Commutator and other second order estimates in real interpolation theory

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1. Introduction and summary

The main theme of this paper is that an operator T which is bounded on a parametrized family of Banach spaces will exhibit further properties (in addition to boundedness) on individual spaces in the family. In [RW] this theme was developed for families of spaces obtained by complex interpolation. The results there concerned a mapping δ which is the differential of the natural map between the spaces of the complex interpolation family. Although δ itself is generally unbounded and nonlinear, boundedness results for the commutator between δ and various linear maps T were obtained. In this paper similar studies are made for spaces obtained by real interpolation. Although there are strong analogies between the two cases, the details are very different. Hence it was rather a surprise to us that in some (but not all) cases the specific results obtained here are the same as those in [RW]. Our results here involve maps Ω derived from an analysis of the real interpolation process. As was true with δ, Ω is generally unbounded and non-linear. In some cases Ω is the same as the map δ obtained in the complex interpolation theory.

In Section 2 we recall the basics of real interpolation theory. We introduce the K and J functionals and the associated interpolation spaces. We also introduce other functionals E and F which are closely related to K and J and which lead to alternative forms of Ω.

In Section 3 we give our abstract results. The basic philosophy is that interpolation spaces are constructed by finding efficient ways of decomposing functions into pieces which can then be effectively studied separately. If there is also a bounded linear operator T in the picture, the decomposition can be done before or after

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the application of T. The estimates in Section 3 control the difference between the two quantities so obtained. This leads to consideration of operators Ω related to the decomposition process. Our two basic abstract results are that the commutator of T with Ω is bounded on the interpolation spaces and that T is bounded on the domain of definition of Ω.

In Section 4 we compute the specific forms of Ω for the common interpolation situations of classical analysis. Generally the K and E methods give different forms for Ω. (Ω's obtained from the J functional are discussed in Section 5.) In Section 4.1 we show that, for interpolation between a Lebesgue space $L^p(dx)$ and a weighted space $L^p(w\,dx)$, the Ω obtained using the K method is given by multiplication by $\log w$. In this case $\Omega=\delta$. As an example we consider potential spaces based on $L^2(\mathbf{R}^n)$. Fourier transform theory reduces their study to the study of L^2 with weights. We conclude that if an operator is bounded on potential spaces then the commutator of that operator with the logarithm of the operator $I+\Delta=$(Identity+Laplacian) is bounded on $L^2(\mathbf{R}^n)$. An interesting choice for T is composition with a homeomorphism of $\mathbf{R}^n$. This leads to the following:

Theorem 1. *Suppose φ and φ^{-1} are smooth homeomorphisms of $\mathbf{R}^n$. The map of f to $(\log(I+\Delta)f)\circ\varphi-\log(I+\Delta)(f\circ\varphi)$ is a bounded map of $L^2(\mathbf{R}^n)$ to itself.*

When the E method of interpolation is used, we obtain a different form for Ω. To describe this we first introduce a piece of notation. For functions a, b on a measure space $(X, d\mu)$ we denote by $B(a, b)$ the $|a|$-measure of the set on which b is big:

$$B(a,b)(x)=\int_{|b(y)|>|b(x)|}|a(y)|\,d\mu(y).$$

In particular, $B(1,f)(x)=\lambda_f(|f(x)|)$ where λ_f is the distribution function of f. The map obtained by the E method involves $\Omega f=f\log B(|f|^p w, w)$. Here is an example of a particular result that follows from those general considerations. For $1<p<\infty$, $0<\varepsilon$, $f\in L^p(\mathbf{R})$, define $I_{p,\varepsilon}(f)$ by

$$I_{p,\varepsilon}(f)(x)=\Big(\int_{|y|<|x|}|f(y)|^p|y|^\varepsilon\,dy+\int_{|y|>|x|}|f(y)|^p|y|^{-\varepsilon}\,dy\Big)^{1/p}.$$

Let $M_{p,\varepsilon}=\{f\in L^p;\, f\log I_{p,\varepsilon}(f)\in L^p\}$. We measure size in $M_{p,\varepsilon}$ with $\|f\|_{L^p}+\|f\log I_{p,\varepsilon}(f)\|_{L^p}$. $M_{p,\varepsilon}$ can be thought of as a type of potential space.

Theorem 2. *If $1<p<\infty$, $0<\varepsilon<\min(p-1, 2)$, then $M_{p,\varepsilon}$ is a linear space. It is mapped boundedly to itself by the Hilbert transform.*

This will follow from the general results when we relate $M_{p,\varepsilon}$ to the domain of definition of a suitable Ω.

In Sections 4.2 and 4.3 we consider L^p spaces for variable p. In [RW] the map δ obtained in that context was

$$(1.1)\qquad (\delta f)(x) = f(x)\log|f(x)|.$$

This is the same as the map Ω we obtain here with the E method. The map Ω obtained using the K method is

$$(1.2)\qquad (\Omega f)(x) = f(x)\log B(1,f)(x).$$

Similar but more complicated formulas are derived for weighted L^p spaces. Curiously, we obtain a formula involving $\log B(w^\alpha, wf)$. (Note the position of the variables in contrast with the $B(|f|^p w, w)$ mentioned before.)

In Sections 4.4 and 4.5 we study the interaction between our constructions and maximal functions. We look at maximal functions as maps on L^p spaces and we look at the identity operator as a bounded map between L^p spaces and L^p spaces renormed using maximal functions. (The two points of view are roughly equivalent.) A typical result is:

Theorem 3. *Let M be the Hardy—Littlewood maximal function. The maps*

$$Rf = f\log|Mf/f|$$

and

$$Sf = f\log\left(B(1, Mf)/B(1,f)\right)$$

are bounded maps of $L^p(\mathbf{R}^n)$ to itself $1<p<\infty$.

Using the results from Section 4.5 and 4.6 we also obtain the following result involving M and the Hilbert transform H acting on $L^p(\mathbf{R})$.

Theorem 4. *For $1<p<\infty$ the map of f to $f\log|MHf/f|$ is a bounded map of $L^p(\mathbf{R})$ to itself. The same holds for the map of f to $Hf\log|Mf/Hf|$.*

This last result is a quantitative version of the informal notion that Mf and Hf are "big at the same place".

In Section 4.6 we give the explicit forms of Ω obtained when interpolating between various Schatten ideals of compact operators. We obtain a formal analog of (1.1); $\Omega S = S\log|S|$ but now $\log|S|$ must be interpreted using the functional calculus for the positive operator $|S|$.

Interpolation between an L^p space and a weighted L^p space is closely related to interpolation between a Banach space X and the domain of definition of an unbounded positive operator Λ defined on X (i.e. Λ is multiplication by the p^{th} root of the weight). In the more general context the Ω which occurs is given by $\Omega = \log(I+\Lambda)$. This abstract situation is considered in Section 4.7.

Besov spaces can be realized as interpolation spaces between Lebesgue spaces and the domain of definition of certain unbounded differential operators. Then we can study Ω for the Besov spaces using the results of Section 4.7. Alternatively Besov spaces can be studied using molecular decompositions. This reduces the interpolation theory for Besov spaces to the corresponding theory for sequence spaces and leads to a different form for Ω. This is in Section 4.8.

In Section 5 we return to the abstract theory and develop a version of Ω based on the J functional. The formalism of that section shows a very close analogy to the results of [RW]. The Ω obtained is shown to be the differential of a naturally occuring map between the spaces of the real interpolation family.

Our primary aim here is to present a formalism and general results. We present only a few examples involving specific operators. However we should emphasize that these abstract results apply to the commonly studied operators of analysis. For instance, [RW] contains the results of applying conclusions similar to the ones in this paper to Calderón—Zygmund operators and to fractional integral operators.

2. Background on interpolation

In this section we introduce the formalism of real interpolation theory. We will be rather brief. See [BL] for more details.

Suppose $\bar{A}=(A_0, A_1)$ is a compatible pair (or couple) of Banach spaces; that is $(A_i, \|\cdot\|_i)$, $i=0, 1$, are Banach spaces both inside a common large Hausdorff topological vector space. The K-functional corresponding to $\bar{A}$ is given by, for $a\in A_0+A_1$, $t>0$,

$$K(t, a;\ \bar{A}) = K_\infty(t, a;\ \bar{A}) = \inf_{a=a_0+a_1} \max\{\|a_0\|_0, t\|a_1\|_1\}. \tag{2.1}$$

Here the infimum is over all decompositions $a=a_0+a_1$ with $a_i\in A_i$ $i=0, 1$. The J-functional is defined for a in $A_0\cap A_1$ by

$$J(t, a;\ \bar{A}) = \max\{\|a\|_0, t\|a\|_1\}.$$

The norms of the real interpolation spaces are defined by taking certain averages of K or J functionals. More precisely, for $0<\theta<1$, $1\leqq q\leqq\infty$, we let $\bar{A}_{\theta,q,K}$ be the space of all a in A_0+A_1 with

$$\|a\|_{\theta,q,K} = \left(\int_0^\infty (t^{-\theta}K(t, a;\ \bar{A}))^q\, dt/t\right)^{1/q} < \infty.$$

Such integral expressions show up often. Hence, for convenience we define, for any positive function $f(t)$ defined for $t>0$,

$$\Phi_{\theta,q}(f) = \Phi(f) = \left(\int_0^\infty (t^{-\theta} f(t))^q\, dt/t\right)^{1/q}.$$

(The choice of θ and q will generally be clear from the context.) Thus, putting $\Phi = \Phi_{\theta,q}$

$$\|a\|_{\theta,q,K} = \Phi(K(t, a; \bar{A})). \tag{2.2}$$

For the same range of θ, q we let $\bar{A}_{\theta,q,J}$ be the space of all a in A_0+A_1 for which

$$\|a\|_{\theta,q,J} = \inf_{a=\int_0^\infty u(t)dt/t} \{\Phi(J(t, u(t); \bar{A}))\} < \infty \tag{2.3}$$

where the infimum is over all representations $a=\int_0^\infty u(t)\,dt/t$ (convergence in A_0+A_1) with $u(t)$ a measurable function taking values in $A_0 \cap A_1$. A basic result of real interpolation theory is that

$$\bar{A}_{\theta,q,K} = \bar{A}_{\theta,q,J} \tag{2.4}$$

with equivalent norms. Hence we will often simply write $\bar{A}_{\theta,q}$.

We will be most interested in those representations of a in (2.1) and (2.3) for which the infimum is (almost) attained. Given a in A_0+A_1 and given $C>1$ we will say that the decomposition $a=a_0(t)+a_1(t)$, $t>0$ is *almost optimal* if, for all $t>0$

$$K(t, a, \bar{A}) \leqq \max\{\|a_0(t)\|_0, t\|a_1(t)\|_1\} \leqq CK(t, a, \bar{A}).$$

(This notion depends on the choice of C but we will leave that implicit.) We then write

$$D(t)a = D_K(t; \bar{A})a = a_0(t).$$

We call $D_K(t)$ the *almost optimal projection*. Thus, for each t, $D(t)$ maps A_0+A_1 to A_0+A_1. The map is not uniquely determined by $\bar{A}$; all we need is a choice of such a map (such maps certainly exist). $D(t)a$ is not required to be smooth in t (although a smooth choice can always be made ([T] Sections 1.3—1.5)) and will not, in general, be linear in the variable a. A pair (A_0, A_1) for which $D(t)$ can be selected to be linear is called *quasi-linearizable*. We consider such decompositions for (2.3) in Section 5.

Another basic abstract result is that the spaces $\bar{A}_{\theta,q}$ are stable under iteration of the interpolation process. In fact, the proof of this in [BL] shows that this stability occurs at the level of the almost optimal decomposition with respect to the K functional. More precisely, suppose $\bar{A}_{\overline{\theta q}}$ is the pair $(\bar{A}_{\theta_0 q_0}, \bar{A}_{\theta_1 q_1})$ with $0<\theta_0<\theta_1<1$, $1 \leqq q_0, q_1 \leqq \infty$. Set $\Delta = \theta_1 - \theta_0$, then $D_K(t, \bar{A}_{\overline{\theta q}})$ given by

$$D_K(t, \bar{A}_{\overline{\theta q}})(a) = D_K(t^{1/\Delta}, \bar{A})(a), \tag{2.5}$$

is an almost optimal projection for the couple $\bar{A}_{\overline{\theta q}}$. (It is not true, however, that every almost optimal projection for $\bar{A}_{\overline{\theta q}}$ is obtained this way.)

Although the K and J functionals are the most widely used, there are other similar functionals which are at times more natural or convenient. We now introduce some of these. For $\alpha>1$ we define E_α for $t>0$ and a in A_0+A_1 by

$$E_\alpha(t, a;\ \bar{A}) = \inf_{a=a_0+a_1} \max\{(\|a_0\|_0/t)^{1/\alpha}, (\|a_1\|_1/t)^{1/(\alpha-1)}\}.$$

For $\alpha=1$ we set

$$E_1(t, a;\ \bar{A}) = \inf_{\|a_1\|_1\leqq t} \left\{\frac{1}{t}\|a-a_1\|_0\right\}.$$

These are variations on the K functional. E_1 is essentially the "best approximation" functional of approximation theory. Functionals of this sort have been studied before in, for instance [Pe], [HP], [BL] and Section 1.4 of [T]. The corresponding substitutes for J are, for a in $A_0\cap A_1$, $t>0$

$$F_\alpha(t, a;\ \bar{A}) = \max\{(\|a\|_0/t)^{1/\alpha},\quad (\|a\|_1/t)^{1/(\alpha-1)}\}$$

for $\alpha>1$. For $\alpha=1$

$$F_1(t, a;\bar{A}) = \begin{cases} \|a\|_0/t & \text{if } \ t \geqq \|a\|_1 \\ \infty & \text{otherwise.} \end{cases}$$

We now define the interpolation space. Select θ, q with $0<\theta<1$, $1\leqq q\leqq\infty$ and set $\varphi=(\theta-\alpha)^{-1}$, $r=q(\alpha-\theta)$. $\bar{A}_{\theta,q,E_\alpha}$ is defined by requiring finiteness of the norm

$$\|a\|_{\theta,q,E_\alpha} = (\Phi_{\varphi,r}(E_\alpha(t, a;\ \bar{A})))^{(\alpha-\theta)}$$

and $\bar{A}_{\theta,q,F_\alpha}$ is defined by the finiteness of

$$\|a\|_{\theta,q,F_\alpha} = \inf_{a=\int_0^\infty u(t)dt/t} (\Phi_{\varphi,r}(F_\alpha(t, u(t);\ \bar{A})))^{(\alpha-\theta)}.$$

A simple adaptation of the proof of (2.4) given in [BL] shows that for $\alpha\geqq 1$

$$\bar{A}_{\theta,q,E_\alpha} = \bar{A}_{\theta,q,F_\alpha} \tag{2.6}$$

with equivalent norms.

Lemma 2.1. *Let* $\alpha\geqq 1$.

(1) For each a in A_0+A_1 the right continuous inverse of the function of t, $E_\alpha(t, a; \bar{A})$, is $K(t, a; \bar{A})/t^\alpha$.

(2) For each a in $A_0\cap A_1$ the right-continuous inverse of $F_\alpha(t, a, \bar{A})$ is $J(t, a; \bar{A})/t^\alpha$.

Proof. (1) let $\Gamma(a)$ be the Gagliardo set of a, $\Gamma(a)=\{(x_0, x_1)\in\mathbf{R}^2;$ there is a decomposition $a=a_0+a_1$ with a_i in A_i and $\|a_i\|_i\leqq x_i\}$. It is easy to check that for any t, $(K(t), K(t)/t)$ is on the boundary of the convex set $\Gamma(a)$ and also that for any $r>0$, $(rE_\alpha(r)^\alpha, rE_\alpha(r)^{\alpha-1})$ is on this boundary. Comparing coordinates gives $t=E_\alpha(r)$, $K(t)/t^\alpha=r$ as required.

(2) is direct.

In fact this proof shows a bit more. It shows that the almost optimal decompositions for the K and E_α methods are obtained from each other by change of variable. Precisely

(2.7) $$D_K(t;\bar{A})a = D_{E_\alpha}(r;\bar{A})a;\; E_\alpha(r) = t, \quad r = K(t)/t^\alpha.$$

This lemma, the change of variable in (2.2) of t to $E_\alpha(r)$, followed by integration by parts shows $\bar{A}_{\theta,q,E_\alpha} = \bar{A}_{\theta,q;K}$ (with norms in fact constant multiples of each other). Combining this with (2.4) and (2.6) we obtain

(2.8) $$\bar{A}_{\theta,q} = \bar{A}_{\theta,q,E_\alpha} = \bar{A}_{\theta,q,F_\alpha}.$$

The corresponding almost optimal decompositions $a=a_0(t)+a_1(t)$ and almost optimal projection $D(t)(a)=a_0(t)$ are defined by the requirement that, for some c independent of t, a

$$E_\alpha(t, a;\; \bar{A}) \leqq \max\{(\|a_0(t)\|_0/t)^{1/\alpha},\; (\|a_1(t)\|_1/t)^{1/(\alpha-1)}\} \leqq E_\alpha(t/c, a;\; \bar{A})$$

if $\alpha>1$. For $\alpha=1$

$$E_1(t, a;\; \bar{A}) \leqq \|a_0(t)\|_0/t \leqq E_1(t/c, a;\; \bar{A})$$

and $\|a_1(t)\|_1 \leqq t$. As with the operators D_K, these operators have a certain stability under iteration. The analog of (2.5) is

(2.9) $$D_{E_{(\alpha-\theta_0)/\Delta}}(t, \bar{A}_{\overline{\theta q}})(a) = D_{E_\alpha}(t, \bar{A})(a).$$

The proof of this is a simple modification of the proof of (2.5). Alternatively (2.9) follows from (2.5) and (2.7).

3. Commutators

Throughout this and the following sections we will follow the custom of using the letter c to denote a positive quantity which will vary from occurrence to occurrence.

Suppose $\bar{A}=(A_0, A_1)$ and $\bar{B}=(B_0, B_1)$ are two couples of Banach spaces. Suppose T: $\bar{A}\to\bar{B}$; that is, suppose T is a linear operator from A_0+A_1 to B_0+B_1 and for $i=0, 1$ $T|_{A_i}$ is a bounded map of A_i to B_i. Suppose D is one of the almost optimal projection operators introduced in the previous section. Our first result concerns the local commutators $[T, D(t)]$ defined for a in A_0+A_1, $t>0$, by

$$[T, D(t)]a = (TD(t)-D(t)T)(a) = (TD(t, \bar{A})-D(t, \bar{B})T)(a).$$

Proposition 3.1. *Suppose* T: $\bar{A}\to\bar{B}$ *then*

(1) $$J(t, [T, D(t)]a; \bar{B}) \leqq cK(t, a; \bar{A}) \quad \text{with} \quad D(t) = D_K(t).$$

(2) $$F_\alpha(ct, [T, D(t)]a;\; \bar{B}) \leqq E_\alpha(t/c, a;\; \bar{A}) \quad \text{with} \quad D(t) = D_{E_\alpha}(t)$$

for all $\alpha \geqq 1$.

Here c is independent of t and of a in A_0+A_1.

Proof. We start with (1). T is bounded from A_0+B_0; hence

$$\|[T, D(t)]a\|_0 \leqq \|T(D(t)a)\|_0+\|D(t)Ta\|_0 \leqq c(\|D(t)a\|_0+\|D(t)Ta\|_0) \leqq c(K(t, a)+K(t, Ta))$$

where the last inequality follows from the definition of $D(t)$. Since $T\colon \bar{A}\to\bar{B}$, we have $K(t, Ta)\leqq cK(t, a)$. Thus we continue the previous estimate to

$$\|[T, D(t)]a\|_0 \leqq cK(t, a). \tag{3.1}$$

To estimate the A_1 norm of $[T, D(t)]$, we rewrite the commutator as $[T, D(t)]a=(I-D(t))Ta-T(I-D(t))a$ (here I is the identity operator). Now, using the fact that T is bounded from A_1 to B_1 and estimating as before,

$$t\|[T, D(t)]a\|_1 \leqq ct(\|a-D(t)a\|_1+\|Ta-D(t)Ta\|_1)\leqq cK(t, a). \tag{3.2}$$

Combining (3.1) and (3.2) gives the required estimate.

The proof of (2) is similar. This time the boundedness of T from $\bar{A}$ to $\bar{B}$ implies that

$$\begin{aligned}(\|[T, D(t)]a\|_0/t)^{1/\alpha} &\leqq c((\|TD(t)a)\|_0/t)^{1/\alpha}+(\|D(t)Ta\|_0/t)^{1/\alpha})\\ &\leqq c((\|D(t)a\|_0/t)^{1/\alpha}+(\|D(t)Ta\|_0/t)^{1/\alpha}) \leqq cE_\alpha(t/c, a).\end{aligned} \tag{3.1$'$}$$

Rewriting the commutator as above, we also find that

$$\begin{aligned}(\|[T, D(t)]a\|_1/t)^{1/(\alpha-1)} &\leqq c(\|a-D(t)a\|_1/t)^{1/(\alpha-1)}+(\|Ta-D(t)Ta\|_1/t)^{1/(\alpha-1)}\\ &\leqq cE_\alpha(t/c, a)\end{aligned} \tag{3.2$'$}$$

for $\alpha>1$ and

$$\|[T, D(t)]a\|_1 \leqq c(\|a-d(t)a\|_1+\|Ta-D(t)Ta\|_1) \leqq ct \tag{3.2$''$}$$

for $\alpha=1$. Putting (3.1) and (3.2) (or (3.2)″ if $\alpha=1$) together, gives us (2) and finishes the proof.

We now integrate this to get our main result. For any a in A_0+A_1, define $[T, \Omega]a$

$$[T, \Omega](a) = \int_0^\infty [T, D(t)]a(dt/t).$$

The domain of $[T, \Omega]$ is defined to be those a for which the integral converges in B_0+B_1.

Theorem 3.2. *Suppose* $T\colon \bar{A}\to\bar{B}$, $0<\theta<1$, $1\leqq q\leqq\infty$, *then* $[T, \Omega]$ *is a bounded operator from* $\bar{A}_{\theta,q}$ *to* $\bar{B}_{\theta,q}$.

Proof. If $[T, \Omega]$ is based on D_K then we set $u(t)=[T, D(t)]a$ in (2.3) and use part (1) of the previous proposition and (2.2) to conclude $\|[T, \Omega](a)\|_{\theta,q,J}\leqq$

$c\,\|a\|_{\theta,q,K}$. We then invoke (2.4). If $[T,\Omega]$ is based on E_α we use part (2) of the proposition, the definitions of the E_α and F_α interpolation spaces, and the equivalence (2.8).

In fact a bit more is true. In the terminology of DeVore, Riemenschneider and Sharpley $[T,\Omega]$ is of *generalized weak type* $((1,1),(\infty,\infty))$. That is, for all $t>0$

$$K(t,[T,\Omega]a)\leqq c\int_0^\infty \min(1,t/s)K(s,a)\,ds/s.$$

This is a direct consequence of the definitions of K and J, the triangle inequality for $\|\cdot\|_0$ and $\|\cdot\|_1$, and Proposition 3.1; all applied to the decomposition $[T,\Omega]a=\int_0^\infty=\int_0^t+\int_t^\infty$. This type of estimate, which is also satisfied by more familiar operators such as the Hilbert transform, is an efficient way of summarizing information. Once it is established Theorem 3.2 follows as do various refinements of the theorem (involving, for instance results at the end points $\theta=0$ and $\theta=1$). An introduction to this point of view is given in [BS].

Formally we have just established the boundedness of the commutator of T with $\int_t^\infty D(t)\,dt/t$. However, for large t, $D(t)$ is almost equal to the identity and thus the integral diverges. We salvage the situation by a renormalization. Let I be the identity and define $\Omega=\Omega_K$

$$\begin{aligned}\Omega a=\Omega_K a&=\int_0^\infty \big(D_K(t)-I\cdot\chi_{(1,\infty)}(t)\big)a\,(dt/t)\\&=\int_0^1 D_K(t)a(dt/t)-\int_1^\infty \big(a-D_K(t)\,a\big)\,dt/t\\&=\int_0^1 a_0(t)\,dt/t-\int_1^\infty a_1(t)\,dt/t\end{aligned}\tag{3.3}$$

where the $a_i(t)$, $i=0,1$ are obtained from the almost optimal decomposition used to define D_K. If a is in some $\bar A_{\theta,q}$, $0<\theta<1$, $1\leqq q\leqq\infty$, then $K(t,a)=\mathcal{O}(t^\theta)$ ([BL] page 45). With this estimate we check that Ωa is in A_0+A_1:

$$\Big\|\int_0^1 a_0(t)\,dt/t\Big\|_0\leqq\int_0^1\|a_0(t)\|_0dt/t\leqq c\int_0^1 K(t,a)\,dt/t\leqq c\int_0^1 t^\theta dt/t<\infty,$$

$$\Big\|\int_1^\infty a_1(t)\,dt/t\Big\|_1\leqq\int_1^\infty\|a_1(t)\|_1\,dt/t\leqq c\int_1^\infty t^{-1}K(t,a)\,dt/t\leqq c\int_1^\infty t^{-1}t^\theta dt/t<\infty.$$

Thus the map of a to Ω_a is well defined as a map from, say, $\bigcup_{0<\theta<1}\bar A_{\theta q}$ to A_0+A_1. Similarly we define

$$\Omega a=\Omega_{E_\alpha}a=-\int_0^1\big(I-D_{E_\alpha}(t)\big)(a)\,dt/t+\int_1^\infty D_{E_\alpha}(t)(a)\,dt/t.\tag{3.4}$$

For Ω's defined this way $[T,\Omega]$ is actually a commutator.

Corollary 3.3. $[T,\Omega]=T\Omega-\Omega T$.

Note. Here and below we are suppressing some of the variables. We really have $[T, \Omega]=T\Omega_{\bar{A},K}-\Omega_{\bar{B},K}T$ and similarly for E_α.

There were several choices in the formula (3.3). Ω could have been defined using $\int_0^c+\int_c^\infty$ for some $c\neq 1$. This would change Ω by a constant multiple of the identity and $[T, \Omega]$ would be unchanged. Changes would occur if we change the choice of almost optimal projection from $D_K(t)$ to $\tilde{D}_K(t)$. However, the difference is not large. Let Ω_K and $\tilde{\Omega}_K$ be the two maps obtained.

Corollary 3.4. *For* $0<\theta<1$, $1\leqq q\leqq\infty$, $\Omega_K-\tilde{\Omega}_K$ *is a bounded map of* $\bar{A}_{\theta,q}$ *to itself.*

Proof. Apply the theorem with $\bar{A}=\bar{B}$, $D_K(t, \bar{A})=D_K(t)$, $D_K(t, \bar{B})=\tilde{D}_K(t)$, and $T=I$.

Of course there is a similar result with $\Omega_{E_\alpha}-\tilde{\Omega}_{E_\alpha}$. It is not generally true that $\Omega_{E_\alpha}-\Omega_K$ or $\Omega_{E_\alpha}-\Omega_{E_\beta}$, $\alpha\neq\beta$, is bounded. We give an explicit example in Section 4.3. Here we use (2.7) to give the exact relation between these operators. By (3.4)

$$\Omega_{E_\alpha}=\int_0^\infty \left(D_{E_\alpha}(r)-I\cdot\chi_{(0,1)}(r)\right)d\log r.$$

Guided by (2.7) we make the change of variables $r=K(t)/t^\alpha$; we then use (2.7) to go from D_{E_α} to D_K

$$\Omega_{E_\alpha}=\int_0^\infty \left(D_K(t)-I\chi_{(c,\infty)}(t)\right)d\log\left(K(t)/t^\alpha\right),$$

where $K(c)/c^\alpha=1$. Thus, since $\int_c^1 d\log\left(K(t)/t^\alpha\right)=\log K(1, a)$

$$\Omega_{E_\alpha}=\int_0^\infty \left(D_K-I\chi_{(1,\infty)}\right)d\log\left(K(t)/t^\alpha\right)-\log K(1, a)I.$$

Hence

$$\Omega_{E_\alpha}-\Omega_{E_1}=-(\alpha-1)\int_0^\infty \left(D_K-I\chi_{(1,\infty)}\right)d\log t$$

so

$$\Omega_{E_\alpha}-\Omega_{E_1}=-(\alpha-1)\,\Omega_K.$$

Thus there are two basic quantities, Ω_K based on integrals with respect to $d\log t$ and Ω_{E_1} based on integrals against $d\log\left(K(t)/t\right)$. The change from t to $K(t)/t$ is closely related to the change from function to inverse function (e.g. from the non-increasing rearrangement to the distribution function). We will see explicit examples in which these two points of view produce what could be considered "dual" results.

We now describe the way in which the theorem is stable under iteration. We start with a couple $\bar{A}$ and almost optimal projection $D_K(t)=D(t)$. For $0<\theta_0<\theta_1<1$, $1\leqq q_0, q_1\leqq\infty$ let $\bar{A}_{\overline{\theta,q}}$ be the couple $(\bar{A}_{\theta_0,q_0}, \bar{A}_{\theta_1,q_1})$ and $\tilde{D}(t)$ be the almost optimal projection for this new couple obtained using equation (2.5); that is,

$\tilde{D}(t)(a)=D(t^{1/\Delta})(a)$ with $\Delta=\theta_1-\theta_0$. The map $\tilde{\Omega}$ obtained from $\tilde{D}$ is given by

$$\tilde{\Omega} = \int_0^1 \tilde{D}\,dt/t - \int_1^\infty (I-\tilde{D})\,dt/t = \int_0^1 D(t^{1/\Delta})\,dt/t - \int_1^\infty (I-D(t^{1/\Delta}))\,dt/t$$
$$= \Delta\Big(\int_0^1 D(s)\,ds/s - \int_1^\infty (I-D(s))\,ds/s\Big) = \Delta\Omega.$$

Under the hypotheses of Theorem 3.2, we can also conclude that T is a bounded operator on the domain of definition of the (generally unbounded) operator Ω. We now make that precise. Given a couple $\bar{A}$ we fix $X=\bar{A}_{\theta,q}$ for some θ, q with $0<\theta<1$, $1\leqq q\leqq\infty$. Pick and fix a choice of Ω given by (3.3) or (3.4). We define $D(\Omega)=D(\Omega; X)$ to be the domain of Ω on X; that is, the set of a in X for which

$$\|a\|_{D(\Omega)} = \|a\|_X + \|\Omega a\|_X$$

is finite. Because Ω need not be linear it is not clear that this is indeed a linear space. First we consider scalar multiplication. For λ in $\mathbf{C}$ define $M_\lambda(a)=\lambda a$. Then

$$\|\lambda a\|_{D(\Omega)} = \|\lambda a\|_X + \|\Omega(\lambda a)\|_X = |\lambda|\|a\|_X + \|\lambda\Omega(a) - [M_\lambda, \Omega](a)\|_X$$
$$\leqq c(|\lambda| + \|[M_\lambda, \Omega]\|)\|a\|_{D(\Omega)}.$$

By Theorem 3.2 we conclude that $\|[M_\lambda, \Omega]\|$ is finite and hence that $\lambda a\in D(\Omega)$ if $a\in D(\Omega)$. Suppose now that a and b are in $D(\Omega)$; we wish to show $a+b\in D(\Omega)$:

$$\|a+b\|_{D(\Omega)} \leqq \|a\|_X + \|b\|_X + \|\Omega(a+b) - \Omega(a) - \Omega(b)\|_X + \|\Omega a\|_X + \|\Omega b\|_X.$$

We would be done if we knew that

$$\|\Omega(a+b) - \Omega(a) - \Omega(b)\|_X \leqq c(\|a\|_X + \|b\|_X). \tag{3.5}$$

Suppose that $\Omega=\Omega_K$ (the Ω_{E_α} case being similar). Arguing as in the proof of Proposition 3.1, $J(S, D(s)(a+b)-D(s)a-D(s)b)\leqq c(K(s,a)+K(s,b))$. Applying Φ and using (2.4) gives (3.5).

If $\Omega=\Omega_K$ we can in fact go one step further in the analysis. We can select D_K to satisfy $D_K(t)(\lambda a)=\lambda D_K(t)(a)$. In this case $\Omega(\lambda a)=\lambda\Omega(a)$, $\|\lambda a\|_{D(\Omega)}=|\lambda|\,\|a\|_{D(\Omega)}$, and thus $\|\cdot\|_{D(\Omega)}$ is a quasinorm.

Finally note that, by Corollary 3.4, the space $D(\Omega)$ is unchanged if Ω is replaced by another choice, $\tilde{\Omega}$. In summary:

Proposition 3.6. *$D(\Omega)=\{a\in\bar{A}_{\theta,q};\ \|a\|_{D(\Omega)}<\infty\}$ is a linear space. The space obtained may depend on the interpolation method but does not depend on the choice of the almost optimal decomposition. If $\Omega=\Omega_K$ then $D(\Omega)$ has a quasinorm equivalent to $\|\cdot\|_{D(\Omega)}$.*

If $\Omega=\Omega_E$, the situation is more complicated. For instance, with the choice given by (4.2), $\Omega a=a\log|a|$, we find that the norm of the operator of multiplication by positive λ is of the order $|\lambda|(1+|\log\lambda|)$. In this case $\|\cdot\|_\Omega$ is not a quasinorm.

Theorem 3.7. *Suppose* $X=\bar{A}_{\theta,q}$, $Y=\bar{B}_{\theta,q}$ *for some* $0<\theta<1$, $1\leqq q\leqq\infty$. *Suppose* $T\colon \bar{A}\to\bar{B}$. *Then* T *is a bounded operator from* $D(\Omega, X)$ *to* $D(\Omega, Y)$.

Proof. By interpolation T is bounded from X to Y. By Theorem 3.2 $[T, \Omega]$ is bounded from X to Y. Hence

$$\|Ta\|_{D(\Omega)} = \|Ta\|_Y+\|\Omega Ta\|_Y \leqq \|Ta\|_Y+\|T\Omega a\|_Y+\|[T, \Omega]a\|_Y$$

$$\leqq c(\|a\|_X+\|\Omega a\|_X+\|a\|_X) \leqq c\|a\|_{D(\Omega)}.$$

There are a number of variations possible on these themes. It is possible to develop an operator Ω_J based on the almost optimal decomposition in (2.3); we do that in Section 5. In fact these techniques can be applied to most of the common variations of real interpolation theory. In particular they apply to the discrete analogues of the K and J methods (in which $a=\int_0^\infty u(t)\,dt/t$ in (2.3) is replaced by a sum) and hence can be extended to quasi-normed spaces (e.g. L^p, $p<1$) or further (quasi- normed Abelian groups...).

Generally Ω is not linear (although it is close, as shown by (3.5)). If, however, $\bar{A}$ is quasi-linearizable then we can choose Ω to be linear. In this case we can iterate the results to obtain information about higher commutators. More precisely, after using Theorem 3.2 to conclude $[T, \Omega]$ is bounded from $\bar{A}_{\theta,q}$ to $\bar{B}_{\theta,q}$ we can then apply Corollary 3.5 to $[T, \Omega]$ to obtain boundedness of $[[T, \Omega], \Omega]$ etc. We don't know what the analogs are of these higher commutator results are if $\bar{A}$ is not quasi-linearizable.

4. Examples

In this section we compute Ω for various commonly encountered couples of Banach spaces. By Corollary 3.4, once the interpolation method is selected, the different possible choices of Ω are equivalent; hence, it suffices to find a single one. Often we will not rewrite the conclusions of Theorems 3.2 and 3.4; however, we do give some specific choices of T. Other interesting choices of T and $\bar{A}$ to which these conclusions apply are in [RW].

4.1 L^p with variable weights

Suppose p is given $1\leqq p<\infty$. We consider L^p of some measure space X. Given positive functions w_i, $i=0, 1$ on X we define the weighted space $L^p(w_i\,dx)$ by the norm $\left(\int |f|^p w_i\,dx\right)^{1/p}$.

Let $\bar{A}=(L^p(w_0\,dx), L^p(w_1\,dx))$. The definition of D_K is not changed if we replace K by an equivalent quantity. A convenient choice of equivalent functional is

$$K_p(t, a) = \inf_{a=a_0+a_1}\left(\int_X |a_0|^p w_0 dx + t^p \int_X |a_1|^p w_1 dx\right)^{1/p}.$$

Using K_p it is easy to see (and well-known) that an almost optimal decomposition is given by

$$a(x) = a(x)\chi_t(x) + a(x)(1-\chi_t(x))$$

where χ_t is the characteristic function of the set $\{x;\, w_0(x) \leqq t^p w_1(x)\}$. Thus $D(t)a = a\chi_t$ and Ωa is given by

$$(\Omega a)(x) = \int_0^1 a(x)\chi_t(x)\,dt/t - \int_1^\infty a(x)(1-\chi_t(x))\,dt/t.$$

so

$$(\Omega a)(x) = \left(\frac{1}{p}\log\frac{w_1(x)}{w_0(x)}\right) a(x).$$

When we apply the theorems of the previous section we recapture some of the results of [RW] and obtain the direct extension of those results to Lorentz spaces.

Here is a further application. Let A_0 be the inhomogeneous Sobolev space (potential space) H^2 consisting of those functions f in $L^2(\mathbf{R}^n)$ with the property that the Laplacian of f, Δf is also in $L^2(\mathbf{R}^n)$, $\|f\|_0 = \|f\|_{L^2} + \|\Delta f\|_{L^2}$. Let $\hat{A}_0 = \{\hat{f};\, f\in A_0\}$ where $\hat{}$ denotes the Fourier transform. Using $(\Delta f)\hat{} = |x|^2\hat{f}$ we see $\hat{A}_0 = \{F;\, (1+|x|^2)F \in L^2\}$. Using this Fourier transform point of view it is easy to see (and well-known) that if f is in A_0 then all the first and second order partial derivatives of f are in L^2. Suppose now that φ and φ^{-1} are smooth homeomorphisms of $\mathbf{R}^n$. Let c_φ be composition with φ: $c_\varphi f = f\circ\varphi$. If f is in A_0 then, using the fact that the first and second order partials of f are in L^2; we conclude by direct computation that $c_\varphi f$ is in A_0 and that c_φ is bounded on A_0.

For A_1 we select H^{-2} defined by $\hat{A}_1 = \{F;\, (1+|x|^2)^{-1} F \in L^2(\mathbf{R}^n)\}$. A_1 is the dual of A_0 with respect to the duality pairing given by the inner product on $L^2(\mathbf{R}^n)$. To see that c_φ is bounded on A_1 we note that $c_\varphi^{-1} = c_{\varphi^{-1}}$ is bounded on A_0 and hence $(c_\varphi^{-1})^*$ is bounded on $(A_0)^* = A_1$. By direct computation $(c_\varphi^{-1})^* = Mc_\varphi$ where M is given by multiplication by a smooth positive function (i.e. the Jacobian determinant). M is bounded and invertible on A_1 (this is most easily checked for M^* on A_0) and thus c_φ is bounded on A_1.

The computations we just did show that Ω for the pair $(\hat{A}_0, \hat{A}_1)$ is given by multiplication by $c\log(1+|x|^2)$. Using the inverse Fourier transform to return to (A_0, A_1) and using the functional calculus for the positive operator Δ we obtain

$$\Omega f = c\log(I+\Delta)f.$$

Since $(A_0, A_1)_{1/2,2} = L^2(\mathbf{R}^n)$ (see, for example, [BL] for this identification) we have

Theorem 4.1. *(= Theorem 1 of Section 1.) Suppose φ and φ^{-1} are smooth homeomorphisms of $\mathbf{R}^n$. $[c_\varphi, \log(I+\Delta)]$ is bounded on $L^2(\mathbf{R}^n)$.*

Many variations on this are possible. First of all, using θ and q other than $1/2, 2$ we obtain a similar result on certain Besov spaces. Second, working on the torus, we obtain results involving $\log \Delta$ (rather than $\log(I+\Delta)$). Finally, c_φ is also bounded on the potential spaces based on L^p for $1<p<\infty$. The duality argument then shows that c_φ is also bounded on the H^{-2} type spaces based on $L^{p'}$, $1/p'+1/p=1$. Then, using complex interpolation and the results of [RW], we can obtain an analog of Theorem 4.1 for $L^p(\mathbf{R}^n)$, $1<p<\infty$. (The difference here is that for $p\neq 2$, $L^p(\mathbf{R}^n)$ is not obtained by real interpolation between potential spaces.)

We also use the formula for D_K to compute Ω_E. Using the formula after Corollary 3.4

$$\Omega = \Omega_{E_1} = \int_0^\infty \big(D_K(t) - I\chi_{(1,\infty)}(t)\big)\, d\log\big(K(t)/t\big)$$

$$= -a(x)\log\left[\left(\frac{w_1}{w_0}\right)^{1/p}(x) K_p\left(\left(\frac{w_0}{w_1}\right)^{1/p}(x)\right)\right] + a(x)\log K_p(1).$$

Using the almost optimal decomposition we find explicitly

$$K_p\left(\left(\frac{w_0}{w_1}\right)^{1/p}(x)\right) \sim \left(B\left(|a|^p w_0, \frac{w_0}{w_1}\right) + B\left(|a|^p w_1, \frac{w_0}{w_1}\right)\right)^{1/p}.$$

As an application we consider $L^2(\mathbf{R})$ and $w_0 = w_1^{-1} = |x|^\varepsilon$, for some $\varepsilon>0$. Thus

$$K_2(|x|^\varepsilon)^2 \sim \tilde{K}_2(|x|^\varepsilon)^2 \equiv \int_{|y|<|x|} |a(y)|^2 |y|^\varepsilon\, dy + \int_{|y|>|x|} |a(y)|^2 |y|^{-\varepsilon}\, dy.$$

Define Ω_* by $\Omega_*(a) = a\log\big(\tilde{K}_2(|x|^\varepsilon)^2\big)$. If T is a linear operator which is bounded on $L^2(|x|^{\pm\varepsilon}\, dx)$ then, by Theorem 3.2, $[T, \Omega_{E_1}]$ and $[T, \Omega_K]$ are both bounded on the intermediate spaces. Subtracting we find $[T, \Omega_* + \Lambda]$ is bounded where $\Lambda a = a\log \tilde{K}_2(1, a)$. Suppose now that $T^n = I$ for some positive integer n. It follows that $\tilde{K}_2(1, Ta)/\tilde{K}_2(1, a)$ is bounded above and below. Hence $[T, \Lambda]$ is bounded. Thus $[T, \Omega_*]$ is bounded. Now, by the same proof as Theorem 3.7, T is bounded on $D(\Omega_*)$, the domain of Ω_*. For instance we can select T to be the Hilbert transform. If $\varepsilon<1$ then the theory of weighted norm inequalities applies and we conclude that T is bounded on $L^2(|x|^{\pm\varepsilon}\, dx)$ (see [G]). All the previous discussion applies and T is a bounded map of $\{a;\ a\in L^2,\ \Omega_* a\in L^2\}$ to itself. The space of those a in L^2 for which $\Omega_* a$ is in L^2 is not an easy one to understand. Note, however, that $\Omega_* a = a\log \tilde{K}_2(|x|^\varepsilon)^2$ and, if a is supported on $(0,\infty)$ then $(d/dx)\,\tilde{K}_2(|x|^\varepsilon)^2 = |a(x)|^2(x^\varepsilon + x^{-\varepsilon})$. Thus $\tilde{K}_2(|x|^\varepsilon)^2$ is an integral operator and the condition on $\Omega_* a$ can be regarded as defining a generalized potential space.

4.2 L^p with variable weights, variable p

Suppose $1 \leqq p_0 < p_1 \leqq \infty$ and w_i, $i=0,1$, are positive functions. Let $\bar{A}=(L^{p_0}(w_0\,dx), L^{p_1}(w_1\,dx))$. This is an example where the K functional is a bit awkward but E_α with $\alpha=p_1/(p_1-p_0)$ is easy to work with.

$$E_\alpha(t, a;\ \bar{A}) = \inf_{a=a_0+a_1} \max\left\{\left(\frac{1}{t^{p_0}}\int_X |a_0|^{p_0} w_0 dx\right)^{1/\alpha p_0}, \left(\frac{1}{t^{p_1}}\int_X |a_1|^{p_1} w_1 dx\right)^{1/\alpha p_1}\right\}.$$

This quantity is comparable to

$$\inf_{a=a_0+a_1} \left(\int \left|\frac{a_0}{t}\right|^{p_0} w_0 + \left|\frac{a_1}{t}\right|^{p_1} w_1\right)^{1/\alpha p_0}. \tag{4.1}$$

Let χ_t be the characteristic function of $\{x;\ |a(x)|(w_1(x)/w_0(x))^{1/(p_1-p_0)} > t\}$. It is easy to check, using the fact that $E_\alpha(t)$ is comparable to the expression (4.1), that $a=a\chi_t+a(1-\chi_t)$ is an almost optimal decomposition. Hence $D(t)a=a\chi_t$ and thus, doing a direct computation in (3.3)

$$\Omega a = \Omega_{E_\alpha} a = a(x) \log\left(|a(x)|\left(\frac{w_1(x)}{w_0(x)}\right)^{1/(p_1-p_0)}\right).$$

In particular, when $w_0=w_1=1$ we have

$$\Omega a = a \log |a| \tag{4.2}$$

for the couple $\bar{A}=(L^{p_0}, L^{p_1})$. Here we recapture Proposition 3.34 of [RW].

4.3 L^p with variable p, the K-method

For $p_0<p_1$, let $\bar{A}=(L^{p_0}, L^{p_1})$. For a function $a(x)$ defined on the underlying measure space X let $a^*(t)$ be the non-decreasing rearrangement of a. It is well-known that an almost optimal decomposition of $a(x)$ is given by $a(x)=a(x)\chi_t+a(x)(1-\chi_t)$ where χ_t is the characteristic function of the set

$$\{x;\ |a(x)| > a^*(t^\alpha)\}, \quad 1/\alpha = 1/p_0 - 1/p_1.$$

(In fact we demonstrate something more general than this in the next section.) Using this we compute directly

$$\Omega a = -\frac{1}{\alpha} a \log B(1, a). \tag{4.3}$$

We now check explicitly that there is no analog of Corollary 3.4 which involves Ω_E of (4.2) and Ω_K of (4.3) at the same time. Suppose $p_0=1$, $p_1=\infty$. $(\Omega_E-\Omega_K)(a)= a \log |a| + a \log B(1, a) = a \log (|a| B(1, a))$. If the functions considered are on

$(0, \infty)$ and a is positive and decreasing then $B(1, a)(x)=x$. Thus we need to know if $f \log |xf|$ is in $L^p((0, \infty), dx)$ whenever f is in L^p. To see this is not so, let $p=2$ and set $f(x)=x^{-1/2}(\log x)^{-3/4}$ (for large x).

Here is another observation which shows how differently the operators in (4.2) and (4.3) can behave. Let T be the operator of multiplication by a bounded function m. By Theorem 3.2 $[T, \Omega]$ is bounded on, say, L^2. We compute

$$[T, \Omega_E]a = cam \log |m|, \tag{4.4}$$

$$[T, \Omega_K]a = cam \log \big(B(1, am)/B(1, a)\big) \tag{4.5}$$

using the Ω's from (4.2) and (4.3) respectively. In fact the result that the operator in (4.4) is bounded is trivial, because $m \log |m|$ is bounded if m is bounded. The fact that (4.5) gives a bounded map on L^2 seems less obvious. We don't know how to give a direct proof of this without dividing X according to the size of a and of ma (thus imitating the proof of Theorem 3.2).

Because the operators Ω_{E_α} and Ω_K behave so differently in this context we include an explicit description of K, Ω_K, and D_K for couples of weighted spaces. By change of notation we may assume one of the weights is 1. First we consider the couple $\bar{A}=\big(L^p(dx), L^\infty_w(dx)\big)$ where $L^\infty_w=\{f; fw\in L^\infty\}$. L^∞_w is normed by $\|fw\|_\infty$. For h defined on X we denote by h^* the non-increasing rearrangement of h regarded as a function on $\big(X, w(x)^{-p}\, dx\big)$. We will show that

$$K(t, f, \bar{A}) \sim \Big(\int_0^{t^p} (wf)^{*p}(s)\, ds\Big)^{1/p} = L(f)(t)$$

and that an almost optimal decomposition is given by

$$f = f_0 + f_1 = f\chi_t + f(1-\chi_t)$$

where χ_t is the characteristic function of the set

$$\{x;\ |w(x)f(x)| > (wf)^*(t^p)\}.$$

First note that $L(f)(t) \leqq \|f\|_p$. This is because $L(f)(t)^{1/p} \leqq L(f)(\infty)^{1/p} = \big(\int |wf|^p w^{-p}\, dx\big)^{1/p} = \|f\|_p$. Next note that $L(f)(t) \leqq t\|f\|_{A_1} \leqq t\|wf\|_\infty$. This is because $L(f)(t)/t$ is decreasing and hence

$$\frac{1}{t} L(f)(t) \leqq \lim_{t\to 0} \frac{1}{t} L(f)(t) = \sup |fw| = \|wf\|_\infty.$$

Finally note that $L(f+g) \leqq c\big(L(f)+L(g)\big)$. Thus, if $f=f_0+f_1$

$$\begin{aligned} L(f) = L(f_0+f_1) &\leqq c(L(f_0)+L(f_1)) \\ &\leqq c(\|f_0\|_{A_0} + t\|f_1\|_{A_1}). \end{aligned}$$

Taking the infimum over all decompositions gives $L(f)(t) \leqq cK(t, f)$. To establish the other inequality we set $f_0=f\chi_t$ so

$$\|f_0\|^p_{A_0}=\|f\chi_t\|^p_{L^p}=\int|f|^p\chi_t dx=\int|fw|^p w^{-p}\chi_t\,dx$$
$$=\int_0^{t^p}(wf)^{*p}(s)\,ds=L(f)(t)^p.$$

Also

$$t\|f_1\|_{A_1}=t\|wf(1-\chi_t)\|_\infty=t(wf)^*(t)$$
$$\leq t\left(\frac{1}{t^p}\int_0^{t^p}(wf)^*(s)\,ds\right)^{1/p}\leq L(f)(t).$$

Combining these estimates shows $K\leq cL$. This also shows that the decomposition f_0+f_1 is almost optimal. Hence

$$(\Omega f)(x)=\int_0^1 f_0(x,t)\,dt/t-\int_1^\infty f_1(x,t)\,dt/t$$
$$=f(x)\left[\int_0^1\chi_t(x)\,dt/t-\int_1^\infty(1-\chi_t(x))\,dt/t\right]$$
$$=cf\log\left[w^{-p}dx\text{ measure of }\{y;\ |wf|(y)>|wf|(x)\}\right]$$
$$=cf\log B(w^{-p},wf).$$

A completely analogous argument works for $(L^{p_0}(dx),L^{p_1}_w(dx))$ where $L^{p_1}_w(dx)=\{f;fw\in L^{p_1}\}$ and we use the norm $\|fw\|_{L^{p_1}}$. (The reason for including the weight with the function rather than with the measure is to obtain a formulation which has the same form for $p_1=\infty$ and $p_1<\infty$.) Let $\varepsilon=p_1/(p_1-p_0)$. χ_t is selected to be the characteristic function of the set

$$\{x;\ |w^\varepsilon f|(x)>(w^\varepsilon f)^*(t^{\varepsilon p_0})\}$$

where $(\ )^*$ is the rearrangement using $w^{\varepsilon p_0}dx$. The resulting Ω is

$$\Omega f=cf\log B(w^{-\varepsilon p_0},wf).$$

We have not yet described the intermediate space $\bar{A}_{\theta,p}$. If $1/p=(1-\theta)/p_0+\theta/p_1$, then $\bar{A}_{\theta,p}$ is the "expected" weighted L^p space $L^p(w^{p\theta}\,dx)$. The off diagonal case is more complicated. (See Chapter 5 of [BL].)

4.4 L^p spaces and maximal functions, the E-method

In this and the next section we use the results of Section 3 to study maximal functions. In this section we use maximal functions to define alternative almost optimal decompositions, use the E-method of interpolation, and then use Corollary 3.4. In the next section we use maximal functions to produce a new couple with an alternative norm, use the K method of interpolation and use Theorem 3.2

to study the identity map between the two couples. These two approaches are only minor variations on each other; the main difference in the results is in the form of Ω obtained by the different interpolation methods.

Our results are fairly complete for maximal functions which satisfy the usual boundedness and sublinearity conditions and also satisfy the pointwise estimate in (4.6) below. Thus our results apply, for example, to the Hardy—Littlewood maximal function, strong maximal function, non-tangential maximal function as well as, of course, the identity operator. We do not have any systematic approach which would include various square functions and sharp functions which fail to satisfy pointwise estimates. However, we do give the computation to show that our results hold for the sharp function.

Let $\bar{A}=(L^{p_0}(w_0\,dx), L^{p_1}(w_1\,dx))$ $1\leqq p_0<p_1\leqq\infty$. Suppose M is an operator which satisfies, for a, a_1, a_2 in A_0+A_1, x in X

$$\begin{cases} M\colon \bar{A}\to\bar{A} \quad \text{boundedly} \\ |a(x)|\leqq |Ma(x)| \\ |M(a_1+a_2)(x)|\leqq |Ma_1(x)|+|Ma_2(x)|. \end{cases} \tag{4.6}$$

Let χ_t be the characteristic function of the set

$$\{x;\ |Ma(x)|(w_1(x)/w_0(x))^{1/(p_0-p_1)}>t\}.$$

We will show that $a=a\chi_t+a(1-\chi_t)$ is an almost optimal decomposition with respect to E_α for $\alpha=p_1/(p_1-p_0)$. We will do the computation for $\alpha>1$, $p_1<\infty$ leaving the case $\alpha=1$, $p_1=\infty$ for the reader. By our assumptions on M

$$\begin{aligned}(E_\alpha(t,a))^{p_0\alpha} &\leqq (\|a\chi_t\|_0/t)^{p_0}+(\|a(1-\chi_t)\|_1/t)^{p_1} \\ &\leqq (\|(Ma)\chi_t\|_0/t)^{p_0}+(\|(Ma)(1-\chi_t)\|_1/t)^{p_1}.\end{aligned}$$

From Section 4.2 we know that $Ma=(Ma)\chi_t+(Ma)(1-\chi_t)$ is an almost optimal decomposition of Ma. Hence the last expression is dominated by $E_\alpha(t/c, Ma)^{p_0\alpha}$. We now use the fact that M is bounded on $\bar{A}$ to conclude that this is dominated by $E_\alpha(t/c, a)^{p_0\alpha}$ which completes the required estimates. The corresponding Ω_{E_α} is given by

$$(\tilde{\Omega}a)(x)=a(x)\log\left(|Ma(x)|(w_1(x)/w_0(x))^{1/(p_1-p_0)}\right). \tag{4.7}$$

We now apply Corollary 3.4 (rather, the version of that corollary for E_α) and conclude that $\tilde{\Omega}-\Omega$ is bounded for this $\tilde{\Omega}$ and the Ω given in Section 4.2. Thus,

$$(\tilde{\Omega}-\Omega)(a)=a\log|Ma/a|$$

is bounded on the intermediate $\bar{A}_{\theta,q}$. Furthermore, if M' is another such operator then we obtain the boundedness of

$$(\tilde{\Omega}'-\tilde{\Omega})(a)=a\log|M'a/Ma|. \tag{4.8}$$

As was the case with the estimate (4.4), this particular result can be obtained much more directly. $x\log|x|$ is bounded for $0\leqq x\leqq 1$ and, by hypothesis, $|a/Ma|\leqq 1$; thus $\|a\log|Ma/a|\|=\|Ma\,(a/Ma)\log|Ma/a|\|\leqq c\,\|Ma\|$. However, this elementary point of view doesn't seem to apply to the sharp function considered below or to the Ω obtained in the next section.

We now specialize to the case of $\bar{A}=(L^{p_0}(\mathbf{R}^n), L^{p_1}(\mathbf{R}^n))$ for some $1<p_0<p_1<\infty$ and consider the sharp function $M^{\#}f$ defined by

$$M^{\#}f(x)=\sup_{x\in Q}\inf_{c\in\mathbf{R}}\frac{1}{|Q|}\int_Q|f(x)-c|\,dx$$

where Q are cubes in $\mathbf{R}^n$ with sides parallel to the axes. Let $\alpha=p_1/(p_1-p_0)$ and let χ_t be the characteristic function of the set $\{x;\,M^{\#}f(x)>t\}$. We wish to show that $f=f\chi_t+f(1-\chi_t)$ is an almost optimal decomposition with respect to $E_\alpha(t,\cdot\,;\bar{A})$. For convenience we only write down the case $t=1$. Trivially

$$E_\alpha(1,f,\bar{A})^{p_0\alpha}\leqq\int|f\chi_1|^{p_0}dx+\int|f(1-\chi_1)|^{p_1}dx$$

$M^{\#}f(x)\leqq\tilde{M}f(x)$ where $\tilde{M}$ is the Hardy—Littlewood maximal function. Thus by what we just saw about $\tilde{M}$,

$$\int_{\{M^{\#}f>1\}}|f|^{p_0}\leqq\int_{\{\tilde{M}f>1\}}\leqq E_\alpha(1/c,f;\,\bar{A})^{p_0\alpha}.$$

To finish we need to estimate $\int|f(1-\chi_1)|^{p_1}$. To do this we use (a minor modification) of the Bennett and Sharpley result [BS]. They show that, denoting by BMO the space of functions of bounded mean oscillation, $(L^{p_0},\mathrm{BMO})_{\theta,p_0}=L^{p_\theta}$ with $1/p_\theta=(1-\theta)/p_0$. Furthermore, an almost optimal decomposition with respect to $E_1(1,\cdot\,;(L^{p_0},\mathrm{BMO}))$ is given by $f=f_0+f_1$

$$f_0=f\chi_1-\Sigma\left(\frac{1}{|Q_i|}\int_{Q_i}f\right)\chi_{Q_i},$$

$$f_1=f(1-\chi_1)+\Sigma\left(\frac{1}{|Q_i|}\int_{Q_i}f\right)\chi_{Q_i}$$

where $\{Q_1\}$ are the cubes in the Whitney decomposition of $\{x;\,M^{\#}f(x)>1\}$. (See [BS] for more details and definitions. The work there is modulo constants which doesn't affect this argument.) By (2.9) we conclude that this same decomposition is almost optimal for $E_\alpha(1/c,f,\bar{A})$ thus

$$\int_{\{M^{\#}f\leqq 1\}}|f|^{p_1}\leqq\int_{\{M^{\#}f\leqq 1\}}|f_1|^{p_1}\leqq E_\alpha(1/c,f;\,\bar{A})^{\varepsilon p_0}.$$

Thus, in particular, the operator (4.8) is bounded on $L^p(\mathbf{R}^n)$ $1<p<\infty$ if M, M' are drawn from the set {identity, sharp function, any maximal function which satisfies (4.6)}. This gives half of Theorem 3 of the introduction. The general conclusion that maximal functions must be comparable to each other where either is large has a suggestive (and not well understood) resemblance to results in [FGSS] comparing the nontangential maximal function and the Lusin area function.

To get Theorem 4 we need to know that if M is the Hardy—Littlewood maximal function and H is the Hilbert transform then the map of f to $H(f\log|Mg|)-(\log|Mg|)Hf$ is a bounded map of $L^p(\mathbf{R})$ to itself $1<p<\infty$. (This for g in $\bigcup_{1<p<\infty} L^p$). This is because $\log|Mg|$ is in BMO. (See, for instance, Proposition (3.43) of [RW]; also see the argument in Section (4.1).) If we set $g=f$ we get the boundedness on $L^p(\mathbf{R})$ of

$$H(f\log|Mf|)-(\log|Mf|)Hf.$$

When we use formula (4.7) (with $w_1=w_0\equiv 1$), the fact that H is bounded on L^p, $1<p<\infty$, and Theorem 3.2, we obtain the boundedness of

$$H(f\log|Mf|)-(\log|MHf|)Hf.$$

Theorem 3 of the introduction, applied to Hf, gives the boundedness of

$$(Hf)(\log|Hf|)-(\log|MHf|)Hf.$$

Arithmetic combination of these three expressions gives half of the Theorem 4 involving $(Hf)\log|Mf/Hf|$. The other half follows on writing $f=Hg$ and using $H^2=-I$.

We could obtain similar results with Ω given by (4.3) if we knew $\log B(1,f)$ is in BMO. For instance, if f is positive, symmetric, and for $x>0$ is decreasing then $\log B(1,f)=\log|2x|$ which is a typical BMO function. However, easy examples show that some conditions are needed.

Question. What conditions on f insure $\log B(1,f)$ is in BMO?

4.5 L^p spaces and maximal functions, the K-method

Suppose $1\leqq p_0<p_1\leqq\infty$ and $\bar{A}=(L^{p_0}, L^{p_1})$. Suppose M is an operator which satisfies (4.6). For p with $p_0\leqq p\leqq p_1$ define the "maximal L^p" space L^p_M to be those a in A_0+A_1 for which $\|a\|_{L^p_M}=\|Ma\|_{L^p}<\infty$. Let $\bar{B}$ be the couple $(L^{p_0}_M, L^{p_1}_M)$. We wish to compute Ω_K for the couple $\bar{B}$. Pick a in $L^{p_0}_M+L^{p_1}_M$ and let χ_t be the characteristic function of the set $\{x;\ |(Ma)(x)|>(Ma)^*(t^\alpha)\}$. (Again $\alpha=p_0p_1/(p_1-p_0)$.) We wish to show that $a=a\chi_t+a(1-\chi_t)$ is an almost optimal decomposition:

$$\max\left(\|a\chi_t\|_{L^{p_0}_M},\ t\|a(1-\chi_t)\|_{L^{p_1}_M}\right)=\max\left(\|M(a\chi_t)\|_{L^{p_0}},\ t\|M(a(1-\chi_t)\|_{L^{p_1}}\right).$$

Since M is bounded on $\bar{A}$ we can estimate this by

$$\max \leqq c \max \left(\|a\chi_t\|_{L^{p_0}},\ t\|a(1-\chi_t)\|_{L^{p_1}} \right).$$

We now increase all the functions involved and continue the estimate to

$$\max \leqq c \max \left(\|(Ma)\chi_t\|_{L^{p_0}};\ t\|(Ma)(1-\chi_t)\|_{L^{p_1}} \right).$$

Now use what we know from Section 4.3 about an explicit almost optimal decomposition to estimate this by

$$\max \leqq cK(t, Ma, \bar{A}).$$

Since M is bounded on $\bar{A}$ we next get

$$\max \leqq cK(t, a, \bar{A})$$

and finally, by (4.6) again, the norms of A_i are comparable to those of B_i $i=0, 1$. We conclude that

$$\max \leqq cK(t, a, \bar{B})$$

as required. Using this decomposition we compute

$$\Omega_{K,B}(a)(x) = ca(x) \log B(1, Ma)(x)$$

with the same c as in Section 4.3. We now apply Theorem 3.2 to the identity operator mapping $\bar{A}$ to $\bar{B}$. Comparing the previous formula with (4.3) we conclude that the map of a to

$$(\Omega_{\bar{A}} - \Omega_B)(a) = ca \log \left(B(1, Ma)/B(1, a) \right)$$

is a bounded operator from $\bar{A}_{\theta q}$ to $\bar{B}_{\theta q}$. Next note that the conditions on M insure that $\bar{A}=\bar{B}$ up to equivalent norms and, hence, $\bar{A}_{\theta q}=\bar{B}_{\theta q}$. Finally note that, as in the previous section, these arguments extend to pairs of maximal operators M, M'. In summary, the map

$$a \to a \log \left(B(1, Ma)/B(1, M'a) \right)$$

is a bounded map of L^p to itself $1<p<\infty$ when M, M' are drawn from {identity, strong maximal function, Hardy—Littlewood maximal function, ... }.

4.6 The Schatten ideals

For a compact operator S on a Hilbert space H we write $|S|$ for $(S^*S)^{1/2}$. $|S|$ is a positive compact operator and hence $|S|=\sum_1^\infty \sigma_n \langle \cdot\, ; \varphi_n \rangle \varphi_n$ for an orthonormal set $\{\varphi_n\}$ in H and scalars σ_n which can be assumed positive and decreasing to zero. Then $S=V|S|$ for some partial isometry V and

(4.9) $$S = V(\textstyle\sum_1^\infty \sigma_n\langle\cdot\,;\ \varphi_n\rangle\varphi_n).$$

For $1\leqq p\leqq\infty$, S is said to be in the Schatten ideal $\mathcal{S}^p$ if $\{\sigma_n\}=\{\sigma_n(S)\}\in\ell^p$. $\mathcal{S}^p$ is a Banach space with norm $\|\{\sigma_n(\cdot)\}\|_{\ell^p}$.

We now consider the couple $\bar{A}=(\mathcal{S}^{p_0},\mathcal{S}^{p_1})$, $1\leqq p_0<p_1<\infty$. It is a consequence of the general theory of $\mathcal{S}^p$ spaces that optimal decompositions for S in $\mathcal{S}^{p_0}+\mathcal{S}^{p_1}$ will be obtained by forming the optimal decomposition of the sequence $\{\sigma_n(S)\}$ with respect to the couple (ℓ^{p_0},ℓ^{p_1}). (This is a consequence of the description of the $\sigma_n(s)$ as approximation numbers.) Thus, following the pattern of Section 4.2, an almost optimal splitting for the E_α method, $\alpha=p_1/(p_1-p_0)$, is given by

$$S = V(\textstyle\sum_{\sigma_n>t})+V(\sum_{\sigma_n\leqq t})$$

and an almost optimal splitting for the K method is

$$S = V(\textstyle\sum_{n<t^\beta})+V(\sum_{n\geqq t^\beta}),\quad \beta=p_0p_1/(p_1-p_0).$$

This leads to

(4.10) $$\Omega_{E_\alpha}S = V(\Sigma\sigma_n\log\sigma_n\langle\cdot\,,\varphi_n\rangle\varphi_n)$$

and

(4.11) $$-\frac{1}{\beta}\Omega_K S = V(\Sigma\sigma_n\log n\langle\cdot\,,\varphi_n\rangle\varphi_n).$$

(4.11) is an analog of (4.3) and we do not understand it very well. (4.10) is an analog of (4.2) and we can analyze it further. $|S|$ is a positive operator and as such admits a nice functional calculus. In particular we can define $\log|S|=\sum\log\sigma_n\langle\cdot\,,\varphi_n\rangle\varphi_n$. Thus the right side of (4.10) is $V|S|\log|S|$. Recall $V|S|=S$. Thus, in stronger analogy with (4.2),

(4.12) $$\Omega S = \Omega_{E_\alpha}S = S\log|S|.$$

Suppose R is a bounded map of H to itself. $T(S)=RS$ defines a bounded operator on $\bar{A}$ and hence, by Theorem 3.2 $[T,\Omega]$ is bounded in $\bar{A}_{\theta,q}$ $0<\theta<1$, $1\leqq q\leqq\infty$.

$$[T,\Omega]S = RS(\log|S|-\log|RS|).$$

In contrast to (4.4), we cannot simplify further. (It need not hold that $\log|RS|=\log|R|+\log|S|$.) Thus the boundedness of this operator is more delicate than was the case in (4.4).

4.7 Domains of definition of positive operators

Suppose A is a Banach space and Λ is a unbounded closed linear map of A to itself which has a dense domain $D(\Lambda)$. We give $D(\Lambda)$ the graph norm, $\|a\|_{D(\Lambda)}=\|a\|_A+\|\Lambda a\|_A$. We will consider the couple $\bar{A}=(A, D(\Lambda))$.

Such a theory is simplest if A is a Hilbert space and Λ is a positive operator on A. Spectral theory can then be used to reduce to the case of L^2 with variable weights. A simple but typical and important example of this was given in Section 4.1. There $A_0=H^2$ was the domain of Δ acting on $L^2(\mathbf{R}^n)$ and we used the Fourier transform (rather than abstract spectral theory) to reduce the couple $(D(\Lambda), D(\Lambda^{-1}))$ to a couple of L^2 spaces with weights. We computed that $\Omega=C\log(I+\Delta)$.

Similar results can be obtained if we suppose merely that Λ is a positive operator in the following abstract sense. We suppose that there is a C so that for all $t\geqq 0$, $\|(\Lambda+tI)^{-1}\|\leqq C/(1+t)$. In this case it can be shown that

$$a=\left(\frac{t\Lambda}{t\Lambda+I}\right)a+\left(\frac{1}{t\Delta+I}\right)a$$

is an almost optimal decomposition with respect to $K(t, a; \bar{A})$. (See Section 1.14 of [T].) Thus, formally,

$$\Omega=\int_0^1\frac{\Lambda}{t\Lambda+I}\,dt-\int_1^\infty\frac{1}{t\Lambda+I}\,\frac{dt}{t}$$

$$=\int_0^1 d(\log(t\Lambda+I))-\int^\infty d\left(\log\frac{t}{t\Lambda+I}\right)=c\log(I+\Lambda).$$

To go further with this analysis we would need a general theory of powers and logarithms of positive operators. Parts of such a theory are given in Section 1.13—1.15 [T] and the references given there. See also [H] and [P] and the references there.

4.8 Retracts and Besov spaces

In this section we consider homogeneous Besov spaces. For definitions and background on these spaces we refer to [BL], [T], [P] and [FJ].

Let $\bar{A}=(\dot{B}^{s_0 p_0}_{p_0}, \dot{B}^{s_1 p_1}_{p_1})$, $1\leqq p_0\leqq p_1\leqq\infty$, $-\infty<s_0, s_1<\infty$ be a couple of homogeneous Besov spaces based on $\mathbf{R}^n$. We could base our analysis of Ω for this couple on the ideas of the previous section. That's because the Besov spaces can be obtained as interpolation spaces from such couples. Fourier transform considerations would let us reduce to considerations of spaces with weights of the sort $|x|^\alpha$ and we would obtain formulas for Ω involving the logarithm of the Laplace operator. (We obtain $|x|^\alpha$ rather than $(1+|x|^2)^{\alpha/2}$ because we are working with the homogeneous Besov spaces.) Here we take another path. We use the theory of molecular decompositions for these spaces given by Frazier and Jawerth in [FJ] to reduce the problem to one for weighted Lebesgue sequence spaces and then use the results about ℓ^p spaces.

Given two couples $\bar{A}$ and $\bar{B}$, we say that $\bar{A}$ is a *retract* of $\bar{B}$ if there are linear maps In (think of it as an injection), $In\colon \bar{A}\to\bar{B}$, and Pr (think of it as a projection), $Pr\colon \bar{B}\to\bar{A}$, so that $Pr\, In=$Identity on $\bar{A}$.

We now summarize what we need from [FJ]. Suppose φ is a test function and $\hat{\varphi}$ is supported on a ring about the origin. Precisely, φ is in the Schwartz class, $\operatorname{supp}\hat{\varphi}\subseteq\{x;\ 1/2<|\xi|<2\}$, $|\hat{\varphi}(\xi)|\geqq C>0$ on $\{\xi;\ 3/5<|\xi|<5/3\}$, and for any $\xi\neq 0$ $\sum_{k=-\infty}^{\infty}|\hat{\varphi}(2^k\xi)|^2=1$. For v in $\mathbf{Z}$, k in $\mathbf{Z}^n$ set $\varphi_{vk}(x)=\varphi(2^v(x-k))$. In [FJ] it is shown that $\bar{A}$ is a retract of $\bar{B}=(\ell^{p_0}_{s_0}, \ell^{p_1}_{s_1})$ where ℓ^p_s is the weighted Lebesgue sequence space $\ell^p_s=\left\{\{\lambda_{vk}\}_{v\in\mathbf{Z},k\in\mathbf{Z}^n};\ \sum 2^{v(sp-n)}|\lambda_{vk}|^p<\infty\right\}$. In fact we may select $In(a)=\{\langle a, \varphi_{vk}\rangle\}$ (using the ordinary inner product on $L^2(\mathbf{R}^n)$) and $Pr(\{\lambda_{vk}\})=\sum\lambda_{vk}\varphi_{vk}$.

Clearly if $D(t,\bar{B})$ are almost optimal decomposition operators for $\bar{B}$ then $PrD(t,\bar{B})In$ are almost optimal decomposition operators for $\bar{A}$. Thus $\Omega_{\bar{A}}=Pr\Omega_{\bar{B}}In$. We can now use the results of Sections 4.1, 4.2 and 4.3 to read off various forms for Ω.

If $p_0=p_1=p$ then, by Section 4.1,

$$\Omega_B(\{\lambda_{vk}\}) = \{cv\lambda_{vk}\}$$

with c the constant $1/p\log 2$. Hence

$$a = \Sigma\langle a, \varphi_{vk}\rangle\varphi_{vk}$$

and

(4.13) $$\Omega_{\bar{A}}a = c\Sigma v\langle a, \varphi_{vk}\rangle\varphi_{vk}.$$

If $p_0<p_1$ the results of Section 4.2 give

$$\Omega_{\bar{A}}a = C\Sigma v\langle a, \varphi_{vk}\rangle\log|\langle a, \varphi_{vk}\rangle|\varphi_{vk}$$

with $C=(s_1p_1-s_0p_0)/(p_1-p_0)$.

Note that derivatives of φ_{vk} involve factors of 2^v. Hence the factors of v in these two formulas are closely related to the logarithms of (Identity+Laplacian) which arose in Section 4.1 when considering the couple formed by inhomogeneous potential spaces. Had we looked at homogeneous potential spaces there (or inhomogeneous spaces here), we would have seen that $\Omega_{\bar{A}}$, the operator in (4.13), differs from $C\log\Delta$ by a bounded operator.

5. Ω_J and comparison with the complex method

In [RW] results similar to those of Sections 3 and 4 were obtained using the complex method of interpolation. Both sets of results are based on estimates of the commutator between a linear operator and the fundamental decomposition process of the interpolation scheme. Here we describe two analogies between real and complex interpolation which suggest deeper similarities between the two sets of results. Our goal here is to display similarities in the formalisms — we will ignore most technical details.

We begin by recalling the formalism of complex interpolation. Let D be the open unit disk in $\mathbf{C}$ and T the unit circle, $T=\partial\bar{D}$. We are given boundary spaces $\{A_{e^{i\theta}}\}_{e^{i\theta}\in T}$. The complex interpolation spaces $\{A_z\}_{z\in D}$ are defined by the norms

(5.1) $$\|a\|_z=\inf\{\sup_\theta \|F(e^{i\theta})\|_{e^{i\theta}};\ F \text{ an analytic vector valued function on } D \text{ which satisfies } F(z)=a\}.$$

Associated with this construction are functions which solve the extremal problem. That is, we denote by $A(\cdot, z)(a)$ the function F for which the infimum in (5.1) is attained. (The existence of A is one of the issues we ignore here.) Among the basic properties of A are

(5.2) $$A(z_0, z_0)(a) = a \quad z_0\in D,$$

(5.3) $$A(z_2, z_1)A(z_1, z_0)(a) = A(z_2, z_0)(a) \quad z_0, z_1, z_2\in D,$$

(5.4) $$\|A(z, z_0)a\|_z = \|a\|_{z_0} \quad z_0, z\in D.$$

The results of [RW] involve the commutator with δ_z given by

(5.5) $$\delta_z(a) = (1-|z|^2)\frac{\partial}{\partial\zeta}(A(\zeta, z)(a))_{\zeta=z}.$$

(The factor $(1-|z|^2)$ is a normalization.)

One analogy between real and complex interpolation is based on the analogy between affine and holomorphic convexity. This analogy is developed in detail in Section 5 of [R]. There the definition of intermediate norms given by (5.1) is replaced by one involving affine vector valued functions defined on the unit ball of $\mathbf{R}$ (i.e. $\{s;\ -1<s<1\}$). The role of $T=\partial\bar{D}$ is taken by the boundary of the unit ball of $\mathbf{R}$. (Note that this boundary consists of two points — corresponding to the two spaces of a given couple.) When these ideas are carried out in detail in [R] and in Section 6 of [RW2], the resulting intermediate norms are a (minor) variation on the idea of norming A_0+A_1 by $K(t, \cdot, \bar{A})$. When the analogs of the constructions of Section 2 of [RW] are carried out in this context of affine functions defined on the unit ball of $\mathbf{R}$ the result is a version of the local commutator estimate of Proposition 3.1: $J(t, [T, D(t)]a; \bar{A})\leq CK(t, a; \bar{A})$.

We now give a different analogy. For $0<\varphi<1$ and $v(t)$ a function taking values in $A_0\cap A_1$, let $G(v)(\varphi)=\int_0^\infty t^\varphi v(t)\,dt/t$. Setting $u=t^\varphi v$ in (2.3) we rewrite the definition of the norm on $\bar{A}_{\theta,q,J}$ as

$$\|a\|_{\theta,q,J} = \inf\left\{\left(\int_0^\infty J(t, v(t), \bar{A})^q\,dt/t\right)^{1/q};\ a = G(v)(\theta)\right\}.$$

Using the fact that $J(t,a)=\max(\|a\|_0, t\|a\|_1)\sim\|a\|_0+t\|a\|_1$ we get

$$\|a\|_{\theta,q} \cong \inf\left\{\max_{i=0,1}\left(\int_0^\infty \|t^i v(t)\|_i^q\,dt/t\right)^{1/q};\ a = G(v)(\theta)\right\}. \tag{5.6}$$

This is analogous to (5.1). The norm is defined as the inf over a function class of a sup of boundary norms. The function class is defined by requiring that a weighted mean value give the element of interest. (The requirement $F(z)=a$ in (5.1) is a statement about a Poisson integral of the boundary values of F.) We now construct the analog of A. Let v be a function so that $G(v)(\theta)=a$ and the inf in (5.6) is almost attained, i.e. for this v

$$\max\left\{\left(\int_0^\infty \|t^i v(t)\|_i^q\,dt/t\right)^{1/q}\right\} \leqq C\|a\|_{\theta,q}. \tag{5.7}$$

We use this (choice of) v to define B by $B(\varphi,\theta)(a)=G(v)(\varphi)$. In analogy with (5.2) and (5.4)

$$B(\theta,\theta)(a) = a, \tag{5.8}$$

$$\|B(\varphi,\theta)(a)\|_{\bar{A}_{\varphi,q,J}} \leqq C\|a\|_{\theta,q},\quad 0<\varphi<1. \tag{5.9}$$

(5.8) follows from the definition. (5.9) is an immediate consequence of the fact that that θ does not appear on the left side of (5.7) and that v is a competing function in computing the left hand side of (5.9). Note that we do not have an analog of the propagator equation (5.3). Nor do we have a two sided estimate in (5.9). (However, Zafran [Z] studied maps similar to $B(\varphi,\theta)$ (also while considering an analogy between real and complex interpolation). His Lemma 9.4 is enough to obtain reverse estimates in (5.9) with constants which depend on θ and φ (but not $\bar{A}$ or a).) $B(\varphi,\theta)$ is a relatively natural map from $\bar{A}_{\theta,q}$ to $\bar{A}_{\varphi,q}$. In analogy with (5.5) we set

$$\Omega_{J,\theta,q}a = \frac{\partial}{\partial\varphi}B(\varphi,\theta)(a)|_{\varphi=\theta}. \tag{5.10}$$

Since $B(\varphi,\theta)(a)=\int_0^\infty t^\varphi v(t)\,dt/t$ we have

$$\Omega a = \int_0^\infty t^\theta v(t)\log t\,dt/t = \int_0^\infty u(t)\log t\,dt/t \tag{5.1)}$$

with $u=t^\theta v$ the almost optimal J decomposition of a. Again we see the ubiquitous logarithm. The analog of the commutator results of [RW] is

Theorem 5.1. *For* $0<\theta<1$, $1\leqq q\leqq\infty$; *if T is bounded on $\bar{A}$ then* $[T,\Omega_{J,\theta,q}]$ *is bounded on* $\bar{A}_{\theta,q}$.

Before proving this we give some heuristics relating this result to those of Section 3. Suppose we start with a choice of almost optimal projection $D_K(t;\bar{A})(a)=D(t)(a)$ which is smooth as a function of t. It is then true that $u(t)=t\,\partial/\partial t\, D(t)a$ is an almost optimal choice for $u(t)$ in (2.3). (See the proof of the fundamental lemma in [BL].) With this u we obtain as v

$$v(t) = t^{1-\theta}\frac{\partial}{\partial t}D(t)a$$

and thus

$$\Omega_{J,\theta,q}a = \int_0^\infty \frac{\partial}{\partial t}D(t)a \log t\,dt.$$

In particular, for this choice $\Omega_{J,\theta,q}$ doesn't depend on θ or q. This Ω is formally related to Ω_K by integration by parts

$$\Omega_J = -\int_0^\infty D(t)\,dt/t = -\Omega_K.$$

Although this formal integration by parts is wrong (due to boundary terms that don't drop out) it is correct at the commutator level.

It is straightforward to give a direct proof of Theorem 5.1. We first prove

Proposition 5.2. *Suppose that there is an h in $\bar{A}_{\theta,q}$ and that there are $h_i(t)$, $i=1$, 2, which satisfy*

$$\int_0^\infty h_i(t)\,dt/t = h;\ \Phi(J(t,h_i(t)) = \left(\int_0^\infty (t^{-\theta}J(t,h_i(t)))^q dt/t\right)^{1/q} \leq C\|h\|_{\theta,q},$$

then

(5.12) $$H = \int_0^\infty (h_1(t)-h_2(t))\log t\,dt/t \in \bar{A}_{\theta,q}$$

and $\|H\|_{\theta,q}\leq C\|h\|_{\theta,q}$. Here H is to be interpreted via integration by parth;

(5.12)′ $$H = \int_0^\infty H(t)\,dt/t;\ H(t) = \int_0^t (h_1(s)-h_2(s)\,ds/s.$$

Theorem 5.1 follows from this with the choices $h_1(t)=t^{-\theta}T(v(a)(t))$, $h_2(t)=t^{-\theta}v(T(a))(t)$. The proposition requires the following version of Hardy's inequality:

Suppose $k(t)$ is a positive function for $t>0$ then

(5.13) $$\Phi\left(\int_0^t k(s)\frac{ds}{s}\right) \leq C\Phi(k(t)).$$

(5.13)′ $$\Phi\left(\int_t^\infty \frac{t}{s}k(s)\frac{ds}{s}\right) \leq C\Phi(k(t)).$$

Proof of Proposition 5.2.

$$\begin{aligned}\|H\| &\leqq \Phi(J(t, H(t))\\ &\leqq \Phi(\|H(t)\|_0+t\|H(t)\|_1)\\ &\leqq \Phi(\|H(t)\|_0)+\Phi(t\|H(t)\|_1).\end{aligned}$$

We estimate the two terms separately

$$\begin{aligned}\Phi(\|H(t)\|_0) &\leqq \Phi\left(\int_0^t (\|h_1(s)\|_0+\|h_2(s)\|_0)\, ds/s\right)\\ &\leqq \Phi\left(\int_0^t (J(h_1(s))+J(h_2(s)))\, ds/s\right)\\ &\leqq C\Phi\big(J(h_1(t))+J(h_2(t))\big)\\ &\leqq C\Phi\big(J(h_1(t))+C\Phi(J(h_2(t)))\big)\\ &\leqq C\|h\|_{\theta,q}.\end{aligned}$$

Here the inequality which introduced the C is (5.13). Similarly using (5.13)′ and writing $H(t)=-\int_t^\infty (h_1(s)-h_2(s))\, ds/s$

$$\begin{aligned}\Phi(t\|H(t)\|_1) &\leqq \Phi\left(t\int_t^\infty (\|h_1(s)\|_0+\|h_2(s)\|_1)\, ds/s\right)\\ &\leqq \Phi\left(\int_t^\infty \frac{t}{s}(J(h_1)+J(h_2))\frac{ds}{s}\right)\\ &\leqq C\Phi(J(h_1))+C\Phi(J(h_2))\\ &\leqq C\|h\|_{\theta,q}\end{aligned}$$

and we are done.

There are many questions related to this construction of $\Omega_{J,\theta,q}$. For example, how does $\Omega_{J,\theta,q}$ depend on θ and q in general? Also, which (if any) interesting operators are obtained by putting positive weights $\omega(t)$ in (5.12)′: $H_\omega=\int_0^t H(t)\omega(t)\, dt/t$. (The proof of Proposition 5.2 is unchanged with any bounded $\omega(t)$.) The functional calculus developed this way may be related to the general theory of unbounded, but slowly growing, functions of positive operators as in [P].

Here is an explicit computation of B in one case. Suppose $\bar{A}$ is the couple $(A, D(A))$ considered in Section 4.7; that is, $D(A)$ is the domain of the operator A. We have $B(\varphi,\theta)(a)=\int_0^\infty t^\varphi v(t)\, dt/t$. If we take $v=t^{-\theta}u=t^{-\theta}t\,(d/dt)\,K$ and use the formula for K in Section 4.7 we obtain

$$B(\varphi,\theta)=C\int_0^\infty t^{\varphi-\theta}\frac{\Lambda}{(t\Lambda+I)^2}\, dt.$$

By the functional calculus for positive operators (see page 8 of [T]) this gives

$$B(\varphi, \theta)(a) = C_{\varphi,\theta}\Lambda^{\varphi-\theta}a.$$

Hence, in this case at least, there is much more structure than is indicated by (5.8) and (5.9).

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