi$$

in some interval $(0, \varepsilon_0)$, $\varepsilon_0>0$, where c is a positive constant.

In fact, we can choose c as

$$c = \frac{|(\log M_\psi)'(\varepsilon_0)|}{\log M_\psi(\varepsilon_0)+\varepsilon_0 |(\log M_\psi)'(\varepsilon_0)|}$$

if $M_\psi(\varepsilon_0)>1$, which is shown by. Since

$$\frac{d}{dx}\Big(x - \frac{\log M_\psi(x)}{(\log M_\psi)'(x)}\Big) = \frac{(\log M_\psi)''(x) \log M_\psi(x)}{((\log M_\psi)'(x))^2} \geqq 0$$

in the interval $(0, \varepsilon_0)$, we have that

$$\frac{\log M_\psi(x)}{-(\log M_\psi)'(x)} \leqq x - \frac{\log M_\psi(x)}{(\log M_\psi)'(x)} \leqq \varepsilon_0 - \frac{\log M_\psi(\varepsilon_0)}{(\log M_\psi)'(\varepsilon_0)} = \frac{1}{c}$$

with our particular choice of c, for all $0<x<\varepsilon_0$. Hence

$$\lambda' \leqq -c$$

in some (possibly smaller) interval $(0, \varepsilon_1)$, $\varepsilon_1>0$, which implies that

$$-\frac{1}{c} \leqq \theta' \leqq 0$$

in some interval (A, ∞). Finally, we show that

$$\int_A^\infty \frac{(\theta'(x))^2}{\pi+2\theta(x)}\,dx<\infty.$$

To see this, we make the following estimate:

$$\int_A^\infty \frac{(\theta'(x))^2}{\pi+2\theta(x)}\,dx \leqq \frac{1}{\pi c}\int_A^\infty |\theta'(x)|\,dx = -\frac{1}{\pi c}\int_A^\infty \theta'(x)\,dx = \frac{1}{\pi c}\,\theta(A)<\infty,$$

where we have used that $-\frac{1}{c}\leqq\theta'\leqq 0$ on (A, ∞).

We have now verified all conditions for Warschawski's Theorems III(a) and IV(a).

Let $dQ_\Omega(z, \zeta)$, $\zeta\in\partial\Omega$, denote the Poisson measure in Ω. Then each bounded harmonic function h on Ω, which extends continuously to the boundary, satisfies

$$h(z) = \int_{\zeta\in\partial\Omega} h(\zeta)\,dQ_\Omega(z, \zeta), \quad z\in\Omega.$$

Let $r(z)=\theta(|x|)e^{|x|}$, for $z\in\partial\Omega$, and let $\varrho(z)$ be the Poisson integral of r in Ω, that is,

(3.8) $$\varrho(z) = \int_{\zeta\in\partial\Omega} r(\zeta)\,dQ_\Omega(z, \zeta).$$

We have to check that (3.8) converges. To this end, we map Ω onto S^0 by φ and then onto the open upper half plane Π by $z\mapsto ie^z$.

The Poisson kernel $P_\Pi(z, \zeta)$ for Π can be estimated from above and below by $\frac{1}{1+\xi^2}$ times constants depending on z. If $P_S(z, \zeta)$ denotes the Poisson kernel in S^0, this means that there exists a constant C_z only depending on z, such that

(3.9) $$P_S(z, \zeta) \leqq C_z e^{-|\xi|}, \quad z\in S^0, \quad \zeta\in\partial S.$$

By (3.8—9),

$$\varrho(z) = \int_{\zeta\in\partial\Omega} r(\zeta)\,dQ_\Omega(z, \zeta) = \int_{\partial S} P_S(\varphi(z), \zeta)\, r\circ\varphi^{-1}(\zeta)\,|d\zeta|$$
$$\leqq C_{\varphi(z)}\int_{\partial S} e^{-|\xi|}\, r\circ\varphi^{-1}(\zeta)\,|d\zeta| = C_{\varphi(z)}\int_{\partial\Omega} e^{-|\mathrm{Re}\,\varphi(\zeta)|}\, r(\zeta)\,|d\varphi(\zeta)|.$$

Let us put $\theta(\zeta)=\theta(|\xi|)$ for $\zeta\in\mathbf{C}$. Then, by (3.7),

$$\int_{\partial\Omega} e^{-|\mathrm{Re}\,\varphi(\zeta)|}\, r(\zeta)\,|d\varphi(\zeta)|$$
$$= \int_{\partial\Omega} e^{-|\mathrm{Re}\,\varphi(\zeta)|} e^{|\xi|}\,\theta(\zeta)\,|d\varphi(\zeta)| \leqq C\int_{\partial\Omega} \theta(\zeta)\,|d\varphi(\zeta)|,$$

where C is a positive constant. Since φ maps $\partial\Omega$ homeomorphically onto ∂S, $d\varphi(z)$ is a positive measure on $\partial\Omega$ if we assume that the integration on $\partial\Omega$ is directed from

$-\infty$ to $+\infty$. By (3.5), (3.7), and partial integration

$$\int_{\partial\Omega}\theta(\zeta)\,|d\varphi(\zeta)| = \int_{\partial\Omega}\theta(\zeta)\,d\varphi(\zeta) = -\int_{\partial\Omega}\operatorname{Re}\varphi(\zeta)\,d\theta(\zeta) = \int_{\partial\Omega}\operatorname{Re}\varphi(\zeta)(-d\theta(\zeta))$$

$$\leqq K+\int_{\partial\Omega}\xi(-d\theta(\zeta)) = K+\int_{\partial\Omega}\theta(\zeta)\,d\xi = K+2\int_{-\infty}^{\infty}\theta(|\xi|)\,d\xi < \infty,$$

where K is a positive constant.

The term from infinity vanishes because of (3.7) and (3.5), which implies that $x\theta(x)\to 0$, as $x\to+\infty$.

Combining the above estimates we have now shown that ϱ is a well-defined harmonic function.

By the construction ϱ coincides with r on $\partial\Omega$. Let R denote the holomorphic function which satisfies

$$\begin{cases}\operatorname{Re} R(z) = \varrho(z), & z\in\Omega \\ \operatorname{Im} R(0) = 0.\end{cases} \tag{3.10}$$

Thus $\operatorname{Re} R(z)\geqq 0$ in Ω.

Put $\delta=\delta_1+\delta_2$. Define $\hat{v}$ to be the holomorphic function

$$z \mapsto \exp\left(-2e^{\frac{3}{4}z}-2e^{-\frac{3}{4}z}\right)\exp\left(-2\delta R(z)\right), \quad z\in\Omega.$$

We will now show that $\hat{v}$ is the Fourier transform of a function v in $L^1_w(\mathbf{R})$. By the Fourier inversion formula,

$$v(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{v}(x)e^{ixt}\,dx.$$

Changing the path of integration, which evidently is allowed, we obtain

$$v(t) = \frac{1}{2\pi}\int_{\Gamma_1}\hat{v}(z)e^{itz}\,dz = \frac{1}{2\pi}\int_{\Gamma_2}\hat{v}(z)e^{itz}\,dz,$$

where Γ_1 and Γ_2 are the two components of $\partial\Omega$, directed from $-\infty$ to $+\infty$, Γ_1 being the upper one.

We get, by (3.6),

$$\int_0^{\infty}|v(t)|\,w(t)\,dt \leqq C_0\int_0^{\infty}\int_0^{\infty}\exp\left(-\frac{1}{2}e^{\frac{3}{4}x}\right)e^{-t\theta(x)+\psi(t)}\,dx\,dt$$

$$\leqq C_0\int_0^{\infty}\exp\left(-\frac{1}{2}e^{\frac{3}{4}x}\right)M_{\psi}(\theta(x))\,dx = C_0\int_0^{\infty}\exp\left(-\frac{1}{2}e^{\frac{3}{4}x}+e^{x/2}\right)dx < \infty.$$

We have used the simple numerical fact that $2\cos\frac{3\pi}{8}>\frac{1}{2}$. Here and in the following, letters C with indices are positive constants.

The corresponding result for $(-\infty, 0)$ yields that $v \in L^1_w(\mathbf{R})$. Let

$$K(z) = \exp\left(-e^{\frac{3}{4}z} - e^{-\frac{3}{4}z}\right) \exp\left(-\delta R(z)\right) \exp\left(-\delta_1 e^z - \delta_2 e^{-z}\right), \quad z \in \Omega.$$

We now intend to prove that $z \mapsto K(z)$, $z \in S$, too, is the Fourier transform of a function $k \in L^1_w(\mathbf{R})$.

First we will show that

$$K_0(z) = \exp\left(-\delta R(z)\right) \exp\left(-\delta_1 e^z - \delta_2 e^{-z}\right)$$

is bounded in Ω. By (3.10), this expression is bounded on $\partial\Omega$:

$$|K_0(z)| \leqq \exp\left(-\delta e^{|x|} \theta(|x|)\right) \exp\left(\delta e^{|x|} \sin \theta(|x|)\right) \leqq 1$$

for $z \in \partial\Omega$. Since

$$|\exp\left(-\delta_1 e^z - \delta_2 e^{-z}\right)| \leqq \exp\left(\delta e^{|x|} \sin \theta(|x|)\right) \leqq \exp\left(\delta e^{|x|} \theta(|x|)\right), \quad z \in \Omega,$$

and $\operatorname{Re} R(z) \geqq 0$ in Ω, we have that

$$|K_0(z)| \leqq \exp\left(\delta e^{|x|} \theta(|x|)\right) \quad \text{for} \quad z \in \Omega.$$

The generalized Phragmén—Lindelöf argument using distortion inequalities, used for $\mathscr{G}_0$ in the proof of Theorem 3.3, applies to K_0, too, and together with the fact that $\theta(|x|) \to 0$ as $|x| \to \infty$, it shows that K_0 is bounded in Ω, since K_0 is bounded on $\partial\Omega$. Hence

$$|K(z)| \leqq C_1 \exp\left(-\frac{1}{4} e^{\frac{3}{4}|x|}\right), \quad z \in \Omega. \tag{3.11}$$

Changing the path of integration in the Fourier inversion formula, which is allowed by (3.11), we obtain

$$k(t) = \frac{1}{2\pi} \int_{\Gamma_1} K(z) e^{itz}\, dz = \frac{1}{2\pi} \int_{\Gamma_2} K(z) e^{itz}\, dz.$$

We get by (3.6) and (3.11) that

$$\int_{-\infty}^{\infty} |k(t)|\, w(t)\, dt \leqq C_2 \int_{-\infty}^{\infty} \int_0^{\infty} \exp\left(-\frac{1}{4} e^{\frac{3}{4}x}\right) e^{-\theta(x)|t| + \psi(t)}\, dx\, dt$$

$$= C_2 \int_0^{\infty} \exp\left(-\frac{1}{4} e^{\frac{3}{4}x}\right) M_\psi(\theta(x))\, dx = C_2 \int_0^{\infty} \exp\left(-\frac{1}{4} e^{\frac{3}{4}x} + e^{x/2}\right) dx < \infty.$$

We conclude that $k \in L^1_w(\mathbf{R})$. We will soon establish that to each $f \in I^+_{\delta_1} \cap I^-_{\delta_2}$ there exists a unique $\tilde{f} \in L^1_w(\mathbf{R})$ such that

$$v * f = k * \tilde{f}. \tag{3.12}$$

Taking this for granted for the moment, we will show how the theorem follows.

Let $g\in L^\infty_w(\mathbf{R})$ satisfy

$$g*\check{f}=0$$

for all $f\in M$. Then

$$g*\check{f}*\check{v}=0,$$

so that

$$g*\check{k}*\check{\check{f}}=0$$

for $f\in M$.

By the definition of $K=\hat{k}$,

$$\begin{cases}\lim\limits_{x\to+\infty} e^{-x}\log|K(x)| = -\delta_1,\\ \lim\limits_{x\to-\infty} e^{x}\log|K(x)| = -\delta_2.\end{cases} \tag{3.13}$$

To see this, we will show that

$$\varrho(x)=o(e^{|x|}),\quad |x|\to\infty, \tag{3.14}$$

from which (3.13) immediately follows. ϱ was defined as the Poisson integral in Ω of the positive function r. Mapping Ω onto the open upper half-plane Π by $z\mapsto ie^{\varphi(z)}$ and $z\mapsto ie^{-\varphi(z)}$, it suffices, by (3.7), to show that every positive function $f_0\in L^1\left(\frac{dt}{1+t^2}\right)$, which the Poisson integral formula extends harmonically to Π, satisfies

$$f_0(iy)=o(y),\quad y\to+\infty.$$

Let $y\geqq 1$. Then

$$f_0(iy)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f_0(t)}{t^2+y^2}\,dt=\frac{y}{\pi}\int_{-\sqrt{y}}^{\sqrt{y}}\frac{f_0(t)}{t^2+y^2}\,dt+\frac{y}{\pi}\int_{\mathbf{R}\setminus(-\sqrt{y},\sqrt{y})}\frac{f_0(t)}{t^2+y^2}\,dt.$$

The first term satisfies

$$\frac{y}{\pi}\int_{-\sqrt{y}}^{\sqrt{y}}\frac{f_0(t)}{t^2+y^2}\,dt\leqq\frac{1}{\pi}\int_{-\sqrt{y}}^{\sqrt{y}}\frac{f_0(t)}{1+t^2}\,dt\leqq\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{f_0(t)}{1+t^2}\,dt<\infty.$$

The second term satisfies

$$\frac{y}{\pi}\int_{\mathbf{R}\setminus(-\sqrt{y},\sqrt{y})}\frac{f_0(t)}{t^2+y^2}\,dt\leqq\frac{y}{\pi}\int_{\mathbf{R}\setminus(-\sqrt{y},\sqrt{y})}\frac{f_0(t)}{1+t^2}\,dt=o(y),\quad y\to+\infty.$$

Combining these estimates, we obtain the desired result

$$f_0(iy)=o(y),\quad y\to+\infty.$$

This completes the verification of (3.13—14). By (3.14),

$$\lim_{|x|\to+\infty} e^{-|x|}\log|\hat{v}(x)|=0,$$

and taking (3.12—13) into account we obtain the relations

$$\delta_+(\tilde{f}) = \delta_+(f) - \delta_1,$$
$$\delta_-(\tilde{f}) = \delta_-(f) - \delta_2,$$

so that Theorem 3.3 implies that

$$g * \check{k} = 0,$$

hence

$$g * \check{f} * \check{v} = 0$$

for all $f \in I^+_{\delta_1} \cap I^-_{\delta_2}$. Since $\delta_+(v) = \delta_-(v) = 0$, by (3.14), we can apply Theorem 3.3 again to obtain

$$g * \check{f} = 0$$

for all functions $f \in I^+_{\delta_1} \cap I^-_{\delta_2}$, which was the desired conclusion.

All that remains to us now is to establish the existence of $\tilde{f}$ in (3.12) for every $f \in I^+_{\delta_1} \cap I^-_{\delta_2}$.

(3.12) can be reformulated in the following way:

(3.15) $$\hat{\tilde{f}}(z) = \hat{f}(z) \exp(\delta_1 e^z + \delta_2 e^z) \exp\left(-e^{\frac{3}{4}z} - e^{-\frac{3}{4}z}\right) \exp(-\delta R(z)), \quad z \in S.$$

By the Phragmén—Lindelöf principle,

$$\hat{f}(z) \exp(\delta_1 e^z + \delta_2 e^{-z})$$

is bounded on S. Combining this with the expression (3.15) and the Fourier inversion formula, we obtain

(3.16)
$$\tilde{f}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\tilde{f}}\left(x + i\frac{\pi}{2}\right) e^{i\left(x + i\frac{\pi}{2}\right)t} \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\tilde{f}}\left(x - i\frac{\pi}{2}\right) e^{i\left(x - i\frac{\pi}{2}\right)t} \, dx.$$

Put

$$\psi_1(t) = \begin{cases} \psi(t), & t \geqq 0 \\ 0, & t < 0 \end{cases}$$

and

$$\psi_2(t) = \psi(t) - \psi_1(t).$$

By the assumptions on ψ, the functions ψ_1 and ψ_2 are subadditive, making $L^1_{\exp\psi_1}(\mathbf{R})$ and $L^1_{\exp\psi_2}(\mathbf{R})$ into Banach algebras under convolution multiplication. If we can show that $\hat{\tilde{f}}\left(x + i\frac{\pi}{2}\right)$ and $\hat{\tilde{f}}\left(x - i\frac{\pi}{2}\right)$ are Fourier transforms of functions in $L^1_{\exp\psi_1}(\mathbf{R})$ and $L^1_{\exp\psi_2}(\mathbf{R})$, respectively, this will be sufficient by (3.16) to prove that $\tilde{f} \in L^1_w(\mathbf{R})$.

Since $f \in L^1_w(\mathbf{R})$, we have that $\hat{f}\left(x + i\frac{\pi}{2}\right)$ and $\hat{f}\left(x - i\frac{\pi}{2}\right)$ are Fourier transforms of functions in $L^1_{\exp\psi_1}(\mathbf{R})$ and $L^1_{\exp\psi_2}(\mathbf{R})$, respectively.

By (3.15), it will therefore suffice to show that

$$H_1(x) = \exp\left(-e^{\frac{3}{4}\left(x+i\frac{\pi}{2}\right)} - e^{-\frac{3}{4}\left(x+i\frac{\pi}{2}\right)}\right)$$

$$\cdot \exp\left[-\delta R\left(x+i\frac{\pi}{2}\right)\right] \exp\left(\delta_1 e^{x+i\frac{\pi}{2}} + \delta_2 e^{-x-i\frac{\pi}{2}}\right)$$

is the Fourier transform of a function $h_1 \in L^1_{\exp\psi_1}(\mathbf{R})$, and that the corresponding expression for $y=-\frac{\pi}{2}$ is the Fourier transform of an element $h_2 \in L^1_{\exp\psi_2}(\mathbf{R})$. We shall make the necessary verifications for h_1. For h_2 the process is analogous. $H_1(z)$ is analytic in

$$\{z \in \mathbf{C}:\ -\pi-\theta(|x|) < y < \theta(|x|)\}.$$

One easily sees that

$$\exp\left[-\delta R\left(z+i\frac{\pi}{2}\right)\right] \exp\left(\delta_1 e^{z+i\frac{\pi}{2}} + \delta_2 e^{-z-i\frac{\pi}{2}}\right)$$

is bounded on

$$\{z \in \mathbf{C}:\ -e^{-|x|} < y < \theta(|x|)\},$$

since $\operatorname{Re} R \geqq 0$. Hence we can change the path of integration in the Fourier inversion formula to obtain

$$h_1(t) = \frac{1}{2\pi}\int_{\gamma_1} e^{itz} H_1(z)\, dz = \frac{1}{2\pi}\int_{\gamma_2} e^{itz} H_1(z)\, dz,$$

where γ_1 and γ_2 are the curves

$$y = \theta(|x|) \quad \text{and} \quad y = -e^{-|x|},$$

respectively, both oriented from $-\infty$ to $+\infty$.

We get by (3.6) that

$$\int_0^\infty |h_1(t)| \exp\psi(t) \leqq C_3 \int_0^\infty \int_0^\infty \exp\left(-\frac{1}{4} e^{\frac{3}{4}x}\right) e^{-t\theta(x)+\psi(t)}\, dx\, dt$$

$$\leqq C_3 \int_0^\infty \exp\left(-\frac{1}{4} e^{\frac{3}{4}x}\right) M_\psi(\theta(x))\, dx = C_3 \int_0^\infty \exp\left(-\frac{1}{4} e^{\frac{3}{4}x} + e^{x/2}\right) dx < \infty.$$

On the left semi-axis, we get

$$\int_{-\infty}^0 |h_1(t)|\, dt \leqq C_4 \int_{-\infty}^0 \int_0^\infty \exp\left(-\frac{1}{4} e^{\frac{3}{4}x}\right) e^{te^{-x}} \exp(\delta e^x \sin e^{-x})\, dx\, dt$$

$$\leqq C_5 \int_0^\infty \exp\left(-\frac{1}{4} e^{\frac{3}{4}x}\right) e^x\, dx < \infty,$$

hence $h_1 \in L^1_{\exp\psi_1}(\mathbf{R})$.

This finishes the proof of the theorem.

Remark 3.6. Theorem 3.5 is another way of stating that the chain $\{I_\alpha^+ \cap I_\beta^-\}_{\alpha,\beta\geqq 0}$ contains all closed ideals primary at infinity.

It is easily established that

$$\bigcap_{\alpha>0} I_\alpha^+ = \bigcap_{\beta>0} I_\beta^- = \{0\}.$$

In the last proof we constructed some elements in $L_w^1(\mathbf{R})$, among them the function k, which by (3.13) belongs to $I_{\delta_1}^+ \cap I_{\delta_2}^-$. In fact, k generates $I_{\delta_1}^+ \cap I_{\delta_2}^-$, by (3.13). Since $\hat{k}(z)$ has no zeros in S, $I_{\delta_1}^+ \cap I_{\delta_2}^-$ is a (closed) primary ideal at ∞ for each pair $\delta_1, \delta_2 \geqq 0$.

Naturally, under the restraints imposed on ψ in the formulation of Theorem 3.5, Theorem 3.3 is a simple consequence of Theorem 3.5.

References

1. Ahlfors, L., Untersuchungen zur Theorie der konformen Abbildungen und der ganzen Funktionen. *Acta Soc. Sci. Fenn. N. S.* 1. **9** (1930).
2. Ahlfors, L. and Heins, M., Questions of regularity connected with the Phragmén—Lindelöf principle. *Ann. of Math.* (2) **50** (1949), 341—346.
3. Beurling, A., Sur les intégrales de Fourier absolument convergentes et leur application à une transformation fonctionelle. *9ème Congrès des Mathématiciens Scandinaves,* août 1938. Helsinki (1939), 345—366.
4. Beurling, A., Analytic continuation across a linear boundary. *Acta Math.* **128** (1972), 153—182.
5. Boas, R. P., Jr., *Entire functions.* Academic Press, New York 1954.
6. Dales, H. G. and Hayman, W. K., Esterle's proof of the Tauberian theorem for Beurling algebras. *Ann. Inst. Fourier* **31** (1981), 141—150.
7. Domar, Y., On the existence of a largest subharmonic minorant of a given function. *Ark. Mat.* **3** (1957), 429—440.
8. Domar, Y., On the analytic transform of bounded linear functionals on certain Banach algebras. *Studia Math.* **53** (1975), 203—224.
9. Domar, Y., Bilaterally translation invariant subspaces of weighted $L^p(R)$. *Springer Lecture Notes* **975** (1983), 210—213.
10. Gelfand, I., Raĭkov, D. and Šilov, G., *Commutative normed rings.* Chelsea, 1964.
11. Gurariĭ, V. P., Harmonic analysis in spaces with a weight. *Trudy Mat. Obšč.* **35** (1976), 21—76. (English translation in *Trans. Moscow Math. Soc.* **35** (1979), 21—75.)
12. Haliste, K., Estimates of harmonic measures. *Ark. Mat.* **6** (1965), 1—31.
13. Koosis, P., *Introduction to H_p spaces.* London Math. Soc. Lectures Notes **40** (1980).
14. Korenblum, B. I., A generalization of Wiener's Tauberian theorem and harmonic analysis of rapidly increasing functions. (Russian) *Trudy Moskov Mat.* Obšč. **7** (1958), 121—148.
15. Levinson, N., Gap and density theorems. *Amer. Math. Soc., Providence,* R.I. 1940.
16. Nyman, B., On the one-dimensional translation group and semigroup in certain function spaces. Thesis. Uppsala 1950.
17. Rudin, W., The closed ideals in an algebra of analytic functions. *Canad. J. Math.* **9** (1957), 426—434.

18. SJÖBERG, N., Sur les minorantes sousharmoniques d'une fonction donné. *9ème Congrès des Mathématiciens Scandinaves,* Helsinki (1939), 309—319.
19. VRETBLAD, A., Spectral analysis in weighted L^1 spaces on **R**. *Ark. Mat.* **11** (1973), 109—138.
20. WARSCHAWSKI, S. E., On conformal mapping of infinite strips. *Trans. Amer. Math. Soc.* **51** (1942), 280—335.
21. WIENER, N., *The Fourier integral and certain of its applications.* Cambridge 1933.
22. ŻELAZKO, W., *Banach algebras.* Elsevier 1973.

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