The Weyl calculus with locally temperate metrics and weights

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1. Introduction

The Weyl calculus of operators, defined by

$$a^w(x, D)u(x) = (2\pi)^{-n} \iint a(1/2(x+y), \xi) \exp(i\langle x-y, \xi\rangle) u(y)\, dy\, d\xi \tag{1.1}$$

was developed with general classes of symbols by Hörmander [7], generalizing the calculus of Beals and Fefferman [1], [2], [3]. Both the Weyl calculus and the Beals—Fefferman calculus require that the symbols are temperate, so they cannot grow faster than a polynomial at infinity. Thus one can't use the calculus to study, for example, the operator $-\Delta+\exp(|x|^2)$ on $\mathbf{R}^n$, where Δ is the Laplacean. In [5], Feigin introduces symbol classes corresponding to the weight $f(x)^2+|\xi|^2$, where $0<c<f(x)$ satisfies

$$|\mathrm{grad}\, f(x)| \leqq Cf(x)^{1+\delta}, \quad \delta < 1.$$

The symbols may therefore grow exponentially in the x variables. The corresponding operators are required to be properly supported, so that the Schwartz kernels are supported where

$$|x-y| \leqq C(f(x)+f(y))^{-\gamma}, \quad \delta < \gamma.$$

This condition makes it possible to get a calculus for the operators.

In this paper, we generalize the results of the Weyl calculus to locally temperate symbols, which are temperate in the ξ variables only. In order to do that we introduce a metric in the x variables, to define neighborhoods over which the symbols are temperate. We use cut-off functions χ supported in the corresponding neighborhood of the

* Research supported in part by the U. S. National Science Foundation under grant MCS-8202241, while the author was at the Massachusetts Institute of Technology.

diagonal, to define the operators

$$(1.2)\qquad a_\chi^w(x, D)u(x) = (2\pi)^{-n}\iint a\big(1/2(x+y), \xi\big)\chi(x, y)\exp\big(i\langle x-y, \xi\rangle\big)u(y)\,dy\,d\xi,$$

where $a(x, \xi)$ is locally temperate.

In section 2 we show that a_χ^w is independent of the choice of χ modulo lower order terms, if $\chi=1$ in a neighborhood of the diagonal. In section 3 we develop the Weyl calculus for the operators a_χ^w, under certain restrictions on the support of χ. C^∞ and $\mathscr{D}'$ continuity for these operators are proved in section 4, where we also show that a_χ^w is continuous on L^2 when a is bounded, compact on L^2 when $a\to 0$ at ∞. In section 5 we derive conditions for the operators to be Hilbert—Schmidt or of trace class and prove an estimate of the trace class norm. These results are used in section 6 to improve and generalize Feigin's estimate [4] for the error term in the asymptotic formula for the number $N(\lambda)$ of eigenvalues $\leqq\lambda$ of certain pseudodifferential operators p_χ^w in $\mathbf{R}^n$,

$$N(\lambda) \cong (2\pi)^{-n}\iint_{p(x,\xi)\leqq\lambda} dx\,d\xi,$$

in the same way Hörmander [8] improved and generalized the estimate of Tulovskii and Šubin [9]. In fact, the proof in [8] goes through with minor changes for the locally temperate case. For some temperated symbol classes, sharper estimates for the error term are known — see [6] and references there.

2. Locally σ temperate metrics

Let V be an n dimensional vector space with a slowly varying Riemannean metric G. (See Definition 2.1 in [7].)

Let g be a slowly varying Riemannean metric on $W=V\oplus V'$, where V' is the dual of V. W is a symplectic vector space with the standard symplectic form

$$\sigma(x, \xi;\ y, \eta) = \langle\xi, y\rangle-\langle x, \eta\rangle;\ (x, \xi),\ (y, \eta)\in W.$$

The dual metric of g with respect to σ is defined by

$$(2.1)\qquad g_w^\sigma(x, \xi) = \sup_{(y,\eta)}\frac{|\sigma(x, \xi;\ y, \eta)|^2}{g_w(y, \eta)},\quad w\in W.$$

The metric g is σ temperate if there exist constants C, N such that

$$g_{x,\xi} \leqq Cg_{y,\eta}\big(1+g_{x,\xi}^\sigma(x-y, \xi-\eta)\big)^N.$$

We shall now localize this definition by using the metric G, which is assumed to be fixed in what follows.

Definition 2.1. *We say that g is locally σ temperate if g is slowly varying,*

$$G_x(t) \leqq g_{x,\xi}(t,\tau) \quad \forall (x,\xi), \quad (t,\tau)\in W, \tag{2.2}$$

and there exist positive constants c, C and N such that

$$g_{x,\xi} \leqq C g_{y,\eta}\big(1+g^\sigma_{x,\xi}(x-y,\xi-\eta)\big)^N \tag{2.3}$$

when $G_x(x-y)\leqq c$. *We say that the positive function m on W is locally σ, g temperate if it is g continuous and there exist positive constants c, C and N such that*

$$m(x,\xi) \leqq C m(y,\eta)\big(1+g^\sigma_{x,\xi}(x-y,\xi-\eta)\big)^N \tag{2.4}$$

when $G_x(x-y)\leqq c$.

Condition (2.2) means that the g neighborhoods in W are refinements of the liftings of the G neighborhoods in V. Observe that, by the slow variation of G, one can make (2.3) and (2.4) hold when $\min\big(G_x(x-y), G_y(x-y)\big)<c$, with a smaller c.

Let g be a slowly varying metric on W and m be a g continuous function. We are going to use the symbol classes $S(m,g)$ of [7]. In order to have a calculus of pseudo-differential operators with symbols in $S(m,g)$, where m and g are locally σ temperate, it seems necessary to make the operators properly supported. For that purpose we shall need cut-off functions supported in a neighborhood of the diagonal in $V\oplus V$. The neighborhoods are to be defined by the metric

$$\tilde G_{x,y}(t,s) = G_x(t)+G_y(s) \quad (x,y),(t,s)\in V\oplus V, \tag{2.5}$$

on $V\oplus V$, which is obviously slowly varying. The following lemma shows that g (or m) satisfies the estimate (2.3) (or (2.4)) in a $\tilde G$ neighborhood of the diagonal.

Lemma 2.2 *Let G be slowly varying, and let*

$$D(x,y) = \inf_{x_0} \tilde G_{x_0,x_0}(x-x_0, y-x_0) \tag{2.6}$$

be the squared $\tilde G$ *distance of* (x,y) *to the diagonal, where* $\tilde G$ *is defined by* (2.5). *Then there exist constants* $C,\varepsilon>0$ *such that*

$$\min\big(G_x(x-y), D(x,y)\big) \leqq \varepsilon \Rightarrow C^{-1} \leqq G_x(x-y)/D(x,y) \leqq C. \tag{2.7}$$

Proof. By the slow variation of G we find that

$$G_{x_0}(x-x_0) \leqq \tilde G_{x_0,x_0}(x-x_0, y-x_0) \leqq \varepsilon$$

implies

$$G_x(x-y) \leqq 2\big(G_x(x-x_0)+G_x(x_0-y)\big) \leqq 2C\varepsilon$$

if ε is small enough. Conversely, if $G_x(x-y)\leqq\varepsilon$ is small enough, then

$$G_{\frac{x+y}{2}}\big(\tfrac{1}{2}(x-y)\big) \leqq C\varepsilon/4,$$

which gives $D(x,y)\leqq C\varepsilon/2$. This gives (2.7) with a smaller ε and proves the lemma.

To constrain the supports of the operators, we shall use cut-off functions in $S(1, \tilde{G})$ supported near the diagonal. By using partial sums of partitions of unity in $V\oplus V$ with respect to $\tilde{G}$, for sufficiently small and positive ε, one can construct $\chi\in S(1, \tilde{G})$ with support where $D(x,y)<\varepsilon$ so that $\chi=1$ where $D(x,y)<\dfrac{\varepsilon}{2}$. (See Lemma 2.5 in [7].) In what follows, we shall denote by $\tilde{G}$ neighborhoods of the diagonal the sets $\{(x,y)\in V\oplus V;\ D(x,y)<c\}$. If χ has support in a sufficiently small $\tilde{G}$ neighborhood of the diagonal, then Lemma 2.2 shows that χ is properly supported.

Let $a(x,\xi)\in\mathscr{S}'(W)$ and $\chi\in S(1,\tilde{G})$ be properly supported. We define the operator a^w_χ by

$$(2.8)\qquad a^w_\chi u(x) = (2\pi)^{-n}\iint a\left(\tfrac{1}{2}(x+y),\xi\right)\chi(x,y)\exp(i\langle x-y,\xi\rangle)u(y)\,dy\,d\xi,$$

$u\in C^\infty(V)$, which maps $C^\infty(V)$ into $C^\infty(V)$ and $C_0^\infty(V)$ into $C_0^\infty(V)$. When $a\in S(m,g)$, m and g are locally temperate, then (since χ is properly supported)

$$(2.9)\quad \langle a^w_\chi u, v\rangle = (2\pi)^{-n}\iint a\left(\tfrac{1}{2}(x+y),\xi\right)\chi(x,y)\exp(i\langle x-y,\xi\rangle)u(y)v(x)\,dx\,dy\,d\xi,$$

$u\in C^\infty(V)$, $v\in C_0^\infty(V)$, gives a well-defined mapping of $C_0^\infty(V)$ into $\mathscr{E}'(V)$ and $C^\infty(V)$ into $\mathscr{D}'(V)$.

We shall study how the operator a^w_χ changes for different choices of χ. Let $a(x,\xi)$, $b(x,\xi)\in\mathscr{S}'(W)$ and let $\chi,\varphi\in S(1,\tilde{G})$ be properly supported such that $|\varphi|\geqq c>0$ on supp χ, which implies $\psi=\chi/\varphi\in S(1,\tilde{G})$.

We have $a^w_\chi=b^w_\varphi$, if

$$(2.10)\qquad \hat{a}\left(\tfrac{1}{2}(x+y), y-x\right)\chi(x,y) = \hat{b}\left(\tfrac{1}{2}(x+y), y-x\right)\varphi(x,y).$$

Dividing by φ and taking the inverse Fourier transform, we obtain (2.10) if

$$(2.11)\qquad b(x,\xi) = (2\pi)^{-n}\iint \exp(i\langle t,\eta-\xi\rangle)\psi\left(x+\frac{t}{2}, x-\frac{t}{2}\right)a(x,\eta)\,dt\,d\eta$$

$$= \exp(-i\langle D_t, D_\eta\rangle)\psi\left(x+\frac{t}{2}, x-\frac{t}{2}\right)a(x,\eta)\Big|_{\substack{t=0\\ \eta=\xi}}.$$

We shall show that (2.11) can be extended to a weakly continuous map $S(m,g)\ni a\to b\in S(m,g)$ when ψ has sufficiently small support, m and g are locally σ temperate and $g\leqq g^\sigma$.

First, we study the integrand in (2.11). If $a\in S(m,g)$ and $\chi\in S(1,\tilde{G})$ has support where $D(x,y)<\varepsilon$, and ε is small enough, then Lemma 2.2 and the slow variation of G imply

$$(2.12)\qquad (t,\tau)\to\chi\left(x+\frac{t}{2}, x-\frac{t}{2}\right)a(x,\tau)\in S(\tilde{m},\tilde{g})$$

uniformly in x. Here

(2.13) $$\tilde{m}(t, \tau) = m(x, \tau)$$

and

(2.14) $$\tilde{g}_{t,\tau}(y, \eta) = G_x(y) + g_{x,\tau}(0, \eta)$$

are constant in the t variables. Obviously, $\tilde{g}$ is slowly varying and $\tilde{m}$ is $\tilde{g}$ continuous. Let A be the quadratic form on W defined by

$$A(x, \xi) = \langle x, \xi\rangle, \quad (x, \xi)\in W.$$

Let

(2.15) $$\tilde{g}^A_{t,\tau}(y, \eta) = \sup_{(r,\varrho)} \frac{|\langle r, \eta\rangle + \langle y, \varrho\rangle|^2}{\tilde{g}_{t,\tau}(r, \varrho)} = g^B_{x,\tau}(y) + G^B_x(\eta),$$

be the dual metric of $\tilde{g}$ with respect to A, where

(2.16) $$G^B_x(\eta) = \sup_r \frac{|\langle r, \eta\rangle|^2}{G_x(r)}$$

and

(2.17) $$g^B_{x,\tau}(y) = \sup_\varrho \frac{|\langle y, \varrho\rangle|^2}{g_{x,\tau}(0, \varrho)}.$$

In order to estimate (2.11) we have to prove that $\tilde{g}$ is uniformly A temperate, i.e. there exist constants C, N such that

$$\tilde{g}_{t,\tau} \leqq C\tilde{g}_{r,\varrho}\big(1 + \tilde{g}^A_{t,\tau}(r-t, \varrho-\tau)\big)^N$$

uniformly in x.

Lemma 2.3. *If g is locally σ temperate, m is locally σ, g temperate and $g \leqq g^\sigma h^2$, then $\tilde{g}$ is A temperate, $\tilde{m}$ is $A, \tilde{g}$ temperate and*

(2.18) $$\tilde{g}_{t,\tau} \leqq h^2(x, \tau)\,\tilde{g}^A_{t,\tau}.$$

The estimates are uniform in x.

Proof. Since

$$G_x(r) \leqq g_{x,\tau}(r, \varrho) \quad \forall (r, \varrho), \quad (t, \tau)\in W,$$

we obtain that

(2.19) $$G^B_x(\eta) \geqq g^\sigma_{x,\tau}(0, \eta) \geqq h^{-2}(x, \tau)\, g_{x,\tau}(0, \eta).$$

Thus

$$g^B_{x,\tau}(y) \geqq h^{-2}(x, \tau) \sup_\varrho \frac{|\langle y, \varrho\rangle|^2}{g^\sigma_{x,\tau}(0, \varrho)} \geqq h^{-2}(x, \tau)\, G_x(y),$$

which gives (2.18). Since g is locally σ temperate, m locally σ, g temperate, (2.19) implies that $\tilde{g}$ is A temperate, and m is $A, \tilde{g}$ temperate, which proves the lemma.

Proposition 2.4. *Let g be a locally σ temperate metric, m be a locally σ, g temperate function and $g/g^\sigma \leqq h^2 \leqq 1$. There exists $\varepsilon > 0$, so that if $\chi \in S(1, \tilde{G})$ has support where $D(x, y) < \varepsilon$, then the mapping $C_0^\infty(W) \ni a(x, \xi) \to b(x, \xi)$ defined by* (2.11) *has a unique extension to a weakly continuous linear mapping of $S(m, g)$ into itself. The remainder term*

$$b(x,\xi) - \sum_0^N (-i\langle D_t, D_\eta\rangle)^j \chi\left(x+\frac{t}{2},\ x-\frac{t}{2}\right) a(x,\eta)/j!\Big|_{\substack{t=0\\ \eta=\xi}}, \tag{2.20}$$

where $b(x, \xi)$ is defined by (2.11), *is weakly continuous with values in $S(mh^{N+1}, g)$.*

Proof. Since $\tilde{m}(0, \xi) = m(x, \xi)$, Theorem 3.5′ in [7] and Lemma 2.2 immediately imply

$$|b(x,\xi)| \leqq Cm(x,\xi)$$

with C independent of x. To obtain bounds on the derivatives of b, we observe that differentiation commutes with the convolution operator $\exp(-i\langle D_t, D_\eta\rangle)$, and $a \in S(m, g)$ implies $\langle w, D\rangle a \in S(m_1, g)$ where $m_1 = mg(w)^{1/2}$. Taking $w \in W$ so that $g_{x,\xi}(w) \leqq 1$ we obtain that

$$|\langle w, D\rangle b(x,\xi)| \leqq C' m(x,\xi),$$

since $G_x(t) \leqq g_{x,\xi}(t, \tau)$. Repeating this argument gives that $b \in S(m, g)$. Using the corresponding argument with Theorem 3.6 in [7], we obtain that (2.20) is bounded in $S(mh^{N+1}, g)$, which proves the proposition.

Corollary 2.5. *Let $a \in S(m, g)$ where g is locally σ temperate, m is locally σ, g temperate and $g/g^\sigma \leqq h^2 \leqq 1$. Let $\chi, \varphi \in S(1, \tilde{G})$ be properly supported such that $|\varphi| \geqq c > 0$ on $\operatorname{supp}\chi$ and $\chi/\varphi - 1$ vanishes of order N on the diagonal. If χ has support where $D(x, y) < \varepsilon$, ε given by Proposition 2.4, then*

$$a_\chi^w = a_\varphi^w + r_\varphi^w, \tag{2.21}$$

where $r \in S(mh^N, g)$.

Proof. Let $\psi = \chi/\varphi \in S(1, \tilde{G})$. We have that the equality (2.21) holds if

$$a(x,\xi) \cong \exp(-i\langle D_t, D_\eta\rangle)\psi\left(x+\frac{t}{2},\ x-\frac{t}{2}\right) a(x,\eta)\Big|_{\substack{t=0\\ \eta=\xi}}$$

$$\cong \sum_0^{N-1} (-i\langle D_t, D_\eta\rangle)^j \psi\left(x+\frac{t}{2},\ x-\frac{t}{2}\right) a(x,\eta)/j!\Big|_{\substack{t=0\\ \eta=\xi}}$$

modulo $S(mh^N, g)$, which holds since $\psi - 1$ vanishes of order N on the diagonal.

Thus the operator a_χ^w does not depend on the choice of χ, if $a \in S(m, g)$ is defined modulo $S(mh^N, g)$ and $\chi = 1$ in a neighborhood of the diagonal.

3. The calculus

We shall now develop a calculus for the operators defined in section 2. First we consider the case when the symbols are in $\mathcal{S}(W)$. Let $a, b\in\mathcal{S}(W)$ and $\chi, \varphi\in S(1, \tilde{G})$ be properly supported. Then $(2\pi)^{-n}\hat{a}\left(\frac{1}{2}(x+y), y-x\right)\chi(x,y)$ and $(2\pi)^{-n}\hat{b}\left(\frac{1}{2}(x+y), y-x\right)\varphi(x,y)$ are the Schwartz kernels for the operators a_χ^w and b_φ^w, where $\hat{a}, \hat{b}$ are the Fourier transforms in the ξ variables. The composition $a_\chi^w b_\varphi^w$ has Schwartz kernel equal to

$$(2\pi)^{-2n}\int \hat{a}\left(\tfrac{1}{2}(x+z), z-x\right)\hat{b}\left(\tfrac{1}{2}(z+y), y-z\right)\chi(x,z)\varphi(z,y)\,dz \tag{3.1}$$

which is supported in $\{(x,y)\in V\oplus V;\ \exists z\colon \chi(x,z)\varphi(z,y)\neq 0\}$, thus is properly supported. In order to get a bound on the support of (3.1) we need the following simple

Lemma 3.1. *Let $D(x,y)$ be the squared $\tilde{G}$ distance of $(x,y)\in V\oplus V$ to the diagonal, defined by* (2.6). *Then there exist $C, \varepsilon>0$ such that, for any x, y and z,*

$$\max(D(x,z);\ D(z,y)) \leqq \varepsilon \Rightarrow D(x,y) \leqq C\max(D(x,z);\ D(z,y)). \tag{3.2}$$

Proof. According to Lemma 2.2 it suffices to prove that

$$\begin{cases} G_x(x-z) \leqq \varepsilon \\ G_z(z-y) \leqq \varepsilon \end{cases} \tag{3.3}$$

implies

$$G_x(x-y) \leqq C\varepsilon \tag{3.4}$$

if ε is small enough. The slow variation of G and (3.3) imply $G_x\leqq CG_z$ for small ε, so

$$G_x(x-y) \leqq 2\big(G_x(x-z)+G_x(z-y)\big) \leqq 2(1+C)\varepsilon,$$

which proves the result. For later use we observe that (3.4) implies $G_{\frac{x+y}{2}}\leqq CG_x$ if ε is small, which together with (3.3) gives

$$\begin{cases} G_{\frac{x+y}{2}}(x-y) \leqq C'\varepsilon \\ G_{\frac{x+y}{2}}(x-z) \leqq C'\varepsilon. \end{cases} \tag{3.5}$$

Thus Lemma 3.1 gives that (3.1) has support where $D(x,y)<C\varepsilon$ if χ and φ have support where $D(x,y)<\varepsilon$ and ε is small enough. Now choose $\Psi\in S(1, \tilde{G})$ properly supported so that $\Psi=1$ on the support of (3.1). We want to find $c\in\mathcal{S}(W)$, so that

$$a_\chi^w b_\varphi^w = c_\psi^w, \tag{3.6}$$

which is satisfied if

(3.7)

$$\hat{c}\left(\tfrac{1}{2}(x+y), y-x\right) = (2\pi)^{-n}\int \hat{a}\left(\tfrac{1}{2}(x+z), z-x\right)\hat{b}\left(\tfrac{1}{2}(z+y), y-z\right)\chi(x, z)\varphi(z, y)\,dz.$$

By taking the inverse Fourier transform of (3.7) and making a linear change of variables, we obtain

$$\begin{aligned}(3.8)\qquad c(x, \xi) &= \pi^{-2n}\iint \exp\left(2i\sigma(t, \tau;\ z, \zeta)\right)a(x+z, \xi+\zeta)b(x+t, \xi+\tau)\\ &\quad\times\chi(x+z-t, x+z+t)\varphi(x+z+t, x-z+t)\,dz\,d\zeta\,dt\,d\tau\\ &= \exp\left(\frac{i}{2}\sigma(D_z, D_\zeta;\ D_t, D_\tau)\right)a(x+z, \xi+\zeta)b(x+t, \xi+\tau)\\ &\quad\times\chi(x+z-t, x+z+t)\varphi(x+z+t, x-z+t)\Big|_{\substack{t=\tau=0\\ z=\zeta=0}}.\end{aligned}$$

Now we are going to extend (3.8) to general $a\in S(m_1, g_1)$ and $b\in S(m_2, g_2)$, where g_j is locally σ temperate and m_j is locally σ, g_j temperate, $j=1, 2$. According to the proof of Lemma 3.1, the integrand in (3.8), for fixed x, is supported over a fixed bounded $\tilde{G}_{x,x}$ neighborhood of $(x, x)\in V\oplus V$ if χ and φ are supported in a sufficiently small $\tilde{G}$ neighborhood of the diagonal. In fact, if

$$\chi(x+z-t, x+z+t)\varphi(x+z+t, x-z+t) \neq 0$$

then by substituting $x+z-t$, $x+z+t$ and $x-z+t$ for x, z and y, (3.5) gives that $G_x(2t)\leqq C'\varepsilon$ and $G_x(2(z-t))\leqq C'\varepsilon$. Thus, if $\chi, \varphi\in S(1, \tilde{G})$ are supported where $D(x, y)<\varepsilon$ and ε is small enough, then the slow variation of G and the inequalities $G\leqq g_j$, $j=1, 2$, imply that the integrand in (3.8) is a symbol in $S(\tilde{m}, \tilde{g})$, where

$$\tilde{m}(w_1, w_2) = m_1(w_1)m_2(w_2)$$

and

$$\tilde{g}_{w_1, w_2}(t_1, t_2) = g_{1, w_1}(t_1)+g_{2, w_2}(t_2), \quad w_j, t_j\in W,$$

is a metric on $W\oplus W$. Obviously, $\tilde{g}$ is slowly varying, $\tilde{m}$ is $\tilde{g}$ continuous and $\tilde{G}\leqq\tilde{g}$. Let B be the quadratic form on $W\oplus W$ defined by

$$B(w_1, w_2) = 2\sigma(w_1, w_2), \quad (w_1, w_2)\in W\oplus W.$$

The dual metric of $\tilde{g}$ with respect to B is equal to

$$\tilde{g}^B_{w_1, w_2}(t_1, t_2) = \sup_{t_1', t_2'}\frac{|\sigma(t_1, t_1')+\sigma(t_2, t_2')|^2}{g_{1, w_1}(t_2')+g_{2, w_2}(t_1')} = g^\sigma_{1, w_1}(t_2)+g^\sigma_{2, w_2}(t_1).$$

In order to extend (3.8) to general symbols we need to know that $\tilde{g}$ is locally B temperate with respect to the diagonal in $W\oplus W$, $\tilde{m}$ is locally $B, \tilde{g}$ temperate with respect to the diagonal and that $\tilde{g}\leqq\tilde{g}^B$ on the diagonal.

When $w_1=w_2=w$, we find

$$\tilde{g}(t_1, t_2) \leqq \tilde{g}^B(t_1, t_2) \quad \forall t_j \in W \tag{3.9}$$

if and only if

$$g_{1,w}(t) \leqq g^\sigma_{2,w}(t), \quad \forall t \in W, \tag{3.10}$$

which is equivalent to

$$g_{2,w}(t) \leqq g^\sigma_{1,w}(t), \quad \forall t \in W.$$

The conditions for $\tilde{g}$ to be locally B temperate and $\tilde{m}$ locally $B, \tilde{g}$ temperate with respect to the diagonal are

(3.11)
$$g^\sigma_{1,w}(t_1)+g^\sigma_{2,w}(t_2) \leqq C\big(g^\sigma_{1,w_1}(t_1)+g^\sigma_{2,w_2}(t_2)\big)\big(1+g^\sigma_{1,w_1}(w_2-w)+g^\sigma_{2,w_2}(w_1-w)\big)^N$$

and

$$m_1(w_1)m_2(w_2) \leqq Cm_1(w)m_2(w)\big(1+g^\sigma_{1,w_1}(w_2-w)+g^\sigma_{2,w_2}(w_1-w)\big)^N, \tag{3.12}$$

when $G_x(x_1-x)+G_x(x_2-x) \leqq c$; $w=(x, \xi)$ and $w_j=(x_j, \xi_j) \in W$. When $w_j=w$, $j=1, 2$, this reduces to

$$\begin{cases} g^\sigma_{1,w}(t) \leqq Cg^\sigma_{1,w_0}(t)\big(1+g^\sigma_{2,w}(w_0-w)\big)^N \\ g^\sigma_{2,w}(t) \leqq Cg^\sigma_{2,w_0}(t)\big(1+g^\sigma_{1,w}(w_0-w)\big)^N, \end{cases} \tag{3.13}$$

when $G_x(x_0-x) \leqq c$; and

$$\begin{cases} m_1(w_0) \leqq Cm_1(w)\big(1+g^\sigma_{2,w}(w_0-w)\big)^N \\ m_2(w_0) \leqq Cm_2(w)\big(1+g^\sigma_{1,w}(w_0-w)\big)^N, \end{cases} \tag{3.14}$$

when $G_x(x_0-x) \leqq c$; w and $w_0=(x_0, \xi_0) \in W$. Conversely, we shall prove the following result.

Lemma 3.2. *Assume that g_1, g_2 are locally σ temperate and that m_j is g_j continuous, $j=1, 2$. If* (3.13) *and* (3.14) *are satisfied, then $\tilde{g}$ is locally B temperate and $\tilde{m}$ is locally $B, \tilde{g}$ temperate with respect to the diagonal in $W \oplus W$.*

Proof. Put

$$M = 1+g^\sigma_{1,w_1}(w_2-w)+g^\sigma_{2,w_2}(w_1-w),$$

then according to (3.13) and (3.14) it suffices to prove that

$$\begin{cases} g^\sigma_{1,w}(w_2-w) \leqq CM^N \\ g^\sigma_{2,w}(w_1-w) \leqq CM^N, \end{cases} \tag{3.15}$$

when $G_x(x_1-x)+G_x(x_2-x) \leqq c$. If c is small enough we obtain, by the slow variation of G, that

$$\begin{cases} G_{x_1}(x_2-x) \leqq Cc \\ G_{x_2}(x_1-x) \leqq Cc, \end{cases} \tag{3.16}$$

which gives

$$1+g^{\sigma}_{1,w_1+w_2-w}(w_2-w) \leqq C(1+g^{\sigma}_{1,w_1}(w_2-w))^{N+1} \leqq CM^{N+1}$$

and

$$1+g^{\sigma}_{2,w_1+w_2-w}(w_1-w) \leqq C(1+g^{\sigma}_{2,w_2}(w_1-w))^{N+1} \leqq CM^{N+1}$$

for small Cc, since g_j is locally σ temperate. Also (3.13) and (3.16) imply

$$g^{\sigma}_{1,w_2}(w_2-w) \leqq Cg^{\sigma}_{1,w_1+w_2-w}(w_2-w)(1+g^{\sigma}_{2,w_2}(w_1-w))^{N} \leqq CM^{N'}$$

and

$$g^{\sigma}_{2,w_1}(w_1-w) \leqq Cg^{\sigma}_{2,w_1+w_2-w}(w_1-w)(1+g^{\sigma}_{1,w_1}(w_2-w))^{N} \leqq CM^{N'}.$$

Thus we find

$$g^{\sigma}_{1,w}(w_2-w) \leqq C(1+g^{\sigma}_{1,w_2}(w_2-w))^{N+1} \leqq CM^{N''}$$

and

$$g^{\sigma}_{2,w}(w_1-w) \leqq C(1+g^{\sigma}_{2,w_1}(w_1-w))^{N+1} \leqq CM^{N''},$$

when $G_x(x_1-x)+G_x(x_2-x)\leqq c$ and c is small enough. This proves (3.15) and the lemma.

Now by using Theorems 3.5′ and 3.6 in [7], Lemma 3.2 and the fact that

$$\sup \tilde{g}_{w,w}/\tilde{g}^{B}_{w,w} = \sup g_{1,w}/g^{\sigma}_{2,w} = \sup g_{2,w}/g^{\sigma}_{1,w}$$

we obtain the following

Theorem 3.3. *Let g_1 and g_2 be locally σ temperate Riemannean metrics in $W=V\oplus V'$, satisfying* (3.10) *and* (3.13). *Let m_j be g_j continuous functions on W satisfying* (3.14), *$j=1, 2$. There exists $\varepsilon>0$, so that if χ and $\varphi\in S(1, \tilde{G})$ are supported where $D(x, y)<\varepsilon$, then* (3.8) *can be uniquely extended to a weakly continuous bilinear map from $S(m_1, g_1)\times S(m_2, g_2)$ to $S(m_1m_2, g)$, where $g=\max(g_1, g_2)$. If*

$$h^2 = \sup g_1/g^{\sigma}_2 = \sup g_2/g^{\sigma}_1 \tag{3.17}$$

then for any N, the remainder

$$c(x, \xi)-\sum_{j<N}\left(\frac{i}{2}\sigma(D_z, D_\zeta;\; D_t, D_\tau)\right)^j a(x+z, \xi+\zeta)b(x+t, \xi+\tau) \tag{3.18}$$
$$\times\chi(x+z-t, x+z+t)\varphi(x+z+t, x-z+t)/j!\Big|_{\substack{t=\tau=0\\ z=\zeta=0}},$$

where c is given by (3.8), *is weakly continuous with values in $S(m_1m_2h^N, g)$.*

Remark. When χ and $\varphi=1$ in a neighborhood of the diagonal, then (3.18) gives the usual formal Weyl calculus. The $\tilde{G}$ neighborhood, in which χ and φ have to be supported only depends on the constants in the slow variation of G and in Definition 2.1. Also $c(x, \xi)$ in (3.8) has support where x has a fixed G_x neighborhood intersecting both the projection of supp a and supp b on V.

The dual metric to $g=\max(g_1, g_2)$ is

$$g^\sigma(w) = \inf_{w_1+w_2=w} (g_1^\sigma(w_1)^{1/2} + g_2^\sigma(w_2)^{1/2})^2. \tag{3.19}$$

The metric g is obviously slowly varying and $m_1 m_2$ is g continuous, since $g_j \leqq g$, $j=1, 2$. We shall digress to study the conditions for g to be σ temperate and to satisfy $g \leqq g^\sigma$. Observe that $g_1 \leqq g_2^\sigma$ does not imply $g \leqq g^\sigma$, for example when $g_1 \leqq g_2^\sigma < g_2 \leqq g_1^\sigma$. But if $g \leqq g^\sigma$, then

$$g_j \leqq g \leqq g^\sigma \leqq g_k^\sigma, \quad j, k = 1, 2.$$

Conversely, we shall prove the following

Proposition 3.4. *Let g_1, g_2 be σ temperate metrics on W satisfying* (3.13) *for all $w, w_0 \in W$, such that $g_j \leqq g_j^\sigma$, $j=1, 2$, and $g_1 \leqq g_2^\sigma$. Then $g=\max(g_1, g_2)$ is σ temperate, and $g \leqq g^\sigma$. If in addition m_j are σ, g_j temperate, $j=1, 2$, and satisfy* (3.14) *for all $w, w_0 \in W$, then m_j are σ, g temperate, $j=1, 2$.*

Proof. To prove that g is σ temperate, it suffices to show that

$$g_{j,w} \leqq C g_{j,w_0}(1 + g_w^\sigma(w_0 - w))^N \tag{3.20}$$

for all $w, w_0 \in W$, $j=1, 2$. According to (3.19) we can choose $w_1 \in W$ so that

$$g_w^\sigma(w_0 - w)^{1/2} = g_{1,w}^\sigma(w_0 - w_1)^{1/2} + g_{2,w}^\sigma(w_1 - w)^{1/2}. \tag{3.21}$$

If (3.13) holds and g_j is σ temperate, then

$$g_{j,w} \leqq C g_{j,w_1}(1 + g_{2,w}^\sigma(w_1 - w))^N, \quad j = 1, 2$$

and

$$g_{j,w_1} \leqq C g_{j,w_0}(1 + g_{1,w_1}^\sigma(w_0 - w_1))^N, \quad j = 1, 2.$$

Since

$$g_{1,w_1}^\sigma(w_0 - w_1) \leqq C g_{1,w}^\sigma(w_0 - w_1)(1 + g_{2,w}^\sigma(w_1 - w))^N,$$

we obtain (3.20). The same argument works with m_j instead of g_j, so m_j is σ, g temperate.

In order to prove that $g \leqq g^\sigma$, we observe that

$$g_1(t) = \sup_{t'} \frac{|\sigma(t, t')|^2}{g_1^\sigma(t')} \leqq g_2^\sigma(t), \quad \forall t \in W$$

is equivalent to

$$|\sigma(t, t')|^2 \leqq g_1^\sigma(t') g_2^\sigma(t); \ \forall t, t' \in W. \tag{3.22}$$

Now, for every $t, t' \in W$ we can find $w, w' \in W$ such that

$$g^\sigma(t)^{1/2} = g_1^\sigma(t-w)^{1/2} + g_2^\sigma(w)^{1/2}$$

and

$$g^\sigma(t')^{1/2} = g_1^\sigma(t'-w')^{1/2} + g_2^\sigma(w')^{1/2}$$

at w_0. Then, since $g_1 \leqq g_2^\sigma$ and $g_j \leqq g_j^\sigma$, $j=1,2$, we obtain

$$\begin{aligned}|\sigma(t,t')| &= |\sigma(t-w,t'-w')+\sigma(t-w,w')+\sigma(w,t'-w')+\sigma(w,w')| \\ &\leqq g_1^\sigma(t-w)^{1/2}g_1^\sigma(t'-w')^{1/2}+g_1^\sigma(t-w)^{1/2}g_2^\sigma(w')^{1/2} \\ &+g_2^\sigma(w)^{1/2}g_1^\sigma(t'-w')^{1/2}+g_2^\sigma(w)^{1/2}g_2^\sigma(w')^{1/2} = g^\sigma(t)^{1/2}g^\sigma(t')^{1/2}\end{aligned}$$

at w_0, which proves that $g \leqq g^\sigma$ and finishes the proof of the proposition.

In general, we do not expect g to be locally σ temperate when g_1, g_2 are locally σ temperate and satisfy (3.13), since w_1 in (3.21) need not be in a lifted G neighborhood of w and w_0.

Example 3.5. Let $f(x) \in C^1(\mathbf{R}^n)$ satisfy

$$\begin{cases} |\operatorname{grad} f(x)| \leqq Cf(x)^{1+\gamma} \\ 1 \leqq f(x) \end{cases} \tag{3.23}$$

where $0 \leqq \gamma < 1$. Put

$$G_x(t) = |t|^2 f(x)^{2\gamma}$$

and

$$g_{x,\xi}(t,\tau) = |t|^2 \Lambda(x,\xi)^{2\delta} + |\tau|^2 \Lambda(x,\xi)^{-2\varrho}$$

where $\gamma \leqq \delta \leqq \varrho \leqq 1$, $\delta < 1$, and

$$\Lambda(x,\xi) = (f(x)^2+|\xi|^2)^{1/2}.$$

Then G is slowly varying, g is locally σ temperate, and $g/g^\sigma = \Lambda(x,\xi)^{2(\delta-\varrho)} \leqq 1$.

4. Continuity in C^∞ and L^2

In this section we shall prove that the operators a_χ^w are continuous in C^∞ and $\mathscr{D}'$. We then get a calculus for these operators according to Theorem 3.3.

Theorem 4.1. *Let g be a locally σ temperate metric on W, m locally σ, g temperate and $g \leqq g^\sigma$. There exists $\varepsilon > 0$ such that if $\chi \in S(1, \tilde{G})$ has support where $D(x,y) < \varepsilon$ and $a \in S(m,g)$, then a_χ^w is a continuous map from $C^\infty(V)$ to $C^\infty(V)$ and from $\mathscr{D}'(V)$ to $\mathscr{D}'(V)$.*

Proof. Since χ is properly supported if the $\tilde{G}$ neighborhood is small enough, C^∞ continuity implies C_0^∞ continuity, which by duality gives $\mathscr{D}'$ continuity. We are going to prove that, if $\chi(x,y)$ has support where $G_x(x-y) \leqq c$, and c is small enough, then for all N there exists M with the property that

$$\sum_{|\alpha| \leqq N} |D^\alpha a_\chi^w u(x_0)| \leqq C \sum_{|\beta| \leqq M} \sup_{G_{x_0}(x-x_0) \leqq c} |D^\beta u(x)|. \tag{4.1}$$

Here the constant C depends on G_{x_0}, $g_{x_0,0}$ and $m(x_0, 0)$.

Choose a partition of unity $\sum\varphi_j=1$ in W and neighborhoods U'_j of supp φ_j such that

$$\operatorname{supp}\varphi_j \subseteqq \{w\colon\ g_{w_j}(w-w_j) \leqq c_0\} \subseteqq U'_j = \{w\colon\ g_{w_j}(w-w_j) \leqq c_2\},$$

$c_0<c_2$, φ_j is uniformly bounded in $S(1, g_{w_j})$, g_w and $m(w)$ only vary with a fixed factor in U'_j and there is a bound on the number of U'_j having non-empty intersection (see Lemma 2.5 in [7]). Choose $c_0<c_1<c_2$ and put

$$U_j = \{w\colon\ g_{w_j}(w-w_j) \leqq c_1\}.$$

Let $a_j=\varphi_j a$, and consider

$$a^w_{j,\chi}u(x_0) = (2\pi)^{-n}\iint \exp(i\langle x_0-y,\ \xi\rangle)\chi(x_0,\ y)a_j\left(\tfrac{1}{2}(x_0+y),\ \xi\right)u(y)\,dy\,d\xi,$$

Since $\chi(x_0, y)$ has support where $G_{x_0}(x_0-y)\leqq c$, we find that the G_{x_0} distance from x_0 to the projection of U_j is less than $c^{1/2}/2$ when $x_0\in\operatorname{supp} a^w_{j,\chi}u$. Then, for small c,

$$G_{x_0} \leqq CG_x \leqq Cg_{x,\xi} \leqq C' g_{w_j}$$

if $(x, \xi)\in U_j$. Thus $G_{x_0}(x-x_0)\leqq c$ when $(x, \xi)\in U'_j$ and $x_0\in\operatorname{supp} a^w_{j,\chi}u$ if c and c_2 are small enough, which we assume in what follows.

Now, if χ has support in a sufficiently small $\tilde{G}$ neighborhood of the diagonal, it follows from the slow variation of G that

$$C_0^\infty(V)\ni u(y) \to \chi(x_0, y)u(y)\in C_0^\infty(V)$$

is continuous with continuity constants only depending on G_{x_0}. Thus (4.1) follows if we show that for all N there exists M such that

$$\sum\nolimits_{|\alpha|\leqq N} |D^\alpha a^w_j u(x_0)| \leqq C\sum\nolimits_{|\beta|\leqq M}\sup|D^\beta u| \tag{4.2}$$

if $u\in C_0^\infty$ has support when $G_{x_0}(x-x_0)\leqq c$. When proving (4.2) it suffices to consider the case $\alpha=0$. In fact, integration by parts gives

$$\langle t, D_x\rangle a^w_j u = a^w_j(\langle t, D_x\rangle u)+b^w_j u,$$

where $b_j(x, \xi)=\langle t, D_x\rangle a_j(x, \xi)\in S(m_1(w_j), g_{w_j})$ uniformly in j, and $m_1=mg(t, 0)^{1/2}$ satisfies the same conditions as m. When $\alpha=0$ we have

$$|a^w_j u(x_0)| \leqq C\|a_j\|_{L^1}\|u\|_{L^\infty} \leqq C'm(w_j)(\det g_{w_j})^{-1/2}\|u\|_{L^\infty} \tag{4.3}$$

and we shall improve this estimate by using integration by parts.

Let $L(x, \xi)=\langle t, \xi\rangle+\langle\tau, x\rangle$ be a linear form on W. Then

$$-L(y-x, \xi)\exp(i\langle x-y,\ \xi\rangle) = L(D_\xi, D_y)\exp(i\langle x-y,\ \xi\rangle),$$

so integration by parts gives

$$a^w_j u = b^w_j u+c^w_j(\langle t, D_x\rangle u) \quad \text{at} \quad x_0, \tag{4.4}$$

where

$$b_j(x, \xi) = \left(\tfrac{1}{2}\langle t, D_x\rangle + \langle \tau, D_\xi\rangle\right)c_j(x, \xi)$$

and

$$c_j(x, \xi) = a_j(x, \xi)/\tilde{L}(x, \xi)$$

if

$$\tilde{L}(x, \xi) = L(2(x-x_0), \xi) \neq 0 \quad \text{when} \quad (x, \xi) \in U_j. \tag{4.5}$$

Lemma 3.1 in [7] gives

$$\tilde{L}(w)/\tilde{L}(w_j) \in S(1, g_{w_j})$$

uniformly when $w \in \operatorname{supp} a_j$, if $\tilde{L} \neq 0$ in U_j. Thus

$$c_j \in S(m(w_j)/\tilde{L}(w_j), g_{w_j})$$

and

$$b_j \in S\left(m(w_j) g_{w_j}\left(\frac{t}{2}, \tau\right)^{1/2}/\tilde{L}(w_j), g_{w_j}\right)$$

uniformly. By repeating this argument we obtain

$$|a_j^w u(x_0)| \leqq C_N m(w_j) (\det g_{w_j})^{-1/2} R_j^{-N} \sum_{|\beta| \leqq N} \sup |D^\beta u| \tag{4.6}$$

if $R_j \leqq \tilde{L}(w_j)$, $\tilde{L}(x, \xi) = \langle t, \xi\rangle + 2\langle \tau, x - x_0\rangle \neq 0$ in U_j and $g_{w_j}\left(\frac{t}{2}, \tau\right) \leqq 1$, since

$$G_{x_0}(t) \leqq C G_{x_j}\left(\frac{t}{2}\right) \leqq C g_{w_j}\left(\frac{t}{2}, \tau\right) \leqq C$$

when $G_{x_0}(x_j - x_0) \leqq c$. As before, we put

$$g_w^A(y, \eta) = \sup_{t, \tau} |\langle t, \eta\rangle + \langle y, \tau\rangle|^2 / g_w(t, \tau). \tag{4.7}$$

Since

$$\frac{|\tilde{L}(x, \xi)|^2}{g_{w_j}(t/2, \tau)} = \frac{|L(2(x-x_0), \xi)|^2}{g_{w_j}(t/2, \tau)} = 4\frac{|\langle t/2, \xi\rangle + \langle x - x_0, \tau\rangle|^2}{g_{w_j}(t/2, \tau)},$$

the Hahn—Banach theorem gives that we can take R_j equal to 2 times the $g_{w_j}^A$ distance from $(x_0, 0)$ to U_j. Thus we obtain

$$|a_j^w u(x_0)| \leqq C m(w_j) (\det g_{w_j})^{-1/2} (1 + d_j)^{-N} \sum_{|\beta| \leqq N} \sup |D^\beta u|, \tag{4.8}$$

where

$$d_j^2 = \inf_{w \in U_j} g_{w_j}^A(w - (x_0, 0)).$$

Now we need the following

Lemma 4.2. *Under the assumptions above, there exist constants $N, c>0$ with the property that for any $x_0\in V$ there is a constant C such that*

$$g_{x_0,0} \leqq C g_{x,\xi}(1+g^A_{x,\xi}(x-x_0,\xi))^N, \tag{4.9}$$

$$m(x,\xi) \leqq C m(x_0,0)(1+g^A_{x,\xi}(x-x_0,\xi))^N \tag{4.10}$$

when $G_{x_0}(x-x_0)\leqq c$; and

$$\sum (1+d_j)^{-N} \leqq C \tag{4.11}$$

if the sum is taken over those j for which

$$G_{x_0}(x-x_0) \leqq c \quad \text{when} \quad (x,\xi)\in U_j.$$

End of proof of Theorem 4.1. Choose $w'=(x',\xi')\in U_j$ such that

$$d_j^2 = g^A_{w_j}(x'-x_0,\xi').$$

Then (4.9) and the minimax principle imply

$$(\det g_{w_j})^{-1/2} \leqq C(\det g_{w'})^{-1/2} \leqq C'(\det g_{x_0,0})^{-1/2}(1+d_j)^{2nN}$$

when $G_{x_0}(x'-x_0)\leqq c$. Similarly, (4.10) gives

$$m(w_j) \leqq Cm(w') \leqq C'm(x_0,0)(1+d_j)^{2N}.$$

Thus, using (4.11) we obtain from (4.8) for large N

$$\sum |a_j^w u(x_0)| \leqq Cm(x_0,0)(\det g_{x_0,0})^{-1/2} \sum_{|\beta|\leqq N} \sup |D^\beta u|$$

if $u\in C_0^\infty(V)$ has support where $G_{x_0}(x-x_0)\leqq c$, and c is small enough. This completes the proof of the theorem.

Proof of Lemma 4.2. First we observe that since g and m are locally σ temperate, there exist $0<c, C$ such that

$$1/C \leqq g_{x,0}/g_{x_0,0} \leqq C \tag{4.12}$$

and

$$1/C \leqq m(x,0)/m(x_0,0) \leqq C \tag{4.13}$$

when $G_{x_0}(x-x_0)\leqq c$. Here C only depends on $g^\sigma_{x_0,0}$ and G_{x_0}, and c is independent of x_0. Also, we can find C such that

$$g_{x_0,0}(t,0) \leqq CG_{x_0}(t) \quad \forall t\in V. \tag{4.14}$$

Since $g^A(t,\tau)=g^\sigma(t,-\tau)$ and g is locally σ temperate, we obtain by using (4.12) that

$$g_{x_0,0} \leqq Cg_{2x-x_0,0} \leqq C'g_{x,\xi}(1+g^A_{x,\xi}(x-x_0,\xi))^N, \tag{4.15}$$

when $G_{x_0}(x-x_0)\leqq c$, and c is small enough, because

$$G_x((2x-x_0)-x) = G_x(x-x_0) \leqq CG_{x_0}(x-x_0) \leqq Cc.$$

This gives (4.9). Also we find

$$g_{x,\xi} \leqq Cg_{2x-x_0,0}(1+g^A_{x,\xi}(x-x_0,\xi))^N \leqq C'g_{x_0,0}(1+g^A_{x,\xi}(x-x_0,\xi))^N \tag{4.16}$$

when $G_{x_0}(x-x_0)\leqq c$. The same argument works for $m(w)$ instead of g_w, so we get (4.10).

To prove (4.11) we observe that by (4.12) and (4.14) we have

$$\begin{aligned} g_{x_0,0}(x-x_0,\xi) &\leqq 2(g_{x_0,0}(2(x-x_0),0)+g_{x_0,0}(x_0-x,\xi)) \\ &\leqq C(1+g_{2x-x_0,0}(x_0-x,\xi)) \leqq C(1+g^\sigma_{2x-x_0,0}(x_0-x,\xi)) \leqq C'(1+g^A_{x,\xi}(x-x_0,\xi))^{N+1} \end{aligned} \tag{4.17}$$

if $G_{x_0}(x-x_0)\leqq c$ is small enough. Now, the estimates (4.16) and (4.17) and the slow variation of g are sufficient for the proof of [7, Lemma 3.4] to go through in this case, so we get (4.11) for large enough N. The details are left for the reader.

Remark. It is easy to see that the number of derivatives needed in the C^∞ estimates of $a^w_\chi u$ only depends on the constants in Definition 2.1.

Theorem 4.3. *Assume that g is locally σ temperate on W and that $g\leqq g^\sigma$. There exists $\varepsilon>0$ such that if $\chi\in S(1,\tilde{G})$ has support where $D(x,y)<\varepsilon$ and $a\in S(1,g)$, then a^w_χ is L^2 continuous.*

Proof. Choose a partition of unity $\Sigma\varphi_j=1$, $\varphi_j\in S(1,g_{w_j})$ and neighborhoods $U_j\subset U_j'$ of supp φ_j as in the proof of Theorem 4.1. The proof of [7, Lemma 5.1] gives, with L^2 operator norms

$$\|a^w_\chi(x,D)\| \leqq (2\pi)^{-2n}\|\chi\|_{L^\infty}\|\hat{a}\|_{L^1} = \|\chi\|_{L^\infty}\|a\|_{FL^1} \tag{4.18}$$

if $a(x,\xi)\in\mathcal{S}(W)$ and $\chi(x,y)\in C^\infty(V\oplus V)$.
Since the Fourier-L^1 norm is invariant under affine transformations and can be estimated by seminorms in $\mathcal{S}$, this gives

$$\|a^w_{j,\chi}\| \leqq C, \quad \forall j. \tag{4.19}$$

Since we are going to use the lemma of Cotlar, Knapp and Stein, we consider

$$(a^w_{j,\chi})^* a^w_{k,\chi} = \bar{a}^w_{j,\psi} a^w_{k,\chi} \tag{4.20}$$

and

$$a^w_{j,\chi}(a^w_{k,\chi})^* = a^w_{j,\chi}\bar{a}^w_{k,\psi}, \tag{4.21}$$

where $\psi(x,y)=\bar{\chi}(y,x)$. Naturally, it suffices to consider (4.20) in what follows. Choose $\varphi\in S(1,\tilde{G})$ such that $\varphi(x,y)=1$ when there exists $z\in V$ so that either

$$\psi(x,z)\chi(z,y)\neq 0$$

or

$$\chi(x, z)\psi(z, y) \neq 0.$$

Then

$$a^w_{jk,\varphi} = (a^w_{j,\chi})^* a^w_{k,\chi}$$

if

$$a_{jk}(x, \xi) = \exp\left(\frac{i}{2}\sigma(D_z, D_\zeta;\ D_t, D_\tau)\right)\bar{\chi}(x+z+t, x+z-t) \times\chi(x+z+t, x-z+t)\bar{a}_j(x+z, \xi+\zeta)a_k(x+t, \xi+\tau)\Big|_{\substack{z=\zeta=0\\ t=\tau=0}}. \tag{4.22}$$

As in the proof of Theorem 3.3, if χ has support in a sufficiently small $\tilde{G}$ neighborhood of the diagonal, then we can use the estimates (3.10) in [7, p. 369] and (3.11) to obtain

$$|a_{jk}(w)| \leqq C_N(1+\tilde{g}^B(w))^{-N}, \quad \forall N \tag{4.23}$$

where

$$\tilde{g}^B(w) = \min_{w' \in U_j} g^\sigma_w(w-w') + \min_{w'' \in U_k} g^\sigma_w(w-w'').$$

We also obtain that g is σ temperate between $\operatorname{supp} a_{jk}$, U_j and U_k, i.e.,

$$g_{w_1} \leqq Cg_{w_2}(1+g^\sigma_{w_1}(w_1-w_2))^N, \tag{4.24}$$

when $w_1, w_2 \in \operatorname{supp} a_{jk} \cup U_j \cup U_k$ and $a_{jk} \not\equiv 0$.
Now, the estimates (4.18), (4.23) and (4.24) are all that is needed for the proof of [7, Th. 5.3] to go through in this case. The details are left for the reader.

Remark. The $\tilde{G}$ neighborhood in which the cut-off function χ has to have support, only depends on the constants in the slow variation of G and in Definition 2.1. The L^2 operator norm of a^w_χ only depends on the seminorms of a in $S(1, g)$, of χ in $S(1, \tilde{G})$ and the constants in the slow variation of G and Definition 2.1.

Corollary 4.4. *Assume that g is locally σ temperate on W and that $g \leqq g^\sigma$. There exists $\varepsilon > 0$, such that if $\chi \in S(1, \tilde{G})$ has support where $D(x, y) < \varepsilon$, $a \in S(m, g)$, where m is g continuous and $m \to 0$ at ∞, then a^w_χ is compact in $L^2(V)$.*

Proof. Since m is bounded, we find $S(m, g) \subseteqq S(1, g)$ with fixed bounds on every seminorm. Thus, if we choose the $\tilde{G}$ neighborhood as in Theorem 4.3 we obtain that a^w_χ is L^2 continuous. Let $\{\varphi_j\}$ be the partition of unity used in the proof of Theorem 4.1, and put $a_j = \varphi_j a$. Since $m \to 0$ at ∞, we find that for every $\varepsilon > 0$ there exists N_ε such that

$$a - \sum_{j \leqq N} a_j \in S(\varepsilon, g) \quad \text{if} \quad N \geqq N_\varepsilon$$

uniformly in ε. The remark after Theorem 4.3 gives a constant C such that for every $\varepsilon > 0$, the operator norm in L^2,

$$\left\|a^w_\chi - \sum_{j \leqq N} a^w_{j,\chi}\right\| \leqq C\varepsilon \quad \text{if} \quad N \geqq N_\varepsilon$$

so

$$\left\|a_\chi^w - \sum_{j\le N} a_{j,\chi}^w\right\| \to 0 \quad \text{as} \quad N \to \infty.$$

Since $a_{j,\chi}^w$ is compact in $L^2(V)$, we obtain that a_χ^w is compact, which proves the theorem.

5. Hilbert—Schmidt and trace class norms

The Hilbert—Schmidt operators on $L^2(\mathbf{R}^n)$ are those with kernels in $L^2(\mathbf{R}^n \times \mathbf{R}^n)$ and the Hilbert—Schmidt norm is equal to the L^2 norm of the kernel. Thus if $a_\chi^w(x, D)$ is defined by (2.8), then the Hilbert—Schmidt norm is equal to

(5.1)
$$\|a_\chi^w\|_{HS}^2 = (2\pi)^{-2n} \iint \left|\hat{a}\left(\tfrac{1}{2}(x+y), y-x\right)\chi(x, y)\right|^2 dx\, dy \le (2\pi)^{-n} \|\chi\|_{L^\infty}^2 \|a\|_{L^2}^2$$

by Parseval's formula, here $\hat{a}$ is the Fourier transform in the ξ variables.

The trace class operators are those which can be written as a composition of Hilbert—Schmidt operators, and the trace class norm is equal to

$$\|A\|_{tr} = \inf_{A=A_1 A_2} \|A_1\|_{HS} \|A_2\|_{HS}. \tag{5.2}$$

The argument of [7, p 415] gives

$$tr\, a_\chi^w = (2\pi)^{-n} \iint \chi(x, x)\, a(x, \xi)\, dx\, d\xi \tag{5.3}$$

if a_χ^w is of trace class, $a \in L^1(\mathbf{R}^{2n})$ and $\chi \in L^\infty(\mathbf{R}^{2n})$.

We shall now estimate the trace class norm. The proof of [7, Lemma 7.2] easily gives that a_χ^w is of trace class and

$$\|a_\chi^w\|_{tr} \le C \sum_{|\alpha|+\ldots+|\beta'| \le 2k} \|D_x^\alpha \chi\|_{L^\infty} \|x^\beta \xi^{\alpha'} D_\xi^{\beta'} D_x^{\alpha''} a\|_{L^2} \tag{5.4}$$

if the right-hand side is finite and $2k > n$.

This shows that if a and $\chi \in \mathcal{S}(\mathbf{R}^{2n})$ then a_χ^w is of trace class with the norm depending continuously on a and χ in $\mathcal{S}(\mathbf{R}^{2n})$. In the following, the metric g need not be locally σ temperate, but we assume that g is a slowly varying metric on $\mathbf{R}^{2n}$, satisfying

$$G_x(t) \le g_{x,\xi}(t, \tau) \le h^2(x, \xi)\, g_{x,\xi}^\sigma(t, \tau) \tag{5.5}$$

for all $(x, \xi), (t, \tau)$, where $h \le 1$, and m is a g continuous function.

Theorem 5.1. *There exists $\varepsilon > 0$ such that if $\chi \in S(1, \tilde{G})$ has support where $D(x, y) < \varepsilon$ and $a \in S(m, g)$, then for every integer $k > 0$,*

$$\|a_\chi^w\|_{tr} \le C_k(\|a\|_{L^1} + \|h^k m\|_{L^1} \|a\|), \tag{5.6}$$

where $\|a\|$ is a seminorm of a in $S(m, g)$ whose order only depends on k.

Proof. Choose a partition of unity $\Sigma\varphi_j=1$ and neighborhoods U_j of $\operatorname{supp}\varphi_j$ as in the proof of Theorem 4.1, so that $\varphi_j\in S(1, g_{w_j})$ uniformly, $w_j=(x_j, \xi_j)$. By the triangle inequality for trace class norms, we obtain

$$\|a_\chi^w\|_{tr} \leqq \sum \|a_{j,\chi}^w\|_{tr} \tag{5.7}$$

where $a_j=\varphi_j a$. Since $G\leqq g$, we may assume that

$$G_x/C \leqq G_{x_j} \leqq CG_x$$

when $(x, \xi)\in U_j$, by taking a refinement of the partition of unity. Choose $\Psi_j\in S(1, G_{x_j})$ uniformly such that $\Psi_j(x)=1$ when $(x, \xi)\in\operatorname{supp} a_j$ and $\Psi_j(x)=0$ when $(x, \xi)\notin U_j$, $\forall\xi$. This gives

$$a_{j,\chi}^w = a_{j,\chi_j}^w,$$

where

$$\chi_j(x, y) = \chi(x, y)\Psi_j\left(\tfrac{1}{2}(x+y)\right)$$

is uniformly bounded in $S(1, \tilde{G}_{x_j,x_j})$ and has support in a fixed, bounded $\tilde{G}_{x_j,x_j}$ neighborhood of (x_j, x_j) if χ has support in a sufficiently small $\tilde{G}$ neighborhood of the diagonal. We now need the following simple

Lemma 5.2. *If a and $\chi\in\mathcal{S}(\mathbf{R}^{2n})$ then*

$$\|a_\chi^w\|_{tr} \leqq (2\pi)^{-2n}\|\hat{\chi}\|_{L^1}\|a^w\|_{tr} = \|\chi\|_{FL^1}\|a^w\|_{tr}, \tag{5.8}$$

where $\hat{\chi}$ is the Fourier transform of χ.

End of proof of Theorem 5.1. Since the Fourier-L^1 norm is invariant under affine transformations and can be estimated by seminorms in $\mathcal{S}$, we obtain from (5.8) that

$$\|a_{j,\chi_j}^w\|_{tr} \leqq C\|a_j^w\|_{tr}. \tag{5.9}$$

Now, [8, Theorem 3.9] gives

$$\|a_j^w\|_{tr} \leqq C_N\left(\|a_j\|_{L^1}+h(w_j)^k(\det g_{w_j})^{-1/2}\sup|a_j|_N^{g_{w_j}}\right) \tag{5.10}$$

with N depending on k. This implies

$$\sum\|a_j^w\|_{tr} \leqq C_k(\|a\|_{L^1}+\|h^k m\|_{L^1}\|a\|) \tag{5.11}$$

for every $k>0$, where $\|a\|$ is a seminorm of a in $S(m, g)$ only depending on k. Combined with (5.7) and (5.9), this proves the theorem.

Proof of Lemma 5.2. We shall prove (5.8) by Fourier decomposition of $\chi(x, y)\in\mathcal{S}(\mathbf{R}^{2n})$. Let $L(x, y)=L_1(x)+L_2(y)$ be a linear form on $\mathbf{R}^{2n}$ and put

(5.12)
$$a_L^w u(x) = (2\pi)^{-n}\iint\exp\left(i\langle x-y, \xi\rangle+iL(x, y)\right)a\left(\tfrac{1}{2}(x+y), \xi\right)u(y)\,dy\,d\xi,$$

$u\in C_0^\infty(\mathbf{R}^n)$. Then

$$a_L^w = \exp(iL_1(x))\circ a^w\circ\exp(iL_2(x)),$$

which gives

$$\|a_L^w\|_{tr} = \|a^w\|_{tr} \tag{5.13}$$

by (5.2), since multiplication by $\exp(iL_j(x))$ is unitary on $L^2(\mathbf{R}^n)$. Fourier decomposition of $\chi(x,y)$ gives

$$\|a_\chi^w\|_{tr} \leqq (2\pi)^{-2n}\|\hat{\chi}\|_{L^1}\|a^w\|_{tr},$$

since the trace class norm depends continuously on χ in $\mathscr{S}(\mathbf{R}^{2n})$. This proves the lemma.

6. The Weyl formula

In this section we shall generalize Hörmander's estimate [8, Th. 4.1] of the error term in the Weyl formula for the number $N(\lambda)$ of eigenvalues $\leqq\lambda$,

$$N(\lambda) \cong (2\pi)^{-n}\iint_{p(x,\xi)\leqq\lambda} dx\,d\xi$$

for certain pseudodifferential operators with symbol $p(x,\xi)$. In fact, Hörmander's proof of that result goes through for the locally temperate case, with minor changes. We therefore only state the results.

Let g be a metric on $\mathbf{R}^{2n}$ which is locally σ temperate and satisfies $g/g^\sigma\leqq h^2\leqq 1$. Assume that p is a locally σ, g temperate function, such that p is a symbol of weight p, i.e. $p\in S(p,g)$.

In what follows, we assume that the cut-off functions $\chi\in S(1,\tilde{G})$ are supported in a sufficiently small $\tilde{G}$ neighborhood of the diagonal, so that a_χ^w is L^2 continuous when $a\in S(1,g)$.

Proposition 6.1. *Let $p\in S(p,g)$ such that $p\leqq ch^{-N}$ and assume that $\chi(x,x)\equiv 1$ and $\overline{\chi(x,y)}=\chi(y,x)$. Then p_χ^w defines a self-adjoint operator P on L^2 which is bounded from below. If $p(x,\xi)\to\infty$ when $(x,\xi)\to\infty$, then P has discrete spectrum.*

The proof is just a modification of the proof of [8, Th. 3.4]. Observe that we can impose any restriction on the support of χ in the proof. In fact, if $\psi\in S(1,\tilde{G})$ has support in a sufficiently small $\tilde{G}$ neighborhood of the diagonal, $|\chi|\geqq c>0$ on supp ψ and $\psi=\chi$ in a neighborhood of the diagonal, then Corollary 2.5 gives

$$p_\chi^w = p_\psi^w + r_\chi^w,$$

where $r\in S(h^N p, g)\subseteq S(1,g)$, so r_χ^w is L^2 continuous.

Let $p\in S(p,g)$ satisfy

$$\sup g/g^\sigma = h^2 \leqq cp^{-2\gamma},\quad \gamma>0, \tag{6.1}$$

and

$$1+|x|+|\xi| \leqq cp(x, \xi)^N. \tag{6.2}$$

Let $\chi \in S(1, \tilde{G})$ satisfy $\chi(x, x) \equiv 1$ and $\chi(x, y) = \overline{\chi(y, x)}$. Let $N(\lambda)$ be the number of eigenvalues $\leqq \lambda$ of $P = p_\chi^w$ and put

$$W(\lambda) = (2\pi)^{-n} \iint_{p(x, \xi) \leqq \lambda} dx\, d\xi. \tag{6.3}$$

The methods of [8] and the results of the earlier sections give the following result.

Theorem 6.2. *If* $0 < \delta < 2\gamma/3$, *then there exists a constant* C_δ *such that*

$$|N(\lambda) - W(\lambda)| \leqq C_\delta \big(W(\lambda + \lambda^{1-\delta}) - W(\lambda - \lambda^{1-\delta})\big) \tag{6.4}$$

for large λ.

Observe that the right-hand side of (6.4) tends to ∞ with λ (see [8, p. 309]).

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Received May 28, 1984

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