A superharmonic proof of the M. Riesz conjugate function theorem

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Introduction

Let $f=f+i\tilde{f}$ be analytic in the unit disk U with $\tilde{f}(0)=0$. It is known that

$$\|F\|_p^p \leqq C_p \|f\|_p^p, \quad 1<p<\infty. \tag{0.1}$$

The purpose of this note is to give a simple proof of this theorem of M. Riesz, using superharmonic functions. For the related inequality

$$\|\tilde{f}\|_p^p \leqq C_p' \|f\|_p^p, \quad 1<p<\infty, \tag{0.2}$$

the best constant was determined by S. K. Pichoiides [3] and, independently, by B. Cole (cf. Gamelin [2] p. 144). (The relation between (0.1) and (0.2), with best constants, is discussed in Remark 2 at the end of Section 2.) A related result of Cole is given in Theorem 8.3 in [2]. Our proof will also give the best constant C_p for $1<p<\infty$. We do not use duality to go from the case $1<p<2$ to the case $2<p<\infty$. In Section 3 we discuss similar inequalities for other plane domains.

What the earlier work and our work have in common is the use of sub- or super-harmonic functions. What is new in our approach is how we choose the superharmonic functions.

A similar idea can be found as early as 1935 (cf. Section 4). In Section 5, we use this idea of P. Stein to extend (0.1) to higher dimensions in the case $1<p\leqq 2$.

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1. The case $1<p<2$

Let $w=u+iv$ be a complex variable. If $\alpha=\pi/2p$, we define

$$G(w)=\begin{cases} |w|^p-(\cos\alpha)^{-p}|u|^p, & \alpha<|\arg w|<\pi-\alpha, \\ -\tan\alpha\,|w|^p\cos p\theta, & |\theta|<\alpha, \quad \text{where} \quad \theta=\arg w, \\ -\tan\alpha\,|w|^p\cos p(\pi-|\theta|), & 0\leqq\pi-|\theta|<\alpha. \end{cases}$$

The function G is non-positive on the real axis. We claim that

I. G is superharmonic in $\mathbf{C}$.

II. $|w|^p-(\cos\alpha)^{-p}|u|^p\leqq G(w)$, $w\in\mathbf{C}$.

If (II) is true, we have

$$|w|^p \leqq (\cos\alpha)^{-p}|u|^p+G(w),$$

$$|F(re^{i\theta})|^p \leqq (\cos\alpha)^{-p}|f(re^{i\theta})|^p+G\big(F(re^{i\theta})\big).$$

Integrating over θ and using the fact that $G\circ F$ is superharmonic, we obtain

$$\|F\|_p^p \leqq (\cos\alpha)^{-p}\|f\|_p^p+G\big(F(0)\big),$$

where $G\big(F(0)\big)\leqq 0$. Thus, when $1<p<2$, we have proved (0.1) with

$$C_p = \big(\cos(\pi/2p)\big)^{-p}.$$

In a standard way, we can prove that we have found the best constant. Let $s>p$ and consider $F_1(z)=((1+z)/(1-z))^{1/s}$, $z\in U$, where $F_1=f_1+i\tilde{f}_1$ is defined to be real on the real axis. We have

$$\|F_1\|_p^p = \big(\cos(\pi/2s)\big)^{-p}\|f_1\|_p^p.$$

This is clear since F_1 maps U onto the sector $\{w\colon |\arg w|<\pi/2s\}$. Letting $s\to p+$, we see that $\big(\cos(\pi/2p)\big)^{-p}$ is the best constant. The associated extremal case is a mapping of U onto the sector $\{w\colon |\arg w|<\pi/2p\}$.

It remains to prove (I) and (II).

Proof of (I). We first note that the constants in the definition of G have been chosen in such a way that $G\in C^1(\mathbf{C})$. G is harmonic in sectors containing the real axis. In the remaining sectors,

$$\Delta G(w) = p^2|w|^{p-2}-p(p-1)(\cos\alpha)^{-p}|u|^{p-2}. \tag{1.1}$$

In these sectors, we have $|\cos\theta|<\cos\alpha$, and it follows that

$$|u|^{p-2} = |w|^{p-2}|\cos\theta|^{p-2} > |w|^{p-2}(\cos\alpha)^{p-2},$$

$$\Delta G(w) \leqq p^2|w|^{p-2}\big(1-(\cos\alpha)^{-2}(p-1)p^{-1}\big) \leqq 0.$$

In the last step, we used the inequality

$$\cos^2(\pi/2p) \leqq (p-1)/p, \quad 1<p<2. \tag{1.2}$$

Since $G\in C^1(\mathbf{C})$ and $\Delta G\leqq 0$ a.e., it follows from Green's theorem that for any nonnegative $\varphi\in C_0^\infty$, we have

$$\iint_{\mathbf{C}} G\Delta\varphi = \iint_{\mathbf{C}} \varphi\Delta G \leqq 0.$$

Thus G is superharmonic in $\mathbf{C}$ and (I) is proved.

Proof of (II). It is sufficient to prove that

$$h(\theta) = (\cos\theta)^p(\cos\alpha)^{-p} - 1 - \cos p\theta \tan\alpha \geqq 0, \quad 0 \leqq \theta \leqq \alpha. \tag{1.3}$$

We note that $h(\alpha)=0$. We have $h'(\alpha)=0$, where

$$h'(\theta)/p = -(\cos\theta)^{p-1}\sin\theta\,(\cos\alpha)^{-p} + \tan\alpha \sin p\theta.$$

To prove (1.3), it is sufficient to prove that

$$h'(\theta) \leqq 0, \quad 0 < \theta < \alpha. \tag{1.4}$$

To prove (1.4), we multiply $h'(\theta)/p$ by r^p and obtain the function

$$K(w) = \tan\alpha\,(\operatorname{Im} w^p) - (\cos\alpha)^{-p} v u^{p-1}, \quad 0 < \arg w < \alpha.$$

It is easy to check that $K(r)=K(re^{i\alpha})=0$, $r>0$. Furthermore,

$$\Delta K(w) = -(\cos\alpha)^{-p}(p-1)(p-2)u^{p-3}v > 0, \quad u > 0, \quad v > 0,$$

and thus K is subharmonic in $0<\arg w<\alpha$ (note that $1<p<2!$).

The Phragmén—Lindelöf theorem now shows that $K(w)\leqq 0$ in this sector. Since $h'(\theta)=K(e^{i\theta})p$, we obtain in particular (1.4) and we have finished the proof of (II).

2. The case $2<p<\infty$

Put $\beta=\pi(1-p^{-1})/2=\pi/2q$, where $p^{-1}+q^{-1}=1$, and define

$$J(w) = \begin{cases} |w|^p - |u|^p(\cos\beta)^{-p}, & |u| > |w|\cos\beta, \\ \tan\beta\,|w|^p \cos p(|\theta|-\pi/2), & \big||\theta|-\pi/2\big| \leqq \pi/2p. \end{cases}$$

The function J is non-positive on the real axis. We claim that

I. J is superharmonic in $\mathbf{C}$.

II. $|w|^p \leqq (\cos\beta)^{-p}|u|^p + J(w)$.

Repeating the argument in Section 1, we obtain

$$\|F\|_p^p \leqq (\cos\beta)^{-p}\|f\|_p^p + J(F(0)),$$

where $J(F(0))\leqq 0$. Thus, when $2<p<\infty$, we have proved (0.1) with

$$C_p = (\cos(\pi/2q))^{-p}.$$

Also here, a standard argument using conformal mapping will show that no smaller constant will work in general in (0.1).

Let $s>p$ and $t=s(s-1)^{-1}$. Consider the conformal mapping $F_2=f_2+i\tilde{f}_2$ of U onto the sector $\{w\colon |\arg w-\pi/2|<\pi/2s\}$ with $F_2(0)=i$. We have

$$\|F_2\|_p^p = (\cos(\pi/2t))^{-p}\|f_2\|_p^p. \tag{2.1}$$

In (0.1), we have assumed that $\operatorname{Im} F(0)=0$. Applying (0.1) to $F=F_2-F_2(0)$, we see that

$$(2.2)\qquad (\|F_2\|_p-1)^p \leqq \|F_2-F_2(0)\|_p^p \leqq C_p\|f_2\|_p^p.$$

Combining (2.1) and (2.2), we obtain

$$(\cos(\pi/2t))^{-1}-\|f_2\|_p^{-1} \leqq C_p^{1/p}.$$

When $s\to p+$ and $t\to q-$, $\|f_2\|_p\to\infty$ and it follows that

$$(\cos(\pi/2q))^{-p} \leqq C_p.$$

Thus, we have found the best constant also when $2<p<\infty$. The associated extremal case is a mapping F of U onto the sector $\{w\colon |\arg w-\pi/2|<\pi/2p\}$. We note, however, that we have $\operatorname{Im} F(0)\neq 0$.

The proofs of (I) and (II) are similar to what we did in the case $1<p<2$.

Proof of (I). We first note that $J\in C^1(\mathbf{C})$ and that J is harmonic in sectors containing the imaginaiy axis. In the remaining sectors.

$$\Delta J(w) = p^2|w|^{p-2}-p(p-1)|u|^{p-2}(\cos\beta)^{-p} \leqq |w|^{p-2}\left(p^2-p(p-1)(\cos\beta)^{-2}\right) \leqq 0,$$

which is true since (put $x=q^{-1}$)

$$\cos^2(\pi x/2) \leqq x,\quad 1/2<x<1.$$

As before, it follows that J is superharmonic in $\mathbf{C}$.

Proof of (II). If $\theta=\pi/2-\varphi$, it is sufficient to prove that

$$(2.3)\qquad k(\varphi) = (\sin\varphi)^p(\cos\beta)^{-p}+\tan\beta\cos p\varphi-1 \geqq 0,\ 0<\varphi<\pi/2p=\alpha.$$

We note that $k(\pi/2p)=0$. We have $k'(\varphi)=L(e^{i\varphi})p$, where

$$L(w) = v^{p-1}u(\cos\beta)^{-p}-\tan\beta\,(\operatorname{Im} w^p).$$

As before $L(r)=L(re^{i\alpha})=0$ and $\Delta L(w)\geqq 0$, $0<\arg w<\alpha$. Thus L is nonpositive in this sector, $k'(\varphi)$ is negative in the interval $(0,\alpha)$ and (2.3) holds since $k(\pi/2p)=0$.

Remark 1. The following 'calculus' proof of (1.3) and (2.3) is due to W. H. J. Fuchs and to G. Wanby.

The case $1<p<2$: *proof of* (1.3). Since $\cos p\alpha=0$, (1.3) is equivalent to

$$(2.4)\qquad (\cos p\theta-\cos p\alpha)/((\cos\theta)^p-(\cos\alpha)^p) \leqq (\sin\alpha)^{-1}(\cos\alpha)^{1-p},\ 0\leqq\theta\leqq\alpha.$$

Let $g(\theta)=(\sin p\theta)/(\sin\theta\,(\cos\theta)^{p-1})$. From a classical mean-value theorem, we see that the left hand member of (2.4) is equal to $g(\xi)$ for some $\xi\in(\theta,\alpha)$. If g is an increasing function on $(0, \alpha)$, we have

$$g(\xi) \leqq g(\alpha) = (\sin\alpha)^{-1}(\cos\alpha)^{1-p},$$

i.e., we have proved (2.4).

To prove that g' is positive on $(0, \alpha)$, we differentiate $\log g$, use trigonometrical formulas foı the double angle and find that $2\sin p\theta \sin\theta\cos\theta\,(g'(\theta)/g(\theta)) = p(2-p)(S((2-p)\theta)-S(p\theta))$; where $S(t)=\sin t/t$. Since S is a decreasing function on $(0, \pi/2)$ and $p>2-p>0$, we see that g' is positive on $(0, \alpha)$ and the proof of (2.4) is complete.

The case $p>2$: *proof of* (2.3). Since $\alpha+\beta=\pi/2$, (2.3) is equivalent to

$$(\cos p\varphi - \cos p\alpha)/((\sin\alpha)^p-(\sin\varphi)^p) \geqq (\cos\alpha)^{-1}(\sin\alpha)^{1-p}, \quad 0 \leqq \varphi \leqq \alpha.$$

Let $g(\theta)=(\sin p\theta)/((\sin\theta)^{p-1}\cos\theta)$. The left hand member of our inequality is equal to $g(\xi)$ for some $\xi\in(\varphi, \alpha)$. The same type of argument as above will show that g is decreasing on $(0, \alpha)$ and thus that

$$g(\xi) \geqq g(\alpha) = (\cos\alpha)^{-1}(\sin\alpha)^{1-p}.$$

Remark 2. When $1<p<2$, inequality (0.2) is a consequence of inequality (0.1) provided that C_p' and C_p are the best constants. In condensed notation, we write this statement as

$$(0.1)(C_p) \Rightarrow (0.2)(C_p'), \quad 1<p<2. \tag{2.5}$$

To prove (2.5), we let $\alpha\in(0, 1)$ be given and consider the inequality

$$\left(\int (f+g)^\alpha\right)^{1/\alpha} \geqq \left(\int f^\alpha\right)^{1/\alpha} + \left(\int g^\alpha\right)^{1/\alpha}, \tag{2.6}$$

which holds for nonnegative functions f and g (cf. Theorem 8, p. 26 in [1]). Using (2.6) with $\alpha=p/2<1$, we see that

$$C_p^{2/p}\|f\|_p^2 \geqq \|F\|_p^2 \geqq \|f\|_p^2+\|\tilde f\|_p^2. \tag{2.7}$$

Choosing $\|\tilde f\|_p/\|f\|_p$ close to $(C_p')^{1/p}$, we see that

$$C_p^{2/p} \geqq 1+(C_p')^{2/p}.$$

If $C_p^{1/p}=(\cos(\pi/2p))^{-1}$, it follows that $(C_p')^{1/p}\leqq\tan(\pi/2p)$. From the extremal case, it is clear that $(C_p')^{1/p}\geqq\tan(\pi/2p)$. We have proved (2.5).

If $2<p<\infty$, we have

$$\|F\|_p^2 \leqq \|f\|_p^2+\|\tilde f\|_p^2, \tag{2.7'}$$

and the same kind of argument will show that

$$(0.2)(C_p') \Rightarrow (0.1)(C_p), \quad 2<p<\infty. \tag{2.8}$$

3. The Riesz theorem for more general domains

For a simply connected domain in the plane, (0.1) will hold with the H^p-norm taken as integration over the boundary with respect to harmonic measure (with respect to a fixed point inside the domain) and with the same constants C_p as in the disk. This is clear from the Riemann mapping theorem since harmonic measure is a conformal invariant.

We now turn to domains which are not necessarily simply connected. Let $D\subset U$ be a domain and let $d\omega$ be harmonic measure on the boundary ∂D with respect to a fixed point $z_0\in D$. This means that for each function h which is harmonic in D and continuous on $D\cup\partial D$, we have

$$h(z_0)=\int_{\partial D} h(z)\,d\omega(z).$$

Let $F=f+i\tilde{f}$ be analytic in D, continuous in $D\cup\partial D$ and let $\tilde{f}(z_0)=0$. Then $G\circ F$ and $J\circ F$ are superharmonic in D and we see that

$$\text{(3.1)}\quad \int_{\partial D} G(F(z))\,d\omega(z)\leqq G(F(z_0))\leqq 0,\quad \int_{\partial D} J(F(z))\,d\omega(z)\leqq J(F(z))\leqq 0,$$

$$\int_{\partial D}|F(z)|^p\,d\omega(z)\leqq C_p\int_{\partial D}|f(z)|^p\,d\omega(z),\quad 1<p<\infty.$$

We can use the same constants C_p as in the disk.

If we assume more on the domain D, we can show that also in this case, we have found the best constants. Let us assume that for some $a>0$, we have $\{z\in U\colon |z-1|<a\}\subset D$. Let $D_n=D\cap\{z\in U\colon |z-1|>1/n\}$ and let $d\omega_n$ be the harmonic measure for D_n with respect to $z_0\in D_n$ (which will be true for all large values of n). Let $s>p$ and let F_1 be the conformal mapping of U onto the sector $\{w\colon |\arg w|<\pi/2s\}$ with $\tilde{f}_1(z_0)=0$. Let F_2 be the conformal mapping of U onto the sector $\{w\colon |\arg w-\pi/2|<\pi/2s\}$ with $\tilde{f}_2(z_0)=1$. If F is F_1 or F_2, we have

$$\int_{\partial D_n}|F(z)-i\tilde{f}(z_0)|^p\,d\omega_n(z)\leqq C_p\int_{\partial D_n}|f(z)|^p\,d\omega_n(z).$$

Letting $n\to\infty$, we obtain

$$\text{(3.2)}\qquad \int_{\partial D}|F(z)-i\tilde{f}(z_0)|^p\,d\omega(z)\leqq C_p\int_{\partial D}|f(z)|^p\,d\omega(z).$$

Let us first discuss the case $1<p<2$ where we choose $F=F_1$. We note that

$$\text{(3.3)}\qquad |F_1(e^{i\theta})|\cos(\pi/2s)=|f_1(e^{i\theta})|,\ \ 0<|\theta|\leqq\pi.$$

There exist constants $b\in(0,a)$, c and A such that for all $s\in(p,p+c)$, we have

$$|F_1(z)|\leqq A,\quad z\in\partial D,\quad |z-1|\geqq b>0.$$

Let $\Gamma=\partial D\cap\{z\colon |z-1|<b\}$ which is a circular arc. From (3.2) we deduce that

$$\Big\{\int_\Gamma |F_1(e^{i\theta})|^p\, d\omega(e^{i\theta})\Big\}^{1/p}-A \leqq C_p^{1/p}\Big(\Big\{\int_\Gamma |f_1(e^{i\theta})|^p\, d\omega(e^{i\theta})\Big\}^{1/p}+A\Big).$$

Letting $s\to p+$, the p-norms over Γ will tend to infinity and it follows that $(\cos(\pi/2p))\leqq C_p^{1/p}$. We have proved that $C_p=(\cos(\pi/2p))^p$ is the best constant in the case $1<p<2$. A similar argument will take care of the case $2\leqq p<\infty$.

We note that the argument showing that our constants are best possible holds as soon as the domain D is nice near one point on ∂U.

4. A historical note

The first example in the literature of a subharmonic proof of the theorem of M. Riesz seems to be the proof of a result of P. Stein [5] as presented by A. Zygmund in the first edition of his book on Trigonometrical Series (cf. [6], p. 149), which gives (0.1) with $C_p=p/(p-1)$ in the case $1<p<2$. To see this, we consider $H(w)=|w|^p-p(p-1)^{-1}|u|^p$ which is superharmonic in $\mathbf{C}$ when $1<p\leqq 2$. This is clear since $H\in C^1(\mathbf{C})$ and we have, for $u\neq 0$,

$$\Delta H(w) = p^2|w|^{p-2}-p^2|u|^{p-2} \leqq p^2(|w|^{p-2}-|w|^{p-2}) = 0.$$

Integrating the equality

$$|F(re^{i\theta})|^p = p(p-1)^{-1}|f(re^{i\theta})|^p+H\big(F(re^{i\theta})\big),$$

we deduce from the fact that H is superharmonic that

$$\|F\|_p^p \leqq p(p-1)^{-1}\|f\|_p^p+H\big(F(0)\big),$$

where $H\big(F(0)\big)=-|F(0)|^p(p-1)^{-1}<0$. This concludes the short proof.

5. The Riesz theorem in higher dimensions

The argument quoted in Section 4 can be used to deduce an extension of the Riesz inequality to higher dimensions in the case $1<p\leqq 2$. Let D be a bounded domain in $\mathbf{R}^{d+1}=\{x=(x_0, x_1, \dots, x_d)\colon x_i\in\mathbf{R}, 0\leqq i\leqq d\}$ and let $d\omega$ be harmonic measure on ∂D with respect to a fixed point $P_0\in D$. We consider $F=(u_0, u_1, \dots, u_d)$ and $F_0=(0, u_1, \dots, u_d)$ where the functions $\{u_k\}_0^d$ satisfy the generalized Cauchy—Riemann equations

(5.1) $$\textstyle\sum_0^d \partial u_k/\partial x_k = 0,\ \partial u_k/\partial x_j = \partial u_j/\partial x_k,\quad 0\leqq j,\ k\leqq d,$$

in D. We also assume that F is continuous in $D\cup\partial D$ and that $u_0(P_0)=0$. Then, for $1<p\leqq 2$, we have

$$\int_{\partial D}|F(x)|^p\,d\omega(x)\leqq C(p,d)\int_{\partial D}|F_0(x)|^p\,d\omega(x), \tag{5.2}$$

where we can take $C(p,d)=(d+2)/(p-1)$.

Let us assume that we can find a constant $C=C(p,d)\geqq 1$ such that $G=|F|^p-C|F_0|^p$ is superharmonic in D. Then we have

$$\int_{\partial D}G(x)\,d\omega(x)\leqq G(P_0)\leqq 0,$$

and (5.2) will be proved.

In the proof that G is superharmonic, we argue as E. Stein ([4], pp. 217—219). If $F=\{u_k\}_0^d$ and $H=\{v_k\}_0^d$, we use the notation

$$(F,H)=\sum_0^d u_k v_k,\quad \operatorname{grad} u=\{\partial u/\partial x_j\}_0^d,\quad |\nabla F|^2=\sum_0^d|\partial F/\partial x_j|^2.$$

It follows from (5.1) that $\Delta F=\Delta F_0=0$, because

$$(\Delta F)_k=\sum_0^d\partial^2 u_k/\partial x_j^2=(\partial/\partial x_k)\sum_0^d\partial u_j/\partial x_j=0,\quad k=0,1,2,\dots,d.$$

We have (cf. formula (19) p. 217 in [4])

$$\Delta|F|^p\leqq p|F|^{p-2}|\nabla F|^2,$$
$$\Delta|F_0|^p=p|F_0|^{p-4}\big((p-2)\sum_0^d(\partial F_0/\partial x_j,F_0)^2+|F_0|^2|\nabla F_0|^2\big).$$

Once more using (5.1), we deduce

$$(\partial u_0/\partial x_0)^2=\big(\sum_1^d\partial u_j/\partial x_j\big)^2\leqq d\,|\nabla F_0|^2,$$
$$|\operatorname{grad} u_0|^2\leqq d\,|\nabla F_0|^2+\sum_1^d(\partial u_j/\partial x_0)^2\leqq(d+1)|\nabla F_0|^2,$$
$$|\nabla F|^2=|\operatorname{grad} u_0|^2+|\nabla F_0|^2\leqq(d+2)|\nabla F_0|^2.$$

Since $1<p\leqq 2$, we see that

$$\Delta|F_0|^p\geqq p|F_0|^{p-4}\big((p-2)|F_0|^2|\nabla F_0|^2+|F_0|^2|\nabla F_0|^2\big)=p(p-1)|F_0|^{p-2}|\nabla F_0|^2,$$
$$\Delta(|F|^p-C|F_0|^p)\leqq p|F_0|^{p-2}\big(|\nabla F|^2-C(p-1)|\nabla F_0|^2\big)$$
$$\leqq p|F_0|^{p-2}|\nabla F_0|^2\big(d+2-C(p-1)\big).$$

Choosing $C=(d+2)/(p-1)$, we see that G is superharmonic and we have completed the proof of (5.2).

Remark. If u is harmonic in D and continuous on $D\cup\partial D$, we can take $F=\operatorname{grad} u$ and $F_0=\operatorname{grad}^{(0)}u=(0,\partial u/\partial x_1,\dots,\partial u/\partial x_d)$. If we assume that $(\partial u/\partial x_0)(P_0)=0$, it follows from (5.2) that

$$\int_{\partial D}|\operatorname{grad} u|^p\,d\omega(x)\leqq C(p,d)\int_{\partial D}|\operatorname{grad}^{(0)}u|^p\,d\omega(x),\quad 1<p\leqq 2.$$

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