Differentiability properties of Bessel potentials and Besov functions

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1. Introduction

A classical result in the theory of differentiability of functions is the theorem of Rademacher. It states that if the function $f\colon \mathbf{R}^n \to \mathbf{R}^m$ is Lipschitzian, then f is differentiable at almost all points of $\mathbf{R}^n$ with respect to Lebesgue measure. We will study similar theorems for functions belonging to the Bessel potential spaces $L_k^p(\mathbf{R}^n)$ and the Besov spaces $\Lambda_k^{p,q}(\mathbf{R}^n)$.

In this paper we always consider differentiability in the ordinary sense. The main result is that if the function $f \in L_k^p(\mathbf{R}^n)$ (or $\Lambda_k^{p,q}(\mathbf{R}^n)$) and if f satisfies a condition of Lipschitz type, then f is differentiable except on a set of appropriate capacity zero, v. Section 3, Theorem 2 and Corollary 1. If $kp \leqq n$ there exist e.g. Bessel potentials $f \in L_k^p(\mathbf{R}^n)$ which are essentially unbounded in the neighbourhood of every point. An additional condition is therefore necessary to assure ordinary differentiability. Examples showing that our conditions and exceptional sets are best possible in a certain sense are given in Section 6. If $(k-1)p > n$ it is well known that if $f \in L_k^p(\mathbf{R}^n)$ or $f \in \Lambda_k^{p,q}(\mathbf{R}^n)$, then f has continuous derivatives. Therefore the investigation is restricted to the case $(k-1)p \leqq n$. Ordinary differentiability of Bessel potentials except on sets that are small in capacity have been studied also by Calderón, Fabes and Rivière [8] and by Sjödin [15]. Aronszajn, Mulla and Szeptycki [5] have studied a special pointwise derivative of a function redefined on an exceptional set. Bagby, Deignan, Meyers, Neugebauer and Ziemer have treated derivatives in the sense of Calderón and Zygmund, v. [6], [9], [12] and [13].

In this paper notation and basic definitions can be found in Section 2, together with a survey of various forms of differentiability. The main results on differentiability are stated in Section 3 and proved in Section 5. Section 4 contains a study of the Besov capacities. In Section 6 we give examples showing that the exceptional sets are essentially as small as possible with respect to the capacity. In Section 7 we discuss generalized Morrey spaces.

2. Notation and preliminaries

2.1. Notation and basic definitions

Let $\mathbf{R}^n$, $n \geqq 1$, denote the n-dimensional Euclidean space. Let $L^p(\mathbf{R}^n)$, $1 \leqq p \leqq \infty$, denote the usual Lebesgue space of measurable functions on $\mathbf{R}^n$ and let $\|\cdot\|_p$ denote the norm. The space of Bessel potentials $L_k^p(\mathbf{R}^n)$ is defined by

$$L_k^p(\mathbf{R}^n) = \{G_k * g \mid g \in L^p(\mathbf{R}^n)\}.$$

Here G_k denotes the Bessel kernel of order $k>0$, v. [15]. The norm in $L_k^p(\mathbf{R}^n)$ is given by $\|G_k * g\|_{k,p} = \|g\|_p$. For $0<k<\infty$, $1<p<\infty$, we define, for a compact set $E \subset \mathbf{R}^n$ the Bessel capacity

$$B_{k,p}(E) = \inf \|f\|_{k,p}^p$$

where the infimum is taken over all functions $f \in C_0^\infty(\mathbf{R}^n)$ such that $f \geqq 1$ on E. C_0^∞ denotes the infinitely differentiable functions on $\mathbf{R}^n$ with compact support. The Bessel capacity has several equivalent definitions, v. [3] and [11]. A relation which holds except on a set E, $B_{k,p}(E)=0$, is said to hold $B_{k,p}$-almost everywhere ($B_{k,p}$-a.e.).

For $0<k<1$ the Besov or Lipschitz space $\Lambda_k^{p,q}(\mathbf{R}^n)$ consists of all functions f for which the norm is finite, i.e.

$$\|f\|_{k,p,q} = \|f\|_p + \left\{\int_{\mathbf{R}^n} \frac{\|f(x+t)-f(x)\|_p^q}{|t|^{n+kq}}\, dt\right\}^{1/q} < \infty.$$

For $1 \leqq k < 2$ the first difference is replaced by the second difference. For $k>1$ the space $\Lambda_k^{p,q}(\mathbf{R}^n)$ consists of those f for which

$$\|f\|_{k,p,q} = \|f\|_p + \sum_{j=1}^n \left\|\frac{\partial f}{\partial x_j}\right\|_{k-1,p,q} < \infty.$$

Here the derivatives are taken in the sense of distribution, v. [16], Chapter 5.

For $0<k<\infty$, $1<p<\infty$, $1<q \leqq \infty$, we define for a compact set $K \subset \mathbf{R}^n$ the Besov capacity

$$A_{k,p,q}(K) = \inf \|f\|_{k,p,q}^p$$

where the infimum is taken over all $f \in C_0^\infty(\mathbf{R}^n)$ such that $f \geqq 1$ on K. The extension to all sets is made by

$$A_{k,p,q}(E) = \sup_{K \subset E} A_{k,p,q}(K).$$

If the function $\Psi \in \Lambda_\gamma^{p,q}(\mathbf{R}^n)$, $0<\gamma<k$, then $f = G_{k-\gamma} * \Psi$ is well defined, i.e.

$$\int G_{k-\gamma}(x-y)\,|\Psi(y)|\, dy < \infty$$

a.e. in $\mathbf{R}^n$ and $A_{k,p,q}$-a.e. when $p>1$, v. Section 4. Corresponding results apply to Bessel potentials. We restrict this investigation to the case $p>1$, as other methods are required for $p=1$ and weaker results expected, cf. [5], Theorem 13.5.

2.2. Definitions of and relations between different kinds of differentiability

The following concepts of differentiability are to be used presently. A function f is differentiable (or has an ordinary derivative) at a point $x_0 \in \mathbf{R}^n$, if f is defined in a neighbourhood of x_0 and there exists a linear function $\Lambda = \Lambda_{x_0}$ from $\mathbf{R}^n$ to $\mathbf{R}^1$, such that

$$\sup_{0<|t|\leq r} \frac{|f(x_0+t)-f(x_0)-\Lambda(t)|}{|t|} \tag{2.1}$$

tends to zero with r.

The following definitions are generalizations of the definition of differentiability at a point:

A function f has a derivative at x_0 in the L^p-sense, $1 \leq p < \infty$, if there is a polynomial P_{x_0} of degree one such that

$$\left(r^{-n}\int_{|y|\leq r} |f(x_0+y)-P_{x_0}(y)|^p\, dy\right)^{1/p} = o(r)$$

as $r \to 0$. This is equivalent to f belonging to the Calderón—Zygmund class $t_p^1(x_0)$, which is the original concept from [7].

Let $B(a, r) = \{x|\ |x-a|<r\}$ denote the open ball and $m(E)$ denote the n-dimensional Lebesgue measure of a set E. A function f is approximately differentiable at x_0 if we can write $f(x_0+y) = P_{x_0}(y) + R_{x_0}(y)$ where $P_{x_0}(y)$ is a polynomial of degree one and if, for each $\varepsilon>0$, the set $E_\varepsilon = \{y|\ |R_{x_0}(y)| \geq \varepsilon |y|\}$ has zero as a point of dispersion, i.e.

$$m(E_\varepsilon \cap B(0, r))/r^n = o(1) \quad \text{as} \quad r \to 0.$$

It is obvious that ordinary differentiability at a point implies L^p-differentiability at that point.

2.3. Definitions of conditions of Lipschitz type

The conditions of Lipschitz type will be defined in this section. We begin with the property used in Theorem 1.

Definition 1. Let $x_0 \in \mathbf{R}^n$ and let f be defined in a neighbourhood of x_0. The function f has property (l_1) at x_0 if for every $\varepsilon>0$ there are a $\delta>0$ and an η, $0<\eta\leqq\varepsilon$, such that $0<|x-x_0|<\delta$ and $|y-x|<\eta|x-x_0|$ imply

$$|f(y)-f(x)|\leqq\varepsilon|x-x_0|. \tag{2.2}$$

If a function f has property (l_1) except on a set of capacity zero, then property (l_1) is weaker than f having a uniform Lipschitz condition everywhere. On the other hand, there exist functions satisfying a nonuniform Lipschitz condition, but not being differentiable on a set of positive capacity, v. Section 6.

For the study of $L_k^p(\mathbf{R}^n)$ and $\Lambda_k^{p,q}(\mathbf{R}^n)$ we define conditions corresponding to (l_1) for $f=G_k*g$ and $f=G_{k-\gamma}*\Psi$, in terms of the functions g and Ψ.

Definition 2. Let $g\in L^p(\mathbf{R}^n)$ and let $f=G_k*g$ be well defined everywhere in a neighbourhood of $x_0\in\mathbf{R}^n$. The function f has property (l_2) at x_0 if for every $\varepsilon>0$ there are a $\delta>0$, and an η, $0<\eta\leqq\varepsilon$, such that

$$\left|\int_{|y|\leqq|t|/2}G_k(y)\big(g(x_0+t-y)-g(x_0+t'-y)\big)\,dy\right|\leqq\varepsilon|t| \tag{2.3}$$

whenever $0<|t|<\delta$ and $|t-t'|<\eta|t|$.

Definition 3. Let $\Psi\in\Lambda_\gamma^{p,q}(\mathbf{R}^n)$ and let $f=G_{k-\gamma}*\Psi$ be well defined everywhere in a neighbourhood of $x_0\in\mathbf{R}^n$. The function f has property (l_2') at x_0 if for every $\varepsilon>0$ there are a $\delta>0$ and an η, $0<\eta\leqq\varepsilon$, such that

$$\left|\int_{|y|\leqq|t|/2}G_{k-\gamma}(y)\big(\Psi(x_0+t-y)-\Psi(x_0+t'-y)\big)\,dy\right|\leqq\varepsilon|t| \tag{2.4}$$

whenever $0<|t|<\delta$ and $|t-t'|<\eta|t|$.

If condition (2.3) in Definition 2 is replaced by the stronger one

$$\int_{|y|\leqq|t|/2}G_k(y)|g(x_0+t-y)-g(x_0+t'-y)|\,dy\leqq\varepsilon|t| \tag{2.5}$$

then $f=G_k*g$ has property (l_3) at x_0. Property (l_3) together with approximate differentiability at x_0 ensures that f is well defined in a neighbourhood of x_0. This can be done in the same way for $f=G_{k-\gamma}*\Psi$.

3. Main results

The differentiability results are of the Rademacher—Stepanoff type. Theorem 1 provides a simple criterion of differentiability. It also serves as an explanation of the role played by the conditions of Lipschitz type, defined in Section 2.3, and as an illustration of the method of proof of the following theorems. The technique was used by Federer in [10], Lemma 3.1.5.

Theorem 1. *Let the function f be defined everywhere in a neighbourhood of $x_0 \in \mathbf{R}^n$. If*

(i) *f is approximately differentiable at x_0,*

(ii) *f has property (l_1) at x_0*

then f is differentiable at x_0.

Theorem 1 is applied to Bessel potentials and Besov functions in the following.

Theorem 2. *Let $f \in L_k^p(\mathbf{R}^n)$, $k>1$, $(k-1)p \leqq n$, $1<p<\infty$. Let $f=G_k*g$, $g \in L^p(\mathbf{R}^n)$. For $B_{k-1,p}$-a.e. $x \in \mathbf{R}^n$ let f be well defined in a neighbourhood of x. Then, for $B_{k-1,p}$-a.e. $x \in \mathbf{R}^n$, the following properties are equivalent.*

(i) *f has property (l_1) at x,*

(ii) *f has property (l_2) at x,*

(iii) *f is differentiable at x.*

Theorem 3. *Let $f \in \Lambda_k^{p,q}(\mathbf{R}^n)$, $k>1$, $(k-1)p \leqq n$, $1<p<\infty$ and $1<q \leqq \infty$. Let $f=G_{k-\gamma}*\Psi$, $\Psi \in \Lambda_\gamma^{p,q}(\mathbf{R}^n)$, $0<\gamma<\min(1, k-1)$. For $A_{k-1,p,q}$-a.e. $x \in \mathbf{R}^n$ let f be well defined in a neighbourhood of x. Then, for $A_{k-1,p,q}$-a.e. $x \in \mathbf{R}^n$, the following properties are equivalent.*

(i) *f has property (l_1) at x,*

(ii) *f has property (l_2') at x,*

(iii) *f is differentiable at x.*

For $kp>n$, the Bessel potential is a continuous function and Theorem 2 has the following corollary.

Corollary 1. *Let $f \in L_k^p(\mathbf{R}^n)$, $k>1$, $n<kp \leqq n+p$, $1<p<\infty$. Then f is differentiable at $B_{k-1,p}$-a.e. x.*

The problem of showing differentiability $A_{k-1,p,q}$-a.e. without an additional condition on f remains open for $kp>n$ and $p \neq q$. In [1], Theorem 5.3, Adams shows that Besov functions, $f \in \Lambda_k^{p,q}(\mathbf{R}^1)$, $1<kp<1+p$, are differentiable a.e. with respect to a related capacity. We summarize the results obtained for Besov capacity and Hausdorff measuıe.

Corollary 2. *Let $f \in \Lambda_k^{p,q}(\mathbf{R}^n)$, $k>1$, $n<kp<n+p$, $1<p<\infty$, $1<q \leqq \infty$.*

(a) *If $q=p$ then f is differentiable at $A_{k-1,p,p}$-a.e. x.*

(b) *For every $\varepsilon>0$ there exists a set E such that f is differentiable for all $x \in \mathbf{R}^n$ except on E, $H_{n-(k-1)p+\varepsilon}(E)=0$.*

The Bessel case $kp \leqq n$ has been studied by Sjödin in [15], Section 4, with conditions similar to (l_3) but of L^p-type. The Bessel case $kp>n$ was treated by Calderón, Fabes and Rivière in [8] with Hausdorff measure as in Corollary 2(*b*). We have generalized their result in Corollary 1.

Remark 1. Let $kp \leqq n$. If property (l_2) in Theorem 2 is changed to (l_3) then f is well defined and this assumption on f unnecessary, v. [17], p. 10. It also applies to the Besov case, v. [18], p. 16.

Remark 2. All results in Theorem 2 and 3 can be given pointwise instead of a.e. with respect to capacity, cf. [17] and [18].

4. Properties of Besov capacities

4.1. For $0<k<\infty$, $1<p<\infty$, and $1<q\leqq\infty$ we define the Besov capacity for a compact set K as

$$A_{k,p,q}(K) = \inf \|f\|_{k,p,q}^p$$

where the infimum is taken over all $f\in C_0^\infty(\mathbf{R}^n)$ such that $f\geqq 1$ on K. The extension to all sets is made by

$$A_{k,p,q}(E) = \sup_{K\subset E} A_{k,p,q}(K) \tag{4.1}$$

where K is compact. An outer extension of $A_{k,p,q}$ is defined as

$$A^*_{k,p,q}(E) = \inf_{G\supset E} A_{k,p,q}(G)$$

where G is an open set. We now give a different definition of Besov capacity,

$$A'_{k,p,q}(K) = \inf \|f\|_{k,p,q}^p$$

where the infimum is taken over all $f\in A_k^{p,q}$ such that $f\geqq 1$, $A^*_{k,p,q}$-a.e. on K. These capacities are equivalent. This follows from Proposition 5.3 in [1], which we state here.

Proposition A. *For all compact sets* $K\subset \mathbf{R}^n$

$$A_{k,p,q}(K) \sim A'_{k,p,q}(K).$$

For $0<\gamma<k$ we define for a set E a third form of capacity

$$A^\gamma_{k,p,q}(E) = \inf \|\Psi\|_{\gamma,p,q}^p$$

where the infimum is taken over all nonnegative Ψ such that $G_{k-\gamma}*\Psi\geqq 1$ on E. We are going to use $A^\gamma_{k,p,q}$ in much the same way as is done in [13].

It is easy to verify that $A_{k,p,q}$, $A'_{k,p,q}$ and $A^\gamma_{k,p,q}$ are capacities, as defined by Meyers in [11], p. 257, with the exception that they are only quasisubadditive. If the sets E_1 and E_2 belong to a σ-additive class of sets which contain the compact sets then

$$A^\gamma_{k,p,q}(E_1\cup E_2) \leqq \text{const}\cdot\left(A^\gamma_{k,p,q}(E_1)+A^\gamma_{k,p,q}(E_2)\right). \tag{4.2}$$

This is easy to show, using the method in [11]. Now let $A'_{k,p,q}$ be extended to all sets as in (4.1). Then these two capacities have the following relation with regard to null sets, cf. [13], Lemma 3 and 5.

Lemma 1. *Let E be a set in $\mathbf{R}^n$. If there exists a constant γ, $0<\gamma<k$, such that $A^{\gamma}_{k,p,q}(E)=0$ then $A'_{k,p,q}(E)=0$.*

Lemma 2. *Let $f\in\Lambda^{p,q}_k(\mathbf{R}^n)$ and let $f=G_{k-\gamma}*\Psi$ where $\Psi\in\Lambda^{p,q}_{\gamma}(\mathbf{R}^n)$, $0<\gamma<\min(1,k)$. Then $G_{k-\gamma}*\Psi$ is well defined, i.e.*

$$\int_{\mathbf{R}^n} G_{k-\gamma}(x-y)\,|\Psi(y)|\,dy<\infty$$

for all x except on a set E with $A'_{k,p,q}(E)=0$.

Proof of Lemma 1. Let $\varepsilon>0$. If $A^{\gamma}_{k,p,q}(E)=0$ then there exists a function $g\geqq 0$, $g\in\Lambda^{p,q}_{\gamma}$, where $G_{k-\gamma}*g\geqq 1$ on E and $\|g\|_{\gamma,p,q}<\varepsilon$. Now $\|G_{k-\gamma}*g\|_{k,p,q}<\text{const}\cdot\varepsilon$. Hence $A'_{k,p,q}(K)<\text{const}\cdot\varepsilon$ for every compact set $K\subset E$ and $A'_{k,p,q}(E)=0$.

Proof of Lemma 2. Let $E=\{x|(G_{k-\gamma}*|\Psi|)(x)=\infty\}$. Now

$$A^{\gamma}_{k,p,q}(E)\leqq A^{\gamma}_{k,p,q}\{x|(G_{k-\gamma}*|\Psi|)(x)\geqq a\} \tag{4.3}$$

$$=A^{\gamma}_{k,p,q}\left\{x\middle|\left(G_{k-\gamma}*\frac{|\Psi|}{a}\right)(x)\geqq 1\right\}\leqq\frac{1}{a}\,\|\,|\Psi|\,\|_{\gamma,p,q}\leqq\frac{1}{a}\,\|\Psi\|_{\gamma,p,q}$$

which tends to zero when $a\in\mathbf{R}^1$ goes to infinity. The result follows from Lemma 1.

To construct a function $f\in\Lambda^{p,q}_k$ which is essentially unbounded in the neighbourhood of every point, v. Section 6.5, we need a function $\varphi\geqq 0$, $\varphi\in\Lambda^{p,q}_{\gamma}$ such that $(G_{k-\gamma}*\Psi)(0)=\infty$. Take a sequence of test functions $\{\varphi_i\}$ for $A^{\gamma}_{k,p,q}(E)$, where $A^{\gamma}_{k,p,q}(E)=0$, such that

$$\|\varphi_i\|_{\gamma,p,q}\leqq 2^{-i}$$

v. (4.3). Now set $\varphi=\sum_{i=1}^{\infty}\varphi_i$. If $A^{\gamma}_{k,p,q}(E)=0$ then there exists $\varphi\in\Lambda^{p,q}_{\gamma}$, $\varphi\geqq 0$, such that $(G_{k-\gamma}*\varphi)(x)=\infty$ on E.

In [13], the following variant of Besov capacity is defined for a compact set $E\subset\mathbf{R}^n$ as

$$B^{p,q}_{\gamma}(E)=\inf\|g\|_{\gamma,p,q}$$

where the infimum is taken over all nonnegative g such that $G_{k-\gamma}*g\geqq 1$ on E. As the difference between the $A^{\gamma}_{k,p,q}$-and the $B^{p,q}_{\gamma}$-capacity is only a power of p,

$$B^{p,q}_{\gamma}(E)=0\quad\text{is equivalent to}\quad A^{\gamma}_{k,p,q}(E)=0. \tag{4.4}$$

4.2. In this section we are going to work out the relations we need between Hausdorff measure and Besov capacity. This has been done for Bessel capacity in [11], Section 8.

Lemma 3. *If* $1<p<\infty$, $1<q\leqq\infty$, *and* $kp<n$, *then there exists a finite positive constant* c, *independent of* r, *such that*

$$A_{k,p,q}(B(0,r)) \leqq cr^{n-kp} \tag{4.5}$$

for $0<r\leqq 1$.

Proof of Lemma 3. Let f be a test function for $A_{k,p,q}(B(0,1))$. Then we have $f(x)\geqq 1$ on $B(0,1)$. This means that $f(x/r)\geqq 1$ on $B(0,r)$. Put $h(x)=f(x/r)$ and observe that for $0<k<2$

$$\|h\|_{k,p,q} = \|h\|_p + \left(\int_{\mathbf{R}^n} \frac{\|h(x+t)+h(x-t)-2h(x)\|_p^q}{|t|^{n+kq}}\,dt\right)^{1/q}.$$

By change of variables, we have

$$\|h\|_p^p = \int_{\mathbf{R}^n} \left|f\left(\frac{x}{r}\right)\right|^p dx = \int_{\mathbf{R}^n} |f(y)|^p r^n\,dy = r^n \|f\|_p^p$$

and

$$\left(\int_{\mathbf{R}^n} \frac{\|h(x+t)+h(x-t)-2h(x)\|_p^q}{|t|^{n+kq}}\,dt\right)^{p/q}$$

$$= \left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n}\left|f\left(\frac{x+t}{r}\right)+f\left(\frac{x-t}{r}\right)-2f\left(\frac{x}{r}\right)\right|^p dx\right)^{q/p} \frac{dt}{|t|^{n+kq}}\right)^{p/q}$$

$$= \left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^n}\left|f\left(y+\frac{t}{r}\right)+f\left(y-\frac{t}{r}\right)-2f(y)\right|^p r^n\,dy\right)^{q/p} \frac{dt}{|t|^{n+kq}}\right)^{p/q}$$

$$= r^{n-kp}\left(\int_{\mathbf{R}^n} \frac{\|f(y+t)+f(y-t)-2f(y)\|_p^q}{|t|^{n+kq}}\,dt\right)^{p/q}.$$

Hence $h(x)\geqq 1$ on $B(0,r), 0<r<1$, and since

$$\|h\|_{k,p,q}^p \leqq \text{const}\cdot r^{n-kp}\|f\|_{k,p,q}$$

it follows that

$$A_{k,p,q}(B(0,r)) \leqq \text{const}\cdot r^{n-kp} A_{k,p,q}(B(0,1)).$$

The desired result follows. For $k\geqq 2$ the noım is defined as

$$\left\|f\left(\frac{x}{r}\right)\right\|_{k,p,q} = \left\|f\left(\frac{x}{r}\right)\right\|_p + \sum_{i=1}^n \left\|\frac{\partial f}{\partial x_i}\left(\frac{x}{r}\right)\right\|_{k-1,p,q}$$

which gives the same result.

The classical Hausdorff measure is defined as follows: Let $h(r)$ be a positive, increasing function in some interval $0<r<r_1$ and let $\lim_{r\to 0} h(r)=0$. If E is an arbitrary set, then the Hausdorff measure with respect to h of E is given by

$$H_h(E) = \lim_{R\to 0}\left\{\inf \sum_{i=1}^\infty h(r_i)\right\}$$

where the above infimum is taken over all countable coverings of E by spheres $B(x_i, r_i)$, $r_i \leqq R$. We note that H_h is a capacity, v. [11]. We do not know if the capacity $A_{k,p,q}$ is countably subadditive, but from Proposition 5.1 in [1], countable quasisubadditivity for $p \leqq q$ is easily deduced. Then using Lemma 3 we find that

$$A_{k,p,q}(E) \leqq \text{const} \cdot H_h(E)$$

if $E \subset \mathbf{R}^n$ and $h(r) = r^{n-kp}$. The following lemma shows the connection between Hausdorff measure and capacity, cf. [1], Proposition 5.4.

Lemma 4. *Let* $kp < n$.

i) *If* $H_h(E) = 0$, $h(r) = r^{n-kp}$, *then* $A_{k,p,q}(E) = 0$ *when* $p \leqq q$.
(ii) *If* $A_{k,p,q}(E) = 0$ *then* $H_h(E) = 0$, $h(r) = r^{n-kp+\varepsilon}$ *where* $\varepsilon > 0$.

Proof of Lemma 4 (ii). As $A_k^{p,q}(\mathbf{R}^n) \subset L^p_{k-\varepsilon/2p}(\mathbf{R}^n)$ for $\varepsilon > 0$, v. [19], p. 478, we have

$$B_{k-\varepsilon/2p,p}(K) \leqq A_{k,p,q}(K) \tag{4.6}$$

for all compact sets K. Hence $B_{k-\varepsilon/2p,p}(E) = 0$. From [11], Theorem 22, p. 290, we deduce that $H_h(E) = 0$, $h(r) = r^{n-(k-\varepsilon/2p)p+\varepsilon/2} = r^{n-kp+\varepsilon}$ and the lemma follows.

5. Proofs of the theorems

5.1. Proof of Theorem 1

Let x_0 be a point where f has property (l_1) and f is approximately differentiable. Let $0 < \varepsilon < 1/2$. Then from condition (l_1) we have a $\delta_1 > 0$ and an η, $0 < \eta \leqq \varepsilon$, such that $|x - x_0| < \delta_1$ and $|y - x| < \eta |x - x_0|$ imply

$$|f(y) - f(x)| \leqq \varepsilon |x - x_0|.$$

Let

$$W = \{y \mid |f(y) - f(x_0) - A(y - x_0)| \leqq \varepsilon |y - x_0|\}.$$

There is a $\delta_2 > 0$ such that

$$\frac{m(B(x_0, r) \setminus W)}{m(B(x_0, r))} < \left(\frac{\eta}{2}\right)^n$$

whenever $0 < r < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$ and take $x \in B(x_0, \delta - \eta\delta)$ and

$$r = \frac{|x - x_0|}{1 - \eta} < \delta.$$

Observe that $B\left(x, \frac{\eta r}{2}\right) \subset B(x_0, r)$ as $|x-x_0|+\frac{\eta r}{2}<r$ and that $B\left(x, \frac{\eta r}{2}\right) \cap W \neq \emptyset$. Now we can choose $y \in B\left(x, \frac{\eta r}{2}\right) \cap W$ and we infer $x \in B\left(y, \frac{\eta r}{2}\right) \subset B(y, \eta|x-x_0|)$

$$|y-x| \leqq \frac{\eta r}{2} = \frac{\eta}{2} \frac{|x-x_0|}{1-\eta} < \eta|x-x_0| \leqq \varepsilon|x-x_0|.$$

Then

$$\begin{aligned} |f(x)-f(x_0)-A(x-x_0)| &\leqq |f(y)-f(x_0)-A(y-x_0)| \\ &+|f(x)-f(y)|+|A(x-y)| \leqq \varepsilon|y-x_0|+\varepsilon|x-x_0| \\ &+c|x-y| \leqq 2\varepsilon|x-x_0|+(\varepsilon+c)\varepsilon|x-x_0| \end{aligned}$$

which shows that f is differentiable at x_0.

5.2. Lemmas for the proofs of Theorem 2 and 3

The notion of approximate differentiability is essential as a technical aid when we prove differentiability. The following lemmas contain a combination of results by Bagby, Ziemer and Neugebauer.

Lemma 5. (a) *Let* $g \in L^p(\mathbf{R}^n)$, $1<p<\infty$ *and* $k>1$. *If* $(G_\beta * |g|)(x_0)<\infty$, $\beta=k-1, k$ *and if*

$$\frac{1}{r^{n-(k-1)p}} \int_{|x-x_0|<r} |g(x)|^p \, dx \to 0 \quad as \quad r \to 0$$

then $f=G_k * g$ *has a derivative at* x_0 *in the* L^p*-sense.*

(b) *If the function* f *has a derivative at* x_0 *in the* L^p*-sense,* $p \geqq 1$, *then* f *is approximately differentiable at* x_0.

The corresponding "quasi everywhere" result is stated in Lemma 6.

Lemma 6. *If* $f \in L^p_k(\mathbf{R}^n)$ *(or* $\Lambda^{p,q}_k(\mathbf{R}^n)$*),* $1<p<\infty$, $1<q\leqq\infty$, $k>1$, *then* f *is approximately differentiable* $B_{k-1,p}$*-a.e. (or* $A_{k-1,p,q}$*-a.e).*

Lemma 5 (a) follows from the remark on p. 198 in [7], v. also [6], p. 133. Lemma 5 (b) is Lemma 4.4 in [6], p. 140.

Proof of Lemma 6. If $f \in \Lambda^{p,q}_k(\mathbf{R}^n)$, then $f \in t^1_1(x)$ except on E, $B^{p,q}_\gamma(E)=0$. This is shown in the proof of Theorem 3, p. 301 in [13] V. also [18]. By (4.4) the corresponding capacity $A^\gamma_{k,p,q}(E)=0$. Then by Lemma 1 and Proposition A, $A_{k,p,q}(E)=0$. If $f \in t^p_1(x)$, $p \geqq 1$, then f is approximately differentiable at x. This is Lemma 4.4, p. 140 in [6] and Lemma 6 is proved.

To prove differentiability "quasi everywhere", we use the following well-known lemma. For $k=0$ it can be found in [16], Ch. 1 and for other values of k in [6], p. 134.

Lemma 7. *Let $f\in L^p(\mathbf{R}^n)$, $p\geqq 1$ and $0\leqq kp<n$, then there exists a set $E\subset\mathbf{R}^n$ with $H_h(E)=B_{k,p}(E)=0$ where $h(r)=r^{n-kp}$, having the following property:*

(i) *If $k>0$,*

$$\lim_{r\to 0}\frac{1}{r^{n-kp}}\int_{B(x_0,r)}|f(x)|^p\,dx = 0 \quad \text{for all} \quad x_0\in\mathbf{R}^n\setminus E.$$

(ii) *If $k=0$,*

$$\lim_{r\to 0}\frac{1}{r^{n}}\int_{B(x_0,r)}|f(x)-f(x_0)|^p\,dx = 0 \quad \text{for all} \quad x_0\in\mathbf{R}^n\setminus E.$$

5.3. Proof of Theorem 3

5.3.1. We choose to prove Theorem 3, as the proof of Theoıem 2 is similar, but less interesting. Let $f\in\Lambda_k^{p,q}(\mathbf{R}^n)$, $k>1$, $(k-1)p\leqq n$, $1<p<\infty$ and $1<q\leqq\infty$. Let $f=G_{k-\gamma}*\Psi$, $\Psi\in\Lambda_\gamma^{p,q}(\mathbf{R}^n)$ where $0<\gamma<\min(1, k-1)$. Lemma 2, Section 4.1, shows that $(G_{k-\gamma-\beta}*|\Psi|)(x_0)<\infty$ for $\beta=0, 1$ except on a set E where $A'_{k-1,p,q}(E)=0$. Then from Proposition A we get $A_{k-1,p,q}(E)=0$. Using Lemma 6 in Section 5.2, we can also choose E such that f is approximately differentiable at every point in $\mathbf{C}E$.

5.3.2. From Theorem 1 and Lemma 6 it follows that (i) implies (iii). In this section we are going to prove that (ii) implies (iii). Consider the set E in Section 5.3.1. We take an $x_0\in\mathbf{C}E$, fixed but arbitrary, such that f is well defined in a neighbourhood of x_0 and

(5.1) $$(G_{k-\gamma-\beta}*|\Psi|)(x_0)<\infty \quad \text{for} \quad \beta=0,1;$$

(5.2) f is approximately differentiable at x_0,

(5.3) f has property (l'_2) at x_0.

The method of pıoof is basically the same as in Theorem 1. Choose ε, $0<\varepsilon<1/2$ arbitrarily. From (l'_2) we have a $\delta_1>0$ and an η, $0<\eta\leqq\varepsilon$, such that $|t|<\delta_1$ and $|t-t'|<\eta|t|$ imply

$$\left|\int_{|y|<|t|/2}G_{k-\gamma}(y)\big(\Psi(x_0+t-y)-\Psi(x_0+t'-y)\big)\,dy\right|\leqq\varepsilon|t|.$$

Let

$$E_\varepsilon=\{x_0+t'\,|\,|f(x_0+t')-f(x_0)-A(t')|\leqq\varepsilon|t'| \quad\text{and}\quad (G_{k-\gamma}*|\Psi|)(x_0+t')<\infty\}.$$

As in the proof of Theorem 1 we choose t' such that $x_0+t'\in B(x_0+t, \eta|t|)\cap E_\varepsilon$, v. [18].

To show that f is differentiable at x_0 we have to show that

$$|t|^{-1}|f(x_0+t)-f(x_0+t')| \tag{5.4}$$

tends to zero as $|t|$ tends to zero. By assumption, we have absolute convergence of the integrals in (5.4). Decomposition of the integrals gives

$$\begin{aligned}|t|^{-1}|f(x_0+t)-f(x_0+t')| &= |t|^{-1}|G_{k-\gamma}*\Psi)(x_0+t)-(G_{k-\gamma}*\Psi)(x_0+t')| \\ &\leqq |t|^{-1}\left|\int_{|y|\leqq 2|t|}\Psi(x_0+y)\big(G_{k-\gamma}(t-y)-G_{k-\gamma}(t'-y)\big)\,dy\right| \\ &+|t|^{-1}\left|\int_{|y|>2|t|}\Psi(x_0+y)\big(G_{k-\gamma}(t-y)-G_{k-\gamma}(t'-y)\big)\,dy\right| = I_1+I_2.\end{aligned} \tag{5.5}$$

First we consider I_2, cf. [2], p. 878. We get

$$I_2 \leqq |t|^{-1}\int_{|y|>2|t|}|\Psi(x_0+y)|\,|G_{k-\gamma}(t-y)-G_{k-\gamma}(t'-y)|\,dy$$

$$\leqq |t|^{-1}\int_{|y|>2|t|}|\Psi(x_0+y)|\sum_{i=1}^{n}|t_i-t_i'|\sup_{0<\theta_i<1}\left|\frac{\partial}{\partial x_i}G_{k-\gamma}(-y+t+\theta_i(t-t'))\right|dy$$

$$\leqq \varepsilon c_1\int_{2|t|<|y|<1}|\Psi(x_0+y)|\frac{1}{|y|^{n-(k-\gamma-1)}}\,dy+\varepsilon c_2\int_{|y|\geqq 1}|\Psi(x_0+y)|e^{-c|y|}\,dy$$

observing that $|t_i-t_i'|\leqq|t-t'|\leqq\varepsilon|t|$ where $i=1, \dots, n$. Each term tends to zero when ε tends to zero as the integrals are convergent.

The integral I_1 remains to be considered. Change of variables gives

$$\begin{aligned}I_1 &= |t|^{-1}\left|\int_{|t-y|\leqq 2|t|}\Psi(x_0+t-y)G_{k-\gamma}(y)\,dy-\int_{|t'-y|\leqq 2|t|}\Psi(x_0+t'-y)G_{k-\gamma}(y)\,dy\right| \\ &\leqq |t|^{-1}\left|\int_{|y|\leqq|t|/2}\big(\Psi(x_0+t-y)-\Psi(x_0+t'-y)\big)G_{k-\gamma}(y)\,dy\right| \\ &+|t|^{-1}\int_{|t|/2\leqq|y|\leqq 3|t|}|\Psi(x_0+t-y|G_{k-\gamma}(y)\,dy \\ &+|t|^{-1}\int_{|t|/2\leqq|y|\leqq 7|t|/2}|\Psi(x_0+t'-y)|\,G_{k-\gamma}(y)\,dy = I_{11}+I_{12}+I_{13}.\end{aligned}$$

Well known properties of the Bessel kernel (v. [16], p. 132) and a change of variables give the estimate

$$\begin{aligned}I_{12} &\leqq c_1|t|^{-1}\int_{|t|/2\leqq|y|\leqq 3|t|}|\Psi(x_0+t-y)|\,|y|^{-n+k-\gamma}\,dy \\ &\leqq c_2|t|^{-1}\int_{|t|/2\leqq|y|\leqq 3|t|}|\Psi(x_0+t-y)|\left(\frac{|t|}{2}\right)^{-n+k-\gamma}dy \\ &\leqq c_3\int_{|y|\leqq 4|t|}|\Psi(x_0-y)|\,|y|^{-n+k-\gamma}\,dy.\end{aligned}$$

As $(G_{k-\gamma-1}*|\Psi|)(x_0)$ is convergent, I_{12} tends to zero with $|t|$. The integral I_{13} is estimated in the same way. As f has property (l_2'), I_{11} tends to zero when $|t|$ tends to zero. This completes the proof of the fact that (ii) implies (iii).

5.3.3. In this section we prove that (iii) implies (ii) in Theorem 2, i.e. differentiability of $f=G_{k-\gamma}*\Psi$ at x_0 implies that f has property (l_2') at x_0. That (iii) implies (i) follows in the same way. Choose $\varepsilon>0$ arbitrarily.

There exist a linear function Λ and to the $\varepsilon>0$ a $\delta>0$ so that $|x-x_0|<\delta$ implies

$$|f(x)-f(x_0)-\Lambda(x-x_0)| \leqq \varepsilon|x-x_0|.$$

Now take t such that $|t|<\delta(1+\varepsilon)^{-1}<\delta$ and then t' such that $|t-t'|<\eta|t|$, where η is chosen such that $0<\eta\leqq\varepsilon$. We find that

$$|t'| \leqq (\eta+1)|t| \leqq (\varepsilon+1)|t| < \delta.$$

Consider

$$\Big|\int_{|y|\leqq|t|/2} G_{k-\gamma}(y)(\Psi(x_0+t-y)-\Psi(x_0-t'-y))\,dy\Big|$$

$$\leqq \Big|\int_{\mathbf{R}^n} G_{k-\gamma}(y)\Psi(x_0+t-y)\,dy-\int_{\mathbf{R}^n} G_{k-\gamma}(y)\Psi(x_0-y)\,dy-\Lambda_{x_0}(t)\Big|$$

$$+\Big|-\int_{\mathbf{R}^n} G_{k-\gamma}(y)\Psi(x_0+t'-y)\,dy+\int_{\mathbf{R}^n} G_{k-\gamma}(y)\Psi(x_0-y)\,dy+\Lambda_{x_0}(t')\Big|$$

$$+\Big|-\int_{|y|>|t|/2} G_{k-\gamma}(y)\Psi(x_0+t-y)\,dy+\Lambda_{x_0}(t)$$

$$+\int_{|y|>|t|/2} G_{k-\gamma}(y)\Psi(x_0+t'-y)\,dy-\Lambda_{x_0}(t')\Big| \leqq \varepsilon|t|+\varepsilon|t'|+c\varepsilon|t| \leqq c_1\varepsilon|t|.$$

The estimate

$$\Big|\int_{|y|>|t|/2} G_{k-\gamma}(y)\Psi(x_0+t-y)\,dy-\int_{|y|>|t|/2} G_{k-\gamma}(y)\Psi(x_0+t'-y)\,dy\Big| \leqq c\varepsilon|t|$$

follows from Section 5.3.2 in the proof of Theorem 3. This completes the proof of the theorem.

5.4. Proof of Corollary 1

We consider the Bessel potentials $f=G_k*g$, $g\in L^p(\mathbf{R}^n)$, $kp>n$ and $1<p<\infty$. Then $G_k\in L^{p'}$, $1/p+1/p'=1$, so that $G_k*|g|$ are finite continuous functions, v. [2], p. 879. Now we can use the method of the proof of Theorem 3, v. Section 5.3.2. Only the integral corresponding to I_{11} needs to be considered. This is the "bad" part of

$$(5.4)\qquad |t|^{-1}|f(x_0+t)-f(x_0+t')| = |t|^{-1}|(G_k*g)(x_0+t)-(G_k*g)(x_0+t')|.$$

Estimating the integrals above as in Section 5.3.2, the only remaining part is I_{11}, i.e.

$$|t|^{-1}\left|\int_{|y|\leqq|t|/2}(g(x_0+t-y)-g(x_0+t'-y))\,G_k(y)\,dy\right|$$
$$\leqq |t|^{-1}\int_{|y|\leqq|t|/2}|g(x_0+t-y)|\,G_k(y)\,dy+|t|^{-1}\int_{|y|\leqq|t|/2}|g(x_0+t'-y)|\,G_k(y)\,dy.$$

The last two terms are treated in the same way. Using Hölder's inequality we get

$$|t|^{-1}\int_{|y|\leqq|t|/2}|g(x_0+t-y)|\,G_k(y)\,dy$$
$$\leqq c|t|^{-1}\Big(\int_{|y|\leqq|t|/2}|g(x_0+t-y)|^p\,dy\Big)^{1/p}\Big(\int_{|y|\leqq|t|/2}|y|^{(-n+k)p'}\,dy\Big)^{1/p'}$$
$$\leqq c\Big(|t|^{-n+(k-1)p}\int_{|s|\leqq 2|t|}|g(x_0+s)|^p\,ds\Big)^{1/p},$$

which tends to zero when $|t|$ tends to zero, v. Lemma 5. The case $k\geqq n$ can be taken care of as in [17], p. 15.

5.5. Proof of Corollary 2

For the case $kp>n$ we consider the Besov function $f=G_{k-\gamma}*\Psi$, $\Psi\in\Lambda_\gamma^{p,q}(R^n)$. Then $\Psi\in L_\beta^p$, $\gamma>\beta$ (v. [19], p. 441), and hence by Sobolev's theorem (v. [4], p. 97)

$$\Psi\in L^r,\quad \frac{1}{r}=\frac{1}{p}-\frac{\beta}{n}>\frac{1}{p}-\frac{\gamma}{n}.$$

When $(k-\gamma)r>n$ we have $G_{k-\gamma}\in L^{r'}$, $1/r+1/r'=1$. These two inequalities involving r give $\dfrac{n}{k-\gamma}<r<\dfrac{np}{n-\gamma p}$. Such an r exists when $kp>n$. We see that $G_{k-\gamma}*\Psi$ is a finite, continuous function, cf. [2], p. 879.

Part (a) is proved for $n>1$ as in [1], Theorem 5.3 for $n=1$. Part (b) is proved as Corollary 1 using $\Psi\in\Lambda_\gamma^{p,q}(\mathbf{R}^n)\subset L^p(\mathbf{R}^n)$. For an x_0 where $(G_{k-\gamma-\beta}*|\Psi|)(x_0)<\infty$, $\beta=0, 1$ and where f is approximately differentiable we prove differentiability as in Section 5.3.2. For the "bad" part of the integral we estimate

$$|t|^{-1}\int_{|y|\leqq|t|/2}|\Psi(x_0+t-y)|\,G_{k-\gamma}(y)\,dy$$
$$\leqq c\Big(|t|^{-n+(k-1-\gamma)p}\int_{|s|\leqq 2|t|}|\Psi(x_0+s)|^p\,ds\Big)^{1/p}.$$

This tends to zero with $|t|$ according to Lemma 7 except on a set E, $H_{n-(k-1-\gamma)p}(E)=0$. Here γ can be chosen as small as we wish. The assumptions on f are valid $A_{k-1,p,q}$-a.e., hence $H_{n-(k-1)p+\varepsilon}$-a.e. This completes the proof of the corollary.

6. Examples and counterexamples

Examples showing that our conditions and exceptional sets are essentially as good as possible are presented in this section. The examples are stated in the Besov case. The Bessel case is almost identical cf. [17].

6.1.1. The first example shows that if $f\in L_k^p(\mathbf{R}^n)$ or $f\in \Lambda_k^{p,q}(\mathbf{R}^n)$, $kp\leqq n$, the Rademacher—Stepanoff Theorem cannot be generalized to yield differentiability $B_{k-1,p}$-a.e. or $A_{k-1,p,q}$-a.e. We construct a function $f\in \Lambda_k^{p,q}(\mathbf{R}^n)$ satisfying a uniform Lipschitz condition on a generalized Cantor set E of positive capacity. The function is not differentiable at any point of E.

6.1.2. We start with the constiuction of the set E. Given $k>1, p>1$ and $(k-1)p\leqq n$, we use the method in [2], p. 899, to construct the set E with $B_{k-1,p}(E)>0$ and $A_{k-1,p,q}(E)>0$, where $q\leqq p$, v. [1], p. 64.

Let $E_0=[0, 1]$ and for $w<(k-1)p<n$ let $l_0=1$, $l_j-2l_{j+1}>0$ and

$$l_j = 2^{-j\frac{n}{n-w}}, \quad j = 0, 1, 2, \dots .$$

Given E_j, E_{j+1} is formed by removing from the center of each intervall E_j an open intervall I_j of length

$$r_j = l_j - 2l_{j+1}.$$

We arrive at a set $E=\bigcap_{j=0}^{\infty} E_j^n$ where E_j^n is the Cantor product of n copies of E_j. Denote each of the 2^{nj} "cubes" I_j^n by R_{ij} and the center of R_{ij} by x_{ij}.

6.1.3. To construct a function which is non-differentiable on the set E we use an example by Sjödin. Choose functions f_{ij} in $C_0^\infty(\mathbf{R}^n)\subset L_k^p(\mathbf{R}^n)\cap \Lambda_k^{p,q}(\mathbf{R}^n)$ such that $0\leqq f_{ij}\leqq r_j$ with the maximum value

$$f_{ij}(x_{ij}) = r_j \tag{6.1}$$

and the properties

$$\|f_{ij}\|_{k,p,q} = c2^{-j\left(n+\frac{n(n+p)}{n-w}+1\right)} \tag{6.2}$$

$$\operatorname{supp} f_{ij} \subset B(x_{ij}, r_j/4). \tag{6.3}$$

To get these conditions, take $\{x_{ij}\}$ as the set K in the definition of the capacity $A_{k,p,q}$ in Section 4.1. As $A_{k,p,q}(\{x_{ij}\})=0$, $kp\leqq n$, the norm $\|f_{ij}\|_{k,p,q}$ is at our disposal, cf. [18].

Define $f=\sum_{j=0}^{\infty}\sum_{i=1}^{2^{nj}} f_{ij}$. Then $f\in \Lambda_k^{p,q}$, as (6.2) yields

$$\|f\|_{k,p,q} \leqq \sum_{j=0}^{\infty}\sum_{i=1}^{2^{nj}} \|f_{ij}\|_{k,p,q} \leqq \sum_{j=0}^{\infty} 2^{-j} < \infty. \tag{6.4}$$

When $x\in E$ we have $f(x)=0$. When $x\notin E$ at most one function $f_{ij}(x)\neq 0$. Take $x_0\in E$ arbitrarily and let $t\neq 0$. Then for some i, j

$$|f(x_0+t)-f(x_0)| = |f_{ij}(x_0+t)| \leqq r_j \leqq 4|t| \tag{6.5}$$

as $|t|\geqq r_j/4$ when f_{ij} differs from zero. Hence f satisfies a Lipschitz condition on E.

6.1.4. We prove that f is not differentiable on E. Take $x_0\in E$ arbitrarily and take a subsequence of $\{x_{ij}\}_{i,j}$, defined in Section 6.1.2, converging to x_0. For every chosen x_{ij} we choose the next from $x_{i+1,1}, \ldots, x_{i+1,2^{n(i+1)}}$, such that the distance to x_0 is as small as possible. Hence

$$\frac{|f(x_{ij})-f(x_0)|}{|x_{ij}-x_0|} = \frac{|f_{ij}(x_{ij})|}{|x_{ij}-x_0|} \geqq \frac{r_j}{\sqrt{n}l_j} = (1-2^{-\frac{w}{n-w}})\frac{1}{\sqrt{n}} > 0.$$

We have used (6.1) and $|x_{ij}-x_0|\leqq\sqrt{n}l_j$ where $(1-2^{-\frac{w}{n-w}})/\sqrt{n}>0$ as $w<n$. This proves that the differential of f cannot be zero. On the other hand the L^p derivative is zero for each $x_0\in E$. To show this, take r such that $2^{-M}\leqq r\leqq 2^{-M+1}$ and observe that in $B(x_0, r)$, $x_0\in E$, there are only functions $f_{ij}\neq 0$ in cubes R_{ij} with side length $r_j\leqq 2^{-M+2}$. Hence

$$j \geqq \frac{n-w}{n}(M-2-{}^2\log(1-2^{-\frac{w}{n-w}})) = M_0$$

and we see that $M_0\to\infty$ as $M\to\infty$. We have

$$r^{-n-p}\int_{B(x_0,r)}|f(x)|^p\,dx \leqq \sum_{j=0}^{\infty}\sum_i r^{-n-p}\int_{B(x_0,r)}|f_{ij}(x)|^p\,dx$$

$$\leqq \sum_{j=M_0}^{\infty} 2^{nj+M(n+p)}\|f_{jj}\|_p^p \leqq c\sum_{j=M_0}^{\infty} 2^{-j}$$

using (6.2). This tends to zero as $M\to\infty$, i.e. $r\to 0$. If f were differentiable on E, then the L^p and the ordinary derivative would be equal. Since they are not, f is nowhere differentiable on E.

6.2. The second example is a function $f\in A_k^{p,q}(\mathbf{R}^n)$, $n\geqq kp$, $q\geqq\max(p,2)$, which is nondifferentiable on a set E where $A_{k-1,p,q}(E)=0$ and $A_{k-1+\varepsilon,p,q}(E)>0$. On the set E f satisfies a uniform Lipschitz condition. In $\mathbf{R}^n\setminus E$ f is differentiable and has property (l_2) at points where $f=G_{k-\gamma}*\Psi$ is well defined in a neighbourhood. The construction of the set is made as in Section 6.1.2, with $w=(k-1)p$, v. [18], Section 5.1. Then $H_h(E)<\infty$ where $h(r)=r^{n-(k-1)p}$, $(k-1)p<n$, and $B_{k-1,p}(E)=0$, v. [11], p. 288, Theorem 21. From (3.4) and Lemma 4 in [13], p. 297, it follows that $A_{k-1,p,q}^{\gamma}(E)=0$ at least for $q\geqq\max(p,2)$. Proposition A in Section 3 gives $A_{k-1,p,q}(E)=0$. For the generalized Cantor set E it follows from Theorem 5.1 in [2], p. 899, that $B_{k-1+\varepsilon/2,p}(E)>0$. Then $\sup_{K\subset E} A_{k-1+\varepsilon,p,q}(K)>0$ and thus $A_{k-1+\varepsilon,p,q}(E)>0$.

The function f is constructed as in Section 6.1.3 and has the same properties. It is easy to see that f has property (l_2') where $f=G_{k-\gamma}*\Psi$ is well defined.

6.3. The third example is a function $f\in\Lambda_k^{p,q}(\mathbf{R}^n)$ which is differentiable, but which does not, for any $\delta>0$, satisfy a condition like

$$|f(x_0+t)-f(x_0)-A(t)|\leqq O(|t|^{1+\delta})$$

on a set of positive $A_{k-1,p,q}$-capacity. The construction is the same as in 6.1, except that the maximum values of $f=\sum_{j=0}^{\infty}\sum_{i=1}^{2^{nj}}f_{ij}$ are

$$f_{ij}(x_{ij})=r_j\left(\ln\frac{1}{r^j}\right)^{-1}=o(r_j)$$

as $r_j\to 0$, v. [18], p. 28.

6.4. The fourth example is a function $\varphi\in\Lambda_k^{p,q}(\mathbf{R}^n)$, $(k-1)p<n$, which is non-differentiable on a set E where $A_{k-1,p,q}(E)=0$ and $A_{k-1+\varepsilon,p,q}(E)>0$, $\varepsilon>0$, v. [18], p. 29. Note that the construction of the set E in our first example is possible for $(k-1)p<n$, i.e. also for $kp>n$. Here l_j is defined by

$$l_j^{n-(k-1)p}\left(\ln\frac{1}{l^j}\right)^s=2^{-nj},\quad s>p.$$

We have to modify our earlier construction of the function φ as $A_{k,p,q}(\{x_{ij}\})>0$ for $kp>n$, v. [11], Theorem 20. Choose $\varphi_{ij}=0$ for $|x-x_{ij}|>r_j/4$. Take $\varphi_{ij}\in C_0^\infty$ and $\varphi_{ij}(x_{ij})=r_j$. The derivatives are bounded, say $|D^l\varphi_{ij}(x)|\leqq 8$ and

$$|D^l\varphi_{ij}(x+t)-D^l\varphi_{ij}(x)|\leqq 8\frac{|t|}{r^j}.$$

Now for $\varphi(x)=\sum_{j=0}^{\infty}\sum_{j=1}^{2^{nj}}\varphi_{ij}(x)$ we can prove non-differentiability on E and that $\varphi\in\Lambda_k^{p,q}(\mathbf{R}^n)$, v. [18], p. 29.

6.5. Let $1<p,q<\infty$, $0<kp\leqq n$. There is a function $f\in\Lambda_k^{p,q}(\mathbf{R}^n)$ which is essentially unbounded in the neighbourhood of every point and then f is non-differentiable everywhere. Choose $\varphi\in\Lambda_\gamma^{p,q}$, $\varphi\geqq 0$, such that $(G_{k-\gamma}*\varphi)(0)=\infty$, $0<\gamma<\min(1,k-1)$, cf. Section 4.1. Put $\Psi(y)=\sum_{i=1}^{\infty}2^{-i}\varphi(y-a_i)$ where $\{a_i\}$ is a dense set in $\mathbf{R}^n$ and put $f=G_{k-\gamma}*\Psi$, cf. [16], p. 159.

7. A remark about differentiability conditions in terms of generalized Morrey spaces

Consider $g\in L^p(\mathbf{R}^n)$ and $f=G_k*g\in L_k^p(\mathbf{R}^n)$, $k>1$. Let $g\in E^{\alpha,p}(\mathbf{R}^n)$, a generalized Morrey space, see [14] for the definition.

Convolution of a function $g\in E^{\alpha,p}$, $-n/p\leqq\alpha<1$, $\alpha\geqq 1-k$ and $kp\leqq n$, with the Bessel kernel G_k gives that the resulting function G_k*g is differentiable $(k-1,p)$-a.e. This is proved by showing that for $g\in E^{1-k,p}$, G_k*g has property (l_2), v. [17], p. 28. Hence G_k*g is differentiable for every x_0 where $(G_\beta*|g|)(x_0)<\infty$, $\beta=k-1,k$, v. [17].

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