On spaces of maps from Riemann surfaces to Grassmannians and applications to the cohomology of moduli of vector bundles

Frances Kirwan

Introduction

The cohomology groups of the moduli space of complex vector bundles of coprime rank n and degree d over a compact Riemann surface M have been computed by Harder and Narasimhan [H & N] and also by Atiyah and Bott [A & B]. The basic idea of Atiyah and Bott is to apply equivariant Morse theory to the Yang—Mills functional on the infinite-dimensional affine space of connections on a fixed C^∞ bundle. However they avoid the analytic problems involved in infinite-dimensional Morse theory by giving an alternative definition of the Morse stratification in terms of the canonical filtrations introduced by Harder and Narasimhan. This stratification turns out to be equivariantly perfect relative to the gauge group of the bundle, which means that the equivariant Morse inequalities are in fact equalities. These then provide an inductive formula for the equivariant cohomology of the minimal or semistable stratum, from which the cohomology of the moduli space can be calculated.

Atiyah observed that the same idea could be applied to the action of a complex reductive group G on a finite-dimensional nonsingular complex projective variety X. Such a variety has a symplectic structure which is preserved by a maximal compact subgroup K of G. To this symplectic action there is associated a moment map. The Yang—Mills functional can be regarded as an analogue of the norm-square of the moment map. In [K] it is shown that the norm-square of the moment map always induces an equivariantly perfect stratification of X. In good cases when every semistable point of X is stable, this stratification can be used to obtain a formula for the Betti numbers of the geometric invariant theory quotient of X by G.

One method of constructing the moduli spaces of vector bundles over M is to identify them with quotients of certain finite-dimensional quasi-projective varieties, which are spaces of holomorphic maps from M into Grassmannians. The question arises whether Atiyah and Bott's formula for the cohomology of the moduli space can be rederived by applying the theory of [K] to these varieties. One of the two aims of this paper is to show that this can be done.

The major obstacle is that we are dealing with varieties which are not projective. This causes technical problems of convergence. Moreover a property of projective varieties which was important in [K] is that their equivariant cohomology is easy to compute. In fact it is just the tensor product of the ordinary cohomology of the variety with that of the classifying space BG. This fails in general for quasi-projective varieties.

Another problem is that the stratifications of these varieties do not correspond precisely with Atiyah and Bott's stratifications using canonical filtrations. However they correspond outside a subset whose codimension tends to infinity with the degree d. Such a subset will only affect cohomology in high dimensions. We then observe that the moduli space of bundles of rank n and degree d can be identified with the moduli space of bundles of rank n and degree $d+ne$ for any integer e by tensoring with a fixed line bundle of degree e. By choosing e large we get an approximate solution giving the cohomology of the moduli space up to some arbitrarily high dimension.

The inductive formula which we obtain in this way involves the cohomology of a space $\tilde{R}$ of holomorphic maps of degree $d+ne$ from the Riemann surface M to an infinite Grassmannian $G(n, \infty)$. The inductive formulas of Atiyah and Bott look very similar except that they involve the cohomology of the entire space of continuous maps from M to $G(n, \infty)$. From this it is possible to deduce inductively that $\tilde{R}$ and the space of all continuous maps have the same cohomology up to an arbitrarily high dimension provided the degree is high enough.

This is not an entirely unexpected result. Indeed Graeme Segal has shown in [S] that the inclusion of the space of all holomorphic maps of degree d from M to any complex projective space $\mathbf{P}_m$ into the space of all continuous maps of the same degree is a homology equivalence up to dimension $(d-2g)(2m-1)$ where g is the genus of M. He conjectures that analogous results hold for maps into a much larger class of algebraic varieties including Grassmannians and flag manifolds. The first part of this paper gives a proof of this conjecture in the case of Grassmannians $G(n, m)$ for finite n and m. (For related results about maps into flag manifolds see [G].) The proof is by induction and relies heavily on Segal's original theorem. It uses the fact that any nonsingular variety has the property that its cohomology groups in low dimensions are unchanged when a closed subvariety of high codimension is removed. This follows trivially from the existence of the Thom—Gysin sequence. Moreover it is necessary to know that certain singular varieties have a similar property (see § 6).

In the case $m=\infty$ one finds that the inclusion of the space $\tilde{R}$ mentioned above in the space of all continuous maps is a cohomology equivalence up to some high dimension depending on d but one does not obtain the same result about the space of all holomorphic maps. The space $\tilde{R}$ is an open subset of the space of all holomorphic maps, but is not dense (see § 7).

Thus we have a triangle of results, any two of which imply the third, but which can all be proved independently. The first is Atiyah and Bott's inductive formula for the cohomology of the moduli space. The second is the formula obtained by finite-dimensional methods, involving the cohomology of a space of holomorphic maps from M to $G(n, \infty)$. The third is the special case $m=\infty$ of the extension of Segal's theorem to maps from M to a Grassmannian $G(n, m)$.

I would like to express my gratitude to Michael Atiyah and Graeme Segal to whom many of the basic ideas of this article are due, and to M. S. Narasimhan for some very helpful comments.

Contents

I. THE EXTENSION OF SEGAL'S THEOREM

1. Statement of results

Let M be a compact Riemann surface of genus g. Let $\mathrm{Map}_d(M, \mathbf{P}_m)$ be the space of continuous maps of degree d from M into the complex projective space $\mathbf{P}_m$. Let $\mathrm{Hol}_d(M, \mathbf{P}_m)$ be the subset of $\mathrm{Map}_d(M, \mathbf{P}_m)$ consisting of holomorphic maps, which is a quasi-projective variety. Segal shows ([S] prop. 1.3) that the inclusion of this subset is a cohomology equivalence up to dimension $(d-2g)(2m-1)$. Note that this dimension tends to infinity with d.

Our aim is to extend Segal's result to the case when the projective space is replaced by an arbitrary Grassmannian $G(n, m)$. However difficulties arise because $\mathrm{Hol}_d(M, G(n, m))$ is a singular variety. Therefore it is technically simplest to show first that the result holds for a certain nonsingular open subset $A_d(n, m)$ of $\mathrm{Hol}_d(M, G(n, m))$. It turns out that the codimension of the complement of this subset tends to infinity with d. From this together with some conditions on the singularities we can deduce that the cohomology of $A_d(n, m)$ is isomorphic to the cohomology of $\mathrm{Hol}_d(M, G(n, m))$ up to some dimension which tends to infinity with d.

The subset $A_d(n, m)$ will reappear in II.

Without any loss of generality we may consider quotient Grassmannians instead of the ordinary ones.

Definition. *Let $G(n, m)$ be the Grassmannian of n-dimensional quotients of $\mathbf{C}^m$. Then there is a natural one-one correspondence between holomorphic maps*

$$h: M \to G(n, m)$$

and holomorphic quotients

$$M \times \mathbf{C}^m \to E$$

with rank n of the trivial complex bundle of rank m over M. The fibre of E at x is the quotient $h(x)$ of $\mathbf{C}^m$.

Let $A_d(n, m)$ be the space of all those holomorphic maps h such that the induced bundle E has degree d and satisfies

$$H^1(M, E) = 0.$$

We shall prove

1.1. Theorem. *Let k be a positive integer with $k \geqq n-2g$. Then the inclusion*

$$A_d(n, m) \to \mathrm{Map}_d(M, G(n, m))$$

induces isomorphisms in cohomology up to dimension k, provided that $d \geqq d_0(k, n)$

where

$$d_0(k, n) = 2n(2g+k+1)+n \max(k+1+n(2g+k+1), \tfrac{1}{4} n^2 g).$$

We shall deduce

1.2. Theorem. *Let k be a positive integer with $k \geqq n-2g$. Then the inclusion*

$$\mathrm{Hol}_d(M, G(n, m)) \to \mathrm{Map}_d(M, G(n, m))$$

induces isomorphisms in cohomology up to dimension $k-2m^2g$, provided that $d \geqq d_0(k, n)$ where $d_0(k, n)$ is defined as in theorem 1.1.

1.3. *Remark.* Of course by the universal coefficient formula it is equivalent to consider homology instead of cohomology.

1.4. *Remark.* There is an anti-holomorphism between the complex manifolds $G(n, m)$ and $G(m-n, n)$ which one obtains by choosing a hermitian inner product on $\mathbf{C}^m$ and taking orthogonal complements. This anti-holomorphism induces a homeomorphism

$$\mathrm{Map}_d(M, G(n, m)) \to \mathrm{Map}_d(M, G(m-n, m))$$

which restricts to a homeomorphism from $\mathrm{Hol}_d(M, G(n, m))$ to the complex conjugate of $\mathrm{Hol}_d(M, G(m-n, m))$ in $\mathrm{Map}_d(M, G(m-n, m))$ and also to a homeomorphism from $A_d(n, m)$ to the complex conjugate of $A_d(m-n, m)$. Therefore the inclusion of $A_d(n, m)$ (or of $\mathrm{Hol}_d(M, G(n, m))$) in $\mathrm{Map}_d(M, G(n, m))$ is a homology equivalence up to dimension k if and only if the same is true when n is replaced by $m-n$.

1.5. *Remark.* When $d > 2g-1$ where g is the genus of M then every line bundle L on M satisfies $H^1(M, L)=0$ and hence $A_d(1, m)=\mathrm{Hol}_d(M, G(1, m))$. Thus both theorems follow immediately from Segal's result in the cases $n=1$ and $n=m-1$.

1.6. *Remark.* The fact that the bound $d_0(k, n)$ in theorem 1.1 depends only on k and n but not on m will be important later when we consider what happens as $m \to \infty$ (see § 7). We shall be able to deduce that the inclusion of $A_d(n, \infty)$ in $\mathrm{Map}_d(M, G(n, \infty))$ is a homology equivalence up to any dimension k provided that d is large enough. The fact that in theorem 1.2 one only has isomorphisms up to dimension $k-2m^2g$ will prevent us from deducing a similar result about the inclusion of $\mathrm{Hol}_d(M, G(n, m))$.

Note that Segal proves that the inclusion of $\mathrm{Hol}_d(M, \mathbf{P}_m)$ in $\mathrm{Map}_d(M, \mathbf{P}_m)$ is a homology equivalence up to dimension k if $d \geqq 2g+(k+1)/(2m-1)$. In particular the same is true if $d \geqq 2g+k+1$, and $2g+k+1$ is independent of m. Of course when m and k are large the bound $2g+(k+1)/(2m-1)$ is much better than the bound $2g+k+1$. In the same way one can obtain better bounds than $d_0(k, n)$ which depend on m.

2. Reduction of Theorem 1.1 to four lemmas

Theorem 1.1 will be proved by induction, using four lemmas. In order to state these we need some definitions.

If $m \geqq n \geqq 1$ let $G(n, m)$ be the subset of $G(n, m) \times G(n-1, m)$ consisting of all pairs (V, W) where V is an n-dimensional quotient of $\mathbf{C}^m$ and W is an $(n-1)$-dimensional quotient of V. Then $\tilde{G}(n, m)$ is a nonsingular complex projective variety, and a holomorphic map $h: M \to \tilde{G}(n, m)$ may be identified with a sequence of surjective holomorphic maps between holomorphic vector bundles over M

$$M \times \mathbf{C}^m \to E \to E'$$

where the ranks of E and E' are n and $n-1$. Let $\mathrm{Hol}_{d,d'}(M, \tilde{G}(n, m))$ be the space of all holomorphic maps h: $M \to \tilde{G}(n, m)$ such that the associated holomorphic vector bundles E and E' have degrees d and d' respectively. Define $\mathrm{Map}_{d,d'}(M, \tilde{G}(n, m))$ similarly as the space of all continuous maps $h: M \to \tilde{G}(n, m)$ such that the associated topological vector bundles E and E' have degrees d and d' respectively.

The obvious maps from $\tilde{G}(n, m)$ to $G(n, m)$ and $G(n-1, m)$ are locally rivial fibrations with fibres $\mathbf{P}_{n-1}$ and $\mathbf{P}_{m-n}$ respectively. They induce maps

$$p\colon \mathrm{Map}_{d,d'}(M, \tilde{G}(n, m)) \to \mathrm{Map}_d(M, G(n, m))$$

and

$$q\colon \mathrm{Map}_{d,d'}(M, \tilde{G}(n, m)) \to \mathrm{Map}_{d'}(M, G(n-1, m)).$$

These restrict to maps between the appropriate spaces of holomorphic maps.

Let $\tilde{A}_{d,d'}(n, m)$ be the space of holomorphic maps h: $M \to \tilde{G}(n, m)$ such that the associated holomorphic vector bundles E and E' over M have degrees d and d' and satisfy

$$H^1(M, E) = 0 = H^1(M, E').$$

Then p and q restrict to maps

$$p_A\colon \tilde{A}_{d,d'}(n, m) \to A_d(n, m)$$

and

$$q_A\colon \tilde{A}_{d,d'}(n, m) \to A_{d'}(n-1, m).$$

Recall that any continuous map $g\colon X \to Y$ has a canonical factorisation

$$\begin{array}{ccc} X & \xrightarrow{s} & P \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & Y & \end{array}$$

as the product of a homotopy equivalence $s\colon X \to P$ and a fibration $\pi\colon P \to Y$. Let I be the unit interval and let Y^I be the space of continuous paths in Y; then

$$P = \{(x, \omega) \in X \times Y^I \mid \omega(0) = g(x)\},$$

$$\pi(x, \omega) = \omega(1)$$

and

$$s(x) = (x, \omega_{g(x)})$$

where ω_y denotes the constant path at y for any y in Y. The fibre $g^{-1}(y)$ of g at any y in Y is embedded in the fibre of π at y by s. The fibre of π is called the homotopy fibre of g. When it is necessary to specify the map g we shall write $P(g)$ for P, π_g for π and $HF(g, y)$ for the homotopy fibre $\pi^{-1}(y)$.

Associated to the fibration $\pi\colon P \to Y$ is a spectral sequence $E^r_{p,q}$ with $E^2_{p,q}$ given by the p^{th} homology group of Y with respect to a local coefficient system with stalk $H_q(HF(g, y))$ at y. This spectral sequence abuts to the homology of P which is isomorphic to the homology of X.

2.1. By standard homotopy theory the inclusion of every fibre of

$$p\colon \mathrm{Map}_{d,d'}(M, \tilde{G}(n, m)) \to \mathrm{Map}_d(M, G(n, m))$$

into the homotopy fibre is a homotopy equivalence.

The same is true of

$$q\colon \mathrm{Map}_{d,d'}(M, \tilde{G}(n, m)) \to \mathrm{Map}_{d'}(M, G(n-1, m)).$$

In particular the homotopy fibres of p and q are homotopically equivalent to $\mathrm{Map}_{d-d'}(M, \mathbf{P}_{n-1})$ and $\mathrm{Map}_{d-d'}(M, \mathbf{P}_{m-n})$ respectively.

Suppose that one knew that the restrictions p_A and q_A to the appropriate spaces of holomorphic maps had the same property. Then one could hope to prove Theorem 1.1 by induction, first using the spectral sequences associated to q and q_A as above to deduce that the inclusion

$$\tilde{A}_{d,d'}(n, m) \to \mathrm{Map}_{d,d'}(M, \tilde{G}(n, m))$$

induces isomorphisms of homology up to some dimension, and then using the spectral sequences associated to p and p_A to deduce that the same is true of the inclusion

$$A_d(n, m) \to \mathrm{Map}_d(M, G(n, m)).$$

In fact p_A and q_A do not possess this property. However it will be shown that they satisfy a weaker condition which ensures that there exists a spectral sequence abutting to the homology of the domain, with $E^2_{p,q}$ term given by the homology of the range with twisted coefficients in the homology of a suitable fibre for $p, q \leqq k$, where k can be taken to be arbitrarily large by appropriate choice of d and d'. This will enable us to prove Theorem 1.1. The basic idea is to show that p_A and q_A behave like fibrations outside a subset of codimension k, and then to use the fact that removing a subset of codimension k from a manifold does not change the homology in dimensions less than k.

We shall call a continuous map $g\colon X \to Y$ a *homology fibration up to dimension* k if for each $y \in Y$ the inclusion of the fibre over y in the homotopy fibre at y induces an isomorphism of homology groups in dimensions $i<k$ and a surjection for $i=k$. This definition coincides with the definition given in [G] and corresponds to the definition of a homology fibration given in [McD]. Alternatively one might generalise the definition of a homology fibration given in [McD & S] (see also [S] definition 4.4, p. 50) by defining g to be a homology fibration up to dimension k if every $y \in Y$ has arbitrarily small contractible neighbourhoods U such that the inclusion of $g^{-1}(y)$ in $g^{-1}(U)$ induces isomorphisms of homology in dimensions less than k and a surjection in dimension k. The latter is a stronger condition; any map which satisfies it is a homology fibration up to dimension k in the sense of this paper (see [G] lemma 4.9).

Given any continuous map $g\colon X \to Y$ we shall call a fibre $g^{-1}(y)$ *k-clean* if its inclusion in the homotopy fibre $HF(g, y)$ induces isomorphisms of homology in dimensions less than k and a surjection in dimension k. Thus g is a homology fibration up to dimension k if and only if every fibre of g is k-clean.

Definition. *A representation of a group π on a vector space V is nilpotent if V has a π-stable filtration such that π acts trivially on the associated graded module.*

Now the four lemmas needed for the proof of Theorem 1.1 can be stated. Let k be any positive integer and suppose that $n \geqq 2$.

2.2. Lemma. *Let* $d_1 = \max(2g-1, 0)$ *and*

$$d_2(k, n, d-d') = n(2g+k+1+d-d'+\max(k+1+n(d-d'), \tfrac{1}{4}n^2)).$$

If $d-d' \geqq d_1$ and $d \geqq d_2(k, n, d-d')$ then the map

$$p_A\colon \tilde{A}_{d,d'}(n, m) \to A_d(n, m)$$

has a k-clean fibre whose inclusion in the corresponding fibre of

$$p\colon \mathrm{Map}_{d,d'}(M, \tilde{G}(n, m)) \to \mathrm{Map}_d(M, G(n, m))$$

induces isomorphisms of homology in dimensions less that k. Moreover $\pi_1(A_d(n,m))$ acts nilpotently on the homology of this k-clean fibre in dimensions less than k.

2.3. Lemma. If $d-d'>2g+k$ *and* $d'>(n-1)(2g+1)$ *then the* map

$$q_A\colon \tilde{A}_{d,d'}(n, m) \to A_{d'}(n-1, m)$$

is a homology fibration up to dimension k. Moreover there is a fibre of q_A *whose inclusion in the corresponding fibre of*

$$q\colon \mathrm{Map}_{d,d'}(M, \tilde{G}(n, m)) \to \mathrm{Map}_{d'}(M, G(n-1, m))$$

induces isomorphisms of homology up to dimension k. Finally $\pi_1(A_{d'}(n-1, m))$ *acts nilpotently on the homology of any fibre of* q_A *up to dimension k.*

2.4. Lemma. *If* $m>2$ *then* $\pi_1(\mathrm{Map}_d(M, G(n, m)))$ *is abelian.*

2.5. Lemma. *If* $m>2$ *and* $m>n>0$ *then* $\pi_1(\mathrm{Hol}_d(G(n, m))$ *is abelian for* $d>2g+1$. *The same is true of* $\pi_1(A_d(n, m))$. *Moreover when* $d>2g+1$ *the inclusion of* $A_d(n, m)$ *in* $\mathrm{Map}_d(M, G(n, m))$ *induces a surjection of fundamental groups.*

Proof of Theorem 1.1 given these four lemmas. The proof is by induction on n. For $n=1$ note that any line bundle L on M of degree strictly greater than $2g-1$ has $H^1(M, L)=0$. Thus $A_d(1, m)=\mathrm{Hol}_d(M, G(1, m))$ when $d>2g-1$. As in Remark 1.4 we may identify $\mathrm{Hol}_d(M, G(1, m))$ with the complex conjugate of $\mathrm{Hol}_d(M, \mathbf{P}_{m-1})$ in $\mathrm{Map}_d(M, \mathbf{P}_{m-1})$ and thus we find that the case $n=1$ follows immediately from Segal's theorem ([S] 1.3).

Now assume that n is greater than 1. We can thus also assume that $m>2$. Suppose that $k\geqq n-2g$ and let $K=k+1$. Suppose also that

$$d \geqq d_0(k, n) = 2n(2g+k+1)+n\max\left(k+1+n(2g+k+1), \tfrac{1}{4}n^2g\right)$$

and let $d'=d-(2g+K+1)$. Then $d'\geqq d_0(K, n-1)$, the conditions of Lemmas 2.2 and 2.3 are satisfied for k and K respectively and d' satisfies the condition $d'>2g+1$ of Lemma 2.5. We have a commutative diagram

$$\begin{array}{ccc} \tilde{A}_{d,d'}(n, m) & \xrightarrow{q_A} & A_{d'}(n-1, m) \\ \downarrow i_1 & & \downarrow i_2 \\ \mathrm{Map}_{d,d'}(M, \tilde{G}(n, m)) & \xrightarrow{q} & \mathrm{Map}_{d'}(M, G(n-1, m)) \end{array}$$

where i_1 and i_2 are the inclusions. Associated to q_A and q are spectral sequences $E^2_{p,q}$ abutting to the homology of $\tilde{A}_{d,d'}(n, m)$ and $\mathrm{Map}_{d,d'}(M, \tilde{G}(n, m))$ respectively such that the $E^2_{p,q}$ term is given by the p^{th} homology group of $A_{d'}(n-1, m)$ and $\mathrm{Map}_{d'}(M, G(n-1, m))$ respectively with twisted coefficients in the q^{th} homology group of the homotopy fibre of q_A and q respectively. It follows immediately from 2.1 and 2.3 that the inclusion of the homotopy fibre of q_A in the homotopy fibre of q induces isomorphisms in homology up to dimension K. Moreover since $d' \geqq d_0(K, n-1)$, by induction i_2 induces isomorphisms in homology up to dimension K.

By Lemmas 2.4 and 2.5 the fundamental groups of $\mathrm{Map}_{d'}(M, G(n-1, m))$ and $A_{d'}(n-1, m)$ are abelian, and hence in particular they are nilpotent. Moreover by Lemma 2.3 $\pi_1(A_{d'}(n-1, m))$ acts nilpotently on the homology of the homotopy fibre of q_A up to dimension K. Since the action of $\pi_1(\mathrm{Map}_{d'}(M, G(n-1, m)))$ on the homology of the homotopy fibre of q up to dimension K is determined by the surjection

$$\pi_1(A_{d'}(n-1, m)) \to \pi_1(\mathrm{Map}_{d'}(M, G(n-1, m))),$$

the isomorphisms

$$H_i(HF(q_A, y)) \to H_i(HF(q, y))$$

for $i \leqq K$ and the action of $\pi_1(A_{d'}(n-1, m))$ on $H_i(HF(q_A, Y))$, this action of $\pi_1(\mathrm{Map}_{d'}(M, G(n-1, m)))$ is also nilpotent. Therefore it follows immediately from Hilton and Roitberg's extension to quasi-nilpotent fibrations of Zeeman's comparison theorem for spectral sequences (see [H & R]) that the inclusion i_1 induces isomorphisms in homology up to dimension $K-1=k$.

We also have a commutative diagram

$$\begin{array}{ccc} \tilde{A}_{d,d'}(n, m) & \xrightarrow{p_A} & A_d(n, m) \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_3} \\ \mathrm{Map}_{d,d'}(M, \tilde{G}(n, m)) & \xrightarrow{p} & \mathrm{Map}_d(M, G(n, m)), \end{array}$$

where i_1 and i_3 are the inclusions. We have shown that i_1 is a homology equivalence up to dimension k. By 2.1 and 2.2 and the definition of a k-clean fibre, the inclusion of the homotopy fibre of p_A in the homotopy fibre of p induces isomorphisms in homology in dimensions less than k. Therefore by the argument used above for q_A and q we may apply Hilton and Roitberg's comparison theorem again to deduce that i_3 induces isomorphisms of homology up to dimension k. This completes the induction step and the proof of the Theorem.

Remark. For the case $M=S^2$ Segal proved that the inclusion of $\mathrm{Hol}_d(M, \mathbf{P}_m)$ in $\mathrm{Map}_d(M, \mathbf{P}_m)$ is a homotopy equivalence up to dimension $(d-2g)(2m-1)$, and he conjectured that the same holds for all Riemann surfaces M. Theorem 1.2 and Lemma 2.5 tell us that the inclusion of $\mathrm{Hol}_d(M, G(n, m))$ in $\mathrm{Map}_d(M, G(n, m))$ is a homology equivalence up to some dimension k which tends to infinity with d, and hence also that it induces an isomorphism of fundamental groups since these are both abelian. Therefore to show that the inclusion is a homotopy equivalence up to dimension k it would be enough (by [H & R]) to show that the fundamental group of each space acts nilpotently on the higher homotopy groups up to dimension k. It seems likely that one could prove this by induction using methods similar to the proof of Theorem 1.1 if the result was known for maps into projective spaces.

To complete this section there follow some results about homology fibrations and k-clean fibres which we shall need later.

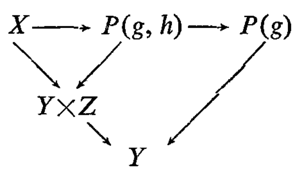
2.6. Proposition. *Let $g\colon X \to Y$ be a smooth map between connected manifolds X and Y, and let k be a non-negative integer.*

a) Suppose $h\colon X \to Z$ is such that $(g, h)\colon X \to Y \times Z$ is a homology fibration up to dimension k. Suppose also that for each $y \in Y$ the restriction of $h\colon X \to Z$ to the fibre $g^{-1}(y)$ is a homology fibration up to dimension k. Then g is a homology fibration up to dimension k. Moreover if $\pi_1(Y)$ acts nilpotently on the homology of the homotopy fibre of the map (g, h) up to dimension k then the same is true of the map g.

b) Suppose that $Z \subseteq X$ is a closed submanifold of codimension d in X with orientable normal bundle such that the restriction $\tilde{g}\colon Z \to Y$ of g to Z is a fibration up to dimension $k-d+1$. Suppose also that the derivative of $\tilde{g}$ is surjective everywhere on Z so that $g^{-1}(y)$ is smooth near $\tilde{g}^{-1}(y) = g^{-1}(y) \cap Z$ and $\tilde{g}^{-1}(y)$ is a closed submanifold of codimension d in $g^{-1}(y)$ for every $y \in Y$. Then $g\colon X \to Y$ is a homology fibration up to dimension k if and only if the restriction of g to $X-Z$ is a homology fibration up to dimension k. Moreover if $\pi_1(Y)$ acts nilpotently on the homology of the homotopy fibres of g (up to dimension k) and of $\tilde{g}$ (up to dimension $k-d+1$) then $\pi_1(Y)$ acts nilpotently on the homology of the homotopy fibre of the restriction of g to $X-Z$ up to dimension k.

c) Suppose that A and B are closed submanifolds of codimension at least $k+1$ in X and Y respectively, such that $g^{-1}(B) = A$. If $y \in Y - B$ then the fibre of g at y is k-clean if and only if the fibre at y of the restriction of g to $X-A$ is k-clean.

Proof. To prove (*a*) first note that there is a homotopy equivalence between $P(g)$ and $P(g, h)$ such that the diagram



commutes. This is because

$$P(g, h) = \{(x, \omega, \tilde{\omega},) \mid x \in X, \omega \in Y^I, \tilde{\omega} \in Z^I, \omega(0) = g(x), \tilde{\omega}(0) = h(x)\}$$

and a homotopy equivalence $P(g, h) \to P(g)$ is given by

$$(x, \omega, \tilde{\omega}) \to (x, \omega)$$

with homotopy inverse

$$(x, \omega) \to (x, \omega, \omega_{h(x)}).$$

For each y in Y this restricts to a homotopy equivalence between the homotopy fibre $HF(g, y)$ and the inverse image

$$A = \{(x, \omega, \tilde{\omega}) | \omega(0) = g(x),\ \omega(1) = y,\ \tilde{\omega}(0) = h(x)\}$$

of $\{y\} \times Z$ under $\pi_{(g,h)}$. There is a fibration $\varphi\colon A \to Z$ given by

$$(x, \omega, \tilde{\omega}) \to \tilde{\omega}(1)$$

whose fibre at z is the homotopy fibre $HF((g, h), (y, z))$ of (g, h) at (y, z). Under the inclusion

$$g^{-1}(y) \to HF(g, y) \to A$$

this fibration restricts to the map $h\colon g^{-1}(y) \to Z$. By assumption this is a homology fibration up to dimension k. Its fibre at any $z \in Z$ is $(g, h)^{-1}(y, z)$. By assumption the inclusion of this in $HF((g, h), (y, z))$ is a homology equivalence up to dimension k. So the inclusion of the homotopy fibre of $h\colon g^{-1}(y) \to Z$ in the homotopy fibre of $\varphi\colon A \to Z$ is a homology equivalence up to dimension k. Therefore there are spectral sequences abutting to the homology of $g^{-1}(y)$ and of A (or equivalently of the homotopy fibre $HF(g, y)$) such that the natural map between the spectral sequences induced by the inclusion induces isomorphisms on the $E^2_{p,q}$ terms with $q \leqq k$. This is because when $q \leqq k$ the $E^2_{p,q}$ terms are given by the p^{th} homology groups of Z with respect to the same local coefficient system. Therefore it is easy to deduce that the inclusion of $g^{-1}(y)$ in $HF(g, y)$ is a homology equivalence up to dimension k (see [Z] or [H & R]). This proves (a), since the part about nilpotent actions is now obvious.

Now consider (b). There is a diffeomorphism from a neighbourhood U of Z in X to the normal bundle N to Z in X, which implies by use of the Thom isomorphism that

$$H^q(X, X-Z) = H^{q-d}(Z)$$

for all $q \geqq 0$. Since the derivative of $\tilde{g}\colon Z \to Y$ is surjective everywhere in Z the fibres $g^{-1}(y) \cap Z$ of $\tilde{g}$ are all closed submanifolds of the fibres $g^{-1}(y)$; and by using the exponential maps associated to the restrictions of some metric on X to these fibres we may choose the diffeomorphism $U \to N$ such that the diagram

$$\begin{array}{ccccc} U & & \xrightarrow{\cong} & & N \\ & \searrow^{g} & & \swarrow & \\ & & & Z & \\ & & \swarrow_{\tilde{g}} & & \\ & & Y & & \end{array}$$

commutes.

For each y in Y

$$HF(\tilde{g}, y) = \{(x, \omega) \in HF(g, y) | x \in Z\}$$

so the subset

$$U' = \{(x, \omega) \in HF(g, y) | x \in U\}$$

is a neighbourhood of $HF(\tilde{g}, y)$ in $HF(g, y)$, and the diffeomorphism $U \to N$ induces a homeomorphism of U' with the pull-back to $HF(g, y)$ of the bundle N. Hence by the Thom isomorphism again we have

$$H^q(HF(g, y), HF(g, y) - HF(\tilde{g}, y)) = H^{q-d}(HF(\tilde{g}, y)).$$

Since $HF(g, y) - HF(\tilde{g}, y)$ is the homotopy fibre of the restriction g'' of g to $X - Z$ we have two long exact sequences of cohomology

$$\begin{array}{ccccccccc} \ldots \to H^{q-d}(HF(\tilde{g}, y)) & \to & H^q(HF(g, y)) & \to & H^q(HF(g'', y)) & \to & H^{q-d+1}(HF(\tilde{g}, y)) \to \ldots \\ \downarrow i_1^* & & \downarrow i_2^* & & \downarrow i_3^* & & \downarrow i_1^* \\ \ldots \to H^{q-d}(\tilde{g}^{-1}(y)) & \to & H^q(g^{-1}(y)) & \to & H^q(g''^{-1}(y)) & \to & H^{q-d+1}(\tilde{g}^{-1}(y)) \to \ldots \end{array}$$

where i_1, i_2 and i_3 are the inclusions. By assumption i_1^* is an isomorphism for $q-d \leqq k-d+1$, so i_2^* is an isomorphism for $q \leqq k$ if and only if i_3^* is an isomorphism for $q \leqq k$ by the five lemma. Finally note that the first of these two long exact sequences is compatible with the action of $\pi_1(Y)$ on the cohomology of the three homotopy fibres. By assumption the action on $H^q(HF(g, y))$ and on $H^{q-d+1}(HF(\tilde{g}, y))$ is nilpotent for $q \leqq k$. Hence the action of $\pi_1(Y)$ on $H^q(HF(g'', y))$ is also nilpotent for $q \leqq k$. This completes the proof of (b).

The proof of (c) involves a similar and straightforward use of Thom—Gysin sequences.

3. Facts about holomorphic bundles

The space $\mathrm{Hol}_d(M, G(n, m))$ is a quasi-projective variety of which $A_d(n, m)$ is a Zariski open subset (cf. [Se] § 6 and also [N] Theorem 5.3). The Zariski tangent space at an element h corresponding to an exact sequence of bundles

$$0 \to K \to M \times \mathbf{C}^m \to E \to 0$$

can be canonically identified with $H^0(M, K^* \otimes E)$.

The variety $\mathrm{Hol}_d(M, \tilde{G}(n, m))$ parametrises in an obvious way a family of bundles of rank n and degree d over M. More precisely, there is a quotient $\tilde{E}$ of the trivial rank m bundle over $\mathrm{Hol}_d(M, \tilde{G}(n, m)) \times M$ such that for every h the restriction of $\tilde{E}$ to $\{h\} \times M$ is just h. (See e.g. [Se] § 6.)

Suppose that $h \in A_d(n, m)$, so that $H^1(M, E) = 0$. Then $K^* \otimes E$ is a quotient of $(M \times \mathbf{C}^m)^* \otimes E$, which is isomorphic to the sum of m copies of E and so has vanishing first cohomology. It follows that $H^1(M, K^* \otimes E) = 0$. Therefore the dimension of the Zariski tangent space $H^0(M, K^*\otimes E)$ is independent of h in $A_d(n, m)$. In fact Seshadri has proved that the variety $A_d(n, m)$ is nonsingular (see [Se] Remark 6.1).

We next need some facts and definitions from [H & N] and [A & B].

3.1. A holomorphic bundle E over M is said to be semistable if it has no proper sub-bundle E' with

$$c_1(E')/rk(E') > c_1(E)/rk(E).$$

Here c_1 is the first Chern class or degree, and rk is the rank.

3.2. Every bundle E has a canonical filtration

$$\mathcal{O} = E_0 \subseteqq \dots \subseteqq E_s = E$$

with semistable quotients

$$D_j = E_j/E_{j-1}$$

such that

$$c_1(D_j)/rk(D_j) > c_1(D_{j+1})/rk(D_{j+1})$$

for $1 \leqq j \leqq s-1$. The degrees d_j and ranks n_j of the quotients D_j determine the type μ of the bundle E, which is an element of the positive Weyl chamber of the group $GL(n)$ for $n=rk(E)$. In fact μ is the vector

$$(d_1/n_1, \dots, d_s/n_s)$$

in which each ratio d_j/n_j appears n_j times. If $c_1(E)=d$ and $rk(E)=n$ we say that μ is an (n, d) type. (See [A & B].)

3.3. If L is a line bundle on M with degree e then the type of $E\otimes L$ is $\mu+e$, where by abuse of notation e stands for the n-vector all of whose coefficients are equal to e. If $\mathcal{U}$ is a set of types we write $\mathcal{U}+e$ for the set $(\mu+e|\mu\in\mathcal{U})$.

3.4. If $d>n(2g-1)$ is a positive integer and E is a semistable bundle of rank n and degree d on M then E is generated by its sections and $H^1(M, E)=0$ (see [N] Lemma 5.2). We deduce that for any fixed type μ there is some integer m_0 such that if $m\geqq m_0$ and E is of type $\mu+m$, then

$$H^1(M, E) = 0$$

and E is generated by its sections.

3.5. From [A & B] we know that there is an infinite-dimensional affine space $\mathcal{C}$ whose points can be regarded as unitary connections on some fixed C^∞ bundle over M, or equivalently as holomorphic structures on this bundle. Atiyah and Bott show that there is a smooth stratification of $\mathcal{C}$ by types so that a holomorphic structure lies in the stratum $\mathcal{C}_\mu$ if and only if it is of type μ.

3.6. Let E be any holomorphic bundle on M representing a point in $\mathscr{C}$ and let $\operatorname{End} E$ be the bundle of endomorphisms of E. Then the normal in $\mathscr{C}$ at E to the submanifold consisting of all bundles isomorphic to E can be identified with $H^1(M, \operatorname{End} E)$. If E is of type μ, let $\operatorname{End}' E$ denote the bundle of endomorphisms of E which preserve the canonical filtration, and define $\operatorname{End}'' E$ by the exact sequence

$$0 \to \operatorname{End}' E \to \operatorname{End} E \to \operatorname{End}'' E \to 0.$$

Then $H^1(M, \operatorname{End}'' E)$ can be identified with the normal to the stratum $\mathscr{C}_\mu$ at E (see [A & B] § 7). So

$$d_\mu = \dim H^1(M, \operatorname{End}'' E)$$

is the complex codimension of $\mathscr{C}_\mu$ in $\mathscr{C}$.

3.7. If we take the dual of the exact sequence

$$0 \to K \to M \times \mathbf{C}^m \to E \to 0$$

corresponding to an element h of $A_d(n, m)$ and then tensor with E we obtain

$$0 \to \operatorname{End} E \to E \oplus E \oplus \ldots \oplus E \to K^* \otimes E \to 0.$$

This gives us a homomorphism

$$\varphi\colon\ H^0(M, K^* \otimes E) \to H^1(M, \operatorname{End} E)$$

which is surjective because $H^1(M, E \oplus E \oplus \ldots \oplus E) = 0$. This homomorphism is the infinitesimal deformation map at h. It can be interpreted as follows in terms of the space $\mathscr{C}$ described at 3.5.

There is a small neighbourhood N of h in $A_d(n, m)$ such that the restriction of the bundle $\tilde{E}$ to $N \times M$ is differentially isomorphic to $N \times E$. If we fix a hermitian metric on E this gives us a family of unitary connections on the underlying C^∞ bundle of E, parametrised by N. So we get a map ψ of N into the space $\mathscr{C}$ of all unitary connections. By [A & B] Lemma 15.5, the infinitesimal deformation map φ is the projection of the differential $d\psi(h)$ of ψ at h onto the normal to the submanifold of $\mathscr{C}$ consisting of bundles isomorphic to E.

Remark. Define a stratification (A_μ) of $A_d(n, m)$ by

$$A_\mu = \{h \mid \text{the corresponding bundle } E \text{ has type } \mu\}.$$

Then since φ is surjective for all h in $A_d(n, m)$, it follows from 3.5 and 3.6 that each stratum A_μ is nonsingular, and if it is nonempty then it has complex codimension d_μ in $A_d(n, m)$.

3.8. By [A & B] 7.16 we have

$$d_\mu = \sum_{i>j} (n_i d_j - n_j d_i + n_i n_j (g-1))$$

if μ is the type $(d_1/n_1, \ldots, d_s/n_s)$ defined as at 3.2 by integers $d_1, \ldots, d_s$ and $n_1, \ldots, n_s > 0$ such that $d_1 + \ldots + d_s = d$, $n_1 + \ldots + n_s = n$ and $d_1/n_1 > \ldots > d_s/n_s$. From this it is easy to see that

$$d_{\mu+m} = d_\mu$$

for any (n, d) type μ and any integer m. It is also easy to see that given any integer k there are only finitely many (n, d) types μ such that

$$d_\mu \leqq k.$$

3.9. Lemma. *Let k and e be any positive integers. Then if*

$$d \geqq n\big(2g + e + \max(k, \tfrac{1}{4}n^2 g, n(1-g))\big)$$

there exists a finite set $\mathscr{U}$ of (n, d) types such that

a) if μ is an (n, d) type and $\mu \notin \mathscr{U}$ then $d_\mu > k$;

b) if E is a bundle of type $\mu \in \mathscr{U}$ and L is a line bundle of degree at most e on M then $H^1(M, L^* \otimes E) = 0$;

c) the union of the strata $\{\mathscr{C}_\mu | \mu \in \mathscr{U}\}$ is an open subset of $\mathscr{C}$; and

d) if $d_1, \ldots, d_n$ are integers such that $d_1 + \ldots + d_n = d$ and $[d/n] \leqq d_j \leqq [d/n] + 1$ for $1 \leqq j \leqq n$ then the type μ of any sum of line bundles on M of degrees $d_1, \ldots, d_n$ is an element of $\mathscr{U}$.

Proof. It is well known that if E is a semistable bundle of rank n and degree d then there are no nonzero holomorphic bundle maps from E to a line bundle L of degree less than d/n (see e.g. [N] § 5). Hence if $L_1, \ldots, L_n$ are line bundles on M each of degree $[d/n]$ or $[d/n]+1$ then the type $\mu = (d_1/n_1, \ldots, d_s/n_s)$ of $L_1 \oplus \ldots \oplus L_n$ satisfies $d_1/n_1 \leqq [d/n] + 1$. From this it follows that s is at most 2 and that $d_s/n_s \geqq [d/n]$. Hence

$$d_\mu = n_1 n_2\big(d_1/n_1 - d_2/n_2 + (g-1)\big) \leqq n_1 n_2 g \leqq \tfrac{1}{4} n^2 g.$$

Without loss of generality we may assume that $k \geqq \frac{1}{4} n^2 g$. Therefore if we define $\mathscr{U}$ to be the set of all (n, d) types μ such that $d_\mu \leqq k$ then it remains only to check that (b) is satisfied.

By 3.4 it suffices to show that if $\mu = (d_1/n_1, \ldots, d_s/n_s)$ is an (n, d) type such that $d_\mu \leqq k$ then $d_j/n_j - e > 2g - 1$ for $1 \leqq j \leqq s$. Since $d_j/n_j \geqq d_s/n_s$ for all j it is enough to show that $d_s/n_s \geqq 2g + e$. We have

$$d_\mu = \textstyle\sum_{i>j} n_i n_j (d_j/n_j - d_i/n_i + g - 1) \leqq k.$$

Moreover $d_j/n_j > d_i/n_i$ when $i > j$, and if $g = 0$ then each d_j/n_j is an integer because every semistable line bundle on $\mathbf{P}_1$ is a sum of line bundles all of the same degree.

Hence

$$d_j/n_j - d_i/n_i + g - 1 \geqq 0$$

whenever $i > j$, and therefore

$$d_1/n_1 - d_s/n_s + s(g-1) \leqq \sum_{i>j} (d_j/n_j - d_i/n_i + g - 1) \leqq k.$$

But $d_1/n_1 \geqq d/n \geqq 2g + k + e$, so

$$d_s/n_s \geqq 2g + e + s(g-1) \geqq 2g + e$$

if $g \geqq 1$. Finally when $g = 0$ then $d/n \geqq k + e + n$ so $d_s/n_s \geqq e + n - s \geqq e = 2g + e$. This completes the proof.

We also need a technical lemma.

3.10. Lemma. *Suppose that T is a nonsingular quasi-projective variety which parametrises a family of bundles of rank $n \geqq 2$ over M, in the sense that there is a holomorphic bundle E over $T \times M$ such that for any t in T the restriction E_t of E to $(t) \times M$ is the bundle parametrised by t. Let k be any non-negative integer. Suppose that for every t in T and every line bundle L over M of degree at most $k+1$, we have*

$$H^1(M, L^* \otimes E_t) = 0.$$

Then there is a bundle W over T such that the fibre W_t at any $t \in T$ can be naturally identified with $H^0(M, E_t)$. Let $\pi\colon \mathbf{P}(W) \to T$ be the projective bundle associated to W. Let S_0 be the image in $\mathbf{P}(W)$ of the nowhere-vanishing sections in W under the identification of points in W_t with sections of E_t. Then the restriction of π to S_0 is a homology fibration up to dimension k. Moreover $\pi_1(T)$ acts nilpotently on the homology of the homotopy fibre of $\pi\colon S_0 \to T$ in dimensions less than k.

Proof. The existence of the bundle W is well known (see e.g. [H] III Corollary 12.9). The proof of the rest of the Lemma is by induction on k. First note that $\pi_1(T)$ acts trivially on the homology of the fibre of the projective bundle $\mathbf{P}(W)$ over T because the homology of any projective space in any dimension has rank at most 1 and $GL^+(1, \mathbf{Z})$ is trivial.

For any $t \in T$ and $s \in W_t$, let

$$D(s) = x_1 + \ldots + x_j$$

be the zero-divisor of s regarded as a section of the bundle E_t over M. For each $j \geqq 0$, let S_j be the image in $\mathbf{P}(W)$ of those sections s which vanish at precisely j points of M with multiplicity. Then the sets $\{S_j \mid j \geqq 0\}$ form a stratification of $\mathbf{P}(W)$. We shall see that the strata S_j for $j \leqq k$ are all smooth, and that the strata S_j for $j > k$ have real codimension at least $k+1$ so we shall be able to ignore them.

Let $M^{(j)}$ be the j^{th} symmetric power of M, which is a smooth projective variety.

Elements of $M^{(j)}$ can be thought of as positive divisors of degree j on M. So for each $j>0$ there is a map

$$\pi_j\colon S_j \to M^{(j)}\times T$$

whose first component sends a point of S_j represented by $s\in W$ to the zero-divisor $D(s)$ of s, and whose second component is the restriction of π to S_j.

The fibre of π_j at (D, t) consists of the image in $\mathbf{P}(H^0(E_t\otimes\mathcal{O}(-D)))$ of the nowhere-vanishing sections of the bundle $E_t\otimes\mathcal{O}(-D)$. The variety $M^{(j)}\times T$ parametrises the family of bundles $E_t\otimes\mathcal{O}(-D)$ over M, and we know that

$$H^1(M, L^*\otimes E_t\otimes\mathcal{O}(-D)) = 0$$

for any line bundle L of degree at most $k+1-j$, because the degree of $\mathcal{O}(-D)$ is $-j$. Therefore it follows by induction that if $k\geqq j>0$ then

$$\pi_j\colon S_j \to M^{(j)}\times T$$

is a homology fibration up to dimension $k-j$, and $\pi_1(M^{(j)}\times T)$ acts nilpotently on the homology of the homotopy fibre of π_j in dimensions less than $k-j$. Similarly for every fixed $t\in T$ the variety $M^{(j)}$ parametrises the family of bundles $E_t\otimes\mathcal{O}(-D)$, and the fibre at D of the map $S_j\cap\pi^{-1}(t)\to M^{(j)}$ given by restricting the first component of π_j is the image in $\mathbf{P}(H^0(E_t\otimes\mathcal{O}(-D)))$ of the nowhere-vanishing sections of $E_t\otimes\mathcal{O}(-D)$. Therefore by the same argument this map is also a homology fibration up to dimension $k-j$. Hence by 2.6(a) the restriction of π to S_j is also a homology fibration up to dimension $k-j$, and $\pi_1(T)$ acts nilpotently on the homology of the homotopy fibre of $\pi\colon S_j\to T$ in dimensions less that $k-j$. Of course this is also trivially true when $j>k$.

Our aim is to apply 2.6(b) to the strata S_j of $\mathbf{P}(W)$ for each $j>0$ in turn in decreasing order of j. (There are only finitely many j such that S_j in nonempty.) For this we need to check that when $0<j\leqq k$ then S_j is smooth of real codimension at least $j+1$ in $\mathbf{P}(W)$ and when $j>k$ then S_j has real codimension greater than $k+1$. It does not matter whether S_j is nonsingular for $j>k$ since we can always refine the stratification so that this becomes true.

If $k\geqq j>0$ then S_j is an open subset of a smooth projective bundle over $M^{(j)}\times T$ and hence it is smooth. Moreover the derivative of π_j and hence also of π at any point of S_j is surjective. In particular the intersection of S_j with the fibre $\mathbf{P}(W_t)$ of π at any $t\in T$ has the same codimension in $\mathbf{P}(W_t)$ as S_j has in $\mathbf{P}(W)$. But if $D\in M^{(j)}$ and $k+1\geqq j>0$ then $H^0(M, E_t\otimes\mathcal{O}(-D))$ has complex codimension nj in $H^0(M, E_t)$ by Riemann—Roch, because the first cohomology of both E_t and $E_t\otimes\mathcal{O}(-D)$ vanishes and the degree of $E_t\otimes\mathcal{O}(-D)$ is $\deg E_t-nj$. Since $M^{(j)}$ has dimension j it follows that the real codimension of $S_j\cap\mathbf{P}(W_t)$ in $\mathbf{P}(W_t)$ is $2(n-1)j$ for $k\geqq j>0$, and at least $2(n-1)(k+1)$ for $j>k$. By assumption $n\geqq 2$

so $2(n-1)j$ is strictly greater than j for all $j \geqq 1$. Repeated application of 2.6(b) now shows that the restriction of π to S_0 is a homology fibration up to dimension k and that $\pi_1(T)$ acts nilpotently on the homology of the homotopy fibre of π: $S_0 \to T$ in dimensions less than k. This completes the induction.

4. Proof of Lemma 2.2

An element h of $\tilde{A}_{d,d'}(n, m)$ can be identified with a pair of bundles E and E' of rank n and $n-1$ and degree d and d' respectively, with surjective maps

$$M \times \mathbf{C}^m \to E \to E'$$

satisfying $H^1(M, E) = 0 = H^1(M, E')$. The kernel of the quotient map $E \to E'$ is then a line bundle $L(h)$ of degree $d-d'$. Let

$$p' \colon \tilde{A}_{d,d'}(n, m) \to A_d(n, m) \times \mathrm{Pic}_{d-d'}(M)$$

be the map whose first component is p_A and whose second sends h to the isomorphism class $[L(h)]$ of $L(h)$ in the Picard variety of M. We shall show that p' has a k-clean fibre (in fact that the generic fibre of p' is k-clean) provided that $d-d' > \max(2g-1, 0)$ and $d \geqq n(2g+k+1+d-d'+\max(k+1+n(d-d'), \frac{1}{4} n^2 g))$. More precisely we shall show that there is an open subset U of $A_d(n, m)$ such that the complements of U in $A_d(n, m)$ and of $P_A^{-1}(U)$ in $\tilde{A}_{d,d'}(n, m)$ have codimension strictly greater than k and such that the restriction

$$p' \colon p_A^{-1}(U) \to U \times \mathrm{Pic}_{d-d'}(M)$$

is a homology fibration up to dimension k. We shall also show that the restriction

$$p' \colon p_A^{-1}(g) \to \{g\} \times \mathrm{Pic}_{d-d'}(M),$$

is a fibration up to dimension k for every g in U. By 2.6(a) and (c) this will be enough to prove that for every $g \in U$ the fibre of p_A at g is k-clean.

Let g be any element of $A_d(n, m)$ represented by a quotient bundle E of the trivial bundle of rank m. Let L be any line bundle of degree $d-d'$. Since $H^1(M, E) = 0$, if

$$0 \to L \to E \to E' \to 0$$

is an exact sequence then $H^1(M, E') = 0$. Moreover embeddings $L \to E$ correspond to nowhere-vanishing sections of $L^* \otimes E$, and the group of automorphisms of L is $\mathbf{C}^*$ acting as multiplication by scalars. Therefore the fibre of p' at $(g, [L])$ can be identified with the image in the projective space

$$\mathbf{P}(H^0(M, L^*\otimes E))$$

of the nowhere-vanishing sections of $L^*\otimes E$.

The nonsingular variety $A_d(n, m)\times \mathrm{Pic}_{d-d'}(M)$ parametrises a family of bundles $L^*\otimes E$ on M, obtained by tensoring the dual of the universal line bundle on $\mathrm{Pic}_{d-d'}(M)\times M$ with the tautological bundle $\tilde{E}$ on $A_d(n, m)\times M$. Similarly for any fixed E the variety $\mathrm{Pic}_{d-d'}(M)$ parametrises the family $L^*\otimes E$ where now only L varies. Therefore by Lemma 3.10 and the remarks made at the beginning of this section, to prove that the fibre of p_A at every g in some nonempty open subset U of $A_d(n, m)$ is k-clean, it is enough to show that the complements of U and of $p_A^{-1}(U)$ both have codimension strictly greater than k and that (*) for any bundles E and L representing elements g of U and [L] of $\mathrm{Pic}_{d-d'}(M)$ and any line bundle H of degree at most $k+1$ we have

$$H^1(M, H^*\otimes L^*\otimes E) = 0.$$

These conditions also imply that the action of $\pi_1(A_d(n, m))$ on the homology of the homotopy fibre of p_A is nilpotent in dimensions less than k. We shall show that it is possible to find such a U provided that $d-d'$ and d are sufficiently large.

First choose $d-d'$ positive and strictly greater than $2g-1$. Then every line bundle of degree greater than or equal to $d-d'$ has a nonzero section. By 3.9 if d is at least

$$n\left(2g+d-d'+k+1+\max(k+1+n(d-d'), \tfrac{1}{4}n^2g)\right)$$

there is a finite set $\mathscr{U}$ of (n, d) types such that

a) whenever μ is an (n, d) type not lying in $\mathscr{U}$ then

$$d_\mu > k+1+n(d-d');$$

b) whenever E is a bundle of type $\mu\in\mathscr{U}$ and L is a line bundle of degree at most $d-d'+k+1$ then $H^1(M, L^*\otimes E)=0$; and

c) the type μ of any sum of n line bundles whose degrees lie between $[d/n]$ and $[d/n]+1$ and add up to d lies in $\mathscr{U}$.

By 3.7 there is a smooth stratification of $A_d(n, m)$ into strata A_μ where an element lies in A_μ if the corresponding bundle E is of type μ. Moreover if A_μ is nonempty then its codimension in $A_d(n, m)$ is d_μ. So (a) implies

d) if $\mu\notin\mathscr{U}$ then the codimension of A_μ is greater than $k+1+n(d-d')$.

Let U be the union of those strata A_μ such that μ belongs to $\mathscr{U}$. By enlarging $\mathscr{U}$ if necessary we may assume that U is open in $A_d(m, n)$. Then since $d-d'$ is positive, by (d) the complement of U has codimension at least $k+1$ in $A_d(n, m)$ and by (b) the condition (*) is satisfied. Therefore it suffices to show that the codimension of the complement of $p_A^{-1}(U)$ in $\tilde{A}_{d,d'}(n,m)$ is greater than $k+1$.

Suppose that L is a line bundle of degree $d-d'$ over M and g in $A_d(n,m)$ corresponds to the bundle E. Then by our assumption L has a nonzero section s. Tensoring by s gives us an embedding of $H^0(M, L^*\otimes E)$ into $H^0(M,E)$. Since $H^1(M,E)=0$, by Riemann—Roch we have

$$\dim H^0(M, L^*\otimes E) \leqq \dim H^0(M,E) = d+n(1-g).$$

So the fibre of p' at $(g\,[L])$ has dimension at most $d+n(1-g)-1$ because it is contained in $\mathbf{P}(H^0(M, L^*\otimes \mathrm{E}))$.

On the other hand, if E is of type $\mu\in\mathcal{U}$, then

$$\dim H^0(M, L^*\otimes E) = n(d'-d)+d+n(1-g),$$

because of (b). Also if $x\in M$ then

$$\dim\ H^0(M,L^*\otimes E\otimes \mathcal{U}(-x)) = n(d'-d)+d+n(1-g)-n$$

by the same argument. Since $n\geqq 2$ and $d'<d$ and $\dim M=1$, it follows that there do exist nonvanishing sections of $L^*\otimes E$. Hence the dimension of the fibre of p' at $(g, [\mathrm{L}])$ is exactly

$$\dim \mathbf{P}\big(H^0(M, L^*\otimes E)\big) = n(d'-d)+d+n(1-g)-1$$

when $g\in U$.

So the dimension of a generic fibre of p' is $d+n(1-g)-n(d-d')-1$, and the dimension of any fibre is at most $d+n(1-g)-1$. Therefore it follows from (d) that the codimension of the complement of $p^{-1}(U)$ in $\tilde{A}_{d,d'}(n,m)$ is greater than $k+1$. This shows that U has all the properties we wanted it to have.

To complete the proof of Lemma 2.2 it is now enough to show that for some particular quotient bundle E of $M\times\mathbf{C}^m$ of type μ belonging to $\mathcal{U}$ the inclusion of the space of all holomorphic line sub-bundles of E in the space of all continuous line sub-bundles is a homology equivalence up to dimension k. The following lemma is needed for this and for the proof of Lemma 2.3.

4.1. Lemma. *Let m, n and d be positive integers such that $m\geqq n+1$ and $d\geqq n(2g+1)$. Then there exists an exact sequence*

$$0\to H_1\oplus\ldots\oplus H_{m-n}\to M\times\mathbf{C}^m\to L_1\oplus\ldots\oplus L_n\to 0$$

where $H_1, \ldots, H_{mn}$ are line bundles on M such that $\deg H_j\leqq 0$ for $1\leqq j\leqq m-n$, and $L_1,\ldots,L_n$ are line bundles on M such that $[d/n]\leqq\deg L_j\leqq[d/n]+1$ for $1\leqq j\leqq n$ and $\deg(L_1\oplus\ldots\oplus L_n)=d$. Moreover $H^1(M, L_1\oplus\ldots\oplus L_n)=0$.

Before proving this lemma, let us use it to complete the proof 2.2. Let $E=L_1\oplus\ldots\oplus L_n$ where $L_1,\ldots,L_n$ satisfy the conditions of 4.1. By (c) above the type of E

lies in $\mathscr{U}$. Therefore the exact sequence of 4.1 defines a point of the open subset U of $A_d(n, m)$.

Clearly if L is any line bundle of degree $[d/n]$, then line sub-bundles of E of degree $d-d'$ correspond one to one with line sub-bundles of $L^*\otimes E$ of degree $d-d'-[d/n]$. But $L^*\otimes E$ is a sum of line bundles each having degree 0 or 1. So the result follows provided it can be shown that if E'' is a sum of line bundles with degrees 0 or 1 then the inclusion of the space of all holomorphic line sub-bundles of E'' of degree $-e$ in the space of all continuous line sub-bundles of the same degree is a homology equivalence up to dimension k provided that e is at least $2g+k+1$. We then only have to set $e=[d/n]-(d-d')$ and note that if $d\geqq n(2g+k+1+d-d')$ then $e\geqq 2g+k+1$.

Holomorphic (respectively continuous) line sub-bundles of degree $-e$ of the trivial bundle of rank n correspond exactly with holomorphic (respectively continuous) maps of degree e from M to $\mathbf{P}_{n-1}$. Therefore when E'' is the sum of trivial line bundles the result we want is just Segal's theorem. But in fact Segal's proof can be adapted without difficulty to the case when E'' is any sum of line bundles to give the following result.

4.2. Theorem. *Let N be a non-negative integer and for $0\leqq j\leqq N$ let L_j be a holomorphic line bundle of degree d_j on N. Then the inclusion of the space $HL_e(L_0\oplus\ldots\oplus L_N)$ of all holomorphic line sub-bundles of $L_0\oplus\ldots\oplus L_N$ of degree $-e$ in the space $CL_e(L_0\oplus\ldots\oplus L_N)$ of all continuous line sub-bundles of the same degree is a homology equivalence up to dimension*

$$(e+\min\{d_j \mid 0\leqq j\leqq N\}-2g)(2N-1).$$

In particular if $d_j\geqq 0$ for $0\leqq j\leqq N$ then this inclusion is a homology equivalence up to dimension $(e-2g)(2N-1)$.

Proof. This theorem is exactly Segal's result in the case when every L_j is the trivial line bundle. To prove the theorem in general one adapts Segal's proof as follows.

Let x_0 be any basepoint of M and let $M'=M-\{x_0\}$. Let $Q_e^{(N)}(M')$ be the space of all sequences $(\xi_0, \ldots, \xi_N)$ of positive divisors of degree e in M' with $\xi_0\cap\ldots\cap\xi_N=\varphi$. Let $j\colon \mathrm{Div}(M)\to\mathrm{Pic}(M)$ be the homomorphism from the group of divisors on M to the Picard variety of M which associates to any divisor the isomorphism class of the line bundle which it determines. Then j restricts to a map $j\colon \mathrm{Div}_0(M)\to J(M)$ where $J(M)=\mathrm{Pic}_0(M)$ is the Jacobian of M and $\mathrm{Div}_0(M)$ is the group of divisors of degree 0. Segal identifies the space of basepoint preserving holomorphic maps $f\colon M\to\mathbf{P}_N$ of degree e with the fibre at 0 of the map $Q_e^{(N)}(M')\to$

$J(M)^N$ which takes $(\xi_0, \ldots, \xi_N)$ to $\big(j(\xi_0-\xi_1), j(\xi_2-\xi_1), \ldots\big)$. He shows that this map is a homology equivalence up to dimension $(e-2g)(2N-1)$.

Let us choose a fixed 1-dimensional complex subspace V_0 of $(L_0)_{x_0}\oplus\ldots\oplus(L_N)_{x_0}$ such that the projection of V_0 onto $(L_j)_{x_0}$ is an isomorphism for each j. Let Q be the space of all sequences $(\xi_0, \ldots, \xi_N)$ of positive divisors of degrees $e+d_0, \ldots, e+d_N$ in M' with empty intersection. Given a holomorphic line sub-bundle L of degree $-e$ in $L_0\oplus\ldots\oplus L_N$ such that $L_{x_0}=V_0$ define an element $(\xi_0, \ldots, \xi_N)$ of Q by taking ξ_j to be the sum (including multiplicities) of those points x in M such that the projection of L_x onto $(L_j)_x$ is zero. In this way we can identify the space of holomorphic line sub-bundles of degree $-e$ in $L_0\oplus\ldots\oplus L_N$ with fixed fibre at the basepoint (as above) with the fibre at the point

$$([L_0\otimes L_1^*],\ [L_1\otimes L_2^*],\ \ldots,\ [L_{N-1}\otimes L_N^*])$$

of the map

$$Q \to \mathrm{Pic}_{d_0-d_1}(M)\times\mathrm{Pic}_{d_1-d_2}(M)\times\ldots\times\mathrm{Pic}_{d_{N-1}-d_N}(M)$$

which takes $(\xi_0, \ldots, \xi_N)$ to $\big(j(\xi_0-\xi_1), j(\xi_1-\xi_2)\ldots\big)$. Segal's proof that this map is a homology fibration up to dimension $(e-2g)(2N-1)$ when each L_j is the trivial line bundle extends with only the most trivial modifications to show that in general this map is a homology fibration up to dimension

$$(e+\min\{d_j \mid 0\leqq j\leqq N\}-2g)(2N-1).$$

The proof of Theorem 4.2 then follows exactly as in the case studied by Segal.

We have thus completed the proof of 2.2 except that it remains to prove Lemma 4.1.

Proof of Lemma 4.1. First note that if $L_1, \ldots, L_n$ are line bundles each of degree at least $[d/n]$ and if $d\geqq n(2g+1)$ then $H^1(M, L_1\oplus\ldots\oplus L_n)=0$ by 3.4. Moreover if

$$0\to H\to M\times\mathbf{C}^{n+1}\to L_1\oplus\ldots\oplus L_n\to 0$$

is an exact sequence with the properties required in the Lemma when $m=n+1$, then the exact sequence

$$0\to H\oplus(M\times\mathbf{C}^{m-n-1})\to M\times\mathbf{C}^m\to L_1\oplus\ldots\oplus L_n\to 0$$

has the properties required in the general case $m\geqq n+1$. Therefore it suffices to consider the case $m=n+1$.

Since $\dim M=1$, it follows from Riemann—Roch that if H is any line bundle of degree $e>2g$ then there is a nowhere-vanishing section of $H\oplus H$. Equivalently there is an embedding of H^* in the trivial rank 2 bundle on M whose quotient is a line bundle L of degree e. Then L is generated by two sections s_1 and s_2 say. By the same argument there is a line bundle L' of degree $e+1$ generated by sections s_1' and s_2'. Let S be the finite set of points x in M such that either $s_1(x)$ or $s_2(x)$ is zero. We can find a, b, c, d in $\mathbf{C}$ such that $ad-bc$ is nonzero and so are $as_1'(x)+bs_2'(x)$ and $cs_1'(x)+ds_2'(x)$ for all x in S. Therefore without loss of generality we may assume that s_1' and s_2' are both nonzero if either s_1 or s_2 is zero.

Now write $d=ne+p$ with $0\leqq p<n$ and define a holomorphic map of vector bundles φ from $M\times\mathbf{C}^m$ to the sum $L_1\oplus\ldots\oplus L_n$ where $L_j=L$ if $j\leqq n-p$ and $L_j=L'$ if $j>n-p$ as follows. Let $e_1, \ldots, e_m$ be the standard basis of $\mathbf{C}^m$. It is enough to specify $\varphi(x, e_j)$ for each $x\in M$ and $1\leqq j\leqq m$. For $1\leqq j\leqq n$ let

$$\chi_j\colon L_j \to L_1\oplus\ldots\oplus L_n$$

be the embedding of L_j as the j^{th} factor of the sum.

Then let

$$\varphi(x, e_1) = \chi_1(s_1(x))$$
$$\varphi(x, e_j) = \chi_{j-1}(s_2(x))+\chi_j(s_1(x)) \quad 2 \leqq j \leqq n-p$$
$$\varphi(x, e_{n-p+1}) = \chi_{n-p}(s_2(x))+\chi_{n-p+1}(s_1'(x))$$
$$\varphi(x, e_j) = \chi_{j-1}(s_2'(x))+\chi_j(s_1'(x)) \quad n-p+1 < j \leqq n$$
$$\varphi(x, e_{n+1}) = \chi_n(s_2'(x)).$$

(Recall that by assumption $m=n+1$). Since s_1 and s_2 generate L they never vanish simultaneously, and nor do s_1' and s_2' for the same reason. Therefore the assumption that s_1' and s_2' are both nonzero whenever either s_1 or s_2 vanishes implies that for all $x\in M$ either $s_1(x)\neq 0$ and $s_1'(x)\neq 0$ or $s_2(x)\neq 0$ and $s_2'(x)\neq 0$. From this it is easy to see that φ is surjective. The result follows.

5. Proof of Lemmas 2.3, 2.4 and 2.5

The proof of 2.3 is similar to the proof of 2.2, but easier. Assume that $d-d'>2g+k$. Let

$$q'\colon \tilde{A}_{d,d'}(n, m) \to A_{d'}(n-1, m)\times\mathrm{Pic}_{d-d'}(M)$$

have first component q_A and second component the map which sends a point represented by a sequence

$$0 \to L \to E \to E' \to 0$$

to the isomorphism class of L in $\mathrm{Pic}_{d-d'}(M)$, as described in § 4. Then points of the fibre of q' at a point $(h, [L])$ represented by a quotient

$$M\times\mathbf{C}^m \to E'$$

and a line bundle L can be identified with isomorphism classes of commutative diagrams

$$\begin{array}{ccccccc} & & & M\times \mathbf{C}^m & & & \\ & & & \downarrow & \searrow & & \\ 0 \to & L & \to & E & \to & E' & \to 0 \end{array}$$

such that $H^1(M, E)=0$. But $H^1(M, E')=0$ because h lies in $A_{d'}(n-1, m)$ and $H^1(M, L)=0$ because $d-d'>2g-1$ so the condition $H^1(M, E)=0$ is satisfied automatically. Hence by arrow-chasing we see that elements of the fibre correspond simply to quotient maps $K\to L$ or equivalently to embeddings $L^*\to K^*$, where K is the kernel of the quotient map $M\times\mathbf{C}^m\to E'$ defining h. Therefore the fibre of q' at $(h, [L])$ can be identified with the image in $\mathbf{P}(H^0(M, L\otimes K^*))$ of the nowhere-vanishing sections of $L\otimes K^*$.

Thus to prove the first part of 2.3 it is enough by 2.6(a) and 3.10 to show that $H^1(M, L\otimes K^*\otimes H^*)=0$ for any line bundle H of degree at most k. But this follows from the long exact sequence of cohomology associated to the exact sequence

$$0\to L\otimes E'^*\otimes H^*\to (H^*\otimes L)^m\to L\otimes K^*\otimes H^*\to 0,$$

on the assumption that $d-d'>k+2g$.

The first and last parts of 2.3 now follow immediately from 2.6(a) and 3.10. To complete the proof we repeat the argument used in the proof of 2.2.

First note that if $d'>(n-1)(2g+1)$ then by Lemma 4.1 there exists an element of $A_{d'}(n-1, m)$ represented by an exact sequence

$$0\to K\to M\times\mathbf{C}^m\to E'\to 0$$

where K^* is a sum of line bundles whose degrees are all non-negative. To prove 2.3 it now suffices to show that the inclusion of the space of all holomorphic line sub-bundles of K^* of degree $-(d-d')$ in the space of all continuous line sub-bundles of the same degree is a homology equivalence up to dimension k provided that $d-d'$ is at least $2g+k+1$. But this follows immediately from Theorem 4.2.

The proof of Lemma 2.3 is now complete.

Proof of Lemma 2.4. We need to prove that $\pi_1(\mathrm{Map}_d(M, G(n, m)))$ is abelian when $m>2$. Clearly we may assume that $0<n<m$.

First consider $\pi_1(\mathrm{Map}_d(M, \mathbf{P}_m))$ where $m\geqq 2$. Let p be a point of $\mathbf{P}_m$. By standard approximation arguments any element of $\pi_1(\mathrm{Map}_d(M, \mathbf{P}_m))$ may be represented by a smooth map $S^1\times M\to\mathbf{P}_m$. Since the real dimension of $S^1\times M$ is 3 and the real codimension of p in $\mathbf{P}_m$ is at least 4, by a general position argument we see that we may assume that the image of $S^1\times M$ in $\mathbf{P}_m$ does not contain p. (First choose the base-point of $\mathrm{Map}_d(M, \mathbf{P}_m)$ to be a smooth map whose image does not contain p.) A similar argument shows that if $m\geqq 3$ then two smooth maps $M\to\mathrm{Map}_d(M, \mathbf{P}_m-\{p\})$ are homotopic as maps into $\mathrm{Map}_d(M, \mathbf{P}_m-\{p\})$ if and only if they are homotopic as maps into $\mathrm{Map}_d(M, \mathbf{P}_m)$. Hence $\pi_1(\mathrm{Map}_d(M, \mathbf{P}_m))$ is isomorphic to $\pi_1(\mathrm{Map}_d(M, \mathbf{P}_m - \{p\}))$ when $m \geqq 3$. As $\mathbf{P}_m - \{p\}$ retracts onto $\mathbf{P}_{m-1}$ we deduce that $\pi_1(\mathrm{Map}_d(M, \mathbf{P}_m))$ is independent of m when $m \geqq 2$.

By 1.4. $\mathrm{Map}_d(M, G(n, m))$ is homeomorphic to $\mathrm{Map}_d(M, G'(n, m))$ where $G'(n, m)$ is the Grassmannian of n-dimensional subspaces of $\mathbf{C}^m$. There is an open subset W of $G'(n, m)$ defined by

$$W = \{V \in G'(n, m) \mid \dim V \cap \mathbf{C}^{m-n+1} = 1\}$$

where $\mathbf{C}^{m-n+1}$ is a fixed $(m-n+1)$-dimensional linear subspace of $\mathbf{C}^m$. The complement of W in $G'(n, m)$ has complex codimension at least

$$n(m-n) - (2(m-n-1) + (n-2)(m-n)) = 2$$

and so the argument used above shows that the inclusion of $\mathrm{Map}_d(M, W)$ into $\mathrm{Map}_d(M, G'(n, m))$ induces a surjection from $\pi_1(\mathrm{Map}_d(M, W))$ to

$$\pi_1(\mathrm{Map}_d(M, G'(n, m)))$$

The map $W \to \mathbf{P}_{m-n}$ given by sending V to $V \cap \mathbf{C}^{m-n+1}$ is a locally trivial fibration with fibre $\mathbf{C}^{(m-n)(n-1)}$ so $\mathrm{Map}_d(M, W)$ retracts onto $\mathrm{Map}_d(M, \mathbf{P}_{m-n})$. Therefore if $\mathrm{Map}_d(M, \mathbf{P}_{m-n})$ is abelian, so is $\mathrm{Map}_d(M, G(n, m))$. We may assume that $m-n \geqq 2$ since otherwise $G(n, m)$ is a projective space itself, so it is now enough to show that $\pi_1(\mathrm{Map}_d(M, \mathbf{P}_2))$ is abelian.

Let Map* denote based maps. Then since $\pi_1(\mathbf{P}_2) = 0$ the fibration sequence

$$\mathrm{Map}_d^*(M, \mathbf{P}_2) \to \mathrm{Map}_d(M, \mathbf{P}_2) \to \mathbf{P}_2$$

induces a surjection $\pi_1(\mathrm{Map}_d^*(M, \mathbf{P}_2)) \to \pi_1(\mathrm{Map}_d(M, \mathbf{P}_2))$. Therefore it suffices to show that $\pi_1(\mathrm{Map}_d^*(M, \mathbf{P}_2))$ is abelian. $\mathrm{Map}_d^*(M, \mathbf{P}_2)$ is independent of d up to homotopy, so we may take $d=0$. The 1-skeleton of M is a wedge VS^1 of $2g$ copies of S^1, and the result of collapsing it to a point in M is S^2. Therefore there is an exact sequence

$$\Omega^2\mathbf{P}_2 = \mathrm{Map}_0^*(S^2, \mathbf{P}_2) \to \mathrm{Map}_0^*(M, \mathbf{P}_2) \to \mathrm{Map}_0^*(VS^1, \mathbf{P}_2)$$

which gives an exact sequence

$$\pi_1(\Omega^2\mathbf{P}_2) \to \pi_1(\mathrm{Map}_0^*(M, \mathbf{P}_2)) \to \pi_1(\Omega^1(\mathbf{P}_2))^{2g}$$

because

$$\mathrm{Map}_0^*(VS^1, \mathbf{P}_2) = \mathrm{Map}_0^*(S^1, \mathbf{P}_2)^{2g} = (\Omega^1(\mathbf{P}_2))^{2g}.$$

Therefore as $\pi_1(\Omega^2\mathbf{P}_2) = \pi_3(\mathbf{P}_2) = 0$ and $\pi_1(\Omega^1(\mathbf{P}_2)) = \pi_2(\mathbf{P}_2) = \mathbf{Z}$ the result follows.

Proof of Lemma 2.5. We need to prove that if $m > 2$ and $0 < n < m$ and $d > 2g+1$ then $\pi_1(\mathrm{Hol}_d(G(n, m)))$ and $\pi_1(A_d(n, m))$ are abelian and the inclusion of $A_d(n, m)$ in $\mathrm{Map}_d(M, G(n, m))$ induces a surjection on fundamental groups.

Let W be the open subset of $G(n, m)$ defined by

$$W = \{\mathbf{C}^m/V \in G(n, m) | \dim V \cap \mathbf{C}^{m-1} = m-n-1\}$$

where $\mathbf{C}^{m-1}$ is a fixed hyperplane in $\mathbf{C}^m$. The complement W^c of W has complex codimension

$$\dim G(n, m) - \dim G(n-1, m-1) = m-n$$

in $G(n, m)$. By standard approximation arguments any element of

$$\pi_1(\mathrm{Hol}_d(M, G(n, m)))$$

may be represented by a smooth based map $S^1 \to \mathrm{Hol}_d(M, G(n, m))$, or equivalently by a smooth map φ: $S^1 \times M \to G(n, m)$ such that the restriction φ_t: $M \to G(n, m)$ of φ to $\{t\} \times M$ is holomorphic of degree d for any $t \in S^1$, and when t is the basepoint of S^1 then φ_t is the chosen basepoint of $\mathrm{Hol}_d(M, G(n, m))$.

Let S be the subgroup of the complex general linear group $GL(m)$ given by

$$S = \{g \in GL(m) | gW = W\}.$$

Then S contains every $g \in GL(m)$ which preserves $\mathbf{C}^{m-1}$ so S is a parabolic subgroup of $GL(m)$ and hence $GL(m)/S$ is compact. Consider the subset

$$\{gS \in GL(m)/S \mid \varphi(S^1 \times M) \nsubseteq gW\}$$

of $GL(m)/S$. This is the image of the submanifold

$$\{(t, x, gS) \in S^1 \times M \times GL(m)/S \mid \varphi_t(x) \notin gW\}$$

of $S^1 \times M \times GL(m)/S$ under the projection to $GL(m)/S$. For any $t \in S^1$ and $x \in M$ we have

$$\{gS \in GL(m)/S \mid \varphi_t(x) \notin gW\} = \{gS \in GL(m)/S \mid \varphi_t(x) \subseteqq g(\mathbf{C}^{m-1})\}$$

and it is easy to see that this has complex codimension $m-n$ in $GL(m)/S$. If $m-n \geqq 2$ this means it has real codimension at least 4. Since $S^1 \times M$ is compact of real dimension 3 it follows that the subset

$$\{g \in GL(m) \mid \varphi(S^1 \times M) \subseteqq gW\}$$

is open and dense in $GL(m)$. In particular we see that by replacing the basepoint φ_0 of $\mathrm{Hol}_d(M, G(n, m))$ by $g\varphi_0$ for a suitable g we may assume that φ_0 lies in $\mathrm{Hol}_d(M, W)$. Moreover we may then choose a continuous map $t \to g_t$ from S^1 to $SL(m)$ which is homotopic to the identity, sends the basepoint of S^1 to 1 and satisfies $g_t \varphi_t(M) \subseteqq W$ for all $t \in S^1$. Since $g\varphi_t$: $M \to G(n, m)$ is holomorphic of degree d for any $g \in GL(m)$ and any $t \in S^1$ we deduce that the inclusion of $\mathrm{Hol}_d(M, W)$ in $\mathrm{Hol}_d(M, G(n, m))$ induces a surjection

$$\pi_1(\mathrm{Hol}_d(M, W)) \to \pi_1(\mathrm{Hol}_d(M, G(n, m))).$$

Moreover a similar argument involving homotopies between maps $S^1 \times M \to G(n, m)$ shows that if $m-n \geqq 3$ then this surjection is an isomorphism.

The map $W \to G(n, m-1)$ given by $V \to V \cap \mathbf{C}^{m-1}$ is a holomorphic vector bundle with fibre $\mathbf{C}^n$, so $\mathrm{Hol}_d(M, W)$ is homotopy equivalent to $\mathrm{Hol}_d(M, G(n, m-1))$. It follows that

$$\pi_1(\mathrm{Hol}_d(M, G(n, m))) \cong \pi_1(\mathrm{Hol}_d(M, G(n, m-1)))$$

when $m-n \geqq 3$, and there is a surjection

$$\pi_1(\mathrm{Hol}_d(M, G(n, m-1))) \to \pi_1(\mathrm{Hol}_d(M, G(n, m)))$$

when $m-n=2$. In particular since by 1.4

$$\pi_1(\mathrm{Hol}_d(M, G(1, m+1))) = \pi_1(\mathrm{Hol}_d(M, \mathbf{P}_m))$$

for all m we have that $\pi_1(\mathrm{Hol}_d(M, \mathbf{P}_m))$ is independent of m when $m \geqq 2$. Moreover by Remark 1.4 we have

$$\pi_1(\mathrm{Hol}_d(M, G(n, m))) \cong \pi_1(\mathrm{Hol}_d(M, G(m-n, m))).$$

Therefore it is now enough to consider those $G(n, m)$ such that $n \leqq \frac{1}{2} m$. If $n > 1$ then $m-n+1 \geqq n+1 > 2$ and there is a surjection

$$\pi_1(\mathrm{Hol}_d(M, G(1, m-n+1))) \to \pi_1(\mathrm{Hol}_d(M, G(n, m))).$$

Hence in order to show that $\pi_1(\mathrm{Hol}_d(M, G(n, m)))$ is abelian when $m > 2$ it suffices to show that $\pi_1(\mathrm{Hol}_d(M, G(1, m)))$ is abelian for all $m > 2$. Moreover since $\pi_1(\mathrm{Hol}_d(M, \mathbf{P}_m))$ is independent of m when $m \geqq 2$ the same is true of

$$\pi_1(\mathrm{Hol}_d(M, G(1, m)))$$

when $m > 2$.

Let $G(1, \infty)$ be the Grassmannian of 1-dimensional quotients of a separable complex Hilbert space $\mathbf{C}^\infty$ (see § 7). Let $\mathrm{Hol}_d(M, G(1, \infty))$ be the union of the subsets $\mathrm{Hol}_d(M, G(1, V))$ of $\mathrm{Map}_d(M, G(1, \infty))$ over all finite-dimensional subspaces V of $\mathbf{C}^\infty$, where $G(1, V)$ is embedded in $G(1, \infty)$ by using the orthogonal projection of $\mathbf{C}^\infty$ onto V. Then $\mathrm{Hol}_d(M, G(1, \infty))$ has the induced limit topology (see § 7 again). The proof that $\pi_1(\mathrm{Hol}_d(M, G(1, m)))$ is independent of m when $m > 2$ shows also that $\pi_1(\mathrm{Hol}_d(M, G(1, m)))$ is isomorphic to $\pi_1(\mathrm{Hol}_d(M, G(1, \infty)))$ when $m > 2$. Therefore it suffices to show that the latter is abelian.

Let $\tilde{R}$ be the subspace of $\mathrm{Hol}_d(M, G(1, \infty))$ consisting of all those maps $h\colon M \to G(1, V)$ for some finite-dimensional subspace V of $\mathbf{C}^\infty$ which induce a quotient line bundle L of $M \times V$ such that the corresponding map of sections $V \to H^0(M, L)$ is surjective. If $d > 2g-1$ then $H^1(M, L) = 0$ and $\dim H^0(M, L) = p$ where $p = d + 1 - g$. Similarly define R to be the subspace of $\mathrm{Hol}_d(M, G(1, p))$ consisting of those

h: $M \to G(1, p)$ such that the corresponding map of sections $\mathbf{C}^p \to H^0(M, L)$ is an isomorphism.

The proof of Corollary 7.2 below shows immediately that the inclusion of $\tilde{R}$ in $\mathrm{Hol}_d(M, G(1, \infty))$ is a weak homotopy equivalence and in particular

$$\pi_1(\mathrm{Hol}_d(M, G(1, \infty))) \cong \pi_1(\tilde{R}).$$

(Note that when $d > 2g-1$ then $\mathrm{Hol}_d(M, G(1, \infty)) = A_d(1, \infty)$).

Let $EGL(p)$ be the space of all quotient maps e: $\mathbf{C}^\infty \to \mathbf{C}^p$. This is a contractible space on which $GL(p)$ acts freely and the quotient $BGL(p)$ is the infinite quotient Grassmannian $G(p, \infty)$ (cf. § 7 and § 10). If we identify elements of R with quotient bundle maps $\mathbf{C}^p \times M \to L$ where L is a line bundle and elements of $\tilde{R}$ with quotient bundle maps $\mathbf{C}^\infty \times M \to L$ then there is a map

$$R \times EGL(p) \to \tilde{R}$$

which sends an element of $R \times EGL(p)$ represented by ψ: $\mathbf{C}^p \times M \to L$ and e: $\mathbf{C}^\infty \to \mathbf{C}^p$ to the element of $\tilde{R}$ represented by the composition $\psi \circ (1d_M \times e)$: $\mathbf{C}^\infty \times M \to L$. It is not hard to check (cf. § 10) that this induces a homeomorphism from $R \times_{GL(p)} EGL(p)$ to $\tilde{R}$.

$GL(p)$ acts freely on R and $R/GL(p)$ is the moduli space of line bundles of degree d on M, so it is isomorphic to the Jacobian $J(M)$. (See § 9, noting that a line bundle is always semistable.) Thus we have a fibration

$$\tilde{R} \cong R \times_{GL(p)} EGL(p) \to J(M)$$

with contractible fibre $EGL(p)$. Hence

$$\pi_1(\tilde{R}) \cong \pi_1(J(M)) = \mathbf{Z}^{2g}.$$

(To make this argument more precise by replacing R and $GL(p)$ by a compact manifold acted on freely by a compact group see e.g. [K].)

This completes the proof that $\pi_1(\mathrm{Hol}_d(M, G(n, m)))$ is abelian for $m > 2$. One can check that exactly the same arguments go through for the open subset $A_d(n, m)$ of $\mathrm{Hol}_d(M, G(n, m))$, or alternatively use the arguments of § 6 to deduce that the inclusion of $A_d(n, m)$ in $\mathrm{Hol}_d(M, G(n, m))$ induces an isomorphism of their fundamental groups.

It remains to prove that the map

$$\pi_1(A_d(n, m)) \to \pi_1(\mathrm{Map}_d(M, G(n, m)))$$

induced by the inclusion is surjective when $d > 2g+1$. When $n = 1$ it follows immediately from Segal's theorem that the map is an isomorphism since we know that both fundamental groups are abelian and hence are isomorphic to the corresponding first homology groups. When $n > 1$ we may assume $n \leq m/2$ as above and we have a commutative diagram

$$\begin{array}{ccc} \pi_1(A_d(n,m)) & \to & \pi_1(\mathrm{Map}_d(M, G(n,m))) \\ \uparrow & & \uparrow \\ \pi_1(A_d(1, m-n+1)) & \underset{\to}{\cong} & \pi_1(\mathrm{Map}_d(M, G(1, m-n+1))) \end{array}$$

where the two vertical maps are surjective by the argument used earlier in the proof of this lemma. The result follows immediately.

This completes the proof of Theorem 1.1.

6. On removing subvarieties of high codimension from singular varieties

It is well-known that the removal of a closed subvariety of codimension k from a nonsingular variety does not change the cohomology in dimensions less than k. In order to deduce Theorem 1.2 from Theorem 1.1 we need to show that a similar result holds for singular varieties satisfying certain conditions. (Clearly some conditions are necessary.)

6.1. Theorem. *Suppose X is of the form*

$$X = \{x \in U \mid f_1(x) = \dots = f_m(x) = 0\}$$

where U is an open subset of $\mathbf{C}^N$ and $f_1, \dots, f_m$ are analytic complex-valued functions on U. Suppose that Y is a closed analytic subvariety of U of complex codimension k. Then

$$H_q(X, X-Y) = 0 = H^q(X, X-Y)$$

for $q<k-m$, where H_q and H^q denote homology and cohomology with integer coefficients.

6.2. *Remark.* By a theorem of Lojasiewicz ([L] thm. 1) there is a locally finite triangulation of $\mathbf{C}^N$ such that both X and Y are simplicial. In particular the simplicial cohomology groups of X and Y coincide with the Čech and Alexander cohomology groups. So we may take either kind of cohomology in 6.1.

Proof of the theorem. By the universal coefficient theorem it is enough to consider homology. Moreover a standard Mayer—Vietoris argument shows that it is enough to prove the result when U is a sufficiently small neighbourhood of any point x of Y (cf. [Sp] 5.7.9).

If the case $m=1$ is true, then the other cases follow by induction as follows. Suppose that $m>1$ and that the result is true for 1 and $m-1$. Let

$$X_1 = \{x \in U \mid f_j(x) = 0, \quad 1 \leqq j \leqq m-1\}$$

and

$$X_2 = \{x \in U | f_m(x) = 0\}.$$

Then

$$X_1 \cup X_2 = \{x \in U | f_m(x) f_j(x) = 0, \quad 1 \leqq j \leqq m-1\} = X_3,$$

say. So by induction if $i=1, 2$ or 3 then

$$H_q(X_i, X_i - Y) = 0$$

for $0 \leqq q \leqq k-m+1$. But since

$$X_1 \cap X_2 = X,$$

there is a Mayer—Vietoris sequence

$$\ldots \to H_{q+1}(X_3, X_3 - Y) \to H_q(X, X-Y) \to$$
$$H_q(X_1, X_1 - Y) \oplus H_q(X_2, X_2 - Y) \to H_q(X_3, X_3 - Y) \to \ldots$$

Hence $H_q(X, X-Y)=0$ for $0 \leqq q \leqq k-m$.

So we need only consider the case

$$X = \{x \in U | f(x) = 0\}.$$

We may assume without loss of generality that $X \cap Y = Y$. There is a stratification of Y into nonsingular strata (the Whitney stratification). By removing strata one by one we may assume that Y is nonsingular in U with codimension k. Thus without loss of generality

$$Y = \{(z_1, \ldots, z_N) \in U \,|\, 0 = z_1 = \ldots = z_k\}.$$

Given a convex open subset V of $\mathbf{C}^{N-k}$ and a real number $t>0$, let $A(V, t)$ be the set of all points $(z_1, \ldots, z_N)$ in $\mathbf{C}^N$ satisfying

$$|z_1|^2 + \ldots + |z_k|^2 < t \quad \text{and} \quad (z_{k+1}, \ldots, z_N) \in V;$$

and let $X(V, t) = A(V, t) \cap X$. Then those $X(V, t)$ for which $A(V, t) \subseteqq U$ form an open cover of X, and the intersection of any two sets of this form is also of the same form because

$$A(V_1, t_1) \cap A(V_2, t_2) = A\big(V_1 \cap V_2, \min(t_1, t_2)\big).$$

Hence by the Mayer—Vietoris sequence argument of [Sp] 5.7.9 it is enough to show that for any $x \in Y$ and any sufficiently small V and t such that $A(V, t)$ is a neighbourhood of x in $\mathbf{C}^N$, the relative cohomology

$$H_q\big(X(V, t),\ X(V, t) - Y\big)$$

vanishes for $q < k-1$.

Without loss of generality $x=0$. It is enough to prove that

a) $\tilde{H}_q\big(X(V, t) - Y\big) = 0$

and

b) $\tilde{H}_q(X(V, t)) = 0$

for sufficiently small V, t and $0 \leqq q < k-1$, where $\tilde{H}$ is reduced homology.

To prove (a) let

$$W(V, t) = (0, t) \times V$$

and define $h: A(V, t) \to W(V, t)$ by

$$h(z_1, \ldots, z_N) = (|z_1|^2 + \ldots + |z_k|^2, z_{k+1}, \ldots, z_N).$$

Let $\tilde{h}: X(V, t) - Y \to W(V, t)$ be the restriction of h to $X(V, t) - Y$ and let F be any fibre of $\tilde{h}$. Then by [M] Theorem 5.2 we have

$$\tilde{H}_q(F) = 0$$

for $0 \leqq q \leqq k-2$. (It should be noted that in [M] Milnor only considers the case when the function f is a polynomial. However the only time he uses this is in the proof of his Curve Selection Lemma. Here he needs the fact that any one-dimensional real or complex algebraic variety is locally the union of finitely many branches intersecting at one point, each branch being homeomorphic to **R** or **C**. This is true for analytic spaces too by resolution of singularities so his proof is valid for any analytic f.)

Milnor's proof that $\tilde{H}_q(F) = 0$ can be adapted to give the following

6.3. Lemma. *Each point of $W(V, t)$ has a base of neighbourhoods U in $W(V, t)$ satisfying $\tilde{H}_q(\tilde{h}^{-1}(U)) = 0$ for $0 \leqq q \leqq k-2$.*

Before proving this lemma let us use it to complete the proof of Theorem 6.1. The direct image sheaf $R^q_{\tilde{h}}(\mathbf{Z})$ is the sheaf on $W(V, t)$ associated to the presheaf

$$U \to H^q(\tilde{h}^{-1}(U))$$

and hence by 6.3 it is 0 when $q > 0$ and is **Z** when $q = 0$. There is a spectral sequence (the Leray spectral sequence) abutting to $H^*(X(V, t) - Y)$ with

$$E_2^{p,q} = H^p(W(V, t), R^q_{\tilde{h}}(\mathbf{Z})) = \begin{cases} \mathbf{Z} & \text{when } p = q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Hence (a) holds.

Applying (a) with Y as the origin we see that $H_q(X(V, t) - (0)) = 0$ for t sufficiently small and $0 \leqq q \leqq k-2$. Hence by excision every inclusion

$$X(V, t) \subset_{*} X(V, t')$$

is a homology equivalence up to dimension $k-2$.

On the other hand by Theorem 1 of [L] X can be triangulated, which implies that there is a base of contractible neighbourhoods of 0 in X. Hence for any V, t

such that $X(V, t)$ contains 0 we can find a contractible open neighbourhood A of 0 in X and V', t' such that

$$X(V', t') \subseteqq A \subseteqq X(V, t).$$

So there are homomorphisms

$$\tilde{H}_q(X(t)) \to \tilde{H}_q(A) = 0 \to \tilde{H}_q(X(t'))$$

whose composition is an isomorphism for $0 \leqq q \leqq k-2$ if V, t are small enough. Thus (b) holds, and the proof of 6.1 is complete except for the proof of Lemma 6.3.

Proof of 6.3. The proof that $\tilde{H}_q(F)=0$ for any fibre F of $\tilde{h}$: $X(t)-Y \to W(t)$ given in [M] by Milnor runs as follows. He considers the function $|f|$ on the fibre containing F of the extension h of $\tilde{h}$ to $V(t)$. This fibre is a sphere S of dimension $k-1$, and the subset of S on which $|f|$ takes its minimum value 0 is $S \cap X = F$. Milnor shows that every critical point of $|f|$: $S \to \mathbf{R}^+$ in $S-F$ has index at least $k-1$ ([M] 5.6). He also shows that the critical points of $|f|$ outside F are contained in a compact subset of $S-F$ ([M] 5.7). Using [Mo] Theorem 8.7 he deduces that there is a smooth mapping s: $S-F \to \mathbf{R}^+$ all of whose critical points are isolated and nondegenerate of index at least $k-1$, such that $s=|f|$ in some neighbourhood of F ([M] 5.8). From this it follows that $\tilde{H}^q(F)=0$ for $q \leqq k-2$.

The deduction of the last statement is standard Morse theory. One method of proof is to put a Riemannian metric on the sphere S and consider the gradient flow of the smooth function s: $S-F \to R^+$. The union of F with the set of all points in $S-F$ whose trajectories under the flow $-\text{grad}\,|f|$ have limit points in F is an open subset S_0 of S. For each critical point x in $S-F$ there is a locally-closed submanifold S_x of $S-F$ consisting of the points whose forward trajectories under the flow $-\text{grad}\,|f|$ converge to x. The codimension of S_x in S is the index of s at x which is at least $k-1$ for each x. The subsets S_x together with S_0 form a smooth stratification of S. This means that we can remove the subsets S_x from S one at a time so that at each stage we are removing a closed submanifold of codimension at least $k-1$ from an open subset of S, until eventually we are left with the open subset S_0. At each stage the Thom—Gysin long exact sequence of cohomology tells us that the cohomology in dimensions up to $k-2$ remains unaltered. We deduce that

$$\tilde{H}^q(S_0) = \tilde{H}^q(S) = 0$$

for $q \leqq k-2$. Finally since S_0 retracts onto arbitrarily small neighbourhoods of F and Čech cohomology is continuous, by 6.2 we have that

$$\tilde{H}^*(S_0) = \tilde{H}^*(F).$$

Now let M be a smooth manifold and let $g\colon S\times M\to \mathbf{R}^+$ be a continuous function whose restriction to the complement of $g^{-1}(0)$ is smooth and satisfies the following property. For each x in M every critical point not belonging to $g^{-1}(0)$ of the restriction of g to $S\times\{x\}$ is isolated and nondegenerate of index at least $k-1$ and there is a neighbourhood of $g^{-1}(0)$ in $S\times M$ whose complement contains all such critical points for every x. Consider the flow on $S\times M-g^{-1}(0)$ given on $S\times\{x\}$ by the projection of $-\operatorname{grad} g$ along $S\times\{x\}$. The fact that the Hessian of g at such a point is nondegenerate in the direction of $S\times\{x\}$ and that no such point lies in some neighbourhood of $g^{-1}(0)$ shows that the set of all fixed points is a disjoint union of closed submanifolds N of $S\times M$ each locally diffeomorphic to M. Moreover the nondegeneracy condition and the assumption on indices imply that $S\times M$ decomposes as the disjoint union of an open subset S_0 together with locally closed submanifolds S_N of codimension at least $k-1$ in S, where (s,x) belongs to S_0 if its forward trajectory $T(s,x)$ under the flow has a limit point in $g^{-1}(0)$ and to S_N if $T(s,x)$ converges to a point of N (see e.g. [H] IX § 5). Then using Thom—Gysin sequences again we see that $H^q(S\times M)=H^q(S_0)$ when q is at most $k-2$.

If we no longer require that the critical points of g restricted to each submanifold $S\times\{x\}-g^{-1}(0)$ are nondegenerate, the result still holds with M replaced by any sufficiently small neighbourhood U of any x in M. For the proof of [Mo] Theorem 8.7 shows that if U is small enough there is a continuous function $s\colon S\times U\to\mathbf{R}^+$ which is smooth away from $g^{-1}(0)$ and satisfies $s=g$ in a neighbourhood of $g^{-1}(0)\cap S\times U$ such that for each y in U the restriction of s to $S\times\{y\}$ has only finitely many critical points and these are all nondegenerate of index at least $k-1$. If we choose U to be contractible then it follows that for Čech cohomology we have

$$\check{H}^q(S\times U\cap g^{-1}(0)) = \check{H}^q(S\times U\cap S_0) = \check{H}^q(S\times U) = 0$$

for $q\leqq k-2$.

There is a diffeomorphism of $V(t)$ onto the product of S and $W(t)$ such that the composition of its inverse with h is projection onto $W(t)$. So if we take M to be $W(t)$ and g to be $|f|$ it follows from 6.2 and the last paragraph that every point of $W(t)$ has a base of neighbourhoods U in $W(t)$ such that

$$\check{H}^q(h^{-1}(U)\cap f^{-1}(0)) = 0$$

for $q\leqq k-2$. Since $h^{-1}(U)\cap f^{-1}(0)=h^{-1}(U)\cap X=\tilde{h}^{-1}(U)$ this completes the proof of 6.3, and hence also of 6.1.

6.4. Corollary. *Let X be a quasi-projective variety and m a non-negative integer such that every $x_0\in X$ has a neighbourhood in X which is analytically isomorphic to an open subset of*

$$\{x\in\mathbf{C}^N \mid f_1(x)=\ldots=f_M(x)=0\}$$

for some N, M and analytic functions $f_1, \ldots, f_M$ depending on x_0 with $M \leqq m$. If Y is a closed subvariety of codimension k in X then

$$H_q(X, X-Y) = 0 = H^q(X, X-Y)$$

for $q<k-m$.

Proof. By the Mayer—Vietoris argument used several times in the proof of 6.1 it is enough to prove this locally. Hence it follows immediately from 6.1.

Remark. If X is a local complete intersection of pure dimension n such that for every $x \in X$ the dimension of the Zariski tangent space at x is at most $n+m$ then X satisfies the hypotheses of Corollary 6.4.

In order to apply this corollary to prove Theorem 1.2 we need the following lemma.

6.5. Lemma. *Any point of $\mathrm{Hol}_d(M, G(n, m))$ has a neighbourhood which is analytically isomorphic to an open subset of*

$$\{x \in \mathbf{C}^N | f_1(x) = \ldots = f_M(x) = 0\}$$

where

$$N = (m+1)(d+ne+n(1-g))+n^2(1-g)$$

and

$$M = d+ne(m+1)+n(1-g)$$

for some positive integer e, and $f_1, \ldots, f_M$ are analytic functions.

Proof. Let h_0 be an element of $\mathrm{Hol}_d(M, G(n, m))$ and let

$$M \times \mathbf{C}^m \to E$$

be the corresponding quotient map of bundles over M. Choose a positive integer e sufficiently large that $H^1(M, L \otimes E) = 0$ for every line bundle L of degree at least e. By Lemma 4.1 if e is large enough there is a line bundle L of degree e on M and an exact sequence.

$$0 \to K \to M \times \mathbf{C}^{m+1} \to L^{\oplus m} \to 0$$

where $L^{\oplus m}$ denotes the sum of m copies of L and K is a line bundle of degree $-me$. Define a map

$$t\colon \mathrm{Hol}_d(M, G(n, m)) \to \mathrm{Hol}_{d+ne}(M, G(n, m+1))$$

by sending a point represented by a quotient

$$M \times \mathbf{C}^m \to E'$$

to the point represented by the composition

$$M \times \mathbf{C}^{m+1} \to L^{\oplus m} \to L \otimes E'.$$

The image of t consists precisely of those points h of $\mathrm{Hol}_{d+ne}(M, G(n, m+1))$ represented by quotients E'' of $M\times\mathbf{C}^{m+1}$ such that the composition

$$K \to M\times\mathbf{C}^{m+1} \to E''$$

is zero. This composition can be identified with a section $s(h)$ of the bundle $K^*\otimes E''$.

Since $H^1(M, L\otimes E)=0$ the point $t(h_0)$ which is represented by the quotient $L\otimes E$ of $M\times\mathbf{C}^{m+1}$ lies in the nonsingular open subset $A_{d+ne}(n, m+1)$ of the variety $\mathrm{Hol}_{d+ne}(M, G(n, m+1))$. Moreover since $H^1(M, K^*\otimes L\otimes E)=0$ there is a holomorphic bundle over some neighbourhood of $t(h_0)$ in $A_{d+ne}(n, m+1)$ whose fibre at a point h represented by a quotient E'' of $M\times\mathbf{C}^{m+1}$ is naturally isomorphic to $H^0(M, K^*\otimes E'')$ (cf. 3.10.) This holomorphic bundle has rank M where

$$M = \dim H^0(M, K^*\otimes L\otimes E) = d+ne(m+1)+n(1-g)$$

by Riemann—Roch since $K^*\otimes L\otimes E$ has rank n and degree $d+ne+nme$ and its first cohomology vanishes. Let $\sigma_1, \ldots, \sigma_M$ be holomorphic sections of this bundle over an open neighbourhood U of $t(h_0)$ in $A_{d+ne}(n, m+1)$ such that $\sigma_1(h), \ldots, \sigma_M(h)$ is a basis of the fibre $H^0(M, K^*\otimes E'')$ at h for every $h\in U$. Then the components $f_1(h), \ldots, f_M(h)$ with respect to this basis of the section $s(h)$ of $K^*\otimes E''$ defined above are analytic functions of $h\in U$. Moreover the intersection with U of the image of the map

$$t\colon \mathrm{Hol}_d(M, G(n, m)) \to \mathrm{Hol}_{d+ne}(M, G(n, m+1))$$

is precisely the subset

$$\{h\in U | f_1(h) = \ldots = f_M(h) = 0\}.$$

But U is an open subset of $A_{d+ne}(n, m+1)$ which is a nonsingular variety of dimension

$$N = (m+1)(d+ne+n(1-g))+n^2(1-g)$$

(cf. the computation in [Se] § 6). Therefore it remains only to show that the restriction of t to some neighbourhood of h_0 in $\mathrm{Hol}_d(M, G(n, m))$ is an analytic isomorphism onto its image.

Clearly given the composition

$$M\times\mathbf{C}^{m+1} \to L^{\oplus m} \to L\otimes E'$$

representing $t(h)$ for any h we can recover the map $L^{\oplus m}\to L\oplus E'$ (because $M\times\mathbf{C}^{m+1}\to L^{\oplus m}$ is a fixed surjection) and hence tensoring with L^* we can recover the map $M\times\mathbf{C}^m\to E'$ representing the element h of $\mathrm{Hol}_d(M, G(n, m))$. Hence t is a bijection onto its image, and in fact it is not hard to see that the process just described defines an analytic inverse to t from its image in $\mathrm{Hol}_{d+ne}(M, G(n, m+1))$ to $\mathrm{Hol}_d(M, G(n, m))$.

This completes the proof of the lemma.

6.6. Corollary. *Let k be a positive integer. If $d \geqq n(2g+\max(k, \frac{1}{4}n^2g, n(1-g)))$ then the inclusion of $A_d(n, m)$ in $\mathrm{Hol}_d(M, G(n, m))$ is a cohomology equivalence up to dimension $k-2m^2g$.*

Proof. First note that if

$$0 \to K \to M \times \mathbf{C}^m \to E \to 0$$

represents any element h of $\mathrm{Hol}_d(M, G(n, m))$ then

$$\dim H^1(M, E) \leqq m \dim H^1(M, \mathcal{O}_M) = mg.$$

Moreover from the exact sequence

$$0 \to E^* \otimes E \to E \oplus \ldots \oplus E \to K^* \otimes E \to 0$$

we obtain

$$\dim H^1(M, K^* \otimes E) \leqq m \dim H^1(M, E) \leqq m^2 g.$$

The Zariski tangent space to $\mathrm{Hol}_d(M, G(n, m))$ at h is naturally isomorphic to $H^0(M, K^* \otimes E)$. Since the dimension of this is

$$\dim H^1(M, K^* \otimes E) + m(d+n(1-g)) + (1-g)n^2$$

the dimension of the Zariski tangent space at h is at most

$$m^2 g + m(d+n(1-g)) + (1-g)n^2.$$

By 6.5 we know that there is a neighbourhood of h in $\mathrm{Hol}_d(M, G(n, m))$ which is analytically isomorphic to an open subset of

$$\{x \in \mathbf{C}^N | f_1(x) = \ldots = f_M(x) = 0\}$$

where

$$N = (m+1)(d+ne+n(1-g)) + n^2(1-g)$$

and

$$M = d + ne(m+1) + n(1-g)$$

for some positive integer e, and $f_1, \ldots, f_M$ are analytic. We may assume that h is mapped to the origin in $\mathbf{C}^N$. Then the Zariski tangent space to $\mathrm{Hol}_d(M, G(n, m))$ at h is isomorphic to the linear subspace of $\mathbf{C}^N$ defined by the vanishing of the derivatives $df_1(0), \ldots, df_m(0)$ of $f_1, \ldots, f_m$ at 0. Therefore at least

$$N - (m^2 g + m(d+n(1-g)) + (1-g)n^2) = (m+1)ne + d + n(1-g) - m^2 g = M - m^2 g$$

of these must be linearly independent, say $df_1(0), \ldots, df_{M'}(0)$ where $M - m^2 g \leqq M' \leqq M$. Then the intersection, U, say, of some sufficiently small open neighbourhood of the origin in $\mathbf{C}^N$ with

$$\{x \in \mathbf{C}^N | f_1(x) = \ldots = f_{M'}(x) = 0\}$$

is nonsingular. Thus there is a neighbourhood of h in $\mathrm{Hol}_d(M, G(n, m))$ which is analytically isomorphic to

$$\{x \in U | f_{M'+1}(x) = f_{M'+2}(x) = \dots = f_M(x) = 0\}$$

and this is a subset of a nonsingular complex analytic variety defined by the vanishing of at most m^2g analytic functions. It is now enough by 6.4 to show that if $d \geqq n(2g + \max(k, \frac{1}{4}n^2g, n(1-g)))$, then the complex codimension of the complement of $A_d(n, m)$ in $\mathrm{Hol}_d(M, G(n, m))$ is greater than $k - m^2g$.

Consider the infinitesimal deformation map

$$H^0(M, K^* \otimes E) \to H^1(M, \mathrm{End}\, E)$$

at any $h \in \mathrm{Hol}_d(M, G(n, m))$ represented by an exact sequence

$$0 \to K \to M \times \mathbf{C}^m \to E \to 0$$

as above (see 3.7). From the exact sequence

$$0 \to E^* \otimes E \to E \oplus \dots \oplus E \to K^* \otimes E \to 0$$

we obtain a bound of

$$\dim H^1(M, E \oplus \dots \oplus E) \leqq m^2 g$$

on the complex dimension of its cokernel. This implies that the elements h in $\mathrm{Hol}_d(M, G(n, m))$ which correspond to bundles of any fixed type μ are contained in a subvariety of complex codimension at least

$$d_\mu - m^2 g$$

(see 3.7 again).

By 3.9 if d is greater than or equal to $n(2g + \max(k, \frac{1}{4}n^2g, n(1-g)))$ there is a finite set $\mathscr{U}$ of types such that if $\mu \notin \mathscr{U}$ then

$$d_\mu > k$$

and if E is a bundle of type $\mu \in \mathscr{U}$ then

$$H^1(M, E) = 0$$

Since $A_d(n, m)$ consists precisely of those elements $h \in \mathrm{Hol}_d(M, G(n, m))$ represented by quotient bundles $M \times \mathbf{C}^m \to E$ such that $H^1(M, E) = 0$ the result now follows immediately from 6.4.

This completes the proof of Theorem 1.2.

7. Maps into infinite Grassmannians

Let $G(n, \infty)$ be the Grassmannian of n-dimensional quotients of a fixed separable complex Hilbert space $\mathbf{C}^\infty$, or equivalently the Grassmannian of closed subspaces of codimension n in $\mathbf{C}^\infty$. Give $G(n, \infty)$ the topology coming from the metric δ defined by

$$\delta(P, P') = \sup\left(\varrho(P, P'),\ \varrho(P', P)\right)$$

where

$$\varrho(P, P') = \sup\left(\inf\left(\|x'-x\|, x \in P \cap S^\infty\right), x' \in P' \cap S^\infty\right)$$

for any closed subspaces P and P' of codimension n in $\mathbf{C}$. Here S^∞ is the unit sphere of $\mathbf{C}^\infty$. Then $G(n, \infty)$ is the classifying space for the group $GL(n, \mathbf{C})$, (see e.g. [V & LP], Exposé no. 3). In fact $G(n, \infty)$ is the quotient of $S(\mathbf{C}^\infty, \mathbf{C}^n)$ by the free action of $GL(n)$ where $S(\mathbf{C}^\infty, \mathbf{C}^n)$ is the open subset of the Hilbert space $\mathscr{L}(\mathbf{C}^\infty, \mathbf{C}^n)$ consisting of all surjective maps $\mathbf{C}^\infty \to \mathbf{C}^n$. The projection $S(\mathbf{C}^\infty, \mathbf{C}^n) \to G(n, \infty)$ given by $\varphi \mapsto \ker \varphi$ is a universal principal $GL(n)$-bundle ([V & LP] Exposé 3, Thme 2).

Let $\mathrm{Map}_d(M, G(n, \infty))$ denote the space of all continuous maps of degree d from M to $G(n, \infty)$. As in the case of finite Grassmannians, a point of $\mathrm{Map}_d(M, G(n, \infty))$ defines a surjective map of continuous bundles over M

$$M \times \mathbf{C}^\infty \to E \to 0$$

such that E has rank n and degree d.

We have

$$G(n, \infty) = \cup G(n, V)$$

where the union is over all finite-dimensional subspaces V of $\mathbf{C}^\infty$ such that $\dim V \geqq n$, and $G(n, V)$ denotes the Grassmannian of n-dimensional quotients of V which is embedded in $G(n, \infty)$ via the orthogonal projection of $\mathbf{C}^\infty$ onto V. Therefore if we regard the set F of all finite-dimensional subspaces of $\mathbf{C}^\infty$ as a directed set via inclusion we obtain a natural continuous injection into $\mathrm{Map}_d(M, G(n, \infty))$ of the direct limit $\underrightarrow{\mathrm{Map}}_d(M, G(n, V))$ of the spaces $\mathrm{Map}_d(M, G(n, V))$ as V runs over all elements of F.

7.1. Lemma. *This map*

$$\underrightarrow{\mathrm{Map}}_d(M, G(n, V)) \to \mathrm{Map}_d(M, G(n, \infty))$$

is a homology equivalence.

Proof. Since direct limits commute with homology by [D] 5.20 and 5. 5.23.1, this follows immediately from

7.2. Lemma. *The inclusion*

$$\mathrm{Map}_d(M, G(n, V)) \to \mathrm{Map}_d(M, G(n, \infty))$$

is a homology equivalence up to dimension $2m-n^2-4$ *where* $m = \dim V$.

Remark. It is not hard to improve this bound to $2m-n-3$.

Proof. An element of the homotopy group $\pi_k(\mathrm{Map}_d(M, G(n, \infty)))$ is represented by a continuous based map

$$S^k \to \mathrm{Map}_d(M, G(n, \infty))$$

or equivalently by a continuous map

$$f\colon S^k \times M \to G(n, \infty)$$

such that the restriction $f_t\colon M \to G(n, \infty)$ of f to $\{t\} \times M$ is the chosen base-point of $\mathrm{Map}_d(M, G(n, \infty))$ when t is the base-point of S^k.

Let W_V be the open subset of $G(n, \infty)$ consisting of all closed subspaces U of codimension n in $\mathbf{C}^\infty$ such that the codimension of $V \cap U$ in V is n. Let W_V^* be the inverse image of W_V in the open subset $S(\mathbf{C}^\infty, \mathbf{C}^n)$ of the Hilbert space $\mathscr{L}(\mathbf{C}^\infty, \mathbf{C}^n)$ regarded as a $GL(n)$-bundle over $G(n, \infty)$. Then the complement of W_V in $S(\mathbf{C}^\infty, \mathbf{C}^n)$ consists of all surjective linear maps from $\mathbf{C}^\infty$ to $\mathbf{C}^n$ whose restrictions to V are not surjective. It is contained in $U(n)A$ where $U(n)$ is the unitary group acting on $(\mathbf{C}^\infty, \mathbf{C}^n)$ and A is the closed subspace of $\mathscr{L}(\mathbf{C}, \mathbf{C}^n)$ consisting of all maps $g\colon \mathbf{C}^\infty \to \mathbf{C}^n$ such that $g(V)$ is contained in some fixed hyperplane of $\mathbf{C}^n$. Moreover A has codimension m in $\mathscr{L}(\mathbf{C}^\infty, \mathbf{C}^n)$ so $\mathscr{L}(\mathbf{C}^\infty, \mathbf{C}^n) \cong A \oplus \mathbf{C}^m$.

Locally $f\colon S^k \times M \to G(n, \infty)$ can be lifted to $\tilde{f}\colon S^k \times M \to S(\mathbf{C}_\infty, \mathbf{C}^u)$ since the quotient map $S(\mathbf{C}^\infty, \mathbf{C}^n) \to G(n, \infty)$ is a principal fibration (see [V & LP]). Using the action of $GL(n)$ this extends to a map

$$\hat{f}\colon U(n) \times S^k \times M \to S(\mathbf{C}^\infty, \mathbf{C}^n).$$

In order to ensure that the image of f is contained in W_V it suffices to ensure that the image of the projection of $\hat{f}$ onto the orthogonal complement $\mathbf{C}^m$ to A in $\mathscr{L}(\mathbf{C}^\infty, \mathbf{C}^n)$ does not contain 0. But this can be achieved by standard approximation techniques provided that $2m$ is strictly greater than

$$\dim(U(n) \times S^k \times M) = n^2 + k + 2.$$

This shows that the inclusion

$$\mathrm{Map}_d(M, W_V) \to \mathrm{Map}_d(M, G(n, \infty))$$

induces surjections on homotopy groups up to dimension $2m-n^2-3$. A similar argument involving homotopies between maps $S^k \times M \to G(n, \infty)$ shows that these surjections are isomorphisms up to dimension $2m-n^2-4$.

Finally it is clear that the map from W_V to $G(n, V)$ regarded as a subset of $G(n, \infty)$ which is defined by

$$U \mapsto (U \cap V) \oplus V^{\perp}$$

is a retraction. Hence the induced map from $\mathrm{Map}_d(M, W_V)$ to $\mathrm{Map}_d(M, G(n, V))$ is also a retraction. It follows that the inclusion

$$\mathrm{Map}_d(M, G(n, V)) \to \mathrm{Map}_d(M, G(n, \infty))$$

induces isomorphisms in homotopy up to dimension $2m - n^2 - 4$, and hence by Hurewicz's theorem it is a homology equivalence up to dimension $2m - n^2 - 4$.

Define $\mathrm{Hol}_d(M, G(n, \infty))$ to be the direct limit of the spaces $\mathrm{Hol}_d(M, G(n, V))$ where V runs over the set F of finite-dimensional subspaces of $\mathbf{C}^\infty$. That is,

$$\mathrm{Hol}_d(M, G(n, \infty)) = \bigcup_{\substack{V \subseteq \mathbf{C}^\infty \\ n \leqq \dim V < \infty}} \mathrm{Hol}_d(M, G(n, V))$$

with the induced topology. A point of $\mathrm{Hol}_d(M, G(n, \infty))$ defines an exact sequence of holomorphic bundles

$$0 \to K \to M \times V \to E \to 0$$

where V is a finite-dimensional subspace of $\mathbf{C}^\infty$ and E is a holomorphic bundle of rank n and degree d on M. Conversely any such sequence defines an element of $\mathrm{Hol}_d(M, G(n, \infty))$. We can define an open subset $A_d(n, \infty)$ just as we defined $A_d(n, m)$ in § 1 by imposing the condition that $H^1(M, E) = 0$. Then $A_d(n, \infty) = \cup A_d(n, V)$ with the induced topology. Since direct limits commute with homology and the bound $d_0(k, n)$ in Theorem 1.1 is independent of m it follows from 7.2 that the analogue of 1.1 in the case $m = \infty$ is true.

7.3. Theorem. *Let k be any positive integer with $k \geqq n - 2g$. Then the natural map*

$$A_d(n, \infty) \to \mathrm{Map}_d(M, G(n, \infty))$$

is a homology equivalence up to dimension k provided that $d \geqq d_0(k, n)$ where

$$d_0(k, n) = 2n(2g + k + 1) + n \max(k + 1 + n(2g + k + 1), \tfrac{1}{4} n^2 g).$$

Proof. Suppose that $d \geqq d_0(k, n)$. Then by 1.1 the inclusion

$$A_d(n, V) \to \mathrm{Map}_d(M, G(n, V))$$

is a homology equivalence up to dimension k for every finite-dimensional subspace V of $\mathbf{C}^\infty$. Moreover if $\dim V \geqq k + n^2 + 4$ the inclusion

$$\mathrm{Map}_d(M, G(n, V)) \to \mathrm{Map}_d(M, G(n, \infty))$$

is a homology equivalence up to dimension k by 7.2. It follows that the inclusion of $A_d(n, V)$ in $\mathrm{Map}_d(M, G(n, \infty))$ is a homology equivalence up to dimension k if $\dim V \geqq k+n^2+4$. Since $A_d(n, \infty)$ is the direct limit of the subsets $A_d(n, V)$ as V runs over F the result follows because direct limits commute with homology by [D] 5.20 and 5.23.1.

Remark. Alternatively one can prove Theorem 7.3 directly using exactly the same proof as for Theorem 1.1. This works basically because none of the bounds obtained in the course of that proof depend on m.

Unfortunately in Theorem 1.2 one only has that if $d \geqq d_0(k, n)$ then the inclusion of $\mathrm{Hol}_d(M, G(n, m))$ in $\mathrm{Map}_d(M, G(n, m))$ induces isomorphisms of cohomology up to dimension $k-2m^2g$. This means that one cannot prove that the inclusion of $\mathrm{Hol}_d(M, G(n, \infty))$ in $\mathrm{Map}_d(M, G(n, \infty))$ is a homology equivalence up to any dimension by using the same methods.

Remark. By Theorem 7.3 the inclusion of $\mathrm{Hol}_d(M, G(n, \infty))$ in $\mathrm{Map}_d(M, G(n, \infty))$ is a homology equivalence up to dimension k for $d \geqq d_0(k, n)$ if and only if the same is true of the inclusion of $A_d(n, \infty)$ in $\mathrm{Hol}_d(M, G(n, \infty))$. It seems unlikely that this is true, because the open subset $A_d(n, \infty)$ is not even dense in $\mathrm{Hol}_d(M, G(n, \infty))$ when d is large (in contrast with the case $m < \infty$: cf. Corollary 6.6). Indeed if $h \in \mathrm{Hol}_d(M, G(n, \infty))$ let $s(h)\colon V \to H^0(M, E)$ be the map of sections induced by the quotient map of bundles corresponding to h. Then if d is large enough the subset

$$W = \{h \mid \dim(im\, s(h)) \geqq d+n(1-g)+1\}$$

is a nonempty open subset of $\mathrm{Hol}_d(M, G(n, \infty))$ which is disjoint from $A_d(n, \infty)$.

In part II we shall consider an open subset $\tilde{R}$ of the space $A_d(n, \infty)$ defined as follows. An element of $A_d(n, \infty)$ represented by an exact sequence

$$0 \to K \to M \times V \to E \to 0$$

belongs to $\tilde{R}$ if the induced map on sections

$$V \to H^0(M, E)$$

is surjective. We shall need the following corollary of Theorem 7.3.

7.4. Corollary. *Let k be a positive integer. Then the inclusion of $\tilde{R}$ in $\mathrm{Map}_d(M, G(n, \infty))$ is a homology equivalence up to dimension k if $d \geqq d_0(k, n)$.*

Proof. By 7.3 it suffices to show that the inclusion of $\tilde{R}$ in $A_d(n, \infty)$ is a homology equivalence up to dimension k. But $\tilde{R}$ is the union of its subsets $\tilde{R} \cap A_d(n, m)$ with the direct limit topology. Since direct limits commute with homology by [D] 5.20 and 5.23.1, it suffices to show that the inclusion of $\tilde{R} \cap A_d(n, m)$ in $A_d(n, m)$ is a homology equivalence up to dimension k if n is sufficiently large.

Therefore since $A_d(n, m)$ is nonsingular it suffices to show that the codimension of $\tilde{R}\cap A_d(n, m)$ in $A_d(n, m)$ tends to infinity with m.

Let $0\to K\to M\times \mathbf{C}^m\to E\to 0$ be an exact sequence of bundles which defines a point of $A_d(n, m)$, and suppose that the map on sections $\mathbf{C}^m\to H^0(M, E)$ is not surjective. Then its kernel contains a subspace U of dimension u in $\mathbf{C}^m$ such that $u\geqq m-d-n(1-g)+1$. So E is a quotient of the trivial bundle $M\times(\mathbf{C}^m/U)$ on M. Therefore when m is large the complement of $\tilde{R}\cap A_d(n, m)$ in $A_d(n, m)$ is contained in the union over all integers u satisfying $m\geqq u>m-d-n(1-g)$ of subvarieties $V(u)$ such that

$$\begin{aligned}\dim V(u) &\leqq \dim G(u, m)+\dim A_d(n, m-u)\\ &= u(m-u)+(m-u)\big(d+n(1-g)\big)-n^2(1-g)\end{aligned}$$

(see the computation in [Se] § 6). Since

$$\dim A_d(n, m) = \big(d+n(1-g)\big)-n^2(1-g)$$

it follows that

$$\operatorname{codim} V(u) \geqq u\big(d+n(1-g)-m+u\big) \geqq m+1-d-n(1-g)$$

which tends to infinity with m. The result follows.

II. MODULI SPACES OF BUNDLES OVER M

8. Using stratifications to compute cohomology

In [A & B] the moduli space of semistable bundles on M of coprime rank n and degree d is represented as a quotient of an open subset of an infinite dimensional affine space by an infinite group. This affine space $\mathscr{C}$ is the space of unitary connections, or equivalently of holomorphic structures, on a fixed C^∞ bundle on M of rank n and degree d. Atiyah and Bott show that there is a stratification $\{\mathscr{C}_\mu\}$ of $\mathscr{C}$ such that the stratum $\mathscr{C}_\mu$ consists of the holomorphic structures of type μ (see 3.5 to 3.7 above). The unique open stratum $\mathscr{C}^{ss}$ consists of all semistable holomorphic structures. They show that this stratification is equivariantly perfect, in the sense that the equivariant Morse inequalities are in face equalities. (The equivariant cohomology is taken with respect to the action of the gauge group $\mathscr{G}$.) That is,

$$\begin{aligned}P_t^{\mathscr{G}}(\mathscr{C}) &= \textstyle\sum_\mu t^{2d_\mu} P_t^{\mathscr{G}}(\mathscr{C}_\mu)\\ &= P_t^{\mathscr{G}}(\mathscr{C}^{ss})+\textstyle\sum_{\mu\neq d/n} t^{2d_\mu} P_t^{\mathscr{G}}(\mathscr{C}_\mu)\end{aligned}\tag{8.1}$$

where d_μ is the complex codimension of the stratum $\mathscr{C}_\mu$ in $\mathscr{C}$, and $P_t^{\mathscr{G}}$ denotes the equivariant Poincaré series (over any chosen field of coefficients).

Atiyah and Bott show that the equivariant Poincaré series of the unique open stratum $\mathscr{C}^{ss}$ is the same as the ordinary Poincaré series of the moduli space except for a factor of $(1-t^2)$. They also show that the equivariant cohomology of the higher strata can be computed inductively. In fact if

$$\mu = (d_1/n_1, \ldots, d_s/n_s)$$

as in 3.2 and if $\mathscr{C}(d_j, n_j)$ denotes the space of unitary connections on a fixed C^∞ bundle of rank n_j and degree d_j, then the equivariant cohomology of $\mathscr{C}_\mu$ is isomorphic to the tensor product of the equivariant cohomology of the semistable strata of the spaces $\mathscr{C}(d_j, n_j)$ with respect to the appropriate gauge groups. Moreover since the total space $\mathscr{C}$ is contractible its equivariant cohomology is just the ordinary cohomology of the classifying space of the gauge group. By [A & B] 2.4, this is homotopy equivalent to the space $\mathrm{Map}_d(M, G(n, \infty))$, and its Poincaré series is given at [A & B] 2.15 as

$$\Pi_{1 \leq k \leq n} (1+t^{2k-1})^{2g} / (\Pi_{1 \leq k \leq n-1} (1-t^{2k})^2 (1-t^{2n}).$$

Thus Atiyah and Bott are able to compute the cohomology of the moduli space from the inductive formula

$$(8.2) \quad P_t(\mathrm{Map}_d(M, G(n, \infty))) = P_t^{\mathscr{G}}(\mathscr{C}^{ss}) + \sum_{\mu \neq d/n} t^{2d_\mu} \Pi_{1 \leq j \leq s} P_t^{\mathscr{G}}(C(n_j, d_j)^{ss})$$

where P_t denotes the ordinary Poincaré series and

$$(8.3) \qquad d_\mu = \sum_{i>j} ((n_i d_j - n_j d_i) + n_i n_j (g-1))$$

by [A & B] 7.16.

On the other hand the moduli space can also be represented as the geometric invariant theory quotient of a reductive action on a nonsingular quasi-projective variety R (see § 9 below). The results of [K] show that this variety can also be stratified, and that the equivariant cohomology of the unique open stratum is the same as the ordinary cohomology of the moduli space. Moreover this stratification is also equivariantly perfect over any field of coefficients, at least in an approximate sense (see § 12 below for more details).

The variety R parametrises in a natural way a family of holomorphic bundles over M. We shall see that outside a subset of codimension $k(d, n)$ the stratification of R corresponds precisely with the stratification given by the types of these holomorphic bundles. Moreover outside the subset of codimension $k(d, n)$ the strata have the same codimension in R as the corresponding strata have in $\mathscr{C}$, and their cohomology can be computed inductively just as the cohomology of the strata $\mathscr{C}_\mu$ can be (see § 11 below). Therefore we shall find that the finite-dimensional stratification gives a formula for the equivariant cohomology of R in terms of that of the semistable stratum and the higher strata which corresponds precisely with the right hand side of 8.2 up to dimension $k(d, n)$. Moreover $k(d, n)$ tends to infinity with d.

In § 10 we find that the equivariant cohomology of R can be identified with the ordinary cohomology of the space $\tilde{R}$ defined in § 7 above. We saw there that the inclusion of $\tilde{R}$ in $\mathrm{Map}_d(M, G(n, \infty))$ induces isomorphisms in cohomology up to some dimensions which tends to infinity with d.

We then use the fact that the moduli space of bundles of rank n and degree d is isomorphic to the moduli space of bundles of rank n and degree $d+ne$ for any integer e. (An isomorphism is obtained by tensoring with a fixed line bundle of degree e). By taking arbitrarily large values of e we find that the finite-dimensional stratifications give the same inductive formula for the cohomology of the moduli space as does Atiyah and Bott's infinite-dimensional stratification.

9. Reduction of the moduli problem to finite dimensions

Let us recall how the moduli problem is reduced to a problem of geometric invariant theory, following [N] chap. 5.

By [N] 5.2, if E is a semistable bundle of rank n and degree d greater than $n(2g-1)$ over M, then

9.1. E is generated by its sections, and

$$H^1(M, E) = 0 \tag{9.2}$$

By Riemann—Roch, 9.2 implies that

$$\dim H^0(M, E) = d+n(1-g).$$

Let $p=d+n(1-g)$. Then it follows that there is a holomorphic map from M to the Grassmannian $G(n, p)$ such the induced quotient bundle of $M\times\mathbf{C}^p$ is isomorphic to E.

Now define R to be the subset of $\mathrm{Hol}_d(M, G(n, p))$ consisting of those maps h such that if E is the induced quotient bundle of $M\times\mathbf{C}^p$ then $H^1(M, E)=0$ and the map on sections

$$\mathbf{C}^p \to H^0(M, E)$$

is surjective. (Equivalently we could require that the map on sections be an isomorphism.) This is an open subset of the space $A_d(n, p)$ defined in § 1 above. Therefore it is a nonsingular quasi-projective variety. Provided that $d>n(2g-1)$ there is a quotient $\tilde{E}$ of the trivial bundle of rank p over $R\times M$ with the following properties (see [N] 5.3 and 5.6).

9.3. (i) $\tilde{E}$ has the local universal property for families of bundles over M of rank n and degree d satisfying 9.1 and 9.2. That is, if F is a family of such bundles parametrised by a variety S, then any s in S has a neighbourhood V such that the restriction of F to V is equivalent to the family induced from $\tilde{E}$ by some morphism $V \to R$.

(ii) If h belongs to R then the restriction E^h of $\tilde{E}$ to $\{h\} \times M$ is the quotient of $M \times \mathbf{C}^p$ induced by h.

(iii) If h and g belong to R then E^h and E^g are isomorphic as bundles over M if and only if h and g lie in the same orbit of the natural action of $GL(p)$ on R.

Moreover by Theorem 5.6 and the remark (a) before Theorem 5.8 of [N], if N is any large integer then R can be embedded as a quasi-projective subvariety of the product $(G(n,p))^N$ by a map of the form

$$h \to (h(x_1), \ldots, h(x_N))$$

where $x_1, \ldots, x_N$ are points of M. This embedding gives us a linearisation of the action of $SL(p)$ on R. If N and d are large enough then the following condition is also satisfied.

(iv) A point h of R is semistable in the sense of geometric invariant theory for this linear action of $SL(p)$ on R if and only if E^h is a semistable bundle. Moreover if a point h of the closure of R in $(G(n,p))^N$ is semistable then $h \in R$.

Since n and d are coprime by assumption, it is not hard to check using the definition of semistability for bundles that $PGL(p)$ acts freely on the set R^{ss} of semistable points of R. It then follows from geometric invariant theory that the quotient $R^{ss}/PGL(p)$ is a nonsingular projective variety, which is the moduli space of semistable bundles of rank n and degree d over M. Furthermore its cohomology is the same as the $PGL(p)$-equivariant cohomology of R^{ss}. Thus its Poincaré polynomial over the field of rational coefficients is given by

$$P_t(\text{moduli space}) = (1-t^2) P_t^{GL(p)}(R^{ss}). \tag{9.4}$$

(The factor $(1-g^2)$ occurs because for the sake of convenience later we are using equivariant cohomology with respect to $GL(p)$ (cf. [A & B] § 9).

10. The equivariant cohomology of R

In order to calculate the equivariant cohomology of R^{ss}, we first need to know that of R itself. This is by definition the ordinary cohomology of the space

$$R \times_{GL(p)} EGL(p)$$

where $EGL(p)$ is the total space of a classifying bundle for $GL(p)$. We can take

$EGL(p)$ to be the space of all quotient maps

$$e\colon \mathbf{C}^\infty \to \mathbf{C}^p$$

with the obvious action of $GL(p)$, since this is a contractible space on which $GL(p)$ acts freely. (See § 7 and [V & LP] Exposé No. 3 for more details.) The quotient $BGL(p)$ is the infinite quotient Grassmannian $G(p, \infty)$.

By definition R consists of those holomorphic maps h from M to the quotient Grassmannian $G(n, p)$ such that the induced bundle E on M has degree d and the map on sections

$$\mathbf{C}^p \to H^0(M, E)$$

is an isomorphism. Let $\mathrm{Hol}_d(M, G(n, \infty))$ be defined as in § 7. Given h in R and e in $EGL(p)$ denote by $f(h, e)$ the element of $\mathrm{Hol}_d(M, G(n, \infty))$ which sends x in M to the quotient $\mathbf{C}^\infty/K$ where K is the kernel of the composition of the quotient map

$$e\colon \mathbf{C}^\infty \to \mathbf{C}^p$$

with the projection of $\mathbf{C}^p$ onto its n-dimensional quotient $h(x) \in G(n, p)$. Equivalently when elements of $\mathrm{Hol}_d(M, G(n, \infty))$ are identified as in § 7 with quotient bundle maps $V \times M \to E$ for suitable finite-dimensional subspaces V of $\mathbf{C}^\infty$ then $f(h, e)$ is given by the composition

$$M \times V \to M \times \mathbf{C}^p \to E^h,$$

where V is chosen so that the restriction of $e\colon \mathbf{C}^\infty \to \mathbf{C}^p$ to V is surjective. Thus $f(h, e)$ belongs to the set $\tilde{R}$ (which by definition consists of all elements of $\mathrm{Hol}_d(M, G(n, \infty))$ such that the associated bundle E satisfies $H^1(M, E) = 0$ and the map on sections from $\mathbf{C}^\infty$ to $H^0(M, E)$ is surjective). This gives us a well-defined map

$$f\colon R \times_{GL(p)} EGL(p) \to \tilde{R}.$$

Now suppose that $g\colon M \to G(n, \infty)$ belongs to $\tilde{R}$. Then the associated bundle E is generated by its sections, and by Riemann—Roch $\dim H^0(M, E) = p$. Therefore by choosing a basis of $H^0(M, E)$ we can regard E as a quotient of $M \times \mathbf{C}^p$. This quotient defines an element h of R such that $f(h, e) = g$ for an appropriate e in $EGL(p)$. Therefore f is surjective. Moreover it follows from 9.3(iii) that f is injective. It is now not hard to check that f is an open map, and therefore a homeomorphism. Thus we have proved

10.1. Lemma. *The equivariant cohomology of R under the action of $GL(p)$ is isomorphic to the ordinary cohomology of $\tilde{R}$.*

11. Stratifying R

We saw in Section 9 that R can be embedded as a quasi-projective subvariety of the product $(G(n,p))^N$ for sufficiently large N. So there is a stratification of R which is the intersection with R of the stratification of $(G(n,p))^N$ described in [K] § 16.

Recall that an element $x=(L_1, \dots, L_N)$ of $(G(n,p))^N$ is semistable for the natural linear action of $SL(p)$ if and only if there does not exist a proper subspace K of $\mathbf{C}^p$ such that

$$(\textstyle\sum_j \dim K \cap L_j)/\dim K > (\textstyle\sum_j \dim L_j)/p$$

(see e.g. [Mu] § 4).

It was shown in [K] § 16 that to each sequence $x=(L_1, \dots, L_N)$ of subspaces of $\mathbf{C}^p$ there corresponds a unique filtration

$$0 = K_0 \subseteqq K_1 \subseteqq \dots \subseteqq K_s = \mathbf{C}^p$$

such that

$$k_1/p_1 > k_2/p_2 > \dots > k_s/p_s$$

where

$$p_i = \dim K_i/K_{i-1}$$

and

$$k_i = \textstyle\sum_{1 \leqq j \leqq N} \dim (K_i \cap L_j + K_{i-1})/K_{i-1},$$

and such that for each i the sequence of subspaces

$$(K_i \cap L_j + K_{i-1})/K_{i-1}$$

of K_i/K_{i-1} is semistable. Then the stratum of $(G(n,p))^N$ to which x belongs is indexed by the p-vector

$$\beta = (k_1/p_1, \dots, k_s/p_s)$$

in which k_i/p_i appears p_i consecutive times.

Hence when $h\colon M \to G(n,p)$ is an element of R and $h(x_j)=\mathbf{C}^p/L_j$ for each of the chosen points $x_1, \dots, x_N$, then the index of the stratum of R containing h is the vector β just described. We need to link this β with the type μ of the bundle E^h corresponding to h.

Recall that $p=d+n(1-g)$.

11.1. Definition. *Suppose that*

$$\mu = (d_1/n_1, \dots, d_s/n_s)$$

is an (n,d)*-type (see* 3.2 *above). Suppose also that*

$$d_i > n_i(2g-1)$$

for $1 \leqq i \leqq s$. *Let*

$$k_i = N(d_i - n_i g)$$

and

$$p_i = d_i + n_i(1-g).$$

Then define $\beta(\mu)$ *to be the p-vector*

$$\beta(\mu) = (k_1/p_1, \ldots, k_s/p_s)$$

in which each k_i/p_i *appears* p_i *times.*

Note that $p_i > n_i g$ and so is positive for each i, and that $\sum_i p_i = p$ and $\sum_i k_i = N(d-ng) = N(p-n)$.

11.2. *Remark.* Note that when the denominators are positive

$$N(d-ng)/(d+n(1-g)) > N(d'-n'g)/(d'+n'(1-g))$$

if and only if

$$d/n > d'/n'.$$

The crucial lemma is the following.

11.3. Lemma. (i) *Suppose that* $d > n(2g-1)$. *Let h lie in the stratum of R indexed by* β, *and let* $\mu = (d_1/n_1, \ldots, d_s/n_s)$ *be the type of the bundle* $E = E^h$. *Then the first component of* β *is greater than or equal to* $N(d_1 - n_1 g)/(d_1 + n_1(1-g))$.

(ii) *Moreover if* $d_1, \ldots, d_s$ *are all sufficiently large depending only on* $n_1, \ldots, n_s$ *and g, and if N is sufficiently large depending on* $d_1, \ldots, d_s, n_1, \ldots, n_s$ *and g, then*

$$\beta(\mu) = \beta.$$

Proof. Let

$$0 = E_0 \subsetneqq E_1 \subsetneqq \ldots \subsetneqq E_s = E$$

be the canonical filtration of E (see 3.2 above). By 9.3(iv) we may assume that E is not semistable, so s is greater than 1. Use the map on sections to identify $\mathbf{C}^p$ with $H^0(M, E)$. Then

$$0 = H^0(M, E_0) \subsetneqq H^0(M, E_1) \subsetneqq \ldots \subsetneqq H^0(M, E) = \mathbf{C}^p$$

is a sequence of the subspaces of $\mathbf{C}^p$.

The fibre of E at x_j is $h(x_j) = \mathbf{C}^p/L_j$. The image $(H^0(M, E_1) + L_j)/L_j$ of $H^0(M, E_1)$ in this must have dimension at most the rank of E_1, which is n_1. Moreover by Riemann—Roch

$$\dim H^0(M, E_1) \geqq d_1 + n_1(1-g).$$

These two inequalities together with 11.2 and the equality

$$\dim H^0(M, E_1) \cap L_j = \dim H^0(N, E_1) - \dim \left(H^0(M, E_1) + L_j\right)/L_j$$

and the fact that d_1/n_1 is strictly greater than d/n imply that

$$(11.4)\qquad \sum_j \dim(H^0(M,E_1)\cap L_j)/\dim H^0(M,E_1)$$
$$\geqq N(1-n_1/\dim H^0(M,E_1))$$
$$\geqq N(d_1-n_1 g)/(d_1+n_1(1-g))$$
$$> N(d-ng)/(d+n(1-g)).$$

It now follows from [K] 16.9 that the filtration

$$0 = K_0 \subseteqq K_1 \subseteqq \ldots \subseteqq K_t = \mathbf{C}^p$$

corresponding to h is a refinement of the sequence

$$0 \subseteqq H^0(M,E_1) \subseteqq \mathbf{C}^p,$$

and thus using 11.4 again that the first component of β is greater than or equal to $N(d_1-n_1 g)/(d_1+n_1(1-g))$.

Now assume that $d_i > n_i(2g-1)$ for each i. Then by 3.2 and 3.4 each quotient $D_i = E_i/E_{i-1}$ is generated by its sections and satisfies $H^1(M,D_i)=0$. By induction we see that the same is true of each E_i. Therefore if we set

$$K_i = H^0(M,E_i)$$

then we can identify $H^0(M,D_i)$ with K_i/K_{i-1} and deduce that

$$\dim K_i/K_{i-1} = p_i.$$

Moreover the image

$$(K_i+L_j)/L_j$$

of K_i in the fibre $(\mathbf{C}^p)/L_j$ of E at each point x_j has dimension equal to the rank of E_i. Therefore

$$\dim(K_i\cap L_j+K_{i-1})/K_{i-1} = \dim(K_i\cap L_j)/(K_{i-1}\cap L_j)$$
$$= \dim K_i - rkE - dim\ K_{i-1} + rkE_{i-1}$$
$$= p_i - n_i.$$

Hence

$$\sum_{1\leqq j\leqq N} \dim(K_i\cap L_j+K_{i-1})/(K_{i-1}) = k_i$$

where

$$k_i = N(d_i-n_i g) = N(p_i-n_i),$$

and

$$k_1/p_1 > k_2/p_2 > \ldots > k_s/p_s$$

by 11.2.

Therefore by [K] 16.9 it remains only to show that for each i if d_i and N are sufficiently large then the image of the sequence $(L_1, \ldots, L_N)$ in the appropriate product of Grassmannians of subspaces of $K_i/K_{i-1}=H^0(M, D_i)$ is semistable. But this follows directly from the semistability of D_i and Theorem 5.6 of [N].

11.5. Corollary. *Suppose $\mathscr{U}$ is any finite set of types of bundles of rank n and degree d_0. Let e be a large positive integer, and set $d=d_0+ne$. Let R be embedded in $(G(n,p))^N$ as above, where $p=d+n(1-g)$, so that the action of $SL(p)$ on R extends to a linear action on $(G(n,p))^N$. Then if e and N are sufficiently large and μ is any type in $\mathscr{U}+e$, a point h of R lies in the stratum indexed by $\beta(\mu)$ if and only if the bundle E^h is of type μ.*

Proof. Consider the set of all (n, d) types $(d_1/n_1, \ldots, d_s/n_s)$ whose first component is less than some fixed bound c. There are only finitely many possible values of s and $n_1, \ldots, n_s$ such that a type $(d_1/n_1, \ldots, d_s/n_s)$ belongs to this set because each n_j is a positive integer and $n_1+\ldots+n_s=n$. Also

$$c \geqq d_1/n_1 > \ldots > d_s/n_s$$

so all the d_i are bounded above. But $d_1+\ldots+d_s=d$ so the d_i are also bounded below. Therefore the set of such (n, d) types is finite. Thus we may assume that if $\mu\in\mathscr{U}$ and μ' is an (n, d) type whose first component is less than that of μ, then $\mu'\in\mathscr{U}$. The same will then be true of $\mathscr{U}+e$ for any e.

Since $\mathscr{U}$ is finite, if e and N are sufficiently large then Lemma 11.3(i) applies to all μ in $\mathscr{U}+e$. Therefore if $h\in R$ and E_h is of type $\mu\in\mathscr{U}+e$, then h lies in the stratum indexed by $\beta(\mu)$. Conversely suppose h lies in the stratum indexed by $\beta(\mu)$ for $\mu=(d_1/n_1, \ldots, d_s/n_s)$, and let $\mu'=(d_1'/n_1', \ldots, d_s'/n_s')$ be the type of E_h. By Lemma 11.3(i) the first component $(d_1-n_1g)/(d_1+n_1(1-g))$ of $\beta(\mu)$ is greater than or equal to $(d_1'-n_1'g)/(d_1'+n_1'(1-g))$. This implies that the first component d_1/n_1 of μ is greater than or equal to that of μ', by 11.2. Hence μ' belongs to $\mathscr{U}+e$, and therefore h lies in the stratum indexed by $\beta(\mu')$. So $\beta(\mu)=\beta(\mu')$, which implies that $\mu=\mu'$. The result follows.

Next we need to consider the codimensions of the strata of R. Let k be any integer. By 3.9 we can choose a finite set $\mathscr{U}$ of (n, d_0) types such that $d_\mu>k$ whenever μ is an (n, d_0) type and $\mu\notin\mathscr{U}$. Suppose that e and N are large enough that Lemma 11.5 applies to $\mathscr{U}$. Then we have

11.6. Lemma. *Every stratum of R indexed by $\beta(\mu)$ for some $\mu\in\mathscr{U}+e$ has codimension the integer d_μ defined at 3.6, and every stratum not indexed by $\beta(\mu)$ for some $\mu\in\mathscr{U}+e$ has codimension greater than k.*

Proof. By 3.8 we have

$$d_{\mu+e} = d_\mu$$

for every (n, d_0) type μ. The result now follows immediately from Lemma 11.5 and the Remark at 3.7.

12. Some technicalities

We now have a description of the stratification of R obtained by embedding it in a product $(G(n,p))^N$ for large N and d, provided that we allow ourselves to ignore a subset of dimension $k(n,d)$ in R, where $k(n,d)$ tends to infinity with d. Next we would like to show that this stratification is equivariantly perfect, in the sense that its equivariant Morse inequalities are in fact equalities. If R were a projective variety this would follow from Theorem 5.4 and Section 8 of [K]. Since R is not projective we need to check that condition 9.4 of [K] holds. In fact we shall only show that it holds outside the subset of codimension $k(n,d)$ previously mentioned. This will imply that the Morse inequalities are equalities in dimensions less than $k(n,d)$, which suffices for our purposes.

It is easy to see from [K] 9.4 and 16.9 that the necessary condition can be interpreted as follows. Suppose h is any element of R, represented in $(G(n,p))^N$ as the point $(h(x_1),\dots,h(x_N))$, and suppose that the associated filtration of $\mathbf{C}^p$ is of the form

$$0 \subseteqq V_1 \subseteqq V_1 \oplus V_2 \subseteqq \dots \subseteqq V_1 \oplus \dots \oplus V_s = \mathbf{C}^p$$

where each V_j has a basis B_j such that the union $B_1 \cup \dots \cup B_s$ is the standard ordered basis of $\mathbf{C}^p$. Then there is required to be some h' in R such that, for each j, $h'(x_j)$ is the direct sum over i of the projections of the subspace $h(x_j)$ onto the subspaces V_i with respect to the decomposition $\mathbf{C}^p = V_1 \oplus \dots \oplus V_s$.

This condition is satisfied provided that h belongs to a stratum indexed by $\beta(\mu)$ for some type μ in the set $\mathcal{U}+e$ chosen before 11.6. For it was shown in the proof of 11.4 that the filtration of $\mathbf{C}^p$ associated to h is then of the form

$$0 \subseteqq H^0(M, E_1) \subseteqq \dots \subseteqq H^0(N, E) = \mathbf{C}^p$$

where

$$0 \subseteqq E_1 \subseteqq \dots \subseteqq E_s = E$$

is the canonical filtration of $E = E^h$. Moreover each subquotient $D_i = E_i/E_{i-1}$ is spanned by its sections and satisfies $H^0(M, D_i) = 0$. From this it is straightforward to check that for each $1 \leqq i \leqq s$ the isomorphism $\mathbf{C}^p \to H^0(M, E)$ induces an isomorphism

$$V_i \to H^0(M, D_i)$$

and that the quotient map

$$M \times \mathbf{C}^p = (V_1 \oplus \dots \oplus V_s) \times M \to D_1 \oplus \dots \oplus D_s$$

corresponds to an element h' of R satisfying the required property.

Thus using 11.6 we conclude that condition 9.4 of [K] holds outside a subset of codimension $k(n,d)$, where $k(n,d)$ tends to infinity with d.

12.1. *Remark.* For general quasi-projective varieties the condition 9.4 of [K] only implies that the stratification is equivariantly perfect over the rationals. However by Remark 16.11 of [K], the stratification is equivariantly perfect over all fields of coefficients when $GL(p)$ acts on a subvariety of a product of Grassmannians of the form $(G(n,p))^N$.

From this, together with 10.1, 11.5, 11.6 and 7.4 we obtain the following formula for the equivariant cohomology of R^{ss} up to some dimension $k(n,d)$ which tends to infinity with d.

$$P_t(\mathrm{Map}_d(M, G(n,\infty))) = P_t^{GL(p)}(R^{ss}) + \Sigma t^{2d_\mu} P_t^{GL(p)}(S_{\beta(\mu)}) + 0(t^{k(n,d)}), \tag{12.2}$$

where the sum is over all types μ in $\mathscr{U}+e$ except for $\mu=e+d/n$. Here $P_t^{GL(p)}$ denotes the equivariant Poincaré series with respect to the action of $GL(p)$, and S_β denotes the stratum of R indexed by β.

It therefore remains to consider for each type μ in $\mathscr{U}+e$ the equivariant cohomology of the stratum indexed by $\beta(\mu)$.

When it is necessary to specify the rank n and degree d then R will be denoted by $R(n,d)$. We have

12.3. Lemma. *Let μ be an (n,d) type belonging to the set $\mathscr{U}+e$. Then provided that d is sufficiently large the equivariant cohomology of the stratum indexed by $\beta(\mu)$ is isomorphic to the tensor product of the equivariant cohomology of the semistable strata $R(n_j, d_j)^{ss}$ with respect to the action of $GL(p_j)$ where $p_j=d_j+n_jg$. Thus*

$$P_t^{GL(p)}(S_{\beta(\mu)}) = \Pi_{1\leq j\leq s} P_t^{GL(p_j)}(R(n_j,d_j)^{ss}).$$

Proof. It is shown in [K] that the $GL(p)$-equivariant cohomology of the stratum indexed by $\beta(\mu)$ is the same as the equivariant cohomology of a quasi-projective subvariety $Z_{\beta(\mu)}^{ss}$ of the stratum $S_{\beta(\mu)}$ under the action of a subgroup $\mathrm{Stab}\,\beta(\mu)$ of $GL(p)$. By [K] 16.9 this subgroup is isomorphic to the product $GL(p_1)\times\ldots\times GL(p_s)$ where $p_j=d_j+n_jg$. Moreover $Z_{\beta(\mu)}^{ss}$ consists precisely of the points h' in R of the form described just before Remark 12.1 (see [K] 16.9 again). Thus the elements of $Z_{\beta(\mu)}^{ss}$ are those holomorphic maps

$$h'\colon M \to G(n,p)$$

which factor through the natural embedding of $G(n_1,p_1)\times\ldots\times G(n_s,p_s)$ in $G(n,p)$ and which also belong to $S_{\beta(\mu)}$. From this it is easy to see that $Z_{\beta(\mu)}^{ss}$ is naturally isomorphic to the product

$$R(n_1,d_1)^{ss}\times\ldots\times R(n_s,d_s)^{ss}$$

with the natural action of $GL(p_1)\times\ldots\times GL(p_s)$.

13. Conclusion

From 12.2 and 12.3 we know that

$$(13.1) \qquad P_t(\mathrm{Map}_d(M, G(n, \infty))) = P_t^{GL(p)}(R^{ss}) + \sum_{\mu \neq d/n} t^{d_\mu} \prod_{1 \leq j \leq s} P_t^{GL(p_j)}(R(n_j, d_j)^{ss}) + 0(t^{k(n,d)})$$

where the sum is over all types

$$\mu = (d_1/n_1, \ldots, d_s/n_s) \neq (d/n, \ldots, d/n) = d/n$$

and $k(n, d)$ tends to infinity with d. Moreover by [A & B] 2.15 the Poincaré series of $\mathrm{Map}_d(M, G(n, \infty))$ is

$$\prod_{1 \leq k \leq n} (1+t^{2k-1})^{2g} / (\prod_{1 \leq k \leq n-1} (1-t^{2k})^2 (1-t^{2n}).$$

13.2. *Remark.* The left hand side of 13.1 is thus independent of d. Moreover if e is any integer and μ is an (n, d) type then $\mu+e$ is an $(n, d+ne)$ type and $d_{\mu+e} = d_\mu$. We also know from 9.4 that provided n and d are coprime the Poincaré series $P_t^{GL(p)}(R^{ss})$ is the product of $(1-t^2)$ with the Poincaré polynomial of the moduli space of semistable bundles of rank n and degree d. The latter is unchanged up to isomorphism if d is replaced by $d+ne$ for any integer e.

The inductive formula 13.1 enables us to calculate the $GL(p)$-equivariant cohomology groups of $R(n, d)^{ss}$ up to some dimension which tends to infinity with d. Therefore by Remark 13.2 if n and d are coprime and we replace d by $d+ne$ in 13.1 for some sufficiently large e we obtain a formula for the dimensions of the cohomology groups of the moduli space of bundles of rank n and degree d up to any preassigned dimension. This formula is identical to the one derived by Atiyah and Bott using infinite-dimensional stratifications, and by Harder and Narasimhan using number theory.

Note that even when n and d are coprime there may well be some n_j and d_j occurring in the inductive formula 13.1 which are not coprime. This does not invalidate the computation; compare the argument of [A & B].

References

A & B Atiyah, M. F. and Bott, R., "The Yang—Mills equations over Riemann surfaces", *Phil. Trans. R. Soc. Lond. A* **308** (1982), 523—615.

D Dold, A., *Lectures on algebraic topology,* Grund. der Math. Wiss. Band 200, Springer-Verlag (1972).

G Guest, M. A., "Topology of the space of absolute minima of the energy function", *Amer. J. Math.* **106** (1984), 21—42.

H HARTMAN, P., *Ordinary differential equations,* Wiley, New York (1964).

H & N HARDER, G. and NARASIMHAN, M. S., "On the cohomology groups of moduli spaces of vector bundles on curves", *Math. Ann.* **212** (1975), 215—248.

H & R HILTON, P. and REITBERG, J., "On the Zeeman comparison theorem for the homology of quasi-nilpotent fibrations", *Quart. J. Math.* **27** (1976), 433—444.

K KIRWAN, F. C., *Cohomology of quotients in symplectic and algebraic geometry,* Mathematical Notes 31, Princeton University Press (1985).

L LOJASIEWICZ, S., "Triangulation of semi-analytic sets", *Annali Scu. Norm. Sup. Pisa, Sc. Fis. Mat. Ser. 3, v.* 18, fasc. 4, 449—474 (1964).

McD MCDUFF, D., "Configuration spaces of positive and negative particles", *Topology,* **14** (1975), 91—107.

McD & S MCDUFF, D. and SEGAL, G. B., "Homology fibrations and the 'group completion' theorem", *Invent. Math.* **31** (1976), 279—284.

M MILNOR, J., *Singular points of complex hypersurfaces,* Ann. of Math. Studies 61, P.U.P. (1968).

Mu MUMFORD, D., *Geometric invariant theory,* Springer-Verlag, New York (1965).

Mo MORSE, M., *The calculus of variations in the large,* A.M.S. Colloq. Publ. **18** (1934).

N & R NARASIMHAN, M. S. and RAMANAN, S., "Moduli of vector bundles on a compact Riemann surface", *Ann. Math.* **89** (1969), 14—51.

N NEWSTEAD, P. E., *Introduction to moduli problems and orbit spaces,* Tata Inst. Lectures 51, Springer-Verlag, Heidelberg (1978).

S SEGAL, G. B., "The topology of spaces of rational functions", *Acta Math.* **143** (1979), 39—72.

Se SESHADRI, C. S., "Space of unitary vector bundles on a compact Riemann surface", *Ann. Math.* **85** (1967), 303—336.

Sp SPANIER, E. H., *Algebraic Topology,* McGraw Hill (1966).

V & LP VERDIER, J-L. and LE POTIER, J., editors, Module de fibrés stables sur les courbes algébriques, Notes de l'Ecole Norm. Sup., Printemps 1983, Progress in Math. 54, Birkhäuser (1985).

Z ZEEMAN, E. C., "A proof of the comparison theorem for spectral sequences", *Proc. Camb. Phil. Soc.* **53** (1957), 57—62.

Received May 21, 1984
Received in revised form November 11, 1985

University of Oxford
Mathematical Institute
24-29 St. Giles
Oxford OX1 3LB
England