On the multiplicative properties of the de Rham—Witt complex. I

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Summary

Due to the length this work is published in two parts. The second part will appear in Vol 23: 1 of this journal.

Part 1 has the subtitle "*Duality for the de Rham—Witt complex*" and Part 2 is entitled "*A Künneth formula for the Hodge—Witt complex*".

Abstract

The behaviour of the cohomology of the de Rham—Witt complex of smooth and proper varieties over a perfect field of positive characteristic under products of varieties and duality are considered. A triangulated category is defined and given the structure of a rigid tensor category. A Künneth formula and a duality formula is proved with respect to the tensor product and dual of the tensor category.

Keywords: *Crystalline cohomology, de Rham—Witt complex, duality, Kunneth formulas.*

Both papers are concerned with aspects of the multiplicative structure of the de Rham—Witt complex (cf. [9]). Namely, the behaviour under products of varieties and the behaviour under duality. Usually for cohomology theories the following situation occurs: One has a triangulated category D, an exact bifunctor $(-)\otimes(-)$ on C, having an adjoint $\underline{\mathrm{Hom}}(-,-)$ and an identity for the tensor product 1, giving its dual $\underline{\mathrm{Hom}}(-,1)$. The cohomology of our geometric objects takes its values in C, with the aid of the tensor product the cohomology of a product of two objects are expressed in terms of cohomology of the two objects and with the aid of the dual the cohomology of an object is expressed as the dual of itself together with a shifting depending on the dimension of the object. This will be seen to be true also in the present situation.

In Part I we study the duality aspect of this. In that case it turns out that the full tensor formalism is not necessary and we develop an ad hoc construction sufficient for our needs. We also get a duality formula for the truncated de Rham complex.

In Part II we develop the above-mentioned tensor formalism and apply it to obtain our Künneth formula. We do not, however, obtain a Künneth formula for the truncated de Rham—Witt complex except at level 1 and it seems highly unlikely that such a formula should exist.

The introductions to the individual parts offer a more thorough description of their contents.

Introduction

The purpose of this article is to prove a duality theorem for the de Rham—Witt complex (cf. [8]) of a smooth and proper N-dimensional variety X over a perfect field k of positive characteristic p. As long as we stick to fixed finite levels n things are very much as expected. We will have isomorphisms $H^{N-i}(X, W_n\Omega^{N-j}) \xrightarrow{\sim} \mathrm{Hom}_W(H^i(X, W_n\Omega^j), W_n)$ induced by cup multiplication and a trace map. When we try to patch together these dualities to a duality for the limit $H^i(X, W\Omega^j) = \varprojlim \{H^i(X, W_n\Omega^j)\}$ the situation changes radically. The reason for this is the following. Combining the dualities at each finite level gives us a duality between the prosystem $\{H^i(X, W_n\Omega^j)\}$ and a certain indsystem $\{H^{N-i}(X, W_n\Omega^{N-j})\}$. In order to get an auto-duality for $H^*(X, W\Omega^*)$ we need to be able to express in a covariant way the indsystem $\{H^*(W_n\Omega^*)\}$ with the aid of $H^*(W\Omega^*)$. The means for doing this is the following formula, which appears in the work of Illusie and Raynaud: $R\Gamma(W_n\Omega^\cdot) = R_n \otimes^L_R R\Gamma(W\Omega^\cdot)$, where R is a certain ring operating on $W\Omega^\cdot$ giving the cohomology $R\Gamma(W\Omega^\cdot)$ of $W\Omega^\cdot$ a structure of R-complex and where the cohomology of $R\Gamma(W_n\Omega^\cdot)$ are the $H^*(X, W_n\Omega^*)$. The final result is a canonical isomorphism in $D(R)$: $R\,\mathrm{Hom}_R(R\Gamma(W\Omega^\cdot), \check{R})(-N)[-N] \xrightarrow{\sim} R\Gamma(W\Omega^\cdot)$ where $\check{R}$ is a certain bi-R-module. This should be compared with the crystalline duality formula: $R\,\mathrm{Hom}_W(R\Gamma(X/W), W)[-2N] \xrightarrow{\sim} R\Gamma(X/W)$. There is, however, one major difference. The functor $R\,\mathrm{Hom}_W(-, W)$ is trivial to compute in the cases of interest, whereas there are considerable difficulties in computing $R\mathrm{Hom}_W(-, \check{R})$ in the cases needed for the application of the duality formula. A consequence there of is that the major part of this paper is taken up with the very problem of computing $D(-) := R\mathrm{Hom}_R(-, \check{R})$ on a certain triangulated subcategory of $D(R)$, which by results of Illusie and Raynaud contains $R\Gamma(W\Omega^\cdot)$. One of the obtained results shows that $D(-)$ fulfills the property desired of a dualizing functor, namely biduality. I will show later that we actually get, in the terminology of [11], a rigid tensor category, which is all the more remarkable as R is a non-commutative ring.

I have attempted to compute everything in sight concerning the functor $D(-)$. That this is not an excessive ambition will be seen in the last chapter where we will consider the case of a supersingular $K3$-surface. In that chapter we will also see that the combination of crystalline duality and the present duality for the de Rham—Witt complex sometimes forces $R\Gamma(W\Omega^{\cdot})$ to be more complicated than perhaps expected.

I would like to thank L. Illusie for carefully pointing out obscurities in a version of this manuscript. He is also responsible for the present definition of the trace map, a definition more general and much slicker than my original one. I would also like to thank O. Gabber for putting my attention to a non-proof of mine of I: Lemma 3.4 and to N. Suwa for correcting a mistake in Chapter III. Finally I would like to express my appreciation of the hospitality shown to me by the IHES during the major part of the writing of this article.

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0. Definitions and preliminaries

X will always denote a smooth N-dimensional scheme over $S=\operatorname{Spec} k$, k being a perfect field of positive characteristic p. $|X|$ will denote the underlying topological space. Let us recall (cf. [9]) that there exists an inverse system of sheaves of differential graded algebras on $|X|$:

$$W_{\cdot}\,\Omega_X \xrightarrow{d} W_{\cdot}\,\Omega_X^1 \xrightarrow{d} \dots \xrightarrow{d} W_{\cdot}\,\Omega_X^N \tag{0.1}$$

whose inverse limit is denoted $W\Omega^{\cdot}_X$. $W\mathcal{O}_X$ then is the sheaf of Witt vectors and $W\Omega^{\cdot}_X$ is a commutative graded $W\mathcal{O}_X$-algebra. In particular, the evident morphism $W\to W\mathcal{O}_X$, where W is the constant sheaf of Witt-vectors of k, then makes $W\Omega^{\cdot}_X$ a sheaf of graded W-modules.

There further are (cf. loc. cit.) endomorphisms (as a graded sheaf of groups)

F and V of $W\Omega_X^{\bullet}$ with relations:

(0.2)
$$\omega_1 \cdot V\omega_2 = V(F\omega_1 \cdot \omega_2) \quad F(\omega_1 \cdot \omega_2) = F\omega_1 \cdot F\omega_2$$
$$\omega_1, \omega_2 \in W\Omega_X^{\bullet}$$
$$FV = p = VF; \; FdV = d.$$

As F and V restricted to the constant sheaf W are σ resp. $p\sigma^{-1}$, where σ is the Frobenius automorphism of W, (0.2) shows that $W\Omega_X^{\bullet}$ is a sheaf of graded modules over the Raynaud-ring R, the graded W-ring generated by F and V in degree 0 and d in degree 1 and relations:

(0.3)
$$Fa = a^{\sigma}F \quad aV = Va^{\sigma} \quad FV = VF = p$$
$$da = ad \quad FdV = d \qquad d^2 = 0 \qquad a \in W$$

Every element in R (resp. $R_n := R/dV^nR + V^nR$) may then be written uniquely as a sum:

$$\sum\nolimits_{n>0} a_{-n}V^n + \sum\nolimits_{n\geqq 0} a_nF^n + \sum\nolimits_{n>0} b_{-n}\,dV^n + \sum\nolimits_{n\geqq 0} b_nF^n d \quad a_n, b_n \in W$$

(0.3.1) resp.

$$\sum\nolimits_{n>m>0} a_{-m}V^m + \sum\nolimits_{m\geqq 0} a_mF^m + \sum\nolimits_{n>m>0} b_{-m}\,dV^m + \sum\nolimits_{m\geqq 0} b_mF^m d$$
$$a_{-m}, b_{-m} W/p^{n-m}, \quad n > m > 0, \quad a_m, b_m \in W/p^n, \quad m \geqq 0.$$

Here the description of R is [10, I: 1.1.4—5] and the one of R_n is [10, I: 1.3.3].

From now on, unless otherwise mentioned, all **R**-modules will be graded left modules. We will use the following sign conventions: A complex of R-modules may be regarded as a (naive) double complex of W-modules and sign changes will be made accordingly. That is, $F[i](j)$ which denotes F shifted i times in complex degree and j times in module degree will be affected by a change of d by $(-1)^j$ and by $(-1)^i$ in the differential of F and, furthermore, the sign changes of [2, XVII: 1.1] will be used when applying a functor to F. The reader should note the distinction between the morphism $d\colon F \to F$ of degree 1 and the morphism $d\colon F(-1) \to F$ of degree 0, which on the underlying ungraded object is the negative of the first. The reader should also note that the morphism $d\colon R(-1) \to R$ which is multiplication by d on the left induces $-d\colon F(-1) = R(-1) \otimes_R F \to R \otimes_R F = F$.

If M is an R-module we will denote (cf. [10]):

(0.4.1)
$$\mathrm{Fil}^n M := V^nM + dV^nM$$
$$\mathrm{gr}^n M := \mathrm{Fil}^n M/\mathrm{Fil}^{n+1} M$$
$$\mathrm{gr}_1^n M := \mathrm{Fil}^{n+1} M/p\,\mathrm{Fil}^{n+1} M.$$

Note that $R_n = R/\mathrm{Fil}^n R$. $\mathrm{Fil}^n R$ is a right R-ideal and evidently $p\,\mathrm{Fil}^n M \simeq \mathrm{Fil}^{n+2} M$, so we may define right R-module homomorphisms π and ϱ:

(0.4.2)
$$\begin{array}{ccc} R_{n+1} & \xrightarrow{\pi} & R_n \\ & \searrow^{p} & \downarrow{\scriptstyle\varrho} \\ & & R_{n+1} \end{array}$$

where π is induced from the identity map and ϱ is uniquely defined by the commutativity of the diagram. Using (0.3.1) one sees that ϱ is injective. As $M/\mathrm{Fil}^n M = R_n \otimes_R M$ we get induced maps $M/\mathrm{Fil}^n M \to M/\mathrm{Fil}^{n+1} M$ and $M/\mathrm{Fil}^{n+1} M \to M/\mathrm{Fil}^n M$ which we will also denote ϱ resp. π. From [10, I: Prop. 3.2] we get the following resolution of R_n:

(0.4.3) $$0 \to R(-1) \xrightarrow{(F^n, -F^n d)} R(-1) \oplus R \xrightarrow{dV^n + V^n} R \to R_n \to 0,$$

where F^n, $F^n d$ etc. denote multiplication by F^n, $F^n d$ etc from the left considered as morphisms of degree 0.

The ringed space $(|X|, W_n \mathcal{O}_X)$ is actually a scheme which we will denote $W_n X$. Note that for different n they all have the same underlying topological space but are, of course, all different as schemes.

I will denote the projection $W_n X \to W_n S$ by f_n, the nilimmersion $W_{n-1} X \to W_n X$ induced by the projection $W_n \mathcal{O}_X \to W_{n-1} \mathcal{O}_X$ I will denote j_n and, finally, the nilimmersion $X \to W_n X$ induced by the projection $W_n \mathcal{O}_X \to \mathcal{O}_X$ I will denote j. We will also need the following exact sequence ([9, I: Cor. 3.9]):

(0.5) $$0 \to \Omega^{\cdot}_X \xrightarrow{V^n} \mathrm{gr}^n W\Omega^{\cdot}_X \to (\Omega^{\cdot}_X/Z_n)(-1) \to 0,$$

where the surjection is characterized by the fact that the composite

$$\Omega^{\cdot}(1) \xrightarrow{dV^n} \mathrm{Fil}^n W\Omega^{\cdot} \to \mathrm{gr}^n W\Omega^{\cdot} \to (\Omega^{\cdot}/Z_n(-1)$$

is the natural projection. There is a dual ıesult for $\mathrm{gr}^n_1 W\Omega^{\cdot}_X$ which, even though not found in [9] is easily derivable from it:

Lemma 0.6. *The following two sequences are exact:*

(0.6.1) $$0 \to W_n \Omega^{\cdot}_X \xrightarrow{\varrho} W_{n+1} \Omega^{\cdot}_X \to \mathrm{gr}^n_1 W\Omega^{\cdot}_X \to 0$$

(0.6.2) $$0 \to B_n \Omega^{\cdot}_X(1) \to \mathrm{gr}^n_1 W\Omega^{\cdot}_X \xrightarrow{F^n} Z_n \Omega^{\cdot}_X \to 0,$$

where the injection is characterized by the fact that the composite $B_n \Omega^{\cdot}(1) \to \mathrm{gr}^n_1 W\Omega^{\cdot} \xrightarrow{F^n d} \Omega^{\cdot}(1)$ *is the natural injection.*

Indeed, the only part in (0.6.1) which is not true by definition is the injectivity of ϱ, which is [9, I, Prop. 3.4]. As for (0.6.2) let us first prove that:

(0.6.3) $$\mathrm{Im}\,\varrho = \mathrm{Ker}\, F^n \cap \mathrm{Ker}\, F^n d.$$

As $\operatorname{Im} F^n, \operatorname{Im} F^n d \subseteq \Omega_X^{\cdot}$ which is killed by p we evidently have $\operatorname{Im} \varrho = p \cdot W_{n+1}\Omega^{\cdot} \subseteq \operatorname{Ker} F^n \cap \operatorname{Ker} F^n d$. Let $x \in \operatorname{Ker} F^n \cap \operatorname{Ker} F^n d$. Then $x = Vx'$ for some $x' \in W_n\Omega^{\cdot}$ ([9, I, 3.11.3]) and $0 = F^n dVx' = F^{n-1}dx'$ which gives $x' = Fx''$ for some $x'' \in W_{n+1}\Omega^{\cdot}$ ([9, I, 3.11.4]). Finally $x = Vx' = VFx'' = px'' \in pW_n\Omega^{\cdot}$.

To conclude let us regard the following commutative diagram with exact rows, where the exactness of the lower row comes from [9, I, 3.11.3]:

$$\begin{array}{ccccccccc} 0 & \to & \operatorname{Ker} F^n \cap \operatorname{Ker} F^n d & \to & W_{n+1}\Omega^{\cdot} & \to & \operatorname{gr}_1^n W\Omega^{\cdot} & \to & 0 \\ & & \downarrow{\scriptstyle \text{incl}} & & \| & & \downarrow{\scriptstyle F^n} & & \\ 0 & \longrightarrow & \operatorname{Ker} F^n & \longrightarrow & W_{n+1}\Omega^{\cdot} & \xrightarrow{F^n} & Z_n\Omega^{\cdot} & \longrightarrow & 0. \end{array} \tag{0.6.4}$$

Evidently $\operatorname{coker}(\text{incl}) = \operatorname{Im} F^n d \colon \operatorname{Ker} F^n \to \Omega^{\cdot}(-1)$ and $\operatorname{Ker} F^n = VW_n\Omega^{\cdot}$(loc.cit.) so $F^n d(\operatorname{Ker} F^n) = F^n dV(W_n\Omega^{\cdot}) = F^{n-1}d(W_n\Omega^{\cdot}) = B_n\Omega^{\cdot}(1)$ ([9, I, 3.11.4]). Snake lemma applied to (0.6.4) now gives the desired result.

As $W\Omega_X^{\cdot}$ is a sheaf of R-modules its cohomology has a natural structure of a complex of R-modules. When considered as such we will denote it $R\Gamma(W\Omega_X^{\cdot})$ and its cohomology modules $R^i\Gamma(W\Omega_X^{\cdot})$. Sometimes I will have the occasion to write $R^i\Gamma(W\Omega_X^{\cdot})$ as

$$H^i(W\mathcal{O}_X) \xrightarrow{d} H^i(W\Omega_X^1) \xrightarrow{d} \cdots .$$

On the other hand, $R\Gamma(X/W)$ will denote the hypercohomology of $W\Omega_X^{\cdot}$ considered as a complex of W-modules. According to [9, II, Thm. 1.4] $R\Gamma(X/W)$ is canonically isomorphic to the crystalline cohomology of X/W.

Similarly, $R\Gamma(W_n\Omega_X^{\cdot})$ will denote the cohomology of $W_n\Omega_X^{\cdot}$ considered as a sheaf of graded $W_n[d]$-modules and $R\Gamma(X/W_n)$ the hypercohomology of $W_n\Omega_X^{\cdot}$ considered as a complex of W_n-modules. Again [loc.cit.] shows that $R\Gamma(X/W_n)$ is canonically isomorphic to the crystalline cohomology of X/W_n.

A complex $G \in D^-(R)$ is said to be coherent if $R_n \otimes_R^L G$ is a coherent complex of W_n-modules for every n and the canonical morphism $G \to R\varprojlim \cdot \{R_n \otimes_R G, \pi\}$ is an isomorphism. Here the fact that R_n is a bi-$(W_n[d], R)$-module, where the d in $W_n[d]$ is given by multiplication by d on the left, gives $R_n \otimes_R^L G$ a structure of a $W_n[d]$-complex and a fortiori a W_n-complex and furthermore can $R_n \otimes_R G$, evidently be regarded as a complex of prosystems which gives a sense to the $R\varprojlim$-expression (cf. III, 2.4.2). The argument used in the proof of [10, II, Thm. 2.2] can now be used almost ad verbatim to prove that if G is coherent then $H^i(G)$ is a coherent module in the sense of [10, I, 3] for every i. (For a complete justification of this claim see [5].) Recall [loc.cit.] that any coherent R-module is a successive extension of R-modules of finite type as W-modules together with, shiftings in degree of the R-modules:

$$\mathbf{U}_j\colon \underset{\deg 0}{k[[V]]} \xrightarrow{d} \underset{\deg 1}{k[[T]]}$$

with $V \cdot V^i = V^{i+1}$; $F \cdot V^i = 0$; $V \cdot T^i = 0$; $F \cdot T^i = T^{i-1}$; $dV^i = T^{i+j}$ and with F, d and V extended by (semi-)linearity and continuity to all of $k[[V]]$ and $k[[T]]$.

Conversely, if M is a coherent R-module then [10, I, Thm. 3.8] shows that M is a coherent complex. Therefore a complex $G \in D^-(R)$ is coherent iff $H^i(G)$ is a coherent module for all i. In particular, as the coherent complexes evidently form a triangulated subcategory of $D^-(R)$, the coherent modules form an an abelian subcategory, closed under extensions, of the category of R-modules. Note that in this terminology [10, II: Thm. 2.2] says that if X is proper then $R\Gamma(W\Omega^\cdot_X) \in \mathbf{D}^b_c(R)$, where the subscript c signifies that we are dealing with coherent complexes.

If M is an R-module let us, following [12] put:

$$F^\infty BM := \bigcup_n \operatorname{Im} F^n d$$

$$V^{-\infty} ZM := \bigcup_n \operatorname{Ker} dV^n$$

$$\operatorname{Coeur} M^i := V^{-\infty} ZM^i / F^\infty BM^i$$

$$\tilde{\tau}_{\leqq i} M := \left(\dots \xrightarrow{d} M^{i-1} \xrightarrow{d} V^{-\infty} ZM^i \to 0\right) \tilde{\tau}_{>i} M := M / \tilde{\tau}_{\leqq i} M$$

$$\tilde{\tau}_{<i} M := \left(\dots \xrightarrow{d} M^{i-1} \xrightarrow{d} F^\infty BM^i \to 0\right) \tilde{\tau}_{\geqq i} M := M / \tilde{\tau}_{<i} M$$

$$\operatorname{dom} M^i := (\tilde{\tau}_{<i+1} \tilde{\tau}_{>i} M)(i).$$

We will consider Coeur M^i as well as M^i as an R-module concentrated in degree 0. The following result is then contained in [10, I: Thm. 2.9]:

Lemma 0.8. *If M is a coherent R-module then* Coeur M^i *is a finitely generated W-module and the R-modules* dom M^i *are successive extensions of* $\mathbf{U}_j$: *s.*

We will by the catch phrase "survie du coeur" refer to the fact ([10, II: Thm. 3.4]) that if X is proper then in the first spectral sequence $E_1^{i,j} = H^j(X, W\Omega^i) \Rightarrow H^*(X/W)$ $B_\infty^{i,j} \subseteqq F^\infty BR^j\Gamma(W\Omega^\cdot)^i \subseteqq V^{-\infty} ZR^j\Gamma(W\Omega^\cdot)^i \subseteqq Z_\infty^{i,j}$.

Finally recall that the standard topology on an R-module is the one defined by $\operatorname{Fil}^n M$.

From [10, II: Thm. 1.2] we also draw the result that the natural projection $W\Omega^\cdot \to W_n\Omega^\cdot$ induces an isomorphism (in $D(W_n[d])$):

$$R_n \otimes^L_R R\Gamma(W\Omega^\cdot) \xrightarrow{\simeq} R\Gamma(W_n\Omega^\cdot). \tag{0.9}$$

For general results on coherent duality I will use [7]. This will not always be explicitly mentioned.

If X is proper I propose to call $R\Gamma(W\Omega^\cdot_X)$, considered as an object in $\mathbf{D}^b_c(R)$, the Hodge—Witt cohomology of X.

I. The dualizing complex of $W_n X$

1. The aim of this chapter is to prove that there exists a natural isomorphism:

$$\mathrm{Tr}\colon W_n\Omega^N[N] \to f_n^! W_n,$$

where $f_n^!$ is the functor of [7] for the morphism $f_n\colon W_nX \to W_nS$. To do this one first defines a morphism locally and shows that it glues to a globally defined one and then shows by a dévissage that this morphism is indeed an isomorphism.

2. Let us first note that $f_n^! W_n$ is concentrated in degree $-N$. This is seen as follows. The short exact sequence of W_n-modules;

$$0 \to j_* F_*^{n-1}\mathcal{O} \to W_n\mathcal{O} \to j_{n*}W_{n-1}\mathcal{O} \to 0 \tag{2.1}$$

(cf. (0.5)) gives after applying $R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(-, f_n^! W_n)$ and using duality for the finite morphisms jF^{n-1} and j_n a triangle

$$j_{n*} f_{n-1}^! W_{n-1} \to f_n^! W_n \to (jF^{n-1})_* f_1^! k \to \tag{2.2}$$

and as $f_n^! k = \Omega^N[N]$ induction on n will show that $f_n^! W_n$ is in fact concentrated in degree $-N$. There will therefore be no problems in glueing together the morphism Tr once it is canonically defined locally.

Assume now that $f = f_1\colon X \to S$ admits a smooth lifting $f'\colon X' \to W_nS$, (locally this is, of course, always possible). Recall ([9, II, Thm. 1.4] and [10, III, Prop. 1.4]) that there exists a canonical isomorphism of sheaves of W-algebras:

$$\theta_\cdot\colon W_n\Omega_X^\cdot \xrightarrow{\sim} \sigma_*^n \underline{\mathrm{H}}_{DR}^\cdot(X'/W_n). \tag{2.3}$$

In particular, it gives us a commutative diagram of schemes with a cartesian square:

$$\begin{array}{ccccc} W_nX & \xleftarrow[\theta]{\sim} & (|X|, \underline{\mathrm{H}}_{DR}^0(X'/W_n)) & \xleftarrow{\varepsilon} & X' \\ \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle \tilde{f}'} & \swarrow{\scriptstyle f'} & \\ W_nS & \xleftarrow{\sigma^n} & W_nS & & \end{array} \tag{2.4}$$

where ε is given by the inclusion $\underline{\mathrm{H}}_{DR}^0(X'/W_n) \to \mathcal{O}_{X'}$. I now claim that ε is a finite morphism. Indeed, as this is a local question we may assume that X' lifts to a smooth formal W-scheme X'' and that the Frobenius of X lifts to a morphism $F''\colon X'' \to X''$. It is then clear that $F''\colon \Omega_{X''/W}^1 \to \Omega_{X''/W}^1$ factors as $p \cdot \tilde{F}''$ for some endomorphism $\tilde{F}''$ and that we have $dF'' = p \cdot \tilde{F}'' d$ on $\Omega_{X''/W}^1$. This immediately implies, if we put F' the morphism induced by F'' on $\mathcal{O}_{X'}$, that $\mathrm{Im}\, F'^n\colon \mathcal{O}_{X'} \to \mathcal{O}_{X'}$ is contained in $\underline{\mathrm{H}}_{DR}^0(X'/W_n)$ which, as F' evidently is finite implies that ε is a finite morphism.

As f' is smooth of relative dimension N there exists a canonical isomorphism [7, VII, Cor. 3.4]:

$$\mathrm{Tr}\colon \Omega_{X'/W_n}^N \to f'^! W_n[-N]. \tag{2.5}$$

Using (2.4), the finiteness of θ and ε and the isomorphism $W_n \xrightarrow{\sim} (\sigma^n)^! W_n$ one obtains, from (2.5) and adjunction, morphisms:

(2.5.1) $$\varepsilon_* \Omega^N_{X'/W_n} \to \tilde{f}'^! W_n[-N]$$

(2.5.2) $$(\theta\varepsilon)_* \Omega^N_{X'/W_n} \to f_n^! W_n[-N]$$

I claim that the composite

(2.6) $$(\theta\varepsilon)_* \Omega^{N-1}_{X'/W_n} \xrightarrow{d} (\theta\varepsilon)_* \Omega^N_{X'/W_n} \to f_n^! W_n[-N]$$

is zero. This is seen as follows. All the sheaves involved in (2.6) are in a natural way given as cartesian sections of the fibered category of coherent $W_n\mathcal{O}_X$-modules over the étale site of smooth N-dimensional W_n-schemes X', where X is the reduction modulo p of X', and the morphisms involved are natural transformations of cartesian sections (cf. [6, VI, 5.2]). (The structure of cartesian section on $f_n^! W_n[-N]$ comes from the fact that for an étale morphism g, $g^* = g^!$ [7, VII, Cor. 3.4].) As every smooth N-dimensional W_n-scheme admits, locally in the Zariski topology, an étale morphism to $\mathbf{P}^N_{W_n}$ we may assume that X' is $\mathbf{P}^N_{W_n}$. Using adjunction for a proper morphism we are reduced to showing that the composite $Rf'_* \Omega^{N-1}_{\mathbf{P}^N/W_n} \xrightarrow{d} Rf'_* \Omega^N_{\mathbf{P}^N/W_n} \xrightarrow{d} Rf_{n*} f_n^! W_n[-N] \to W_n[-N]$ is zero or, as W_n is injective as module over itself, that the composite $R^N f'_* \Omega^{N-1}_{\mathbf{P}^N/W_n} \xrightarrow{d} R^N f'_* \Omega^N_{\mathbf{P}^N/W_n} \to W_n$ is zero. This is obvious as $R^N f'_* \Omega^{N-1}_{\mathbf{P}^N/W_n}$ itself is zero.

Using (2.3) and the fact that the composite (2.6) is zero we may now define the map $\mathrm{Tr}_{(f')}$ as the composite:

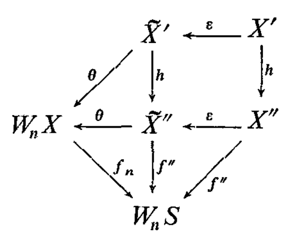
(2.7) $$\mathrm{Tr}_{(f')}\colon W_n\Omega^N_X \xrightarrow{\theta_N} (\theta\varepsilon)_* \Omega^N_{X'/W_n} / d(\theta\varepsilon)_* \Omega^{N-1}_{X'/W_n} \xrightarrow{\mathrm{Tr}} f_n^! W_n[-N].$$

This morphism does in fact not depend on the lifting f'. To prove this suppose that we have another lifting $f''\colon X'' \to W_n S$ and put

(2.8) $$\tilde{X}' := (|X|, \underline{\mathrm{H}}^0_{DR}(X'/W_n)) \quad \tilde{X}'' := (|X|, \underline{\mathrm{H}}^0(X''/W_n)).$$

After possibly localizing on X, which is allowed as $\mathrm{Tr}_{(f')}$ and $\mathrm{Tr}_{(f'')}$ are mappings between actual sheaves and not only morphisms in $D(W_n\mathcal{O}_X)$, we may assume that there exists a W_nS-isomorphism $h\colon X' \to X''$ inducing the identity on X. This gives us a commutative diagram of schemes:

(2.9)



with $f''h=f'$ and $\tilde{f}''\tilde{h}=\tilde{f}'$. Because $\tilde{h}^!\tilde{f}''^!=\tilde{f}'^!$ and because

$$\begin{array}{ccc} (\theta\varepsilon)_*\Omega^N_{X''/W_n} & \searrow^{\theta_N^{-1}} & \\ \wr\uparrow h & \theta_N^{-1}\nearrow & W_n\Omega^N_X \\ (\theta\varepsilon)_*\Omega^N_{X'/W_n} & & \end{array} \tag{2.10}$$

commutes ([9, II, 1.1.8]) we get that $\mathrm{Tr}_{(f')}=\mathrm{Tr}_{(f'')}$.

From this and the observation made above we may now patch these mappings together to obtain for any X a unique mapping

$$\mathrm{Tr}\colon W_n\Omega^N_X \to f_n^! W_n[-N] \tag{2.11}$$

such that on any open subset U of X admitting a smooth lifting U'/W_nS, $\mathrm{Tr}|U$ is given by (2.7). Tr will certainly be compatible with étale mappings in the sense that it will be a natural transformation of cartesian sections of the fibered category of coherent $W_n\mathcal{O}_X$-modules over the étale site of smooth N-dimensional k-schemes X and étale morphisms.

3. Before we can prove that (2.11) is an isomorphism we will need some compatabilities.

From [10, III, Prop. 1.4] we obtain an isomorphism $C^{-n}\colon W_n\Omega^i \to \underline{\mathrm{H}}^i(W_n\Omega^\bullet, d)$ making the following diagram commute

$$\begin{array}{ccc} W_{2n}\Omega^i & \xrightarrow{F^n} & ZW_n\Omega^i \\ \downarrow\pi^n & & \downarrow \\ W_n\Omega^i & \xrightarrow{C^{-n}} & \underline{\mathrm{H}}^i W_n\Omega^\bullet \end{array} \tag{3.1}$$

We will denote C^n the induced mapping $ZW_n\Omega^i \to W_n\Omega^i$. In the case $n=1$, $C^1\colon F_*\Omega^N \to \Omega^N$ is simply the mapping θ_N of (2.3), (in this case $\varepsilon\colon \tilde{X}'\to\tilde{X}'$ is simply the Frobenius $F\colon X\to X$).

Lemma 3.2. *Let* $\omega=\left\{\frac{d\underline{t}_1\wedge d\underline{t}_2\wedge\ldots\wedge d\underline{t}_N}{\underline{t}_1\cdot\underline{t}_2\cdot\ldots\cdot\underline{t}_N}\right\}$ *denote the obvious Čech-N-cocycle for the standard covering of* $W_n\mathbf{P}^N$, *where* $\underline{t}_i$ *is the canonical lifting to* $W_n\mathcal{O}_{\mathbf{P}^N}$ *of the coordinate function* $t_i\in\mathcal{O}_{\mathbf{P}^N}$. *Then*

$$\mathrm{Tr}\,\omega = 1. \tag{3.2.1}$$

Furthermore, this cocycle generates $H^N(\mathbf{P}^N, W_n\Omega^N)$ *as a* W_n*-module.*

Indeed, by definition, the composite $H^N(\mathbf{P}^N_{W_n}, \Omega^N)\to H^N(\mathbf{P}^N, W_n\Omega^N)\xrightarrow{\mathrm{Tr}} W_n$ is the usual trace map. As the cocycle $\left\{\frac{dt_1'\wedge dt_2'\wedge\ldots\wedge dt_N'}{t_1'\cdot t_2'\cdot\ldots\cdot t_N'}\right\}$ for the standard covering of $\mathbf{P}^N_{W_n}$, where t_i' is the ith coordinate function $\in \mathcal{O}\mathbf{P}^N_{W_n}$, has trace 1 and generates $H^N(\mathbf{P}^N_{W_n}, \Omega^N)$ and as $H^N(\mathbf{P}^N_{W_n}, \Omega^N) \to H^N(\mathbf{P}^N, W_n\Omega^N)$ is surjective it suffices to show that this cocycle maps to ω. Let us therefore recall the construction of θ; in the case of $\mathbf{P}^N$. There is a morphism of differential graded algebras

$$\Omega^{\cdot}_{\mathbf{P}^N/W} \to W\Omega^{\cdot}_{\mathbf{P}^N} \tag{3.2.2}$$

compatible with the lifting of Frobenius which takes

$$(x_0 : x_1 : \ldots : x_N) \quad \text{to} \quad (x_0^p : x_1^p : \ldots : x_N^p).$$

This map is a quasi-isomorphism and combined with C^n it gives $\theta_{\cdot}$. According to [9, 0: 1.3.18] it takes the coordinate function $t_i \in \mathcal{O}_{\mathbf{P}}N$ to $\underline{t}_i$. Therefore it takes the element $\frac{dt_1 \wedge dt_2 \wedge \ldots \wedge dt_N}{t_1 \cdot t_2 \cdot \ldots \cdot t_N}$ to $\frac{d\underline{t}_1 \wedge d\underline{t}_2 \wedge \ldots \wedge d\underline{t}_N}{\underline{t}_1 \cdot \underline{t}_2 \cdot \ldots \cdot \underline{t}_N}$ and to prove the lemma it suffices to show that C^n takes ω to itself. As C^n is multiplicative it is enough to show that C^n fixes $\frac{d\underline{x}}{\underline{x}}$ where $x \in \mathcal{O}^x$ and hence, by (3.1), that $F\left(\frac{d\underline{x}}{\underline{x}}\right) = \frac{d\underline{x}}{\underline{x}}$. But $pF(\underline{x}^{-1} d\underline{x}) = \underline{x}^{-p} d(F\underline{x}) = \underline{x}^{-p} \cdot d\underline{x}^p = p\underline{x}^{-1} d\underline{x}$ and $W\Omega^{\cdot}$ is p-torsion free.

Lemma 3.3. *The following diagram commutes*

$$\begin{array}{ccc} j_{n*} W_{n-1}\Omega^N & \xrightarrow{\varrho} & W_n\Omega^N \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ j_{n*} f^!_{n-1} W_{n-1} & \xrightarrow{i} & f^!_n W_n \end{array} \tag{3.3.1}$$

where i is the adjoint of $f^!_{n-1}W_{n-1} = f^!_{n-1}h^! W_n = j^!_n f^!_n W_n$ and $h\colon W_{n-1}S \xrightarrow{j_n} W_nS$.

Proof. Here we have two morphisms between cartesian sections so by using the reasoning above we may assume that $X = \mathbf{P}^N$ and by adjunction it suffices to check commutativity after applying $H^N(-)$ and composing by the adjunction unit. As i clearly induces $\underline{p}\colon W_{n-1} \to W_n$, after applying $H^N(-)$ and composing with the adjunction unit we are reduced to showing the commutativity of the following diagram:

$$\begin{array}{ccc} H^N(W_{n-1}\Omega^N) & \xrightarrow{\varrho} & H^N(W_n\Omega^N) \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \swarrow{\scriptstyle \mathrm{Tr}} \\ W_{n-1} & \xrightarrow{p} & W_n \end{array} \tag{3.3.2}$$

By (3.2) it is sufficient to show that $\varrho(\omega_{n-1}) = p\omega_n$. This is obvious.

Lemma 3.4. *The following diagram commutes*

$$\begin{array}{ccc} W_n\Omega^N & \xrightarrow{F} & (Fj_n)_* W_{n-1}\Omega^N \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ f^!_n W_n[-N] & \xrightarrow{m} & (Fj_n)_* f^!_{n-1} W_{n-1}[-N] \end{array} \tag{3.4.1}$$

where $m\colon f_n^! W_n = R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(W_n\mathcal{O}, f_n^! W_n) \xrightarrow{V^*} R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}}((Fj_n)_* W_{n-1}\mathcal{O}, f_n^! W_n) =$

$$(Fj_n)_*(Fj_n)^! f_n^! W_n = (Fj_n)_* f_{n-1}^! W_{n-1}.$$

Proof. I claim that the following diagrams commute, for a smooth lifting X'/W_n of X:

$$\text{(3.4.2)}\qquad \begin{array}{ccc} W_n\Omega_X^i & \xrightarrow{F} & W_{n-1}\Omega_X^i \\ \downarrow{\scriptstyle\theta_i} & & \downarrow{\scriptstyle\theta_i} \\ H^i\Omega^{\cdot}_{X'/W_n} & \longrightarrow & H^i\Omega^{\cdot}_{X''/W_{n-1}} \end{array} \qquad \begin{array}{ccc} W_{n-1}\Omega_X^i & \xrightarrow{V} & W_n\Omega_X^i \\ \downarrow{\scriptstyle\theta_i} & & \downarrow{\scriptstyle\theta_i} \\ H^i\Omega^{\cdot}_{X''/W_{n-1}} & \xrightarrow{p} & H^i\Omega^{\cdot}_{X'/W_n} \end{array}$$

where X'' is the reduction modulo p^{n-1} of X', the lower horizontal morphism of the first square is induced by the projection and p is induced by multiplication by p. The question is local so we may assume that X' lifts to a smooth formal scheme over W admitting a morphism which lifts Frobenius. We then get a morphism as in (3.2.2) and we may therefore replace $H^i\Omega^{\cdot}_{X'/W_n}$, θ etc. with $H^i W_n\Omega^{\cdot}_X$, C^{-n} etc. The searched for commutativity is now [10, III, 1.4.7 and 1.4.9]. The question of the commutativity of (3.4.1) is a local question so we may assume that X admits a smooth lifting X'/W_n. The fact that $\Omega^N_{X'/W_n} \to W_n\Omega^N_X$ is surjective and (3.4.2) implies that it suffices to show the commutativity of the following diagram:

$$\text{(3.4.3)}\qquad \begin{array}{ccc} (\theta\varepsilon)_*\Omega^N_{X'/W_n} & \longrightarrow & (\theta\varepsilon)_* t_*\Omega^N_{X''/W_{n-1}} \\ \downarrow{\scriptstyle\mathrm{Tr}} & & \downarrow{\scriptstyle\mathrm{Tr}} \\ f_n^! W_n[-N] & \xrightarrow{m} & (Fj_n)_* f_{n-1}^! W_{n-1}[-N] \end{array}$$

where t is the nil-immersion $t\colon X'' \to X'$. Adjunction reduces to showing the commutativity of the following diagram:

$$\text{(3.4.4)}\qquad \begin{array}{ccc} \Omega^N_{X'/W_n} & \longrightarrow & t_*\Omega^N_{X''/W_{n-1}} \\ \downarrow{\scriptstyle\mathrm{Tr}} & & \downarrow{\scriptstyle\mathrm{Tr}} \\ f'^! W_n[-N] & \xrightarrow{r} & (\theta\varepsilon)^!(Fj_n)_* f_{n-1}^! W_{n-1}[-N] \end{array}$$

where $r = (\theta\varepsilon)^!(m)$. Note also that (3.4.2) shows that the following diagram is commutative:

$$\text{(3.4.5)}\qquad \begin{array}{ccc} X'' & \xrightarrow{\theta\varepsilon} & W_{n-1}X \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle Fj_n} \\ X' & \xrightarrow{\theta\varepsilon} & W_n X \end{array}$$

To show that (3.4.4) is commutative it is enough to show that the following diagram commutes:

$$\text{(3.4.6)}\qquad \begin{array}{ccccc} \Omega^N_{X'/W_n} & \longrightarrow & t_*\Omega^N_{X''/W_{n-1}} & & \\ \downarrow{\scriptstyle\mathrm{Tr}} & & \downarrow{\scriptstyle\mathrm{Tr}} & \searrow{\scriptstyle\mathrm{Tr}} & \\ f'^! W_n[-N] & \xrightarrow{a} & t_* f''^! W_{n-1}[-N] & \xrightarrow{b} & (\theta\varepsilon)^!(Fj_n)_* f_{n-1}^! W_{n-1}[-N] \end{array}$$

where $a[N]$ is the composite

$$f'^! W_n = R\,\underline{\mathrm{Hom}}_{\mathcal{O}_{X'}}(\mathcal{O}_{X'}, f'^! W_n) \xrightarrow{p^*} R\,\underline{\mathrm{Hom}}_{\mathcal{O}_{X'}}(t_* \mathcal{O}_{X''}, f'^! W_n) = t_* f''^! W_{n-1}$$

and b is the base change morphism for the square (3.4.5) (cf. [2, XVII, 2.1.3]) and to show that $r = ba$. The commutativity of the right hand triangle is just abstract nonsense. Using Rf_* and $f^!$ we may construct a bifibered category over the category of finite type k-schemes and proper morphisms whose fiber over a scheme V is $D^-_{\mathrm{coh}}(\mathcal{O}_V)$. Then the desired commutativity is just a simple diagram chase and in the end the characterization [2, XVII, Prop. 2.1.3] of the base change map. I leave this as an exercise for the reader. The fact that $r = ba$ is seen as follows. Applying $R\,\underline{\mathrm{Hom}}_{W_n \mathcal{O}_X}(-, f_n^! W_n)$ to the commutative diagram:

(3.4.7)
$$\begin{array}{ccc} (Fj_n)_* W_{n-1}\mathcal{O}_X & \xrightarrow{V} & W_n \mathcal{O}_X \\ \downarrow{\scriptstyle \theta\varepsilon} & & \downarrow{\scriptstyle \theta\varepsilon} \\ (Fj_n)_* (\theta\varepsilon)_* \mathcal{O}_{X''} & \xrightarrow{p} & (\theta\varepsilon)_* \mathcal{O}_{X'} \end{array}$$

we get the commutative square:

(3.4.8)
$$\begin{array}{ccc} f_n^! W_n & \xrightarrow{m} & (Fj_n)_* f_{n-1}^! W_{n-1} \\ \uparrow & & \uparrow \\ (\theta\varepsilon)_* f'^! W_n & \xrightarrow{c} & (\theta\varepsilon)_* t_* f''^! W_{n-1} \end{array}$$

where $c = (\theta\varepsilon)_*(a)$. Adjunction with respect to $\theta\varepsilon$ then gives $r = ba$. There now only remains to show the commutativity of the left hand square. Consider the diagram:

(3.4.9)
$$\begin{array}{ccccc} \Omega^N_{X'/W_n} & \longrightarrow & t_* \Omega^N_{X''/W_{n-1}} & \xrightarrow{p} & \Omega^N_{X'/W_n} \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ f'^! W_n[-N] & \xrightarrow{a} & t_* f''^! W_{n-1}[-N] & \xrightarrow{d} & f'^! W_n[-N] \end{array}$$

where d is defined by adjunction with respect to t. The two composites of the horizontal morphisms are both multiplication by p and d is injective as it is, upto isomorphism, obtained by applying $\underline{\mathrm{Hom}}_{\mathcal{O}_{X'}}(-, f'^! W_n)$ to the surjection $\mathcal{O}_{X'} \to t_* \mathcal{O}_{X''}$. We are therefore finally reduced to showing that the right hand square commutes. This can be done as in the proof of (3.3) — reduce to projective space and compute.

4. We are now prepared to prove the main result of this chapter.

Theorem 4.1. *The morphism* (2.11) *is an isomorphism.*

I first claim that we have a short exact sequence:

(4.1.1)
$$0 \to j_{n*} f_{n-1}^! W_{n-1} \xrightarrow{i} f_n^! W_n \xrightarrow{q} (jF^{n-1})_* f^! k \to 0,$$

where q is m iterated $n-1$ times. Indeed, this is just $R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}_X}(-, f_n^! W_n)$ applied to the the exact sequence

$$(4.1.2)\qquad 0 \to F_*^{n-1}\mathcal{O}_X \xrightarrow{V^{n-1}} W_n\mathcal{O}_X \to j_{n*}W_{n-1}\mathcal{O}_X \to 0$$

plus the fact that $f_{n-1}^! W_{n-1}$ and $f^! k$ are concentrated in degree $-N$. Furthermore by (3.3) and (3.4) the following diagram is commutative:

$$(4.1.3)\qquad \begin{array}{ccccccc} 0 \to j_{n*}W_{n-1}\Omega^N & \xrightarrow{\varrho} & W_n\Omega^N & \xrightarrow{F^{n-1}} & (jF^{n-1})_*\Omega^N \to 0 \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ (0 \to j_{n*}f_{n-1}^! W_{n-1} & \xrightarrow{i} & f_n^! W_n & \xrightarrow{q} & (jF^{n-1})_* f^! k \to 0)[-N] \end{array}$$

and by (0.6) the upper row is exact. We will therefore be able to conclude by induction once we have shown that $\mathrm{Tr}\colon W_1\Omega^N \to f_1^! k[-N]$ is an isomorphism. I claim that this map coincides with the usual trace map which is indeed an isomorphism. We reduce as usual to projective space and calculate. The equality needed is exactly the one obtained in the proof of (3.2).

5. This section discusses the relation with the trace map that has just been defined and the crystalline trace map of Berthelot ([15]). It will not be used anywhere else in this article.

We know ([9, 11: Thm. 1.4]) that $R\bar{u}_*\mathcal{O}_{X/W_nS} = W_n\Omega^\bullet$ so C^n gives an isomorphism $R^N\bar{u}_*\mathcal{O}_{X/W_nS} = W_n\Omega^N$ ($\bar{u}$ is the projection from the crystalline topos to the Zariski one). (2.11) therefore gives us a trace map $R^N\bar{u}_*\mathcal{O}_{X/W_nS} \to f_n^! W_n[-N]$ and, by adjunction, if X is proper a trace map $\mathrm{Tr}_n\colon H^{2N}(X/W_n) \to W_n$. If we know that $\mathrm{Tr}_n/p = \mathrm{Tr}_1$ and Tr_1 is the usual trace map for de Rham cohomology, we may, exactly as Berthelot does, prove the duality theorem for $R\Gamma(X/W_n)$. These two compatabilities follow from Lemma 3.4 and the proof of Thm. 4.1. To be completely at ease we would like to convince ourselves that the map Tr_n coincides with Berthelot's trace map ([15, VII: 1.4.9]). However ([15, VI: Prop. 1.4.5, Prop. 1.6.1]) shows that $R\bar{u}_*\mathcal{O}_{X/W_n}$ is Cohen—Macaulay for the codimension filtration and hence that any trace map is determined by the induced morphisms $H_x^N(R\bar{u}_*\mathcal{O}_{X/W_n}) \to W_n$ where x runs over the closed points of X. In the case of the trace map of the present article this morphism is defined for any pointed smooth k-scheme and is compatible with étale morphisms between pointed smooth k-schemes. In the case of Berthelot's trace map this is also the case as the morphism $H_x^N(R\bar{u}_*\mathcal{O}_{X/W_n}) \to W_n$ is induced by the crystalline residue map ([15, VII: Thm. 1.4.6]) which clearly has this property. We may therefore reduce to projective space, in which case Lemma 3.2 shows that Tr_n equals the trace map for the de Rham cohomology of $\mathbf{P}_{W_n}^N$ which also is true for the trace map of Berthelot ([15, VII: 1.4.11]).

Remark. This coincidence of traces was also proved by Illusie by a slightly different method.

II. Self duality of the de Rham—Witt complex

1. The purpose of this chapter is to show that multiplication induces an isomorphism $W_n\Omega^\bullet \to R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}_X}(W_n\Omega^\bullet_X, W_n\Omega^N_X)(-N)$ of graded sheaves. Using the exact sequences (0.6.1) and (0.6.2) and their duals one is reduced to a duality between higher boundaries and cycles in the de Rham complex of X which is easily proved by induction.

Lemma 2.1. *The following diagrams are commutative*

(2.1.1)
$$\begin{array}{ccc} F^n_* W_n\Omega^N_X & \xrightarrow{C^n} & W_n\Omega^N \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ F^n_* f^!_n W_n[-N] & \xrightarrow{k} & f^!_n W_n[-N] \end{array}$$

(2.1.2)
$$\begin{array}{ccc} (j_n F)_* W_{n-1}\Omega^N_X & \xrightarrow{V} & W_n\Omega^N_X \\ \downarrow{\scriptstyle \mathrm{Tr}} & & \downarrow{\scriptstyle \mathrm{Tr}} \\ (j_n F)_* f^!_{n-1} W_{n-1}[-N] & \xrightarrow{s} & f^!_n W_n[-N] \end{array}$$

where k is adjoint to $f^!_n W_n = f^!_n(\sigma^n)^! W_n = (F^n)^! f^!_n W_n$ and s to $f^!_{n-1} W_{n-1} = f^!_{n-1} h^! W_n = (j_n F)^! f^!_n W_n$ and h is the composite $W_n S \xrightarrow{j_n} W_n S \xrightarrow{F} W_n S$.

Proof. As before we may assume X is $\mathbf{P}^N$ and we need only commutativity after applying $H^N(-)$ and composing with the adjunction unit. In (2.1.1) we need to know that C^n fixes the standard cocycle which is proved in (I, 3.2) and in (2.1.2) we need to know that $V(\omega_{n-1}) = p\cdot\omega_n$ which follows from $VF=p$ and $F(\omega_n)=\omega_{n-1}$ which was proved in (I, 3.2).

Theorem 2.2. *The pairing $W_n\Omega^i \otimes_{W_n\mathcal{O}} W_n\Omega^{N-i} \to W_n\Omega^N$ given by multiplication induces an isomorphism:*

(2.2.1)
$$W_n\Omega^i \to R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(W_n\Omega^{N-i}, W_n\Omega^N).$$

To prove this let us start by putting

(2.2.2)
$$\mathrm{gr}^n := \mathrm{gr}^n(W\Omega^\bullet) \quad \mathrm{gr}^n_1 := \mathrm{gr}^n_1(W\Omega^\bullet).$$

These will be regarded as $W_{n+1}\mathcal{O}$-modules and we will furthermore consider gr^n as a submodule of $W_{n+1}\Omega$.

The following relation will hold

(2.2.3)
$$\varrho(\pi x\cdot y) = x\cdot\varrho y \quad x\in W_{n+1}\Omega^\bullet \quad y\in W_n\Omega^\bullet.$$

Indeed, $y=\pi y'$ for some $y'\in W_{n+1}\Omega^\bullet$ so $\varrho(\pi x\cdot y)=\varrho(\pi x\cdot\pi y')=\varrho\pi(x\cdot y')=p(x\cdot y')=x\cdot py'=x\cdot\varrho y$.

Lemma 2.2.4. *The following diagrams commute:*

$$(2.2.5)\qquad \begin{array}{ccc} j_{n*}W_n\Omega^N & \xrightarrow{\varrho} & W_{n+1}\Omega^N \\ \downarrow\wr & \nearrow{\scriptstyle \mathrm{Tr}_{j_n}} & \\ j_{n*}j_n^! W_{n+1}\Omega^N & & \end{array}$$

$$(2.2.6)\qquad \begin{array}{ccc} (jF^n)_*\Omega^N & \xrightarrow{V^n} & W_{n+1}\Omega^N \\ \downarrow\wr & \nearrow{\scriptstyle \mathrm{Tr}_{jF^n}} & \\ (jF^n)_*(jF^n)^! W_{n+1}\Omega^N & & \end{array}$$

where the vertical isomorphisms come from [I, Thm. 4.1] *and* Tr_{j_n} *resp* Tr_{jF^n} *are the adjunction units.*

Proof. This follows from [1, 3.3] and (2.1.2).

Lemma 2.2.7. $\underline{\mathrm{Ext}}^i_{W_n\mathcal{O}}(W_n\Omega^*, W_n\Omega^N)=0$ *for* $i>0$.

Proof. (0.6) shows that $W_n\Omega^\bullet$ is a successive extension of locally free $\mathcal{O}$-modules. Using this and [I, 3.3] it sufficesto show that $\underline{\mathrm{Ext}}^i_{W_n\mathcal{O}}(j_*M, f_n^! W_n[-N])=0$ for $i>0$ and M a locally free $\mathcal{O}$-module. However, duality for j gives $\underline{\mathrm{Ext}}^i_{W_n\mathcal{O}}(j_*M, f_n^! W_n[-N])=\underline{\mathrm{Ext}}^i_{\mathcal{O}}(M, f^! k[-N])=0$ for $i>0$ because M is locally free.

Remark. $W_n\Omega^\bullet$ is not a locally free $W_n\mathcal{O}$-module so (2.2.7) is not automatic as in the case $n=1$.

Using (2.2.7) we may dispense with the R when proving (2.2.1). (2.2.3) shows that $\mathrm{gr}^n\cdot\varrho W_n\Omega^\bullet=\mathrm{Ker}\,\pi\cdot\varrho W_n\Omega^\bullet=0$ and we get an induced pairing $\mathrm{gr}^n\otimes\mathrm{gr}_1^n\to W_{n+1}\Omega^N$. (0.6.2) and the obvious exact sequence $0\to\mathrm{gr}^n\to W_{n+1}\Omega^\bullet\to W_n\Omega^\bullet\to 0$ therefore gives a commutative diagram with exact rows:

$$(2.2.8)\qquad \begin{array}{ccccccccc} 0 & \to & j_{n*}W_n\Omega^{N-i} & \xrightarrow{\varrho} & W_{n+1}\Omega^{N-i} & \to & \mathrm{gr}_1^n & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \underline{\mathrm{Hom}}_{W_{n+1}\mathcal{O}}(j_{n*}W_n\Omega^i, W_{n+1}\Omega^N) & \xrightarrow{R^*} & \underline{\mathrm{Hom}}_{W_{n+1}\mathcal{O}}(W_{n+1}\Omega^i, W_{n+1}\Omega^N) & \to & \underline{\mathrm{Hom}}_{W_{n+1}\mathcal{O}}(\mathrm{gr}^n, W_{n+1}\Omega^N) & & \end{array}$$

Further, (2.2.3) and (2.2.5) show that the following diagram commute:

$$(2.2.9)\qquad \begin{array}{ccc} & j_{n*}W_n\Omega^{N-i} & \\ {\scriptstyle a}\swarrow & & \searrow{\scriptstyle c} \\ j_{n*}\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(W_n\Omega^i, W_n\Omega^N) & \xrightarrow[\sim]{b} & \underline{\mathrm{Hom}}_{W_{n+1}\mathcal{O}}(j_{n*}W_n\Omega^i, W_{n+1}\Omega^N) \end{array}$$

where a is j_{n*} applied to (2.2.1), b is the duality isomorphism and c is the morphism in (2.2.8). If we proceed by induction on n ($n=1$ being wellknown) it only remains to show that the morphism

$$\mathrm{gr}_1^n \to \underline{\mathrm{Hom}}_{W_{n+1}\mathcal{O}}(\mathrm{gr}^n, W_{n+1}\Omega^N) \tag{2.2.10}$$

is an isomorphism.

Recall that we have the following exact sequences (cf. (0.5), (0.6)):

$$0 \longrightarrow j_*(F_*^n\Omega^i/B_n) \xrightarrow{V^n} \mathrm{gr}^n \longrightarrow j_*(F_*^n\Omega^{i-1}/Z_n) \longrightarrow 0, \tag{2.2.11}$$

$$0 \longrightarrow j_*B_n\Omega^{i+1} \longrightarrow \mathrm{gr}_1^n \xrightarrow{F^n} j_*Z_n\Omega^i \longrightarrow 0 \tag{2.2.12}$$

and if $x\in \mathrm{Im}\, V^n \subseteqq \mathrm{gr}^n$ and $y\in \mathrm{Ker}\, F^n \subseteqq \mathrm{gr}_1^n$ then $x=V^n x'$ for some x' and $x\cdot y= V^n x'\cdot y = V^n(x\cdot F^n y)=0$ (cf. (0.2)) so we get induced pairings:

$$\mathrm{Coim}\, F^n \otimes \mathrm{Im}\, V^n \to W_{n+1}\Omega^N, \tag{2.2.13}$$

$$\mathrm{Ker}\, F^n \otimes \mathrm{Coker}\, V^n \to W_{n+1}\Omega^N \tag{2.2.14}$$

and the proof of (2.2) is reduced to show that the induced mappings:

$$\mathrm{Coim}\, F^n \to \underline{\mathrm{Hom}}_{W_{n+1}\mathcal{O}}(\mathrm{Im}\, V^n, W_{n+1}\Omega)^N), \tag{2.2.15}$$

$$\mathrm{Ker}\, F^n \to \underline{\mathrm{Hom}}_{W_{n+1}\mathcal{O}}(\mathrm{Coker}\, V^n, W_{n+1}\Omega^N) \tag{2.2.16}$$

are isomorphisms.

Lemma 2.2.17. *The following diagrams commute:*

$$\begin{array}{ccc}
\mathrm{Coim}\, F^n & \longrightarrow & \underline{\mathrm{Hom}}_{W_{n+1}\mathcal{O}}(\mathrm{Im}\, V^n, W_{n+1}\Omega^N) \\
\Big| & & \Big\downarrow \wr\, b^* \\
\Big| & & \underline{\mathrm{Hom}}_{W_{n+1}\mathcal{O}}\big(j_*(F_*^n\Omega^i/B_n), W_{n+1}\Omega^N\big) \\
\wr\, a & & \Big\uparrow \wr\, c \\
\Big| & & j_*\,\underline{\mathrm{Hom}}_{\mathcal{O}}(F_*^n\Omega^i/B_n, \Omega^N) \\
\Big\downarrow & & \Big\uparrow j_*((\mathrm{Tr}_{F^n})_*) \\
j_*Z_n\Omega^{N-i} & \xrightarrow{h} & j_*\,\underline{\mathrm{Hom}}_{\mathcal{O}}(F_*^N\Omega^i/B_n, F_*^n\Omega^N)
\end{array} \tag{2.2.18}$$

$$\begin{array}{ccc}
\mathrm{Ker}\, F^n & \longrightarrow & \underline{\mathrm{Hom}}_{W_{n+1}\mathcal{O}}(\mathrm{Coker}\, V^n, W_{n+1}\Omega^N) \\
\Big\uparrow & & \Big\downarrow \wr\, e^* \\
\Big| & & \underline{\mathrm{Hom}}_{W_{n+1}\mathcal{O}}\big(j_*(F_*^n\Omega^{i-1}/Z_n), W_{n+1}\Omega^N\big) \\
\wr\, (-1)^{N-i}g & & \Big\uparrow \wr\, f \\
\Big| & & j_*\,\underline{\mathrm{Hom}}_{\mathcal{O}}(F_*^n\Omega^{i-1}/Z_n, \Omega^N) \\
\Big| & & \Big\uparrow j_*((\mathrm{Tr}_{F^n})_*) \\
j_*B_n\Omega^{N-i+1} & \xrightarrow{k} & j_*\,\underline{\mathrm{Hom}}_{\mathcal{O}}(F_*^n\Omega^{i-1}/Z_n, F_*^n\Omega^N)
\end{array} \tag{2.2.19}$$

Here a and g come from (2.2.12), b and e from (2.2.11), c and f are duality isomorphisms and h and k are induced by multiplication in $\Omega^\cdot$.

Proof. By (2.2.6) and transitivity of trace we have that $\mathrm{Tr}_j \circ j_*(\mathrm{Tr}_{F^n}) = \mathrm{Tr}_{jF^n} = V^n\colon (jF^n)_* \Omega^N \to W_{n+1}\Omega^N$. Further, if $x \in \Omega^\cdot$, $y \in W_{n+1}\Omega^\cdot$ then $V^n x \cdot y = V^n(x \cdot F^n y)$ (cf. (0.2)) and if further $F^n y = 0$ then;

$$V^n(x \cdot F^n dy) = V^n x \cdot dy = (-1)^{\deg} d(V^n x \cdot y) + (-1)^{\deg+1}(dV^n x \cdot y)$$
$$= (-1)^{\deg} dV^n(x \cdot F^n y) + (-1)^{\deg+1}(dV^n x \cdot y) = 0 + (-1)^{\deg+1}(dV^n x \cdot y).$$

Combining these observations and noting that $W_{n+1}\Omega^\cdot \xrightarrow{\pi^n} \Omega^\cdot$ is multiplicative, one concludes that (2.2.18) and (2.2.19) commute, using, of course, that the duality isomorphisms c and f are the composite of "applying j_*" and $(\mathrm{Tr}_j)_*$.

Lemma 2.2.17 now shows that the following lemma establishes Theorem 2.2:

Lemma 2.2.20. The composites

(2.2.21) $$Z_n\Omega^i \xrightarrow{h} \underline{\mathrm{Hom}}_{\mathcal{O}}(F_*^n\Omega^{N-i}/B_n, F_*^n\Omega^N) \xrightarrow{(\mathrm{Tr}_{F^n})_*} \underline{\mathrm{Hom}}_{\mathcal{O}}(F_*^n\Omega^{N-i}/B_n, \Omega^N)$$

(2.2.22) $$B_n\Omega^i \xrightarrow{k} \underline{\mathrm{Hom}}_{\mathcal{O}}(F_*^n\Omega^{N-i}/Z_n, F_*^n\Omega^N) \xrightarrow{(\mathrm{Tr}_{F^n})_*} \underline{\mathrm{Hom}}_{\mathcal{O}}(F_*^n\Omega^{N-i}/Z_n, \Omega^N)$$

are isomorphisms (cf. [12] and [13]).

Recall that by definition (cf. [9, 0, 2.2.2])

$$Z_1\Omega^i = \mathrm{Ker}\, d\colon F_*\Omega^i \to F_*\Omega^{i+1}$$
$$B_1\Omega^i = \mathrm{Im}\, d\colon F_*\Omega^{i-1} \to F_*\Omega^i$$

and

$$C^{-1}(B_n) = B_{n+1}/B_1$$
$$C^{-1}(Z_n) = Z_{n+1}/Z_1$$

where $C^{-1}\colon \Omega^\cdot \xrightarrow{\sim} \underline{\mathrm{H}}^\cdot(F_*\Omega^\cdot)$ is the multiplicative and $\mathcal{O}$-linear Cartier-isomorphism. Recall, [loc. cit.], that B_n, Z_n, $F_*^n\Omega^\cdot/B_n$ and $F_*^n\Omega^\cdot/Z_n$ are all locally free $\mathcal{O}$-modules.

From (I: 2.6) we see that the composite $F_*\Omega^{N-1} \xrightarrow{d} F_*\Omega^N \xrightarrow{\mathrm{Tr}_F} \Omega^N$ is zero and as $(F_*\Omega^\cdot, d)$ is a complex of $\mathcal{O}$-modules we get a perfect pairing $F_*\Omega^\cdot \otimes F_*\Omega^\cdot \xrightarrow{\text{mult}} F_*\Omega^N[-N] \xrightarrow{\mathrm{Tr}_F} \Omega^n[-N]$ of $\mathcal{O}$-complexes. This gives the desired result for $n=1$. Furthermore, by (2.1.1) Tr_F is simply C, the inverse of the Cartier-isomorphism, and as C^{-1} is multiplicative the induced pairing

$$\Omega^i \otimes_{\mathcal{O}} \Omega^{N-i} \xrightarrow{C^{-1}\otimes C^{-1}} \underline{\mathrm{H}}^i(F_*\Omega^\cdot) \otimes \mathrm{H}^{N-i}(F_*\Omega^\cdot) \to \Omega^N$$

is the original pairing.

Using this and the definitions of B_n and Z_n one quickly establishes the desired result by induction.

Corollary 2.2.23. *Suppose that* X *is proper. For all* (i, j) *the mapping ,*

(2.2.24) $$H^i(X, W_n\Omega^j) \xrightarrow{\sim} \mathrm{Hom}_{W_n}(H^{N-i}(X, W_n\Omega^{N-j}), W_n)$$

induced by cup product and the trace morphism $H^N(X, W_n\Omega^N) \to W_n$, *is an isomorphism.*

This follows immediately from I: Thm. 4.1, Thm. 2.2, coherent duality for the proper morphism f_n and the injectivity of W_n as a module over itself.

Remark. O. Gabber has indicated an alternative proof of I: Thm. 4.1 and Thm 2.2. Let me quickly sketch how this is done. Once the trace map is defined the problem is local, so we may assume that there exists a lifting X'/W_n of X. We have isomorphisms $\Omega^i_{X'/W_n} \xrightarrow{\sim} R\,\underline{\mathrm{Hom}}_{\mathcal{O}_{X'}}(\Omega^{N-i}_{X'/W_n}, f'^!W_n[N])$ which by adjunction give (using the notation of I: 2) an isomorphism

$$(\theta\varepsilon)_*\Omega^i_{X'/W_n} \xrightarrow{\sim} R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}_X}((\theta\varepsilon)_*\Omega^{N-i}_{X'/W_n}, f^!W_n[N]).$$

Using the fact that $(\theta\varepsilon)_*\Omega^{\cdot}_{X'/W_n}$ has a $W_n\mathcal{O}_X$-linear differential and I: 2.6 we get an isomorphism in $D(W_n\mathcal{O}_X)$; $(\theta\varepsilon)_*\Omega^{\cdot}_{X'/W_n} \xrightarrow{\sim} R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}_X}((\theta\varepsilon)_*\Omega^{\cdot}_{X'/W_n}[N], f_n^!W_n[N])$. As in (2.2.7) we see that $\underline{\mathrm{Ext}}^i_{W_n\mathcal{O}_X}(W_n\Omega^*, f_n^!W_n[N])=0$ if $i>0$. This togethei with I: 2.3 gives us an isomorphism $W_n\Omega^i \xrightarrow{\sim} R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(W_n\Omega^{N-i}, f_n^!W_n[N])$ and putting $i=N$ we get $W_n\Omega^N \xrightarrow{\sim} f_n^!W_n[N]$.

Even though this method of proof is somewhat quicker than the present one, I have chosen the latter approach finding it more instructive.

III. Duality for the Hodge—Witt cohomology

1.

We will suppose in this chapter that X is smooth and proper over S. Corollary 2.2.23 gives us an auto-duality for $H^*(X, W_n\Omega^*)$ at each finite level n. Using (0.2), Lemma 2.1 and (2.2.5) we see that this auto-duality is, in a certain sense, compatible with F, d and V and exchanges π and ϱ. We thereby get a duality between the prosystem $\{H^*(X, W_n\Omega^*), \pi\}$ and the indsystem $\{H^*(X, W_n\Omega^*), \varrho\}$. As indicated in the introduction we will want to express the indsystem in terms of the prosystem and by passing to the limit obtain an auto-duality for $R\Gamma(W\Omega^{\cdot})$. There is one technical detail. We will want to assure ourselves that this autoduality respects R-module structures in the strongest sense possible. This will be the reason for the introduction of the notion of de Rham—Witt systems on X.

At the end of the chapter we will discuss the relation between the duality for $R\Gamma(X, W\Omega^{\cdot})$ and the ones already obtained for $H^*(X, W_n\Omega^*)$ and $R\Gamma(X/W)$.

2.

Definition 2.1. *An inverse (resp. direct) de Rham—Witt system on X will be a system*

$$(M_n, \pi\colon M_n \to (j_n)_* M_{n-1},\ F\colon M_n \to (Fj_n)_* M_{n-1},$$
$$d\colon F_*^n M_n \to F_*^n M_n(1),\ V\colon (Fj_n)_* M_{n-1} \to M_n)$$

(resp.

$$(M_n, \varrho\colon (j_n)_* M_{n-1} \to M_n,\ F\colon M_n \to (Fj_n)_* M_{n-1},$$
$$d\colon F_*^n M_n \to F_*^n M_n(1),\ V\colon (Fj_n)_* M_{n-1} \to M_n)$$

where M_n is a graded quasi-coherent $W_n\mathcal{O}$-module and (π, F, d, V) (resp. (ϱ, F, d, V) are graded $W_n\mathcal{O}$-homomorphisms and where we have the relations

$$\pi F = F\pi,\quad d\pi = \pi d,\quad \pi V = V\pi \text{ (resp. } \varrho F = F\varrho,\ d\varrho = \varrho d,\ V\varrho = \varrho V)$$
$$FV = p,\quad F dV = d,\quad VF = p,\quad d^2 = 0.$$

(I want to draw the reader's attention to the fact that VF is a morphism $M_{n-1} \to M_{n-1}$, whereas FV is a morphism $(Fj_n)_* M_{n-1} \to (Fj_n)_* M_{n-1}$.)

The same signconventions as for R-modules will be used.

The inverse (resp. direct) de Rham—Witt systems form abelian categories which we will denote

(2.1.1) $$\text{inv}-\text{drw}-X \text{ (resp. dir}-\text{drw}-X).$$

Example 2.1.2: $(W.\Omega^\cdot, \pi, F, d, V)$ (cf. (0.1)) forms an inverse de Rham—Witt system which, by abuse, will be denoted $W.\Omega^\cdot$. $(W.\Omega^\cdot, \varrho, F, d, V)$ forms a direct de Rham—Witt system which will be denoted $W_\infty.\Omega^\cdot$.

Definition 2.2. *A dualizing system will be a system*

$$(M_n, \varrho\colon (j_n)_* M_{n-1} \to M_n,\ C^n\colon F_*^n M_n \to M_n,\ V\colon (Fj_n)_* M_{n-1} \to M_n),$$

where M_n is a quasi-coherent $W_n\mathcal{O}$-module, (ϱ, C^n, V) are $W_n\mathcal{O}$-homomorphisms with the following relations:

$$VC^{n-1} = C^n\varrho,\quad \varrho V = V\varrho$$

and where it is further required that the adjoints

$$M_{n-1} \to j_n^! M_n,\quad M_n \to (F^n)^! M_n \quad \text{and} \quad M_{n-1} \to (Fj_n)^! M_n \quad \text{of} \quad \varrho, C^n \text{ resp. } V$$

be isomorphisms.

Example 2.2.1. $(W_n\Omega^N, \varrho, C^n, V)$ is, by (I:4.1), Lemma 2.1 and (2.2.5), a dualizing system. I claim that (Cousin $(W_n\Omega^N)$, ϱ, C^n, V), where Cousin $(W_n\Omega^N)$ is the Cousin-complex ([7, p. 235]) of $W_n\Omega^N$ and ϱ, C^n and V are defined by functoriality from ϱ, C^n and V of $W_n\Omega^N$, is a complex of dualizing systems. By functoriality the required relations are fulfilled and it remains to check the last condition of (2.2). Using the notation and result of [7, VI: Lemma 4.1] we get that j_n' (Cousin $(W_n\Omega^N)$) is residual, so by [7, IV: Prop. 3.4] there is a unique isomorphism of *complexes* Cousin $(W_{n-1}\Omega^N) \xrightarrow{\sim} j_n'$(Cousin $(W_n\Omega^N)$) making the following diagram commute (in $D(W_{n-1}\mathcal{O})$):

$$\begin{array}{ccc}
W_{n-1}\Omega^N & \xrightarrow{\quad\sim\quad} & j_n^! W_n\Omega^N \\
\downarrow \wr & & \downarrow \wr \\
\text{Cousin}\,(W_{n-1}\Omega^N) & \xrightarrow{\sim} & j_n'\,(\text{Cousin}\,(W_n\Omega^N))
\end{array}$$

We therefore get isomorphisms Cousin $(W_{n-1}\Omega^N)^i \xrightarrow{\sim} j_n'$(Cousin $(W_n\Omega^N)^i$) in all degrees i. As Cousin $(W_n\Omega^N)^i$ is injective and j_n' evidently equals $j_n^!$ on injective modules we get isomorphisms Cousin $(W_{n-1}\Omega^N)^i \xrightarrow{\sim} j_n^!$ (Cousin $(W_n\Omega^N)^i$), which are easily seen to be adjoint to the ϱ^i: s. We finish the proof by doing similarly for C^n and V.

Suppose that (N_n, ϱ, C^n, V) is a dualizing system. We then get a functor

(2.3.1) $$\underline{\text{Hom}}(-, N)\colon (\text{dir}-\text{drw}-X)^0 \to \text{inv}-\text{drw}-X,$$

where $\underline{\text{Hom}}(M, N)_n := \underline{\text{Hom}}_{W_n\mathcal{O}}(M_n, N_n)$ and π, F, V and d are defined by the commutativity of the diagrams:

(2.3.2)
$$\begin{array}{ccc}
\underline{\text{Hom}}_{W_n\mathcal{O}}(M_n, N_n) & \xrightarrow{\quad\varrho^*\quad} & \underline{\text{Hom}}_{W_n\mathcal{O}}((j_n)_* M_{n-1}, N_n) \\
\downarrow \pi & & \uparrow \varrho_* \\
(j_n)_* \underline{\text{Hom}}_{W_{n-1}\mathcal{O}}(M_{n-1}, N_{n-1}) & \longrightarrow & \underline{\text{Hom}}_{W_n\mathcal{O}}((j_n)_* M_{n-1}, (j_n)_* N_{n-1})
\end{array}$$

(2.3.3)
$$\begin{array}{ccc}
\underline{\text{Hom}}_{W_n\mathcal{O}}(M_n, N_n) & \xrightarrow{\quad V^*\quad} & \underline{\text{Hom}}_{W_n\mathcal{O}}((Fj_n)_* M_{n-1}, N_n) \\
\downarrow F & & \uparrow V_* \\
(Fj_n)_* \underline{\text{Hom}}_{W_{n-1}\mathcal{O}}(M_{n-1}, N_{n-1}) & \longrightarrow & \underline{\text{Hom}}_{W_n\mathcal{O}}((Fj_n)_* M_{n-1}, (Fj_n)_* N_{n-1})
\end{array}$$

(2.3.4)
$$\begin{array}{ccc}
F_*^n \underline{\text{Hom}}_{W_n\mathcal{O}}(M_n, N_n) & \longrightarrow & \underline{\text{Hom}}_{W_n\mathcal{O}}(F_*^n M_n, F_*^n N_n) \\
\Big| & & \downarrow (C^n)_* \\
\Big| & & \underline{\text{Hom}}_{W_n}(F_*^n M_n, N_n) \\
\Big| d & & \downarrow d^* \\
\Big| & & \underline{\text{Hom}}_{W_n}(F_*^n M_n, N_n)(1) \\
\downarrow & & \uparrow (C^n)^* \\
F_*^n \underline{\text{Hom}}_{W_n\mathcal{O}}(M_n, N_n)(1) & \longrightarrow & \underline{\text{Hom}}_{W_n\mathcal{O}}(F_*^n M_n, F_*^n N_n)[1]
\end{array}$$

(2.3.5)
$$\begin{array}{ccc}
(Fj_n)_* \underline{\text{Hom}}_{W_{n-1}\mathcal{O}}(M_{n-1}, N_{n-1}) & \longrightarrow & \underline{\text{Hom}}_{W_n\mathcal{O}}((Fj_n)_* M_{n-1}, (Fj_n)_* N_{n-1}) \\
\downarrow V & & \downarrow V_* \\
\underline{\text{Hom}}_{W_n}(M_n, N_n) & \xleftarrow{\quad F^*\quad} & \underline{\text{Hom}}_{W_n}((Fj_n)_* M_{n-1}, N_n)
\end{array}$$

These definitions, of course, require some justification. By the definition of dualizing system the composites

$$(j_n)_* \underline{\mathrm{Hom}}_{W_{n-1}\mathcal{O}}(M_{n-1}, N_{n-1}) \to \underline{\mathrm{Hom}}_{W_n\mathcal{O}}((j_n)_* M_{n-1}, (j_n)_* N_{n-1}) \xrightarrow{\varrho_*} \underline{\mathrm{Hom}}_{W_n\mathcal{O}}((j_n)_* M_{n-1}, N_n)$$

etc. are, by duality, isomorphisms. This shows that there does indeed exist unique π, F, d and V making the diagrams above commute.

Using (2.2.1) we obtain a functor

(2.3.6) $\underline{\mathrm{Hom}}(-, \mathrm{Cousin}(W.\Omega^N))\colon K(\mathrm{dir}-\mathrm{drw}-X)^0 \to K(\mathrm{inv}-\mathrm{drw}-X)$

As

$$\mathrm{inv}-\mathrm{drv}-X \to W_n\mathcal{O}_X-\mathrm{mod}$$

$$(M_n, \pi, F, d, V) \mapsto M_n$$

is a set of exact conservative functors and $\mathrm{Cousin}(W_n\Omega^N)$ is an injective complex, $\underline{\mathrm{Hom}}(-, \mathrm{Cousin}(W.\Omega^N))$ trivially extends to the derived category and we will denote this extension

(2.3.7) $$(-)^{\check{}}\colon D(\mathrm{dir}-\mathrm{drw}-X)^0 \to D(\mathrm{inv}-\mathrm{drw}-X).$$

Remark. I do not know if $(-)^{\check{}}$ is the derived functor of $\underline{\mathrm{Hom}}(-, W.\Omega^N)$ or not, but I expect it not to be.

We clearly also have that

(2.3.8) $((M)^{\check{}})_n = R\,\underline{\mathrm{Hom}}_{W_n\mathcal{O}}(M_n, W_n\Omega^N)$ for $M \in D(\mathrm{dir}-\mathrm{drw}-X)$.

Proposition 2.4. Multiplication in $W.\Omega^{\cdot}$ induces an isomorphism

(2.4.1) $$(W_\infty.\Omega^{\cdot})^{\check{}} \to W\Omega^{\cdot}(N).$$

This follows from (0.1), (0.2), Thm. II: 2.2, the fact that $C^n d=0$ which follows from (I: 3.1) and (2.3.8) paired with the observation, made above, that the $(-)_n$:s form a set of conservative functors on $D(\mathrm{inv}-\mathrm{drw}-X)$.

Taking global sections gives us a functor:

(2.5.1) $$\Gamma\colon \mathrm{inv}-\mathrm{drw}-X \to \mathrm{inv}-\mathrm{drw}-S$$

$$(M_n, \pi, F, d, V) \mapsto (\Gamma(M_n), \Gamma(\pi), \Gamma(F), \Gamma(d), \Gamma(V))$$

and correspondingly for direct systems.

Γ derives to a functor

(2.5.2) $$R\Gamma\colon D^+(\mathrm{inv}-\mathrm{drw}-X) \to D^+(\mathrm{inv}-\mathrm{drw}-S)$$

resp. $$R\Gamma\colon D^+(\mathrm{dir}-\mathrm{drw}-X) \to D^+(\mathrm{inv}-\mathrm{drw}-S).$$

Using again the conservativity of $\{(-)_n\}$ we see that $R\Gamma$ may be computed with the aid of Čech-resolutions, which in turn gives;

(2.5.3) $$(R\Gamma(M))_n = R\Gamma(M_n)$$

which in turn shows that Γ has finite cohomological dimension so we may extend $R\Gamma$ to all of $D(\text{inv}-\text{drw}-X)$ (resp. $D(\text{dir}-\text{drw}-X)$).

[7: VII, Thm. 2.1] provides us with a mapping of complexes $\Gamma(\text{Cousin}(W_{\bullet}\omega^N))[N] \to \text{Cousin}(W_{\bullet})$ wich induces a morphism

(2.5.4) $$R\Gamma((M)^{\vee})[N] \to (R\Gamma(M))^{\vee} \quad \text{for} \quad M \in D(\text{dir}-\text{drw}-X).$$

Again, using (2.3.8), (2.5.3), the conservativity of $\{(-)_n\}$ and duality (cf. [7, VII: Thm 3·3]) for the proper morphisms $f_n\colon W_nX \to W_nS$ one sees that (2.5.4) is an isomorphism. In particular we have an isomorphism (using (2.4.1)):

(2.5.5) $$(R\Gamma(W_{\infty}.\Omega^{\cdot}))^{\vee} \xrightarrow{\sim} R\Gamma(W.\Omega^{\cdot})(N)[N] \quad \text{in} \quad D(\text{inv}-\text{drw}-S).$$

We have a functor

(2.6.1) $$\varprojlim\colon \text{inv}-\text{drw}-S \to R-\text{mod},$$

where the limit is taken along $\pi\colon M_{n+1} \to M_n$. As is easily seen, the standard flasque system associated to $\{M_n, \pi\}$ has a canonical structure of inverse de Rham—Witt system, thereby showing that $\varprojlim$ derives to;

(2.6.2) $$R\varprojlim\colon D(\text{inv}-\text{drw}-S) \to D(R).$$

As $W\Omega^{\cdot} := \varprojlim W_n\Omega^{\cdot}$ and $\varprojlim^1 W.\Omega^{\cdot} = 0$ ($W_{n+1}\Omega^{\cdot} \xrightarrow{\pi} W_n\Omega^{\cdot}$ are surjective and $W_n\Omega^{\cdot}$ is coherent on W_nX) we get an isomorphism in $D(R)$

(2.6.3) $$R\Gamma(W\Omega^{\cdot}) \xrightarrow{\sim} R\varprojlim(R\Gamma(W.\Omega^{\cdot})).$$

The system (R_n, ϱ, F, d, V), where F, d and V are induced by multiplication to the left, is in dir$-$drw$-S$ and all the morphisms involved are right R-module homomorphisms. It therefore gives rise to a functor:

(2.7.1) $$R_{\infty}.\otimes_R(-)\colon R-\text{mod} \to \text{dir}-\text{drw}-S$$
$$M \mapsto \{R_n \otimes_R M\}.$$

As (0.4.3) shows that $R.\otimes_R(-)$ has finite homological dimension it (left-) derives into

(2.7.2) $$R_{\infty}.\otimes_R^L(-)\colon D(R) \to D(\text{dir}-\text{drw}-S)$$

with $(R_{\infty}.\otimes_R(-))_n = R_n \otimes_R^L(-)$. (0.9) and a Čech-resolution argument show that we have an isomorphism

(2.7.3) $$R_{\infty}.\otimes_R^L R\Gamma(W\Omega^{\cdot}) \xrightarrow{\sim} R\Gamma(W_{\infty}.\Omega^{\cdot}).$$

Definition 2.8. *Put*

(2.8.1) $$D(-)\colon D(R)^0 \to D(R)$$

equal to $R\varprojlim\{(R_\infty.\otimes^L_R(-))^\check{}\}$.

Theorem 2.9. *There is a canonical isomorphism*

(2.9.1) $$D(R\Gamma(W\Omega^\cdot))(-N)[-N] \to R\Gamma(W\Omega^\cdot).$$

To prove this we simply combine (2.7.3), (2.6.3) and (2.5.5).

3.

Definition 3.1. *Put*

(3.1.1) $$\check{R} := \varprojlim\{((R_n, \varrho, F, V))^\check{}\}.$$

$\check{R}$ is considered as an R-bi-module, the first structure coming from the fact that (R_n, ϱ, F, d, V) is a right R-module object in dir$-$drw$-S$, so by functionality the limit has a structure of R-module, and the second comes from (2.6.1). We get a bimodule because we get by functoriality a left R-module object in R-mod which is the same thing as an R-bimodule.

We may therefore regard $R\mathrm{Hom}_R(-, \check{R})$ as a functor $D(R)^0 \to D(R)$, where the first R-module structure is used to compute $R\mathrm{Hom}_R(-, \check{R})$ and the second to provide $R\mathrm{Hom}_R(-, \check{R})$ with an R-complex structure.

Proposition 3.2. *There is a canonical isomorphism in* $D(R)$;

(3.2.1) $$D(M) \to R\,\mathrm{Hom}_R(M, \check{R})$$

for all $M \in D(R)$.

Proof. We have

$$D(M) = R\varprojlim\{\mathrm{Hom}^\cdot_{W_n}(R_n \otimes^L_R M, \mathrm{Cousin}(W_n))\}$$

$$= R\varprojlim\{R\,\mathrm{Hom}_R(M, \mathrm{Hom}_{W_n}(R_n, W_n))\} = R\,\mathrm{Hom}_R(M, R\varprojlim\{(R.)^\check{}\}).$$

Using that the injectivity of $\varrho\colon R_n \to R_{n+1}$ implies that $R_\bullet$ has surjective transition morphisms and therefore is $\varprojlim$-acyclic we are finished except for the fact that the isomorphism is not strictly speaking defined as a morphism in $D(R)$. I leave as an exercise to use the technique used in Section 2 to remedy this.

Lemma 3.3. *The ring homomorphism*

(3.3.1) $$\varphi\colon R \to R^{\mathrm{op}}$$

taking F, d, V and $a \in W$ to V, d, F resp. a is an isomorphism with itself as an inverse (considered as a ring homomorphism $R^{op} \to R$).

This is obvious.

Using φ we may pass freely back and forth between left and right R-modules. (0.3.1) allows us to define a W-linear function

(3.4.1) $$\psi\colon R \to W$$

uniquely determined by $\psi(V^n)=\psi(F^n)=0$ for all $n\geqq 0$,

$$\psi(dV^n) = \psi(F^n d) = 0 \quad \text{for} \quad n > 0 \quad \text{and} \quad \psi(d) = 1.$$

Direct verification shows that the pairing

$$R \times R \to W$$

$$(r, s) \mapsto \psi(rs)$$

passes to a pairing

(3.4.2) $$\bar{\psi}\colon R_n \times R \to W_n$$

giving rise to a mapping

(3.4.3) $$R \to \mathrm{Hom}_W(R_n, W_n)$$

$$r \mapsto \big(s \mapsto \bar{\psi}(sr)\big).$$

These mappings give in the limit a map

(3.4.4) $$\alpha\colon R \to \check{R} = \varprojlim \{\mathrm{Hom}_W(R_n, W_n)\}$$

$\mathrm{Hom}_W(R_n, W_n)$ has a natural topology as the Pontryagin dual of the discrete W-module R_n, which in this case is the topology of eventual constantness and therefore $\check{R}$ also has, as an inverse limit of topological groups. As $\check{R}$ is complete α extends to the completion of R in the topology on R induced by α.

Proposition 3.5. *α induces an isomorphism between the completion $\bar{R}$ of R in the above mentioned topology and $\check{R}$. Every element in $\bar{R}$ can be written uniquely as a product*

(3.5.1) $$\Pi_{n>0} V^n a_{-n} + \Pi_{n\geqq 0} a_n F^n + \Pi_{n>0} dV^n b_{-n} + \Pi_{n\geqq 0} b_n F^n d,$$

where $a_n, b_n \in W$ and R embeds in the obvious way.

The bimodule structure on $\bar{R}$ coming from the one on $\check{R}$ (cf. (3.1)) by transport of structure through α has as first structure multiplication by elements in R to the left on products as in (3.5.1), and as second structure multiplication to the right turned into a left action via (3.3).

Proof. That α induces an isomorphism on completions and the description (3.5.1) are shown by a simple calculation using (0.3.1). I leave this verification to the reader. That the first bimodule structure is as claimed is clear as $\psi(rs\cdot t)=\psi(r\cdot st)$ for $r, s, t\in R$. As for the second structure one need only observe, using again (0.3.1), that $\psi(rF)=p\sigma^{-1}\psi(Fr)$ etc. for $r\in R$.

Note. The reader should be aware that as ψ is of degree -1, so is α.

Corollary 3.5.1. *The projection* $\check{R}\to\mathrm{Hom}_W(R_n, W_n)$ *induces an isomorphism, where tensor product is taken wrt the second structure*

(3.5.1.1) $$R_n\otimes^L_R \check{R}\xrightarrow{\sim}\mathrm{Hom}_W(R_n, W_n).$$

Proof. That $R_n\otimes_R\check{R}\xrightarrow{\sim}\mathrm{Hom}_W(R_n, W_n)$ is easily seen and that $\mathrm{Tor}_i^R(R_n, \check{R})=0$ for $i>0$ is proved, using (0.4.3) and (3.5.1), exactly as [10, Prop. 3.2a] with the exception that the sums should be replaced by products.

4.

Identifying graded $W_n[d]$-modules with W_n-complexes we get a functor $\mathrm{Hom}_{W_n}(-, W_n)$: $(W_n[d]\text{-mod})^0\to W_n[d]$-mod which derives to $R\,\mathrm{Hom}_{W_n}(-, W_n)$: $D(W_n[d])^0\to D(W_n[d])$. Using the now familiar technique of Section 2 and II Cor. 2.2.23 we get, for X proper an isomorphism in $D(W_n[d])$;

(4.1) $$D(R\Gamma(W_n\Omega^\cdot))(-N)[-N]\xrightarrow{\sim}R\Gamma(W_n\Omega^\cdot)$$

where I have put $D(-):=R\,\mathrm{Hom}_{W_n}(-, W_n)$.

Proposition 4.2. *There is a canonical isomorphism for* $M\in D(R)$;

(4.2.1) $$R_n\otimes^L_R D(M)\xrightarrow{\sim}D(R_n\otimes^L_R M)$$

in $D(W_n[d])$, *taking, for* X *proper,* (2.9.1) *to* (4.1) *through* (0.9).

Proof. As R_n has a resolution by finitely generated free R-modules (cf. (0.4.3)) we have $R_n\otimes^L_R D(M)=R_n\otimes^L_R R\,\mathrm{Hom}_R(M, \check{R})=R\,\mathrm{Hom}_R(M, R_n\otimes^L_R\check{R})$, where in $R_n\otimes^L_R\check{R}$ the tensor product is taken with respect to the second structure, so by (3.5.1.1) this is equal to $R\,\mathrm{Hom}_R(M, \mathrm{Hom}_{W_n}(R_n, W_n))=R\,\mathrm{Hom}_{W_n}(R_n\otimes^L_R M, W_n)=D(R_n\otimes^L_R M)$. That this isomorphism takes (2.9.1) to (4.1) is clear.

5.

Definition 5.1. *An R-module is said to be of level $N>0$ if it is zero except in degrees i, $0\leqq i\leqq N$ and F is invertible in degree N.*

An F-crystal of level N is a module over the W-ring $W[F, V](N)$ with generators F and V and relations $Fa=a^{\sigma}F$, $aV=Va^{\sigma}$ and $FV=VF=p^N$.

The category of R-modules of level N (resp. F-crystals of level N) will be denoted R-mod-N (resp. F-crys-N). It is clear that R-mod-N is an abelian subcategory of R-mod, closed under kernels, cokernels, sums, products and extensions.

Lemma 5.2. i) *$R(-i)$ $0\leqq i<N$ and $(\sum_{i\in Z} WF^i)(-N)$, with the obvious R-module structure, form a set of projective generators for R-mod-N.*

ii) *If P is projective in R-mod-N then $\mathrm{Tor}_i^R(R_n, P)=0$ for $i, n>0$.*

iii) *If P is projective in R-mod-N then $\mathrm{Ext}_R^i(P, \check{R})=0$ for $i>0$.*

iv) *If $M\in R$-mod-N then $\mathrm{Hom}_R(M, \check{R})(-N)\in R$-mod-$N$.*

Proof. i) is clear, except for the fact that $R(-N+1)\in R$-mod-N but this follows from (0.3.1). As for ii) one may assume that P is one of $R(-i)$ $0\leqq i<N$ or $W[F, F^{-1}](-N):=(\sum_{i\in Z} WF^i)(-N)$ because of i). For $R(i)$ ii) is obvious and for $W[F, F^{-1}](-N)$ one uses (0.4.3) and the fact that as here $d=0$ ii) is equivalent to the bijectivity of F and the injectivity of V which is clear. iii) follows from ii) and the formula $\mathrm{Ext}_R^i(M, \check{R})=\mathrm{Hom}_W(\varinjlim_n \{\mathrm{Tor}_i^R(R_n, M)\}, K/W)$ which follows from (3.2). We may choose an exact sequence $P_1\to P_0\to M\to 0$ with P_1 and P_0 projective in R-mod-N giving an exact sequence $0\to\mathrm{Hom}_R(M, \check{R})(-N)\to \mathrm{Hom}_R(M, \check{R})(-N)\to\mathrm{Hom}_R(M, \check{R})(-N)$ in R-mod and as R-mod-N is closed under kernels we are reduced to M projective and by i) to $M=R(-i)$, $0\leqq i<N$, or $W[F, F^{-1}](-N)$. As $\check{R}$ is concentrated in degree -1 and $0, M=R(-i)$, $0\leqq i<N$, or $W[F, F^{-1}]$ is clear. Proposition 3.5 shows that F (second structure) is bijective on R in degree 0, so $M=R$ is clear. Finally, $R_n\otimes_R W[F, F^{-1}](-N)$ lies in degree N so by the formula above $\mathrm{Hom}_R(W[F, F^{-1}](-N), \check{R})(-N)$ lies in degree 0 and therefore in R-mod-N.

Lemma 5.2 shows that if $M\in D(R\text{-mod-}N)$ then $D(M)(-N)$ may be given a canonical structure of R-mod-N-complex, which we by abuse of notation will denote again $D(M)(-N)$.

If $M\in F-$crys-N then $\mathrm{Hom}_W(M, W)$ may be given a structure of F-crystal of level N as follows: It is a W-module in the obvious way, $F:=V^*\sigma_*$ and $V:=F^*\sigma_*^{-1}$. It therefore derives to a functor $D(-)(-N)\colon D(F\text{-crys-}N)^{\circ}\to D(F\text{-crys-}N)$ and as $W[F, V](-N)$ is free as a W-module the underlying W-complex of $D(M)(-N)$ is $R\,\mathrm{Hom}_W(M, W)$ for $M\in D(F\text{-crys-}N)$, which shows that $D(-)(-N)$ is indeed defined on the whole of $D(F\text{-crys-}N)$.

As R_n is killed by p^n the natural projection $R \to R_n$ induces mappings $R/p^n \to R_n$ which fit together to give morphisms $(R/p^n, \underline{p}, F, d) \to (R_n, \varrho, F, d)$ of right R-module objects in the category "dir$-$drw$-S$ without V" ($V\colon R \to R$ does not take $p^n R$ to $p^{n-1}R$) and we then get an induced morphism of $R-W[V, d]$-modules $\check{R} = \varinjlim_n \{\mathrm{Hom}_{W_n}(R_n, W_n)\} \to \varinjlim_n \{\mathrm{Hom}_{W_n}(R/p^n, W_n)\} = \mathrm{Hom}_W(\varinjlim \{W/p^n \otimes_W R\}, K/W) = \mathrm{Hom}_W(K/W \otimes_W R, K/W) = \mathrm{Hom}_W(R, \mathrm{Hom}_W(K/W, K/W)) = \mathrm{Hom}_W(R, W)$. It is clear that the R-module structure on $\mathrm{Hom}_W(R, W)$ is induced by right multiplication on R and it follows from (2.3.4—5) that the $W[V, d]$-module structure is given by $V = pF^*\sigma_*^{-1}$ and $d = d^*$ where the mappings F and on R are multiplication on the left.

If M is an R-module of level N the we may define morphisms $\underline{F}, \underline{V}\colon M \to M$ of complexes:

(5.3)
$$
\underline{F}\colon\quad
\begin{array}{ccccccc}
M^0 & \xrightarrow{d} & M^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & M^N \\
\downarrow{\scriptstyle F} & & \downarrow{\scriptstyle pF} & & & & \downarrow{\scriptstyle p^N F} \\
M^0 & \xrightarrow{d} & M^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & M^N
\end{array}
$$

$$
\underline{V}\colon\quad
\begin{array}{ccccccc}
M^0 & \xrightarrow{d} & M^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & M^N \\
\downarrow{\scriptstyle p^{N-1} V} & & \downarrow{\scriptstyle p^{N-2} V} & & & & \downarrow{\scriptstyle F^{-1}} \\
M^0 & \xrightarrow{d} & M^1 & \longrightarrow & \cdots & \xrightarrow{d} & M^N
\end{array}
$$

and we clearly have the relations: $\underline{FV} = \underline{VF} = p^N$; $\underline{F}a = a^\sigma \underline{F}$ and $a\underline{V} = \underline{V}a^\sigma$. Similarly, if M is a complex of R-modules of level N we get morphisms $\underline{F}$ and $\underline{V}$ of double complexes $M \to M$ and by the relations just mentioned we get a canonical structure of a complex of F-crystals of level N on $\mathrm{s}(M)$, the associated simple complex, thereby giving us an exact functor;

(5.4) $$\mathrm{s}\colon D(R\text{-mod-}N) \to D(F\text{-crys-}N).$$

Using the mapping $\check{R} \to \mathrm{Hom}_W(R, W)$ from above, we get for a complex M of R-modules of level N a mapping;

$$\mathrm{s}(\mathrm{Hom}^{\cdot}_R(M, \check{R})) \to \mathrm{s}(\mathrm{Hom}^{\cdot}_R(M, \mathrm{Hom}_W(R, W))) = \mathrm{s}(\mathrm{Hom}^{\cdot}_W(M, W)) = \mathrm{Hom}^{\cdot}_W(\mathrm{s}(M), W)$$

which gives us a mapping

(5.5) $$\mathrm{s}(\mathrm{Hom}^{\cdot}_R(M, \check{R})(-N)) \to \mathrm{Hom}^{\cdot}_W(\mathrm{s}(M), W)[-N].$$

So fai it is only given as a morphism of W-complexes, even though both sides have canonical structures of complexes of F-crystals of level N by (5.4). I claim that (5.5) is, in fact, a morphism of complexes of F-crystals of level N. As the right hand side is torsion free it suffices to show that (5.5) commutes with the action of pV. As $\mathrm{Hom}^{\cdot}_W(M, W)(-N)$ is concentrated in degrees i $0 \leq i \leq N$ and V is defined on it we see that $p\underline{V}$ is defined on $\mathrm{Hom}^{\cdot}_W(M, W)(-N)$ and is a mapping of double complexes. It is clear that $\mathrm{Hom}^{\cdot}_R(M, \check{R})(-N) \to \mathrm{Hom}^{\cdot}_W(M, W)(-N)$ commutes with the action of $p\underline{V}$. By the observation made above V on $\mathrm{Hom}^{\cdot}_{W}(M, W)(-N)$ is simply $pF^*\sigma_*^{-1}$ so by the definition of V on $\mathrm{Hom}^{\cdot}_{W}(\mathrm{s}(M), W)[-N]$ $\mathrm{s}(\mathrm{Hom}^{\cdot}_{W}\cdot(M, W)(-N))=\mathrm{Hom}^{\cdot}_{W}(\mathrm{s}(M), W)[-N]$ takes $\mathrm{s}(p\underline{V})$ to pV. This evidently proves the assertion.

Proposition 5.6. *The canonical morphism in* $D(F\text{-crys-}N)$

(5.6.1) $$\mathrm{s}(D(M)(-N)) \to D(\mathrm{s}(M))(-N)[-N]$$

induced from (5.5) *is an isomorphism for* $M\in D^-(R\text{-mod-}N)$ *and takes* (2.9.1) *to (the limit of) the crystalline duality isomorphism* [15, VII, 2.1.3].

To show that we get an isomorphism we may, by Lemma 5.2, assume that M is $R(i)$ or $W[F, F^{-1}]$. As $\mathrm{s}(\mathrm{Hom}_R(M, \check{R}))=\varprojlim \{\mathrm{Hom}_{W_n}(\mathrm{s}(R_n\otimes_R M), W_n)\}$ and similarly for $\mathrm{s}(\mathrm{Hom}_W(M, W))$ it suffices to show, by Lemma 5.2 ii), that $\mathrm{s}(M/p^nM)\to \mathrm{s}(R_n\otimes_R M)$ is an isomorphism for M equal to $R(i)$ or $W[F, F^{-1}]$. For $W[F, F^{-1}]$ it is clear as $p^nW[F, F^{-1}]=V^nF^nW[F, F^{-1}]=V^nW[F, F^{-1}]=V^nW[F, F^{-1}]+dV^nW[F, F^{-1}]$ so that already $W[F, F^{-1}]/p^n\to R_n\otimes_R W[F, F^{-1}]$ is an isomorphism. The case of $R(i)$ is an explicit computation using (0.3.1) which I leave to the reader. That this isomorphism takes (2.9.1) to the crystalline duality isomorphism follows from I: 5.

Let us also note that in the course of the proof we proved the following lemma:

Lemma 5.6.1. *If* $M\in D^-(R)$ *then the projection* $R/p^n\to R_n$ *induces an isomorphism*

(5.6.2) $$W/p^n\otimes^L_W \mathrm{s}(M) = \mathrm{s}(R/p^nR\otimes^L_R M) \xrightarrow{\sim} \mathrm{s}(R_n\otimes^L_R M).$$

Indeed, we reduce to $M=R(i)$ and this is proved in (5.6).

IV. Coherent complexes and duality

0. Having proved the duality formula (III: Thm. 2.9) we would like to be able to use it. In order to do that we must compute $D^i(M):=H^i(D(M))$ for a coherent module M after which we can analyze $D(M)$ for M a coherent complex with the aid of the spectral sequence $D^i(H^{-j}(M))\Rightarrow D^{i+j}(M)$. This will be done in this chapter. It will begin by showing that $D(M)$ is coherent if M is and that in that case we have a biduality isomorphism $M\xrightarrow{\sim} D(D(M))$. We will then carry out the computation of $D^i(M)$ for M a coherent module and spell out the consequences for the calculation of $D(M)$ for M a coherent complex.

The commutativity of the multiplication in $W\Omega^{\cdot}$ implies that (III: 2.9.1) is selfdual. We will point out some implications of this as well as the calculation of the evaluation mapping $M \to D(D(M))$ in terms of the explicit computation of $D^i(M)$, M coherent, already given. This will be applied to supersingular $K3$-surfaces in the next chapter.

Finally we will compute the dual of a modified complex, in the sense of Nygaard ([13]).

1.

Proposition 1.1. i) *$D(-)$ maps $D^b_c(R)$ to itself.*
ii) *There is a natural transformation*

$$\text{ev}: \text{id} \to D(D(-)) \tag{1.1.1}$$

which is an isomorphism on every bounded coherent complex.

Proof. By definition $D(-) = R\varprojlim\{\text{Hom}_{W_n}(R_n \otimes^L_R (-), W_n)\}$ and by (0.4.3) and the fact that $\varprojlim$ has cohomological dimension 1, $D(-)$ has cohomological dimension at most 3 so it takes bounded complexes to bounded complexes. It therefore suffices to show that $D(M)$ is coherent if M is bounded coherent. By (III: Prop. 4.2) $R_n \otimes^L_R D(M) = D(R_n \otimes^L_R M)$ which is a coherent W_n-complex, so the first condition for coherence is fulfilled. Furthermore,

$$D(M) = R\varprojlim\{\text{Hom}_{W_n}(R_n \otimes^L_R M, W_n)\} =: R\varprojlim\{D(R_n \otimes^L_R M)\} = R\varprojlim\{R_n \otimes^L_R D(M)\}$$

also the second condition is fulfilled. This proves i).

To prove ii) let us first note that if we put $\check{\underline{R}}$ the R-bimodule whose underlying set is R but the two bimodule structures are interchanged we have the usual evaluation mapping

$$\text{ev}: \text{id} \to \text{Hom}_R(\text{Hom}_R(-, \check{R}), \check{\underline{R}}) \tag{1.1.2}$$

where, as usual, the first structure for a bimodule is used to compute the Hom-groups and the second to give the Hom-groups an R-module structure.

Lemma 1.1.3. *The mapping taking $\prod_{n>0} V^n a_{-n} + \prod_{n\geq 0} a_n F^n + \prod_{n>0} dV^n b_{-n} + \prod_{n\geq 0} b_n F^n d$ to $\prod_{n>0} V^n a_n + \prod_{n\geq 0} a_{-n} F^n + \prod_{n>0} dV^n b_n + \prod_{n\geq 0} b_{-n} F^n d$ gives an isomorphism $\varphi: \check{\underline{R}} \xrightarrow{\sim} \check{R}$ of bimodules.*

This is clear. (Note that this isomorphism extends φ of (III: Lemma 3.3).)
Using this lemma we may define the evaluation mapping

$$\text{ev}: \text{id} \to D(D(-)) \tag{1.1.4}$$

as the composite of (1.1.2) and $(\varphi)_*$.

Lemma 1.1.5. *The following diagrams commute*

$$\begin{array}{ccc} R_n\otimes^L_R(-) & \xrightarrow{R_n\otimes^L_R \mathrm{ev}} & R_n\otimes^L_R D(D(-)) \\ \| & & \| \\ \| & & D(R_n\otimes^L_R D(-)) \\ \| & & \| \\ R_n\otimes^L_R(-) & \xrightarrow{\mathrm{ev}} & D(D(R_n\otimes^L_R(-))) \end{array} \tag{1.1.5.1}$$

$$\begin{array}{ccc} \mathrm{s}(-) & \xrightarrow{\mathrm{s(ev)}} & \mathrm{s}(D(D(-)(-N))(-N)) \\ \| & & \| \\ \| & & D(\mathrm{s}(D(-)(-N)))(-N)[-N] \\ \| & & \| \\ \mathrm{s}(-) & \xrightarrow{\mathrm{ev}} & D(D(\mathrm{s}(-))(-N)[-N])(-N)[-N] \end{array} \tag{1.1.5.2}$$

where the canonical isomorphisms are those of (III: 4.2.1) *and* (III: 5.6.1).

Proof. Let us explicate the isomorphism $R_n\otimes^L_R D(-)=D(R_n\otimes^L_R(-))$. Let $\delta\colon \check{R}\to W_n$ be the composite of the projection $\check{R}\to R_n\otimes_R\check{R}$, the mapping $R_n\otimes_R\check{R}\to R_n\otimes_R \mathrm{Hom}_{W_n}(R_n, W_n)$ induced by the projection $\check{R}\to\mathrm{Hom}_{W_n}(R_n, W_n)$ and the mapping $R_n\otimes_R\mathrm{Hom}_{W_n}(R_n, W_n)\to W_n$ which is the adjoint of the identity $\mathrm{Hom}_{W_n}(R_n, W_n)\to\mathrm{Hom}_{W_n}(R_n, W_n)$. We then have a commutative diagram;

$$\begin{array}{ccc} \mathrm{Hom}_R(M, R) & \xrightarrow{a} & R_n\otimes_R \mathrm{Hom}_R(M, R) \\ \downarrow{\scriptstyle \delta_*} & & \downarrow{\scriptstyle c} \\ \mathrm{Hom}_{W_n}(M, W_n) & \xrightarrow{b} & \mathrm{Hom}_{W_n}(R_n\otimes_R M, W_n) \end{array} \tag{1.1.5.3}$$

for an R-module M, where a is the projection, b is induced by the projection $M\to R_n\otimes_R M$ and c is uniquely determined by the commutativity. The isomorphism $R_n\otimes^L_R D(-)=D(R_n\otimes^L_R(-))$ is then the derivation of c. To show that (1.1.5.1) is commutative we therefore have to show that for $M\in R$-mod the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\mathrm{ev}} & \mathrm{Hom}_R(\mathrm{Hom}_R(M, \check{R}), \underline{\check{R}}) \\ \downarrow & & \downarrow{\scriptstyle \varphi_*} \\ R_n\otimes_R M & & \mathrm{Hom}_R(\mathrm{Hom}_R(M, \check{R}), \check{R}) \\ \downarrow{\scriptstyle \mathrm{ev}} & & \searrow{\scriptstyle (ac)^*} \\ \mathrm{Hom}_{W_n}(\mathrm{Hom}_{W_n}(R_n\otimes_R M, W_n), W_n) & \xrightarrow{c^*} & \mathrm{Hom}_{W_n}(R_n\otimes_R \mathrm{Hom}_R(M, \check{R}), W_n) \end{array} \tag{1.1.5.4}$$

It is clear, however, that this will be true if $\delta=\delta\varphi$. From (III: 3) it follows that δ takes the product in (III: 3.5.1) to $\bar{b}_0\in W_n$, which makes the equality $\delta=\delta\varphi$ obvious. Similarly, the commutativity of (1.1.5.2) boils down to $\delta'=\delta'\varphi$, where $\delta'\colon\check{R}\to W$ is the composite of $\check{R}\to\mathrm{Hom}_W(R, W)$ of (III: 5) and $\mathrm{Hom}_W(R, W)\to W$, the adjoint of the identity. As $\delta'\colon \check{R}\to W$ takes the product in (III: 3.5.1) to b_0 the desired commutativity is clear.

We can now prove ii) of Proposition 1.1. Indeed, as $M=R\varprojlim\{R_n\otimes^L_R M\}$ if M is a coherent complex, any such complex is zero if $R_n\otimes^L_R M=0$ for all n, which shows that the functors $R_n\otimes^L_R(-)$ on $D^b_c(R)$ form a conservative family. To show that (1.1.1) is an isomorphism on $D^b_c(R)$ it suffices to show that it becomes an isomorphism after applying $R_n\otimes^L_R(-)$, so by Lemma 1.1.5 it suffices to show that ev: $R_n\otimes^L_R M\to D(D(R_n\otimes^L_R M))$ is an isomorphism for $M\in D^b_c(R)$, but $R_n\otimes^L_R M$ is a coherent W_n-complex so this follows from the usual biduality.

2.

Definition 2.1. *If* $M\in D^b_c(R)$ *put*

(2.1.1) $$D^i(M) := H^i(D(M)).$$

This is, by Prop. 1.1, *a coherent R-module.*

Just as for $\varprojlim$: inv$-$drw$-S\to R$-mod we may define a functor $\varinjlim$: dir$-$drw$-S\to R$-mod by giving the direct limit along ϱ a structure of R-module, using the mappings induced from F, d and V.

Lemma 2.2. *We have, for* M *an R-module, an isomorphism*

(2.2.1) $$D^i(M) = \mathrm{Hom}_W(\varinjlim(\mathrm{Tor}^R_i(R_n, M), \varrho, F, d, V), K/W)$$

where $(\mathrm{Tor}^R_i(R_n, M), \varrho, F, d, V):=H^{-i}(R_\infty.\otimes^L_R M)$. *The induced R-module structure on*

$$\mathrm{Hom}_W(\varinjlim(\mathrm{Tor}^R_i(R_n, M), \varrho, F, d, V), K/W)$$

is given by

$$(F, d, V) = (V^*\sigma_*, d^*, F^*\sigma_*^{-1}).$$

Indeed, we have for N a W_n-module, $\mathrm{Hom}_{W_n}(N, W_n)=\mathrm{Hom}_W(N, K/W)$ and as $\varrho\colon W_n\to W_{n+1}$ and $V\colon W_n\to\sigma_* W_{n+1}$ are the morphisms induced by $p\colon K/W\to K/W$ resp. $p\sigma^{-1}\colon K/W\to K/W$, we see that if $(N., \varrho, F, d, V)\in$ dir$-$drw$-S$ then $(N., \varrho, F, d, V)^\vee=(\mathrm{Hom}_W(N_n, K/W), \varrho^*, V^*\sigma_*, d^*, F^*\sigma_*^{-1})$ and therefore $D^i(M)= H^i(D(M))=H^i(R\varprojlim\{\mathrm{Hom}_W(R_n\otimes^L_R M, K/W)\})=H^i(\mathrm{Hom}_W(\varinjlim\{R_n\otimes^L_R M\}, K/W))= \mathrm{Hom}_W(\varinjlim\{H^{-i}(R_n\otimes^L_R M)\}, K/W)=\mathrm{Hom}_W(\varinjlim\{\mathrm{Tor}^R_i(R_n, M)\}, K/W)$ and it is clear that the R-module structure is as claimed.

Corollary 2.2.2. *If* M *is an R-module then* $D^i(M)=0$ *for* $i\neq 0, 1, 2$.

This follows from (2.2.1) and (0.4.3).

Definition 2.3. *Let* M *be a coherent R-module. Define a filtration*

(2.3.1) $$0\subseteq T^2(M)\subseteq T^1(M)\subseteq T^0(M) = M$$

as the filtration coming from the spectral sequence $D^i(D^j(M)) \Rightarrow H^{i-j}(D(D(M)))$ *and the isomorphism* $M \xrightarrow{\sim} D(D(M))$ *of* (1.1.1). *(Note that by Cor.* 2.2.2 *this really is a 3-step filtration.) Put*

$$A^i(M) := T^i(M)/T^{i+1}(M) \quad i = 0, 1, 2. \tag{2.3.2}$$

Note that by Prop. 1.1 (2.3.1) is a filtration by coherent R-modules.

Proposition 2.4. *If* M *is a coherent R-module then*

i) $$D^i(D^j(M)) = 0 \quad \text{if} \quad i \neq j,$$

ii) $$A^j(M) = D^j(D^j(M)) \quad \text{for} \quad j = 0, 1, 2.$$

Proof. We will first need the following lemma:

Lemma 2.4.1. *Let* M *be a coherent R-module and* M' *its torsion submodule.*

i) $D^0(M)$ *is torsion free.*

ii) *The inclusion* $M' \hookrightarrow M$ *induces an isomorphism* $D^2(M) \xrightarrow{\sim} D^2(M')$.

As $D(M) = R\,\mathrm{Hom}_R(M, \check{R})$ we have $D^0(M) = \mathrm{Hom}_R(M, \check{R})$ but by (III: Prop. 3.5) $\check{R}$ is torsion free and then so is $\mathrm{Hom}_R(M, \check{R})$ which proves i). By (0.4.3) $\mathrm{Tor}_2^R(R_n, M) = \mathrm{Ker}\, F^n d \cap \mathrm{Ker}\, F^n$ and if $F^n x = 0$ then $p^n x = V^n F^n x = 0$ so $\mathrm{Tor}_2^R(R_n, M') = \mathrm{Tor}_2^R(R_n, M)$ and ii) follows from Lemma 2.2.

Let us now prove the proposition for $j = 2$. It is clear that $p^n M' = 0$ for some n. (This is a condition stable under extension and true for the modules in Section 0 of which any coherent module is an extension.) Therefore, $M' = \mathrm{Ker}\, p^n \colon M \to M$ and so is coherent. We have a morphism $D^2(D^2(M)) \to A^2(M)$ coming from the spectral sequence

$$D^i(D^j(M)) \Rightarrow M \tag{2.4.2}$$

of above and Cor. 2.2.2, which shows that (2.4.2) is concentrated in a 3×3 square. This morphism is by definition surjective. We therefore see from Lemma 2.4.1 ii) that we may assume that $M' = M$. In this case we have that $p^n D^0(D^i(M)) = 0$ and Lemma 2.4.1 i) shows that $D^0(D^i(M)) = 0$, so (2.4.2) degenerates, giving $D^2(D^i(M)) = 0$ if $i \neq 2$ and $A^2(M) = D^2(D^2(M))$. The fact that the proposition is true for $j = 2$ implies that (2.4.2) degenerates which is equivalent to the proposition.

3.

Lemma 3.1. *If* M *is a coherent R-module then*

$$D^2(M) = \mathrm{Hom}_W\Big(\bigcup_n \mathrm{Ker}\, F^n d \cap \mathrm{Ker}\, F^n, K/W\Big)(1) \tag{3.1.1}$$

with $(F, d, V) = (V^* \sigma_*, d^*, F^* \sigma_*^{-1})$.

By Lemma 2.2 it will suffice to show that $(\mathrm{Tor}_2^R(R_n, M), \varrho, F, d, V)=(\mathrm{Ker}\, F^n d\cap \mathrm{Ker}\, F^n(-1), id, F, d, V)$ and that the W_n-module structure on $\mathrm{Tor}_2^R(R_n, M)$ is the natural one on $\mathrm{Ker}\, F^n d\cap\mathrm{Ker}\, F^n(-1)$. All this is provided by (0.4.3) and the following coverings:

$$\text{(3.1.2)}\quad \begin{array}{ccccccccc}
0 \to & R(-1) & \xrightarrow{(F^n,\, -F^n d)} & R(-1)\oplus R & \xrightarrow{dV^n+V^n} & R & \to & R_n & \to 0 \\
 & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \varrho} & \\
0 \to & R(-1) & \xrightarrow{(F^{n+1},\, -F^{n+1} d)} & R(-1)\oplus R & \xrightarrow{dV^{n+1}+V^{n+1}} & R & \to & R_{n+1} & \to 0
\end{array}$$

$$\text{(3.1.3)}\quad \begin{array}{ccccccccc}
0 \to & R(-1) & \xrightarrow{(F^n,\, -F^n d)} & R(-1)\oplus R & \xrightarrow{dV^n+V^n} & R & \to & R_n & \to 0 \\
 & \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} p & 0 \\ 0 & id \end{smallmatrix}\right)} & & \downarrow{\scriptstyle V} & & \downarrow{\scriptstyle V} & \\
0 \to & R(-1) & \xrightarrow{(F^{n+1},\, -F^{n+1} d)} & R(-1)\oplus R & \xrightarrow{dV^{n+1}+V^{n+1}} & R & \to & R_{n+1} & \to 0
\end{array}$$

$$\text{(3.1.4)}\quad \begin{array}{ccccccccc}
0 \to & R(-1) & \xrightarrow{(F^{n+1},\, -F^{n+1} d)} & R(-1)\oplus R & \xrightarrow{dV^{n+1}+V^{n+1}} & R & \to & R_{n+1} & \to 0 \\
 & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} id & 0 \\ 0 & p \end{smallmatrix}\right)} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle F} & \\
0 \to & R(-1) & \xrightarrow{(F^n,\, -F^n d)} & R(-1)\oplus R & \xrightarrow{dV^n+V^n} & R & \to & R_n & \to 0
\end{array}$$

$$\text{(3.1.5)}\quad \begin{array}{ccccccccc}
0 \to & R(-2) & \xrightarrow{(F^n,\, -F^n d)} & R(-2)\oplus R(-1) & \xrightarrow{dV^n+V^n} & R(-1) & \to & R_n(-1) & \to 0 \\
 & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 0 & 0 \\ -id & 0 \end{smallmatrix}\right)} & & \downarrow{\scriptstyle -d} & & \downarrow{\scriptstyle -d} & \\
0 \to & R(-1) & \xrightarrow{(F^n,\, -F^n d)} & R(-1)\otimes R & \xrightarrow{dV^n+V^n} & R & \to & R_n & \to 0
\end{array}$$

$$\text{(3.1.6)}\quad \begin{array}{ccccccccc}
0 \to & R(-1) & \xrightarrow{(F^n,\, -F^n d)} & R(-1)\oplus R & \xrightarrow{dV^n+V^n} & R & \to & R_n & \to 0 \\
 & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a^{\sigma^n}} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle a} & \\
0 \to & R(-1) & \xrightarrow{(F^n,\, -F^n d)} & R(-1)\oplus R & \xrightarrow{dV^n+V^n} & R & \to & R_n & \to 0
\end{array}\qquad a\in W.$$

Corollary 3.1.7. *$A^0(M)$ and $A^1(M)$ are finitely generated as W-modules and F is injective on them.*

By Prop. 2.4 $D^2(A^0(M))=D^2(A^1(M))=0$ so it will suffice to show that if $D^2(M)=0$ for M a coherent R-module then M fulfills the conclusions of the corollary. But $\mathrm{Hom}_W(\bigcup_n \mathrm{Ker}\, F^n d\cap\mathrm{Ker}\, F^n, K/W)=D^2(M)=0$ implies that $\bigcup_n \mathrm{Ker}\, F^n d\cap \mathrm{Ker}\, F^n=0$ and a fortioii that $\mathrm{Ker}\, Fd\cap\mathrm{Ker}\, F=0$. Let us therefore consider a coherent R-module M such that $\mathrm{Ker}\, Fd\cap\mathrm{Ker}\, F=0$. I claim that $F^\infty BM^i=0$ for all i. If this is false then the R-module $M^{i-1}/V^{-\infty}ZM^{i-1} \xrightarrow{d} F^\infty BM^i$ is non-zero for some i and by (0.8) $F^\infty BM^i$ contains some $\mathbf{U}_j^1$ as a submodule, but $Fd=0$ on $F^\infty BM^i\subseteq V^{-\infty}ZM^i$ and F is certainly not injective on $\mathbf{U}_j^1$. As now $F^\infty BM^i=0$ for all i $d=0$, so $V^{-\infty}ZM^i=M^i$ for all i, which by (0.8) implies that M is finitely generated as a W-module and that $\mathrm{Ker}\, Fd\cap \mathrm{Ker}\, F=\mathrm{Ker}\, F$ showing that F is indeed injective on M.

Definition 3.2. *Let M be a coherent R-module. Put*

$$(3.2.1)\qquad A_s^1(M) := p\text{-tors}\,(A^1(M)) \quad A_n^1(M) := A^1(M)/A_s^1(M)$$
$$A_f^2(M)^i := \text{Coeur}\, A^2(M)^i \quad A_d^2(M)^i := \text{dom}\, A^2(M)^i.$$

Theorem 3.3. *Let M be a coherent R-module.*

i) $A^0(M)$ *and* $D^0(M)$ *are torsion free, finitely generated W-modules with F bijective.*

$$(3.3.1)\qquad D^0(M) = \text{Hom}_W(A^0(M), W)(F, d, V) = (F^{-1*}\sigma_*, 0, pF^{-1}).$$

ii) $A_n^1(M)$ *and* $D^1(M)/p$*-tors are torsion free, finitely generated W-modules with F topologically nilpotent.* $A_s^1(M)$ *and* p*-tors* $(D^1(M))$ *are W-modules of finite length with F bijective.* $A^1(M)=A_n^1(M)\oplus A_s^1(M)$ *as R-modules.*

$$(3.3.2)\qquad D^1(M) = \text{Hom}_W(A_n^1(M), W)(1)\oplus\text{Hom}_W(A_s^1(M), K/W)$$
$$(F, d, V) = (V^*\sigma_*\oplus F^{-1*}\sigma_*, 0, F^*\sigma_*^{-1}\oplus pF^{-1}).$$

iii) $A_f^2(M)^i$ *and* Coeur $D^2(M)^i$ *are W-modules of finite length with F nilpotent.*

$$(3.3.3)\qquad D^2(M) = \text{Hom}_W\Big(\bigcup_n \text{Ker}\, F^n d\cap \text{Ker}\, F^n, K/W\Big)(1)$$
$$(F, d, V) = (V^*\sigma_*, d^*, F^*\sigma_*^{-1})$$
$$(3.3.4)\qquad A_f^2(D^2(M))^i = D^2(A_f^2(M)^{-i-1})(-1)$$
$$(3.3.5)\qquad A_d^2(D^2(M)) = D^2(A_d^2(M)^{-i-2})(-2)$$
$$(3.3.6)\qquad H^i(D^2(M), d) = \text{Hom}_W(H^{-i-1}(A^2(M), d), K/W).$$

iv) $D^i(M) = 0$ *if* $i \neq 0, 1, 2$.

Proof: Let us first prove i) and ii). Prop. 2.4 shows that $D^i(M)=D^i(A^i(M))$ and $D(A^i(M))=D^i(A^i(M))[-i]$. For $i=0, 1$ $A^i(M)$ is a finitely generated W-module by Cor. 3.1.7 and the proof of this corollary shows that $d=0$ on these R-modules (cf. [14, 2.5]). Therefore $A^i(M)=\oplus_j A^i(M)^j(-j)$ as R-module ($i=0, 1$) and we may confine ourselves to a particular degree and, by shifting of degree, to $A^i(M)$ being concentrated in degree 0. In this case $A^i(M)\in R$-mod-1 and by (III: 5) $D^i(A^i(M))(-1)\in R$-mod-1. Again by Prop. 2.4 $D^i(M)=A^i(D^i(M))$ so here we also have $d=0$. Therefore, using (III: Prop. 5.6), for $i=0, 1$;

$$(3.3.7)\qquad D^i(M)^j = H^{j+1}(\text{s}(D^i(A^i(M))(-1))) = H^{j+i+1}(\text{s}(D(A^i(M))(-1))$$
$$= H^{j+i}(D(\text{s}(A^i(M)))(-1)) = H^{j+i}(D(A^i(M))(-1))$$

in F-crys-1. As $D(-)(-1)=R\,\text{Hom}_W(-, W)$ we see that $D^0(M)$ is concentrated in degree 0 and that F is invertible on $D^0(M)$ as $D^0(M)(-1)\in R$-mod-1. This implies, by Prop. 2.4, that F is invertible on $A^0(M)$, and, in view of (III: 5.3) that $D^0(M)=\mathrm{Hom}_W(A^0(M), W)$ with $(F, V)=(F^{-1*}\sigma_*, pF^{-1})$. Similarly we see that $D^1(M)^{-1}=\mathrm{Hom}_W(A^1(M), W)$ and $D^1(M)^0=\mathrm{Ext}^1_W(A^1(M), W)$, that F is invertible on $D^1(M)^0$ and, as $\mathrm{Hom}_W(A^1(M), W)$ is torsion free, whereas $\mathrm{Ext}^1_W(A^1(M), W)$ is torsion, that F is invertible on p-tors$(D^1(M))$. Hence, in view of Prop. 2.4, F is invertible on $A^1_s(M)$. Finally, $\mathrm{Hom}_W(A^1(M), W)=\mathrm{Hom}_W(A^1_n(M), W)$, $\mathrm{Ext}^1_W(A^1(M), W)=\mathrm{Ext}^1_W(A^1_s(M), W)$ and, as $A^1_s(M)$ is of finite length, $\mathrm{Ext}^1_W(A^1_s(M), W)=\mathrm{Hom}_W(A^1_s(M), K/W)$. This, together with (III:5.3), shows (3.3.2). (3.3.2) shows that $D^1(M)=A^1_n(D^1(M))\oplus A^1_s(D^1(M))$, which combined with Prop. 2.4 shows that $A^1(M)=A^1_n(M)\oplus A^1_s(M)$. To show i) and ii) it now only remains to show the first sentence of ii). Prop. 2.4 shows that we need only do it for $D^1(M)/p\text{-tors}=\mathrm{Hom}_W(A^1(M), W)$, but V is topologically nilpotent on $A^1(M)$ so $F=V^*\sigma_*^{-1}$ on $\mathrm{Hom}_W(A^1(M), W)$ is also topologically nilpotent.

Let us now turn to iii). Conforming to notation in the theory of torsion theories we will say that a coherent R-module M is A^2-torsion if $A^2(M)=M$ and A^2-closed if $A^2(M)=0$. We will need the following lemma:

Lemma 3.3.8. *Let M be A^2-torsion. Then $\tilde{\tau}_{<i}M, \tilde{\tau}_{>i}M, \tilde{\tau}_{\leqq i}M, \tilde{\tau}_{\geqq i}M$ and* Coeur M^i *are A^2-torsion. Furthermore, $\tilde{\tau}_{<-i-1}D^2(M)=D^2(\tilde{\tau}_{>i}M)$, $\tilde{\tau}_{>-i-1}D^2(M)=D^2(\tilde{\tau}_{<i}M)$, $\tilde{\tau}_{\leqq -i-1}D^2(M)=D^2(\tilde{\tau}_{\geqq i}M)$ and $\tilde{\tau}_{\geqq -i-1}D^2(M)=D^2(\tilde{\tau}_{\leqq i}M)$.*

To see this note first that images of A^2-torsion modules are A^2-torsion so to show that Coeur M^i is A^2-torsion we may, by factoring out by $\tilde{\tau}_{<i}M$, assume that $M^j=0\, j<i$, in which case Coeur M^i is a submodule of M. As Coeur M^i is torsion, a glance at i) and ii) reveals that Coeur M^i/A^2(Coeur M^i)$=A^1_s$(Coeur M^i) and that D^1(Coeur M^i) is concentrated in degree $-i$ and is non-zero iff Coeur M^i is not A^2-torsion. The exact sequence $0\to$Coeur $M^i\to M\to\tilde{\tau}_{>i}M\to 0$ gives an exact sequence $0\to D^1$(Coeur M^i)$\to D^2(\tilde{\tau}_{>i}M)$. (Note that $D^1(M)=0$ as M is A^2-torsion.) However, as $\tilde{\tau}_{>i}M$ is concentrated in degrees $\geqq i$, (3.1.1) shows that $D^2(\tilde{\tau}_{>i}M)$ is concentrated in degrees $\leqq -i-1$, which shows that Coeur M^i is A^2-torsion. Let N be an A^2-torsion module which is finitely generated as W-module. As all A^2-torsion modules are torsion (Lemma 2.4.1 ii)) N is actually of finite length. Let F-tors(N) denote the submodule consisting of the elements killed by some power of F. Being a quotient of an A^2-torsion module N/F-tors(N) is A^2-torsion but by definition F is injective on it, so (3.1.1) shows that it is also A^2-closed and therefore 0. As N is of finite length F is actually nilpotent on $N=F$-tors(N). Applying this to Coeur M^i for an A^2-torsion module M we see that Coeur M^i in this case is of finite length as W-module with F nilpotent.

Sublemma 3.3.9. *Let M be a coherent R-module such that* Coeur M^i *is of finite length with F nilpotent. Then M is A^2-torsion.*

If not then M has an A^2-closed non-trivial image N and by above an A^2-closed module has $d=0$ so it is the direct sum of its components in the different degrees, so by further factoring we may assume that N is concentrated in a particular degree i, say. We will show that $\mathrm{Hom}_R(M, N)^0=0$ which will give the searched for contradiction. Fix therefore an R-homomorphism $A: M\to N$. Consider first its restriction B to $\tilde{\tau}_{<i}M$. As $\tilde{\tau}_{<i}\tilde{\tau}_{<i}M=\tilde{\tau}_{<i}M$ and $\tilde{\tau}_{<i}(-)$ is a functor B factors through $\tilde{\tau}_{<i}N=0$ and is therefore zero. A therefore factors to give $C: \tilde{\tau}_{\geqq i}M\to N$. However, by (0.8), $M^i/V^{-\infty}ZM^i$ is a successive extension of U_j^0: s and hence F is nilpotent on it. By assumption F is therefore nilpotent on $\tilde{\tau}_{\geqq i}M^i$ and as, by Cor. 3.1.7, F is injective on N, C, and thus A, is 0.

We can now finish the proof of Lemma 3.3.8. As the heart of $\tilde{\tau}_{<i}M, \tilde{\tau}_{>i}M$ etc. in a particular degree is either zero or equal to the heart of M in that degree, we see that, by what was proved above, if M is A^2-closed then the hearts of $\tilde{\tau}_{<i}M$ etc. are of finite length with F nilpotent, so by the sublemma $\tilde{\tau}_{<i}M$ etc. are A^2-closed. The exact sequence $0\to\tilde{\tau}_{\leqq i}M\to M\to\tilde{\tau}_{>i}M\to 0$ of A^2-torsion modules gives an exact sequence $0\to D^2(\tilde{\tau}_{>i}M)\to D^2(M)\to D^2(\tilde{\tau}_{\leqq i}M)\to 0$. By (3.1.1) $D^2(\tilde{\tau}_{\leqq i}M)$ is concentrated in degrees $\geqq -i-1$. Therefore $\tilde{\tau}_{<-i-1}D^2(M)\subseteq D^2(\tilde{\tau}_{>i}M)$ and to show equality we need only show that $\tilde{\tau}_{<-i-1}D^2(\tilde{\tau}_{>i}M)=D^2(\tilde{\tau}_{>i}M)$ or, as $D^2(\tilde{\tau}_{>i}M)$ is, by (3.1.1), concentrated in degrees $\leqq -i-1$, that $\mathrm{Coeur}\, D^2(\tilde{\tau}_{>i}M)^{-i-1}=0$. We have an epimorphism $D^2(\tilde{\tau}_{>i}M)\to\mathrm{Coeur}\, D^2(\tilde{\tau}_{>i}M)^{-i-1}$. Applying $D^2(-)$ gives a morphism $D^2(\mathrm{Coeur}\, D^2(\tilde{\tau}_{>i}M)^{-i-1})\to D^2(D^2(\tilde{\tau}_{>i}M))=\tilde{\tau}_{>i}M$, as $\tilde{\tau}_{>i}M$ is A^2-torsion. By (3.1.1) $\tilde{\tau}_{\leqq i}D^2(\mathrm{Coeur}\, D^2(\tilde{\tau}_{>i}M)^{-i-1})=D^2(\mathrm{Coeur}\, D^2(\tilde{\tau}_{>i}M)^{-i-1})$ and clearly $\tilde{\tau}_{\leqq i}\tilde{\tau}_{>i}M=0$. Functoriality of $\tilde{\tau}_{\leqq i}(-)$ shows that $D^2(\mathrm{Coeur}\, D^2 (\tilde{\tau}_{>i}M)^{-i-1})\to\tilde{\tau}_{>i}M$ is zero and applying again $D^2(-)$ shows that the epimorphism $D^2(\tilde{\tau}_{>i}M)\to\mathrm{Coeur}\, D^2(\tilde{\tau}_{>i}M)^{-i-1}$ is zero concluding the proof that $\tilde{\tau}_{<-i-1}D^2(M)= D^2(\tilde{\tau}_{>i}M)$. This equality implies that $\tilde{\tau}_{\geqq -i-1}D^2(M)=D^2(\tilde{\tau}_{\leqq i}M)$, using the exact sequence $0\to D^2(\tilde{\tau}_{>i}M)\to D^2(M)\to D^2(\tilde{\tau}_{\leqq i}M)\to 0$. The remaining two equalities of the lemma are proved similarly or by applying $D^2(-)$ to the ones already obtained.

We can now finish the proof of part iii) of the theorem:

$$A_f^2(D^2(M))^i = \tilde{\tau}_{\leqq i}\tilde{\tau}_{\geqq i}D^2(M)(i) = D^2(\tilde{\tau}_{\geqq -i-1}\tilde{\tau}_{\leqq -i-1}A^2(M))(i)$$
$$= D^2(A_f^2(M)^{-i-1}(i+1))(i) = D^2(A_f^2(M)^{-i-1})(-1)$$

and

$$A_d^2(D^2(M))^i = \tilde{\tau}_{>i}\tilde{\tau}_{<i+1}D^2(M)(i) = D_2(\tilde{\tau}_{<-i-1}\tilde{\tau}_{>-i-2}A^2(M))(i)$$
$$= D^2(A_d^2(M)^{-i-2}(i+2))(i) = D^2(A_d^2(M))(-2).$$

As for (3.3.6) we may assume that M is A^2-torsion and by shifting of degree that M is an R-module of level N for some large N. Then, by (III: 5.6.1),

$$H^i(D^2(M), d) = H^{i+2+N}(\mathrm{s}(D(M)(-N))) = H^{i+2+N}(R\,\mathrm{Hom}_W(\mathrm{s}(M), W)[-N])$$
$$= \mathrm{Ext}^1_W(H^{-i-1}(M, d), W) = \mathrm{Hom}_W(H^{-i-1}(A^2(M), d), K/W).$$

(Note that $H^*(M, d)$ is torsion so that $\mathrm{Hom}_W(H^*(M, d), W)=0$ and

$$\mathrm{Ext}^1_W(H^*(M, d), W) = \mathrm{Hom}_W(H^*(M, d), K/W).)$$

Finally, iv) is simply Cor. 2.2.2.

Corollary 3.3.10. *Let M be a coherent R-module.*

i) *Suppose that there is a filtration $0 \subseteqq S^2 \subseteqq S^1 \subseteqq S^0 = M$ by coherent submodules, such that* Coeur $(S^2)^i$ *is of finite length as W-module with F nilpotent, that p-*tors (S^1/S^2) *is of finite length as W-module with F bijective, that $(S^1/S^2)/$ p-*tors (S^1/S^2) *has F topologically nilpotent, that S^0/S^1 is without p-torsion with F bijective. Then $S^i = T^i(M)$ for $i=0, 1, 2$.*

ii) *If $A^2(M)=M$ then the same is true for every coherent quotient of M.*

iii) *If $A^1(M)=M$, then the same is true for every (coherent) submodule of M.*

iv) *If $A^0(M)=M$ then the same is true for every (coherent) submodule of M.*

Proof. By Sublemma 3.3.9 S^2 is A^2-torsion, but F is visibly injective on M/S^2, so (3.1.1) shows that M/S^2 is A^2-closed, which implies that $S^2 = T^2(M)$. To show i) it therefore suffices to show that S^1/S^2 has no torsion free quotient with F bijective and that S^0/S^1 has no submodule with F bijective on the torsion and F topologically nilpotent on the submodule modulo the torsion. (Note that as F is injective on S^0/S^1 and S^1/S^2, they are both A^2-closed.) Let $S^1/S^2 \to N$ be a torsion free quotient with F bijective. As it is torsion free the projection factors over $(S^1/S^2)/p$-tors, where by assumption F is topologically nilpotent and therefore so is F on N, which implies that N is zero. Let $N \to S^0/S^1$ be a submodule with F bijective on the torsion of N and F topologically nilpotent on N modulo torsion. As F is bijective on S^0/S^1 it is surjective on $(S^0/S^1)/N$ and because $(S^0/S^1)/N$ is a noetherian W-module F is bijective on it (cf. [3, IV: 5.2]). Therefore F is bijective on N which, by the assumption made, implies that N is torsion, but it is a submodule of a torsion free module so it is zero.

It has already been noted that ii) is true. In iii) and iv) note that if $A^i(M)=M$ $i=0, 1$ then M is finitely generated as W-module, so all its submodules are coherent (cf. [10, I: Thm. 3.8]). iv) is proved as ii): $A^0(M)=M$ iff the projection $M \to A^0(M)$ is injective, a property certainly preserved under submodules. Assume that $N \subseteqq M = A^1(M)$. As a submodule of an A^2-closed module it is A^2-closed, so to prove iii) it suffices to show that $D^0(N)=0$. As $d=0$ on M we may argue degree by degree and assume that M is concentrated in a particular degree, which we may assume to be 0. As $D^0(M)=0$, $D^0(N)$ injects into $D^1(M/N)$. By Thm. 3.3 i) $D^0(N)$ is torsion free and concentrated in degree 0 and by Thm. 3.3 ii) $D^1(M/N)$ is of finite length in all degrees but -1. This implies that $D^0(N)=0$.

Corollary 3.3.11. *Let M be a coherent R-module. Then $T^2(M)$ is the closure, in the standard topology, of $\bigcup_n \operatorname{Ker} F^n d \cap \operatorname{Ker} F^n =: T_0^2(M)$.*

To see this note first that as $T^2(M)$ is a coherent submodule it is closed ([10, I: Prop. 2.3]). Furthermore as F is injective on $M/T^2(M)$, $\bigcup_n \operatorname{Ker} F^n d \cap \operatorname{Ker} F^n \subseteq T^2(M)$. As the closure of $T_0^2(M)$ in M contains the closure of $T_0^2(M)$ in $T^2(M)$ we may assume that $M = T^2(M)$. Clearly the inclusion $T_0^2(M) \hookrightarrow M$ induces an isomorphism $D^2(M) \xrightarrow{\simeq} D^2(T_0^2(M))$ and by assumption $D(M) = D^2(M)[-2]$. (III: Prop. 4.2) therefore gives us a commutative diagram:

$$\begin{array}{ccc} \operatorname{Hom}_W(R_n \otimes_R T_0^2(M), K/W) = \operatorname{Tor}_0^R(R_n, D(T_0^2(M))) & \longrightarrow & \operatorname{Tor}_2^R(R_n, D^2(T_0^2(M))) \\ \uparrow & & \uparrow \wr \\ \operatorname{Hom}_W(R_n \otimes_R M, K/W) = \operatorname{Tor}_0^R(R_n, D(M)) & \xrightarrow{\ \sim\ } & \operatorname{Tor}_2^R(R_n, D^2(M)) \end{array} \tag{3.3.11}$$

(Note that Cor. 2.2.2 shows that we have a morphism $D(N) \to D^2(N)[-2]$ for any N.) This shows that $\operatorname{Hom}_W(R_n \otimes_R M, K/W) \to \operatorname{Hom}_W(R_n \otimes_R T_0^2(M), K/W)$ is injective for all n and therefore that $R_n \otimes_R T_0^2(M) \to R_n \otimes_R M$ is surjective, which implies that $T_0^2(M)$ is dense in M.

Remark. It may help the reader to visualize the results obtained in the following way. Every coherent R-module has a filtration $0 \subseteq S^3 \subseteq S^2 \subseteq S^1 \subseteq S^0 = M$ by coherent submodules, such that M/S^3 is a finitely generated W-module, the hearts of S^3 are of finite length as W-modules with F nilpotent, S^3 is the closure of the F-torsion of M (this is seen by an argument as in Cor. 3.3.9), S^2 is the p-torsion of M, S^2/S^3 is of finite length with F bijective, M/S^2 is a (torsion free) F-crystal in the usual sense, alternatively the covariant Dieudonné-module of a connected p-divisible group, S^0/S^1 is the part of slope zero of M/S^2 (the toroidal part of the Dieudonné-module), S^1/S^2 is the part of positive slope of M/S^2 (the bi-connected part), by the decomposition theorem for F-crystals $M/S^2 = S^0/S^1 \oplus S^1/S^0$. This also motivates the following definition:

Definition 3.3.13. *Let M be a coherent R-module.*

i) *$A^0(M)$ is the slope zero part of M and N is of slope zero if $A^0(M) = M$.*

ii) *$A_n^1(M)$ is the positive slope part of M and M is of positive slope if $A_n^1(M) = M$.*

iii) *$A_s^1(M)$ is the semi-simple torsion of M and M is semi-simple torsion if $A_s^1(M) = M$.*

iv) *$\oplus_i A_f^2(M)(-i)$ is the nilpotent torsion of M and M is of nilpotent torsion if $\oplus_i A_f^2(M)[-i] = M$.*

v) *$A_d^2(M)^i$ is the domino part in degree i of M and M is a domino (in degree i) if $A_d^2(M)^i = M$.*

(Note that the last part of v) is [10, I: Déf. 2.16].)

Remark: Another reasonable definition of the different parts would be to take for instance the positive slope part of M to be the positive slope parts of CoeurM^i: s etc. and the domino part of M to be dom M^i. These two definitions do not in general coincide, e.g. the inclusion $A^2(M)\hookrightarrow M$ will induce a morphism between the two different definitions of the domino part which in general is not an isomorphism. (The two definitions will always coincide up to modules of finite length however.)

Example. Let M be a domino in degree zero. Thm. 3.3 then shows that $D^2(M)(-2)$ is again a domino (in degree 0). Recall ([10, Prop. 2.18]) that $\dim_k M:=\dim_k M^0/VM^0$, that the dimension is additive on exact sequences of dominoes (in degree 0) and that the dominoes (in degree zero) of dimension 1 are exactly the $\mathbf{U}_j$: s. Furthermore, Cor. 3.3.8 and Prop. 2.4 show that $D^2(-)(-2)$ is exact on the category of dominoes (in degree 0), where a short sequence of dominoes is said to be exact if it is exact in R-mod, and that $D^2(M)\neq 0$ if $M\neq 0$. I claim that $\dim_k D^2(M)(-2)=\dim_k M$. Indeed, as $D^2(-)(-2)$ is exact and every domino (in degree 0) is a successive extension of $\mathbf{U}_j$: s and $D^2(\mathbf{U}_j)\neq 0$, we see that $\dim_k D^2(M)(-2)\geqq \dim_k M$ for all dominoes (in degree 0) M. Therefore $\dim_k M=\dim_k D^2(D^2(M)(-2))(-2)\geqq \dim_k D^2(M)(-2)\geqq \dim_k M$ so we must have equality. Therefore $D^2(\mathbf{U}_j)(-2)=\mathbf{U}_{j'}$ for some j'. However among the dominoes of dimension 1 $\mathbf{U}_k$ is characterized by $\dim_k H^*(\mathbf{s}(\mathbf{U}_k))=k$ (cf. [10, I: 2.14.4]). (III: Prop. 5.6) therefore shows that $D^2(\mathbf{U}_j)(-2)=\mathbf{U}_{-j}$. In [5] it is shown that M possesses a unique filtration $M=M_j\supseteqq M_{j+1}\supseteqq\ldots\supseteqq M_{i-1}\supseteqq M_i=0$ by coherent submodules such that for all $k\in[j,i[$, M_k/M_{k+1} is a successive extension of $\mathbf{U}_k$: s. The function $\sigma\colon\mathbf{z}\to\mathbf{N}$, $\sigma(k):=\dim M_k/M_{k+1}$ is called the type of M. The results just shown imply that the type of $D^2(M)(-2)$ is the function $k\mapsto\sigma(-k)$.

Theorem 3.4. *Let* $M\in D_c^b(R)$.

i) $$A^1(D^i(M))=D^1(H^{1-i}(M)).$$

ii) *There is a natural exact sequence of R-modules;*

$$0\to A^0(D^i(M))\to D^0(H^{-i}(M))\xrightarrow{d_2} D^2(H^{1-i}(M))\to A^2(D^{i+1}(M))\to 0,$$

where $\operatorname{Im} d_2$ *is semi-simple torsion.*

iii) *There is natural monomorphism;*

$$A^0(D^i(M))^j\to D^0(A^0(H^{-i}(M))^{-j})$$

whose cokernel is $\operatorname{Im} d_2^j$.

iv)
$$A_n^1(D^i(M))^j=D^1(A_n^1(H^{1-i}(M)^{-j-1}))(-1)\qquad A_s^1(D^i(M))^j=D^1(A_s^1(H^{1-i}(M)^{-j})).$$

v) $$A_f^2(D^i(M))^j=D^2(A_f^2(H^{2-i}(M)^{-j-1}))(-1).$$

vi) *There is a natural epimorphism;*

$$D^2(A^2_d(H^{2-i}(M))^{-j-2})(-2) \to A^2_d(D^i(M))^j$$

whose kernel is $\operatorname{Im} d_2^j(-1)$.

Proof. We will, of course, use the spectral sequence:

$$D^i(H^{-j}(M)) \Rightarrow D^{i+j}(M). \tag{3.4.1}$$

Let $0 \subseteq S^2 \subseteq S^1 \subseteq S^0 = D^*(M)$ be the abutment filtration, which by Cor. 2.2.2 is a 3-step filtration. I claim that it coincides with the T-filtration. Indeed, (3.4.1) shows that S^2 is a quotient of $D^2(H^{*+2}(M))$, that $S^1/S^2 = D^1(H^{*+1}(M))$ and that S^0/S^1 is a submodule of $D^0(H^*(M))$. We then conclude by Prop. 2.4 and Cor. 3.3.10 i), ii) and iv). This immediately shows i) and ii) where d_2 is the differential of (3.4.1), except for the fact that $\operatorname{Im} d_2$ is semi-simple torsion. As $\operatorname{Im} d_2 \subseteq D^2(H^{1-i}(M))$ it is torsion and as it is an image of $D^0(H^{-i}(M))$, which, by Thm. 3.3, is of slope zero, it is of finite length over W with F surjective and therefore bijective, so $\operatorname{Im} d_2$ is semi-simple torsion. Now $D^2(H^{1-i}(M))^j/F^\infty B$ has F nilpotent, as shown in the proof of Sublemma 3.3.9. This implies that $\operatorname{Im} d_2 \subseteq F^\infty BD^2 \cdot (H^{1-i}(M))$ and this together with (3.3.5) shows v). i) and (3.3.2) resp. ii) and (3.3.1) show iv) resp. iii). As $\operatorname{Im} d_2 \subseteq F^\infty BD^2(H^{1-i}(M))$ the induced morphism $\tilde{\tau}_{>j} D^2(H^{1-i}(M)) \to \tilde{\tau}_{>j} A^2(D^{i+1}(M))$ is surjective and injective in degree j. To finish the proof of the theorem we use (3.3.5) and the following lemma:

Lemma 3.4.2. $\tilde{\tau}_{<i}(-)$ *preserves surjections.*

Let $f: N_1 \to N_2$ be a surjection of R-modules and put $N = \operatorname{coker} \tilde{\tau}_{<i} f$. Clearly N is concentrated in degree i, so $\tilde{\tau}_{<i} N = 0$. On the other hand, as $\tilde{\tau}_{<i}\tilde{\tau}_{<i} = \tilde{\tau}_{<i}$ and $\tilde{\tau}_{<i}(id/\tilde{\tau}_{<i}) = 0$, $\tilde{\tau}_{<i} N = N$. Combining Thm. 3.4 and III: Thm. 2.9 we get:

Theorem 3.5. *Let X be a smooth and proper variety of dimension N.*

i) $$A^1(R^i\Gamma(W\Omega^\cdot)) = D^1(R^{N+1-i}\Gamma(W\Omega^\cdot))(-N).$$

ii) *There is a natural exact sequence of R-modules;*

$$0 \to A^0(R^i\Gamma(W\Omega^\cdot)) \to D^0(R^{N-i}\Gamma(W\Omega^\cdot))(-N)$$
$$\xrightarrow{d_2} D^2(R^{N+1-i}\Gamma(W\Omega^\cdot))(-N) \to A^2(R^{i+1}\Gamma(W\Omega^\cdot)) \to 0$$

where $\operatorname{Im} d_2$ *is semi-simple torsion.*

iii) *There is a natural monomorphism;*

$$A^0(R^i\Gamma(W\Omega^\cdot))^j \to D^0(A^0(R^{N-i}(W\Omega^\cdot))^{N-j})$$

whose cokernel is $\operatorname{Im} d_2^j$.

iv) $$A_n^1(R^i\Gamma(W\Omega^\cdot))^j = D^1(A_n^1(R^{N+1-i}\Gamma(W\Omega^\cdot))^{N-j-1})(-1)$$
$$A_s^1(R^i\Gamma(W\Omega^\cdot))^j = D^1(A_s^1(R^{N+1-i}\Gamma(W\Omega^\cdot))^{N-j}).$$

v) $$A_f^2(R^i\Gamma(W\Omega^\cdot))^j = D^2(A_f^2(R^{N+2-i}\Gamma(W\Omega^\cdot))^{N-j-1})(-1).$$

vi) *There is a natural epimorphism;*

$$D^2(A_d^2(R^{N+2-i}\Gamma(W\Omega^\cdot))^{N-j-2})(-2) \to A_d^2(R^i\Gamma(W\Omega^\cdot))^j$$

whose kernel is $\operatorname{Im} d_2^{j+1}(-1)$.

Corollary 3.5.1. *Put* $T^{i,j} := \dim_k \operatorname{dom} R^j\Gamma(W\Omega^\cdot)^i$. *Then*

(3.5.1.1) $$T^{i,j} = T^{N-i-2,N-j+2}.$$

Indeed, as observed above $\operatorname{dom} R^j\Gamma(W\Omega^\cdot)^i$ is, up to finite length, equal to $A_d^2(R^j\Gamma(W\Omega^\cdot))^i$ and so has the same dimension. One then concludes by vi).

Remark. One could paraphrase Thm. 3.5 by saying that the part of $R\Gamma(W\Omega^\cdot)$ in degree (i,j) is, as it should under duality, dual to the part almost in degree $(N-i, N-j)$. One has to modify the degree by $(0,0)$ for the slope zero part, by $(1,0)$ for the semisimple torsion, by $(1,-1)$ for the positive slope part, by $(2,-1)$ for the nilpotent torsion and by $(2,-2)$ for the domino part. In particular, as $R^i\Gamma(W\Omega^\cdot)=0$ for $i>N$ and in negative degrees, one obtains the result, related to [9, II: Cor. 3.11] and [9, II: Cor. 2.17—18], that $\Gamma(W\Omega^\cdot)$ is of slope zero, that $R^1\Gamma(W\Omega^\cdot)$ is A^2-closed and that $R^i\Gamma(W\Omega^\cdot)^N$ is of slope zero and semi-simple torsion.

The following definition will allow us to avoid some cumbersome notation:

Definition 3.6. *Let M be a coherent R-module.*

i) *If* $A^0(M)=M$ *we will write* $M^{i\check{}} := D^0(M)^{-i}$.

ii) *If* $A_n^1(M)=M$ *we will write* $M^{i\check{}} := D^1(M)^{-i-1}$.

iii) *If* $A_s^1(M)=M$ *we will write* $M^{i\check{}} := D^1(M)^{-i}$.

iv) *If* $A^2(M)=M$ *we will write* $\ldots \xrightarrow{d\check{}} M^{i\check{}} \xrightarrow{d\check{}} M^{i-1\check{}} \xrightarrow{d\check{}} \ldots := D^2(M)$.

(Exactly where the different degrees will be put will depend on the circumstances.)

4.

Definition 4.1. *Let N be an R-module. Define new R-modules as follows* (cf. [10, III: 3.2.1—2]) *for* $n>0$;

(4.1.1) $$N(i,n) := (\ldots N^{i-2} \xrightarrow{d} N^{i-1} \xrightarrow{dV^n} \sigma_*^{-n}N^i \xrightarrow{d} \sigma_*^{-n}N^{i+1} \xrightarrow{d} \ldots)$$

(4.1.2) $$N(i,-n) := (\ldots N^{i-2} \xrightarrow{d} N^{i-1} \xrightarrow{F^n d} \sigma_*^{n}N^i \xrightarrow{d} \sigma_*^{n}N^{i+1} \xrightarrow{d} \ldots)$$

where F and V are unchanged.

Evidently, the functors $(-)(i, n)$ and $(-)(i, -n)$ are inverses of each other and therefore adjoints. This gives

Proposition 4.2. *There exists a natural isomorphism*

$$\operatorname{Hom}_R(M(i, n), N) = \operatorname{Hom}_R(M, N(i, -n)) \tag{4.2.1}$$

for all integers i and n and all R-modules M and N.

Proposition 4.3. *There are isomorphisms of R-bimodules*

$$\check{R}(i, n)_1 \xrightarrow{\sim} \check{R}(i, n)_2, \tag{4.3.1}$$

where the subscript i means changing the i:th structure.

Proof: Using (III: 3.5.1) one sees that the mapping $\check{R} \to \check{R}$ which is the identity in degree -1 and multiplication by F^n from the first structure in degree 0 is bijective and it is trivially verified that it gives the required isomorphism.

Combining Props. 4.2 and 4.3 with the obvious fact that $(-)(i, n)(1) = (-(1))(i-1, n)$ we get $\operatorname{Hom}_R(M(i, n), \check{R}) = \operatorname{Hom}_R(M, \check{R})(-i, -n)$ for all R-modules N. As $(-)(i, n)$ derives trivially we get:

Proposition 4.4. $D((-)(i, n)) = D(-)(-i, -n)$.

5.

Lemma 5.1. *Let M be an R-module. The isomorphism $R_n \otimes_R^L D(M) = D(R_n \otimes_R^L M)$ and the fact that $\operatorname{Tor}_i^R(R_n, -) = D^i(-) = 0$, $i \neq 0, 1, 2$, on R-modules, give us morphisms:*

$$D^2(M) \to R_n \otimes_R D^2(M) \to \operatorname{Hom}_W(\operatorname{Tor}_2^R(R_n, M), K/W) \tag{5.1.1}$$

$$\operatorname{Hom}_W(R_n \otimes_R M, K/W) \to \operatorname{Tor}_2^R(R_n, D^2(M)). \tag{5.1.2}$$

These are, through (0.4.3) *and* (3.1.1), *equal to the translate of (resp. -1 times) the evident morphisms:*

$$\operatorname{Hom}_W(\bigcup_n \operatorname{Ker} F^n d \cap \operatorname{Ker} F^n, K/W) \to \operatorname{Hom}_W(\operatorname{Ker} F^n d \cap \operatorname{Ker} F^n, K/W) \tag{5.1.3}$$

$$\operatorname{Hom}_W(R_n \otimes_R M, K/W) \to \operatorname{Hom}_W(R_n \otimes_R T_0^2(M), K/W) \tag{5.1.4}$$

$= \operatorname{Ker} F^n d \cap \operatorname{Ker} F^n \colon \operatorname{Hom}_W(T_0^2(M), K/W)(1)(-1) = \operatorname{Ker} F^n d \cap \operatorname{Ker} F^n \colon D^2(M)(-1)$.

Proof. Recall that the isomorphism $R_n \otimes_R^L D(-) = D(R_n \otimes_R^L (-))$ is defined as follows. We have $\operatorname{Hom}_R(M, \check{R}) = \operatorname{Hom}_W(\varinjlim (R_\infty . \otimes_R M), K/W)$ so that the mapping $R_n \otimes_R M \to \varinjlim (R_\infty . \otimes_R M)$ induces $\operatorname{Hom}_R(M, \check{R}) \to \operatorname{Hom}_W(R_n \otimes_R M, K/W) = D(R_n \otimes_R M)$ and the searched for isomorphism is the derivation of the induced mapping $R_n \otimes_R \mathrm{Hom}_R(M, \check{R}) \to D(R_n \otimes_R M)$. Using (0.4.3) and (3.1.2—6) we may accomplish this derivation by resolving R_n and $R_{\infty}.$ rather than M in the following way. Note that the coverings (3.1.2—6) actually give us a complex $R^{\bullet}_{\infty}.$ of R-module objects in dir—dıw—S (a priori we would only expect e.g. FV to be homotopic to p but we have an actual equality). We therefore get, for F a complex of R-modules, a bicomplex $R^{\bullet}_{\infty}.\otimes_R F$ in dir—dıw—S together with an augmentation $R^{\bullet}_{\infty}.\otimes_R F \to R_{\infty}.\otimes_R F$ making the associated simple complex of $R^{\bullet}_{\infty}.\otimes_R F$ a representative of $R_{\infty}.\otimes^L_R F$. $\mathrm{Hom}^{\bullet}_W(\varinjlim (R^{\bullet}_{\infty}.\otimes_R F), K/W)$ will therefore be a bicomplex of R-modules whose associated simple complex computes $D(F)$. Similarly, we get a double complex $R^{\bullet}_n \otimes_R F$ of W-complexes together with an augmentation $R^{\bullet}_n \otimes_R F \to R_n \otimes_R F$ making the associated simple complex a representative of $R_n \otimes^L_R F$. Thus the triple complex $R^{\bullet}_n \otimes_R \mathrm{Hom}_W(\varinjlim (R^{\bullet}_{\infty}.\otimes_R F), K/W)$ has an associated simple complex computing $R_n \otimes^L_R D(F)$. What we want to do is to find a morphism of double complexes $\mathrm{s}(R^{\bullet}_n \otimes_R \mathrm{Hom}_W(\varinjlim (R^{\bullet}_{\infty}.\otimes_R F), K/W)) \to D(R^{\bullet}_n \otimes_R F)$, where s means taking the associated simple complex in the two directions not involving the degrees of F, such that the following diagram commutes:

(5.1.5)
$$
\begin{array}{ccc}
R_n \otimes_R \mathrm{Hom}^{\bullet}_W(\varinjlim (R_{\infty}.\otimes_R F), K/W) & \longrightarrow & D(R_n \otimes_R F) \\
\downarrow & & \Big| \\
R_n \otimes_R \mathrm{Hom}^{\bullet}_W(\varinjlim (R^{\bullet}_{\infty}.\otimes_R F), K/W) & & \Big| \\
\uparrow & & \downarrow \\
\mathrm{s}(R^{\bullet}_n \otimes_R \mathrm{Hom}^{\bullet}_W(\varinjlim (R^{\bullet}_{\infty}.\otimes_R F), K/W)) & \longrightarrow & D(R^{\bullet}_n \otimes_R F)
\end{array}
$$

Once this has been done, the induced morphism on simple complexes will certainly compute the isomorphism $R_n \otimes^L_R D(F) = D(R_n \otimes^L_R F)$. The desired morphism will be $\mathrm{Hom}_W(\varinjlim \{-\}, K/W)$ applied to the following morphism of double complexes of ind-objects:

$$
\begin{array}{ccc}
F(-2) & \xrightarrow{\begin{pmatrix} F^n \\ -F^n d \end{pmatrix}} & F(-2)\oplus F(-1) \xrightarrow{(dV^n,\, V^n)} \\
\Big\downarrow \mathrm{id} & & \Big\downarrow \begin{pmatrix} F^{m-n}, & 0 \\ \mathrm{id}, & F^{m-n} \\ 0, & -\mathrm{id} \end{pmatrix} \\
F(-2) & \xrightarrow{\begin{pmatrix} F^m \\ -F^m d \\ F^n \\ F^n d \end{pmatrix}} & F(-2)\oplus F(-1)\oplus F(-2)\oplus F(-1) \xrightarrow{\begin{pmatrix} dV^m, & V^m, & 0, & 0 \\ -\mathrm{id}, & 0, & F^{m-n}, & 0 \\ 0, & -p^n, & -F^{m-n}d, & 0 \\ 0, & \mathrm{id}, & 0, & F^{m-n} \\ 0, & 0, & 0, & -F^{m-n}d \\ 0, & 0, & -dV^n, & V^n \end{pmatrix}}
\end{array}
$$

$$
\begin{array}{c}
F(-1) \\
\Big\downarrow \begin{pmatrix} p^{m-n} \\ 0 \\ -F^m \\ 0 \\ F^m d \\ -\mathrm{id} \end{pmatrix} \\
F(-1)\oplus F(-2)\oplus F(-1)\oplus F(1)\oplus F\oplus F(-1) \xrightarrow{\begin{pmatrix} F^n, & dV^{m-n}, & V^{m-n}, & 0, & 0, & 0 \\ -F^n d, & 0, & 0, & dV^{m-n}, & V^{m-n}, & 0 \\ 0, & 0, & -\mathrm{id}, & -p^n, & 0, & F^m \\ 0, & 0, & 0, & 0, & -\mathrm{id}, & -F^m d \end{pmatrix}}
\end{array}
$$

$$
F(-1)\oplus F\oplus F(-1)\oplus F \xrightarrow{(dV^n,\, V^n,\, dV^m,\, V^m)} F.
$$

Here all matrices act on the left and d always denotes the morphism of degree 0. From this the lemma clearly follows.

Corollary 5.1.1. *Let M be A^2-torsion. Then the following diagram commutes:*

$$\begin{array}{ccc} T_0^2(M) & \xrightarrow{\text{ev}} & \mathrm{Hom}_W(\mathrm{Hom}_W(T_0^2(M), K/W), K/W) \\ \downarrow & & \downarrow \\ M & & \mathrm{Hom}_W(T_0^2(\mathrm{Hom}_W(T_0^2(M), K/W)), K/W) \\ \downarrow \text{ev} & & \downarrow \\ D^2(D^2(M)) & = & \mathrm{Hom}_W(T_0^2(\mathrm{Hom}_W(T_0^2(M), K/W))(1), K/W)(1) \end{array} \tag{5.1.1.2}$$

Indeed, according to Lemma 1.1.5 $\text{ev}\colon M \to D^2(D^2(M))$ induces the usual evaluation mapping $\text{ev}\colon R_n \otimes_R M \to D(D(R_n \otimes_R M))$, whereas it is clear that $\text{ev}\colon T_0^2(M) \to \mathrm{Hom}_W(\mathrm{Hom}_W(T_0^2(M), K/W), K/W)$ induces the evaluation mapping $\text{ev}\colon R_n \otimes_R T_0^2(M) \to D(D(R_n \otimes_R T_0^2(M)))$. As $M \xrightarrow{\sim} \varprojlim (R_n \otimes_R M)$ to prove the corollary it suffices to show that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_W(\mathrm{Hom}_W(T_0^2(M), K/W), K/W) & \longrightarrow & D(D(R_n \otimes_R T_0^2(M))) \\ \downarrow a & & \downarrow \\ \mathrm{Hom}_W(T_0^2(\mathrm{Hom}_W(T_0^2(M)(-1), K/W)), K/W)(1) & \xrightarrow{b} & D(D(R_n \otimes_R M)) \end{array} \tag{5.1.1.3}$$

where a is the composite of the canonical isomorphism

$$\mathrm{Hom}_W(\mathrm{Hom}_W(T_0^2(M), K/W), K/W) \xrightarrow{c} \mathrm{Hom}_W(\mathrm{Hom}_W(T_0^2(M)(-1), K/W), K/W)(1)$$

and the morphism induced by

$$T_0^2(\mathrm{Hom}_W(T_0^2(M)(-1), K/W)) \hookrightarrow \mathrm{Hom}_W(T_0^2(M)(-1), K/W)$$

and b is obtained from (5.1.1—4). Our sign-conventions imply that $c = -id$ and the lemma now shows that (5.1.1.3) does indeed commute.

Proposition 5.2. *Let M be A^2-torsion.*

i) *The standard topology on* $D^2(M) = \mathrm{Hom}_W(T_0^2(M), K/W)(1)$ *coincides with the Pontryagin topology as the dual of the discrete W-module* $T_0^2(M)$.

ii) *The composite* $\mathrm{Hom}_W^c(M, K/W) \hookrightarrow \mathrm{Hom}_W(M, K/W) \to \mathrm{Hom}_W(T_0^2(M), K/W)$ *where* $\mathrm{Hom}_W^c(M, K/W)$ *denotes the continuous W-morphisms, M having the standard topology, factors to give an isomorphism:*

$$\mathrm{Hom}_W^c(M, K/W) \xrightarrow{\sim} T_0^2(D^2(M))(-1) = T_0^2(\mathrm{Hom}_W(T_0^2(M), K/W)).$$

To see i) note that the Pontryagin topology on $\mathrm{Hom}_W(N, K/W)$, N a discrete W-module, has as neighbourhoods of 0 the homomorphisms that vanish on finitely generated submodules, whereas the neighbourhoods of 0 in the standard topology of $\mathrm{Hom}_W(T_0^2(M), K/W)$ aie $N_n := V^n \mathrm{Hom}_W(T_0^2(M), K/W) + dV^n \mathrm{Hom}_W$

$(T_0^2(M), K/W)$ for all n. Thm. 3.3 shows that N_n consists of the homomorphisms that vanish on $\mathrm{Ker}\, F^n d \cap \mathrm{Ker}\, F^n$, which, as M is coherent, are finitely generated W-modules and by definition of $T_0^2(M)$ they are cofinal in the set of finitely generated submodules. This shows i). As for ii) let us first note that as a $\varphi \in \mathrm{Hom}_W^c(M, K/W)$ is continuous it will vanish on $V^nM + dV^nM$ for some n and therefore belong to $\mathrm{Ker}\, F^n d \cap \mathrm{Ker}\, F^n: \mathrm{Hom}_W(M, K/W)$. We thus see that the image of $\mathrm{Hom}_W^c(M, K/W)$ in $\mathrm{Hom}_W(M, K/W)$ lies in $T_0^2(\mathrm{Hom}_W(M, K/W))$ so we get a morphism $\mathrm{Hom}_W^c(M, K/W) \to T_0^2(\mathrm{Hom}_W(T_0^2(M), K/W))$. As $\mathrm{Hom}_W(-, K/W)$ reflects isomorphisms to show that this is an isomorphism it will suffice to show that $D^2(D^2(M)) = \mathrm{Hom}_W(T_0^2(\mathrm{Hom}_W(T_0^2(M), K/W), K/W)) \to \mathrm{Hom}_W(\mathrm{Hom}_W^c(M, K/W))$ is an isomorphism. Cor. 5.1.1 shows that the composite $M \xrightarrow{\mathrm{ev}} D^2(D^2(M)) \to \mathrm{Hom}_W(\mathrm{Hom}_W^c(M, K/W), K/W)$ is the evaluation map of Pontryagin duality. By Prop. 2.4 and Pontryagin duality for the linearly compact module M (it is profinite and therefore linearly compact) both evaluation mappings are isomorphisms and therefore so is

$$D^2(D^2(M)) \to \mathrm{Hom}_W(\mathrm{Hom}_W^c(M, K/W), K/W).$$

Proposition 5.3. *Let M be a coherent R-module. Through* Thm. 3.3 $\mathrm{ev}: M \to D(D(M))$ *is identified.*

i) *if* $M = A^0(M)$, *to* $\mathrm{ev}: M \to \mathrm{Hom}_W(\mathrm{Hom}_W(M, W), W)$,

ii) *if* $M = A_n^1(M)$, *to* $\mathrm{ev}: M \to \mathrm{Hom}_W(\mathrm{Hom}_W(M, W), W)$,

iii) *if* $M = A_s^1(M)$, *to* $\mathrm{ev}: M \to \mathrm{Hom}_W(\mathrm{Hom}_W(M, K/W), K/W)$,

iv) *if* $M = A^2(M)$, *through* Prop. 5.2, *to* $\mathrm{ev}: M \to \mathrm{Hom}_W(\mathrm{Hom}_W^c(M, K/W), K/W)$.

Proof. i)—iii) follows from (1.1.5.2) and iv) from Cor. 5.1.1.

Definition 5.4. *Let $M, N \in D_c^b(R)$ and $f: D(N) \to M$ be a morphism. $f^\vee: D(M) \to N$ is defined to be the unique morphism making the following diagram commute:*

(5.4.1)
$$\begin{array}{ccc} D(M) & \xrightarrow{f^\vee} & N \\ \| & & \downarrow \\ D(M) & \xrightarrow{D(f)} & D(D(N)) \end{array}$$

Proposition 5.5. *Let $M, N \in D_c^b(R)$ and $f: D(N) \to M$ be a morphism. Then the following diagrams $(-1)^j$-commute for $i = 0, 1, 2$:*

(5.5.1)
$$\begin{array}{ccccc} A^i(H^j(N)) & \xleftarrow{H^j(f^\vee)} & A^i(D^j(M)) & \xleftarrow{a} & D^i(H^{i-j}(M)) \\ \| & & & & \diagdown\!\!\diagdown \\ D^i(D^i(H^j(N))) & \xleftarrow{b} & D^i(D^{i-j}(N)) & \xleftarrow{D^i(H^{i-j}(f))} & D^i(H^{i-j}(M)) \end{array}$$

Here a (resp. *b*) *is* (resp. *$D(-)$ applied to*) *the morphism coming from the spectral* *sequence* $D^i(H^{i-j}(M)) \Rightarrow D^j(M)$ (resp. $D^i(H^{i-j}(N)) \Rightarrow D^j(N)$). *A slight abuse of notation is used in case* $i=0$, *as* a *and* c *then are defined only on a submodule of finite index.*

Proof. We start by explicating the canonical isomorphism $A^i(H^j(N)) = D^i(D^i(H^j(N)))$ and by inserting two morphisms:

(5.5.2)
$$\begin{array}{ccccc}
A^i(H^j(N)) & \longrightarrow & A^i(D^j(M)) & \longleftarrow & D^i(H^{i-j}(M)) \\
\downarrow{\scriptstyle \mathrm{ev}} & \searrow{\scriptstyle \mathrm{ev}} & & & \| \\
A^i(D(D(H^j(N)))) & & H^j(D(D(N))) & & \| \\
\uparrow{\scriptstyle c} & & \uparrow{\scriptstyle d} & & \| \\
D^i(D^i(H^j(N))) & \longleftarrow & D^i(D^{i-j}(N)) & \longleftarrow & D^i(H^{i-j}(M))
\end{array}$$

where c (resp. d) comes from the spectral sequence $D^i(D^{i-k}(H^j(N))) \Rightarrow H^j(N)$ (resp. $D^i(D^{i-j}(N)) \Rightarrow H^j(D(D(N)))$). As the evaluation maps are isomorphisms it is sufficient to show that the two rectangles commute. The right one commutes by definition of $f^{\check{}}$. The left rectangle is natural in N. As the morphism $N \to \tau_{\leq j} N$ induces an isomorphism on H^j we are reduced to $\tau_{\leq j} N$. Finally, the natuıality of the spectral sequence $D^i(D^{i-j}(-)) \Rightarrow H^j(-)$, shows that the morphism $H^j(N)[-j] \to \tau_{\leq j} N$ induces an epimorphism on the domain of definition of d and b. We are therefore reduced to $H^j(N)[-j]$ which is obvious, using the sign conventions.

Proposition 5.6. *Assume the hypotheses of* 5.5. *Put*

$$F^j := \operatorname{Im} d_2 \colon D^0(H^{-j}(N)) \to D^2(H^{1-j}(N))$$
$$G^j := \operatorname{Im} d_2 \colon D^0(H^{-j}(M)) \to D^2(H^{1-j}(M))$$

so that by Thm. 3.4 *we have exact sequences:*

(5.6.1)
$$0 \to A^0(D^j(N)) \to D^0(H^{-j}(N)) \to F^j \to 0$$
$$0 \to F^j \to D^2(H^{1-j}(N)) \xrightarrow{a} A^2(D^{j+1}(N)) \to 0$$

(5.6.2)
$$0 \to A^0(D^j(M)) \to D^0(H^{-j}(M)) \to G^j \to 0$$
$$0 \to G^j \to D^2(H^{1-j}(M)) \to A^2(D^{j+1}(M)) \to 0$$

Proposition. 5.5. *(and the fact that* $D^1(F^j) \to D^2(A^2(D^{j+1}(N))$ *is injective as* $D^1(D^2(H^{1-j}(N)))=0$*) show that* $D^0(f)$ (resp. $D^2(f)$) *and the long exact sequences of* D^i*: s applied to* (5.6.1) *induce morphisms:*

(5.6.3)
$$G^{-j} \to D^1(F^j)$$

(5.6.4)
$$G^{-j} \to D^1(F^j)$$

a) *The morphisms of* (5.6.3) *and* (5.6.4) *coincide.*

b) *The morphism of* (5.6.3) *may be described as follows: Take* $g \in G^{-j}$ *and lift it to* $g' \in D^0(H^j(M))$. *Choose* n *such that* $p^n g' \in A^0(D^{-j}(M))$. *Then* $f^\vee(p^n g') \in A^0(H^{-j}(N))$ *and gives, through the evaluation map and* Thm. 3.3 *rise to a homomorphism* $D^0(H^{-j}(N)) \to W$. *The composite* $D^0(H^{-j}(N)) \to W \xrightarrow{p^{-n}} K/W$ *vanishes on the image of* $A^0(D^j(N))$ *and so induces a map* $F^j \to K/W$ *and by* Thm. 3.3 *an element of* $D^1(F^j)$.

c) *The morphism of* (5.6.4) *may be described as follows: Take* $g \in G^{-j}$ *and* $h \in F^j$. *For* n *large there is a* $h' \in D^2(H^{1-j}(N))$ *such that* $F^n h = F^n dh'$ *and* $a(h') \in T_0^2(A^2(D^{j+1}(N)))$. *Hence* $f(a(h')) \in T_0^2(H^{j+1}(M))$ *and we may through* Thm. 3.3 *evaluate* g *on* $f(a(h'))$. *For fixed* g *this gives a* W*-homomorphism* $F^j \to K/W$ *and through* Thm. 3.3 *an element in* $D^1(F^j)$.

Proof. Let us first conside1 a). As $D^0(H^j(M)) \to G^{-j}$ is surjective and $D^1(F^j) \to D^2(A^2(D^{j+1}(N)))$ injective it suffices to show that the two composites $\delta, \gamma: D^0(H^j(M)) \to D^2(A^2(D^{j+1}(N)))$ coincide. By definition this amounts to showing that the large rectangle of the following diagram commutes:

$$\begin{array}{ccc} D^0(H^j(M)) & \xrightarrow{d_2} & D^2(H^{1+j}(M)) \\ \downarrow{\scriptstyle D^0(f)} & & \downarrow{\scriptstyle D^2(f)} \\ D^0(D^j(N)) & \xrightarrow{d_2} & D^2(D^{1+j}(N)) \\ \uparrow\wr & & \downarrow\wr \\ D^0(A^0(D^j(N))) & \xrightarrow{\delta^{(2)}} & D^2(A^2(D^{j+1}(N))) \end{array} \tag{5.6.5}$$

(δ is the left hand composite oi the diagram and γ is the right hand composite.) Here $\delta^{(2)}$ is the second connecting homomorphism coming from applying $\{D^i\}$ to the exact sequence

$$0 \to A^0(D^j(N)) \to D^0(H^j(N)) \xrightarrow{d_2} D^2(H^{1+j}(N)) \to A^2(D^{j+1}(N)) \to 0. \tag{5.6.6}$$

By naturality the upper square commutes so it suffices to show that the lower square commutes. Recall that d_2: $D^0(D^j(N)) \to D^2(D^{1+j}(N))$ is induced from the morphism obtained by applying $D(-)$ to the triangle:

$$\to D^j(N)[-j] \to \tau_{\leq j+1}\tau_{\geq j} D(N) \to D^{j+1}(N)[-j-1] \to D^j(N)[-j+1] \tag{5.6.7}$$

whereas $\delta^{(2)}$ is induced from the morphism obtained by applying $D(-)$ to the morphism $A^2(D^{j+1}(N))[-2] \to A^0(D^j(N))$ coming from (5.6.6). The desired commutativity now follows from the following lemma, which also has some independent interest:

Lemma 5.6.8. *Assume the notations of* 5.6. *Then the composite*

$$A^2(D^{j+1}(N))[-2] \to D^{j+1}(N)[-2] \to D^j(N) \to A^0(D^j(N)), \tag{5.6.9}$$

where the second morphism comes from (5.6.7), *equals the morphism coming from* (5.6.6).

Proof. (5.6.6), naturality and Prop. 2.4 show that we first, by truncation and shifting, may assume that N is concentrated in degrees 0 and 1 and then, by taking push-out along $H^0(N) \to A^0(H^0(N))$ and pullback along $A^2(H^1(N)) \to H^1(N)$, we may assume that $A^0(H^0(N)) = H^0(N)$ and $A^2(H^1(N)) = H^1(N)$. Put Y equal to the mapping cone of $d_2 \colon D^0(H^0(N)) \to D^2(H^1(N))$. Then the morphism coming from (5.6.6) fits into a distinguished triangle:

$$\to D^0(N) \to Y \to D^1(N)[-1] \to D^0(N)[1]. \tag{5.6.10}$$

On the other hand the assumptions imply that (5.6.7) becomes

$$\to D^0(N) \to D(N) \to D^1(N)[-1] \to D^0(N)[1] \tag{5.6.11}$$

so we want to prove that Y is isomorphic to $D(N)$ under an isomorphism inducing the identity on $D^0(N)$ and $D^1(N)$. Applying $D(-)$ to the triangle: $\to H^0(N) \to N \to H^1(N)[-1] \to H^0(N)[1]$ gives us a distinguished triangle:

$$\to D(N) \to D^0(H^0(N)) \xrightarrow{d_2} D^2(H^1(N)) \to D(N)[1]. \tag{5.6.12}$$

By definition the morphisms $D^0(N) \to D^0(H^0(N))$ resp. $D^2(H^1(N)) \to D^1(N)$ equals the composite $D^0(N) \to D(N) \to D^0(H^0(N))$ resp. $D^2(H^1(N)) \to D(N)[1] \to D^1(N)$, which together with (5.6.12) show that Y is isomorphic to $D(N)$ undei an isomorphism inducing the identity on $D^0(N)$ and $D^1(N)$.

b) is a simple diagram chase which I leave to the reader. As for c) suppose that F is semi-simple torsion. (3.1.2) shows that $\mathrm{Tor}_1^R(R_{\infty\cdot}, F) = (\sigma_*^n F, F, p, 0, id)$ (except for small n) which as indsystem is canonically isomoiphic to the constant system F. Therefore $D^1(F) = \mathrm{Hom}_W(F, K/W)$ and I claim that this canonical isomorphism coincides with the one of Thm. 3.3. For this it suffices to compute the morphism $\mathrm{Tor}_1^R(R/p^n, F) \to \mathrm{Tor}_1^R(R_n, F)$ and this is done by the following covering:

$$\begin{array}{ccccccccc} & & 0 & \to & R & \xrightarrow{p^n} & R & \to & R/p^nR \to 0 \\ & & & & \downarrow{\scriptstyle F^n} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ 0 \to R(-1) & \xrightarrow{(F^n, -F^n d)} & R(-1) \oplus R & & \xrightarrow{dV^n + V^n} & & R & \to & R_n \to 0 \end{array} \tag{5.6.13}$$

The proof of c) is now another diagram chase, using (3.1.2), which again I leave to the reader.

Lemma 5.7. *Let M be A^2-torsion. Then the following diagram commutes:*

$$\begin{array}{ccc} D^2(M)(1) & = & \mathrm{Hom}_W(T_0^2(M), K/W)(2) \\ \| & & \| \\ D^2(M(-1)) & = & \mathrm{Hom}_W(T_0^2(M(-1)), K/W) \end{array} \tag{5.7.1}$$

Everything clearly boils down to computing the morphism $\mathrm{Tor}_2^R(R_n, M(-1)) = \mathrm{Tor}_2^R(R_n, M)(1)$. This is done through the following covering:

$$
\text{(5.7.2)} \qquad
\begin{array}{ccccccccc}
0 \to & R(-2) & \xrightarrow{(F^n, -F^n d)} & R(-2)\oplus R(-1) & \xrightarrow{dV^n + V^n} & R(-1) & \longrightarrow & R_n(-1) & \longrightarrow 0 \\
& \downarrow{\scriptstyle -\tau} & & \downarrow{\scriptstyle \begin{pmatrix} -\tau & 0 \\ 0 & \tau \end{pmatrix}} & & \downarrow{\scriptstyle \tau} & & \downarrow{\scriptstyle \tau} & \\
0 \longrightarrow & R(-1) & \xrightarrow{(F^n, -F^n d)} & R(-1)\oplus R & \xrightarrow{dV^n + V^n} & R & \longrightarrow & R_n & \longrightarrow 0
\end{array}
$$

Here τ is the canonical isomorphism of degree -1.

Definition 5.8. *Let M be a coherent module such that $M = A^0(M)$* (resp. $A_s^1(M)$, $A_n^1(M)$, $A^2(M)$). *If $m \in M$ and $\varphi \in D^0(M)$* (resp. $D^1(M)$, $D^1(M)$, $T_0^2(D^2(M))$) *we may, through* Thm. 3.3 *and* Prop. 5.2, *evaluate φ on m and obtain an element in W* (resp. K/W, W, K/W). *We will denote this element $\langle \varphi, m \rangle$. Similarly if $m \in T_0^2(M)$ and $\varphi \in D^2(M)$.*

Theorem 5.9. *Let X be smooth and proper of dimension N.*

i) *Let $m \in A^0(R^i\Gamma(W\Omega^\cdot))^j$, $n \in A^0(R^{N-i}\Gamma(W\Omega^\cdot))^{N-j}$ and let*

$$\varphi : A^0(R^*\Gamma(W\Omega^\cdot))^\cdot \to D^0(A^0(R^{N-*}\Gamma(W\Omega^\cdot))^{N-\cdot})$$

be the morphism of Thm. 3.5 iii). *Then*

$$\langle \varphi(m), n \rangle = (-1)^{i+j} \langle \varphi(n), m \rangle; \quad \langle \varphi(Fm), n \rangle = \sigma \langle \varphi(m), F^{-1}(n) \rangle.$$

ii) *Let $m \in A_n^1(R^i\Gamma(W\Omega^\cdot))^j$, $n \in A_n^1(R^{N+1-i}\Gamma(W\Omega^\cdot))^{N-j-1}$ and let*

$$\varphi : A_n^1(R^*\Gamma(W\Omega^\cdot))^\cdot \to D^1(A_n^1(R^{N+1-*}\Gamma(W\Omega^\cdot))^{N-\cdot-1})$$

be the morphism of Thm. 3.5 iv). *Then*

$$\langle \varphi(m), n \rangle = (-1)^{i+j} \langle \varphi(n), m \rangle; \quad \langle \varphi(Fm), n \rangle = \sigma \langle \varphi(m), Vn \rangle;$$

$$\langle \varphi(Vm), n \rangle = \sigma^{-1} \langle \varphi(m), Fn \rangle.$$

iii) *Let $m \in A_s^1(R^i\Gamma(W\Omega^\cdot))^j$, $n \in A_s^1(R^{N+1-i}\Gamma(W\Omega^\cdot))^{N-j}$ and let*

$$\varphi : A_s^1(R^*\Gamma(W\Omega^\cdot))^\cdot \to D^1(R^{N+1-*}\Gamma(W\Omega^\cdot))^{N-\cdot})$$

be the morphism of Thm. 3.5 iv). *Then*

$$\langle \varphi(m), n \rangle = (-1)^{i+j+1} \langle \varphi(n), m \rangle; \quad \langle \varphi(Fm), n \rangle = \sigma \langle \varphi(m), F^{-1}(n) \rangle.$$

iv) *Let $m \in D^2(A^2(R^{N+2-i}\Gamma(W\Omega^\cdot)))^j$, $n \in T_0^2(D^2(A^2(R^i\Gamma(W\Omega^\cdot))))^{N-j-1}$ and let*

$$\varphi : D^2(A^2(R^{N+2-*}\Gamma(W\Omega^\cdot))) \to A^2(R^{*+1}\Gamma(W\Omega^\cdot))$$

be the morphism induced from Thm. 3.5 ii). *Then*

$$\langle n, \varphi(m)\rangle = (-1)^{i+j}\langle m, \varphi(n)\rangle; \quad \langle m, \varphi(n)\rangle = (-1)^{j}\langle dm, \varphi(n')\rangle$$

$$\langle n, \varphi(Fm)\rangle = \sigma\langle Vn, \varphi(m)\rangle; \quad \langle n, \varphi(Vm)\rangle = \sigma^{-1}\langle Fn, \varphi(m)\rangle$$

where n' *is such that* $dn'=n$. *Similarly for* $m\in T_0^2(D^2(A^2(R^{N+2-i}\Gamma(W\Omega^\cdot))))$, $n\in D^2(A^2(R^i\Gamma(W\Omega^\cdot)))^{N-j-1}$.

Proof. Except for the symmetry this is just a reformulation of Thm. 3.3, using the fact that φ is an R-homomorphism. It is clear that the commutativity of multiplication in $W\Omega^\cdot$ implies that the following diagram commutes:

$$\text{(5.9.1)}\qquad \begin{array}{ccc} D(R\Gamma(W\Omega^\cdot)) & \longrightarrow & R\Gamma(W\Omega^\cdot)(N)N \\ \| & & \downarrow{\scriptstyle \mathrm{ev}} \\ D(R\Gamma(W\Omega^\cdot)) & \longrightarrow & D(D(R\Gamma(W\Omega^\cdot)(N)[N])) \end{array}$$

where the horizontal isomorphisms are the duality isomorphism resp. $D(-)$ and shifting applied to the duality isomorphism. We may therefore apply Props. 5.3 and 5.5, Lemma 5.7 and the $D^0(-)$ and $D^1(-)$ analogues of Lemma 5.7, which are obvious. The rest is just a careful application of the sign conventions. I will do this for iv) and leave the rest of the cases to the reader. Indeed, 5.3, 5.5 and 5.7 show that (5.9.1) implies that the following diagram $(-1)^{-N+i}$-commutes:

$$\text{(5.9.2)}\qquad \begin{array}{c} \mathrm{Hom}_W(T_0^2(R^{N+2-i}\Gamma(W\Omega^\cdot)), K/W) \\ \| \\ \mathrm{Hom}_W(T_0^2(R^{N+2-i}\Gamma(W\Omega^\cdot)), K/W) \end{array}$$

$$\xrightarrow{\quad\varphi\quad} A^2(R^i\Gamma(W\Omega^\cdot))(N-1)$$

$$\downarrow{\scriptstyle \mathrm{ev}}$$

$$\xrightarrow{\varphi^*} \mathrm{Hom}_W(\mathrm{Hom}_W^c(A^2(R^i\Gamma(W\Omega^\cdot))(N-1), K/W), K/W)$$

As the sign conventions imply that the evaluation mapping is given the sign $(-1)^j$ in degree j, iv) follows.

V. Examples: Supersingular K3-surfaces and abelian fourfolds

0. In this chapter we will look at two examples. We will redo some of the results of [1], [16], [13] and [17] using the duality for the Hodge—Witt cohomology in the first section and then we will show that for a supersingular abelian fourfold $E_2^{1,3}\neq E_\infty^{1,3}$ in the second section. (Heie $E^{i,j}$ is the first s.s. of the de Rham—Witt complex.)

1. In this section X will denote a supersingular $K3$-surface. Recall that for any $K3$-surface $H^1(\mathcal{O})={}_{\mathrm{tors}}NS=0$ and $b_2=22$ and that, by definition, for a supersingulaı $K3$-surface $H^2(W\mathcal{O})$ is isomorphic to $k_\sigma[[V]]$.

Proposition 1.1

i) $\Gamma(W\Omega^\cdot)=W\cdot 1.$

ii) $R^1\Gamma(W\Omega^\cdot)$ *is a rank* 22 *module of slope zero concentrated in degree* 1.

iii) $R^2\Gamma(W\Omega^\cdot)\xrightarrow{\sim}\mathrm{U}_{\sigma_0}\oplus W(-2)$, *where* $W(-2)$ *is of slope zero and* $0\leqq\sigma_0\leqq 11$.

Indeed, $H^1(W\mathcal{O})$ and $H^0(W\Omega^1)$ aıe zero because $H^1(\mathcal{O})$ is zero, which implies that $(\mathrm{Pic}\,X)^\circ=0$ but $H^1(W\mathcal{O})$ is the typical curves of $(\mathrm{Pic}\,X)^{\mathrm{red},\circ}$ and $H^0(W\Omega^1)=W\otimes_{\mathbf{Z}_p}T_p\mathrm{Pic}X$ (at least after we extend the base to $\bar{k}$), (cf. [9, II: 5.8.2—3]). $H^1(W\Omega^1)$ is torsionfree as its torsion is semi-simple by duality, so its torsion comes from the torsion in NS ([9, II: 5.22.1]) which is zero. $H^0(W\Omega^2)$ is zero because it is of slope zero and its slope zero part is dual to the slope zero part of $H^2(W\mathcal{O})$ which is zero by assumption.

$H^1(W\Omega^2)$ is zero because it consists of slope zero and semi-simple torsion and the slope zero part is dual to the slope zero part of $H^1(W\mathcal{O})$ and its semi-simple torsion is dual to the semi-simple torsion of $H^2(W\mathcal{O})$ and they are both zero.

$H^2(W\Omega^1)$ is torsion by duality as $H^1(W\mathcal{O})=H^0(W\Omega^1)=0$. Its semi-simple (nilpotent) torsion is dual to the semi-simple (nilpotent) torsion of $H^1(W\Omega^1)(H^2(W\mathcal{O}))$ and is theıefore zero. We now see that $H^1(W\mathcal{O})\xrightarrow{d}H^2(W\Omega^1)$ is a domino and as $H^2(W\Omega^2)$ is always equal to W and $H^1(W\mathcal{O})$ equals $k_\sigma[[V]]$, the above mentioned domino is one-dimensional.

Thus it only remains to show that σ_0 lies in the claimed range. Consider the exact sequence:

$$(1.1.1)\qquad \begin{array}{ccccccccc} & & & & 0\to & H^2(W\Omega^1)^\vee & \xrightarrow{\varphi} & H^2(W\mathcal{O}) & \to 0\\ & & & & & \downarrow{\scriptstyle d^\vee} & & \downarrow{\scriptstyle d} & \\ 0\to & H^1(W\Omega^1) & \xrightarrow{\varphi} & H^1(W\Omega^1)^\vee & \xrightarrow{d_2} & H^2(W\mathcal{O})^\vee & \xrightarrow{\varphi} & H^2(W\Omega^1) & \to 0\end{array}$$

Snake lemma gives us an exact sequence

$$0\to\mathrm{Ker}\,d^\vee\to\mathrm{Ker}\,d\to H^1(W\Omega^1)^\vee/H^1(W\Omega^1)\to\mathrm{Coker}\,d^\vee\to\mathrm{Coker}\,d\to 0.$$

If σ_0 is negative then $\mathrm{Ker}\,d^\vee\neq 0$ and $\mathrm{Ker}\,d=0$, which is a contıadiction. Therefore $\sigma_0\geqq 0$ an we get $2\cdot\dim\mathrm{Ker}\,d=\dim H^1(W\Omega^1)^\vee/H^1(W\Omega^1)$ which is less than 23 as $H^1(W\Omega^1)$ is of rank 22. Now dim $\ker d=\sigma_0$.

Let us continue the study of (1.1.1). It is clear that $\mathrm{Ker}\,F^{\sigma_0+1}d^\vee$ is one-dimensional over k. Pick, provisionally, a non-zero element in this space, 1, say. As $F^{\sigma_0}d^\vee$ is an isomorphism $d_i:=F^{\sigma_0}d^\vee V^i1$, $i\geqq 0$ form a base for $T_0^2(H^2(W\mathcal{O})^\vee)$.

As $H^2(W\mathcal{O})$ is the dual of $T_0^2(H^2(W\mathcal{O})^\vee)$ (IV: Prop. 5.2) we may find a unique element $1'$ in $H^2(W\mathcal{O})$ such that $\langle d_i, 1'\rangle=\delta_{0i}$.

Now $1', V1', \ldots, V^{2\sigma_0}1'$ is a base for $\operatorname{Ker} F^{\sigma_0+1}d$ and $\varphi(1)\in\operatorname{Ker} F^{\sigma_0+1}d$ as φ is an R-homomorphism. We may therefore write $\varphi(1)=a_0 1'+a_1 V1'+\ldots+a_{2\sigma_0}V^{2\sigma_0}\eta'$ and as 1 generates $H^2(W\Omega^1)^\vee$ as a $k_\sigma[[V]]$-module and φ is an isomorphism in degree 0 we have that $a_0\neq 0$. If we change 1 by a non-zero scalar, starting with $\bar{1}=\lambda 1$, we will get the following changes: $\bar{d}_i:=F^{\sigma_0}d^\vee V^i\bar{1}=\lambda^{p^{\sigma_0-i}}d_i$. $\bar{1}'$ which is characterized by $\langle\bar{1}',\bar{d}_i\rangle=\delta_{0i}$ will then clearly be $\lambda^{-p^{\sigma^0}}1'$. The new base for $\operatorname{Ker} F^{\sigma_0+1}d$ will be

$$\lambda^{-p^{\sigma_0}}1', \ldots, \lambda^{-p^{\sigma_0-1}}V^{2\sigma_0}1'$$

so therefore we will have

$$\varphi(\bar{1})=\lambda\varphi(1)=\lambda^{1+p^{\sigma_0}}a_0\bar{1}'+\lambda^{1+p^{\sigma_0-1}}a_1V\bar{1}'+\ldots+\lambda^{1+p^{-\sigma_0}}a_{2\sigma_0}V^{2\sigma_0}\bar{1}'.$$

Note that $\langle F^{\sigma_0}d^\vee V^i 1, V^j 1'\rangle=\sigma^{-j}\langle F^jF^{\sigma_0}dV^i 1, 1'\rangle=\sigma^{-j}\langle F^{\sigma_0}dV^{i-j}1, 1'\rangle=\delta_{ij}$. We therefore get, if

$$i\leqq\sigma_0,\ a_i=\langle F^{\sigma^0}d^\vee V^i, \varphi(1)\rangle=\langle 1, \varphi F^{\sigma_0}d^\vee V^i 1\rangle=-\sigma^{\sigma_0-i}\langle F^i d^\vee V^{\sigma_0}1, \varphi(1')\rangle$$

$$=-\sigma^{\sigma_0-i}\langle F^{\sigma_0}dV^{2\sigma_0-i}1, \varphi(1')\rangle=-a_{2\sigma_0-i}^{p^{\sigma_0-i}},$$

so that $\varphi(1)$ will have the form:

$$\varphi(1)=(a_0+a_1V+\ldots+a_{\sigma_0}V^{\sigma_0}-a_{\sigma_0-1}^pV^{\sigma_0+1}-\ldots-a_0V^{2\sigma_0})\cdot 1' \tag{1.1.2}$$

and $a^{\sigma^0}=0$ if $p\neq 2$. Of course, if $k=\bar{k}$ we may choose $\bar{1}=\lambda 1$ such that $\bar{a}_0=1$. It is clear that $\varphi(1)$ determines the square of (1.1.1). In particular it determines the pairing of (IV: 5.6.4), which, by (IV: Prop. 5.6 a)) coincides with the pairing on $H^1(W\Omega^1)^\vee/H^1(W\Omega^1)$ of (IV: 5.6.3). This is the pairing that Ogus considers in ([16]), so we find the relation with this approach. (It is clear from (IV: 4) that our approach is more or less equivalent to the one of ([13]).) One can show that, for $k=\bar{k}$, Artin's period map is obtained by taking the fixed points of F on the diagram (1.1.1) and hence that this map is the morphism $a_0+a_1F+\ldots-a_0^{p^{\sigma^0}}F^{2\sigma_0}$: $G_a^{\text{perf}}\to G_a^{\text{perf}}$. (Details on this will appear elsewhere.) Finally, as duality provides us with a Gysin map we can calculate σ_0 for a Kummer surface (cf. [16]). Let A be a supersingular abelian surface $(p\neq 2)$, put A' the blow up of A at the kernel of multiplication by 2, put $X=A'/\langle -1\rangle$ the associated Kummer surface and $q\colon A'\to X$ the projection. By definition $\operatorname{dom}(R^2\Gamma(W\Omega_A^\cdot))^0\xrightarrow{\sim}\mathbf{U}_{\sigma_0}$ for some $1\leqq\sigma_0\leqq 2$ and as it is easily seen that blowing ups of surfaces only affect $H^1(W\Omega^1)$ we see that $\operatorname{dom}(R^2\Gamma(W\Omega_{A'}^\cdot))^0=\operatorname{dom}(R^2\Gamma(W\Omega_A^\cdot))^0$. Now as in ([15, VII: Prop. 3.2.4]) one sees that $q_*q^*=\deg q=2$, so that q^* is a split injection, but as $H^2(W\mathcal{O}_X)\neq 0$ and $\mathbf{U}_{\sigma_0}$ clearly is indecomposable we see that $\operatorname{dom}(R^2\Gamma(W\Omega_X^\cdot))^0\xrightarrow{\sim}\mathbf{U}_{\sigma_0}$. In particular $\sigma_0(X)=\sigma_0(A)$.

2. Let A be a supersingular abelian fourfold. Recall that this means that $H^1(A/W)$ is of slope 1/2. As $H^*(A/W)=\Lambda^*H^1(A/W)$ (cf. [4, Cor. 2.5.5]) we see that $H^4(A/W)$ is of slope 2 and therefore $R^i\Gamma(W\Omega^\cdot)^{4-i}$ is torsion except for $i=2$ where the torsion free part is of slope zero. If M is a W-lattice with a non-degenerate pairing put $\operatorname{discr} M:=\operatorname{length} M^\vee/M$. We will compute $\operatorname{discr} A^0(R^2\Gamma(W\Omega^\cdot))^2$ in two ways using crystalline duality and the duality for the Hodge—Witt cohomology. Put, for simplicity, $N:=A^0(R^2\Gamma(W\Omega^\cdot))^2$. As $H^4(A/W)$ is without torsion we see that $\operatorname{length} H^4(A/W)/N=\operatorname{length} E_\infty^{0,4}+\operatorname{length} E_\infty^{1,3}$, where $E_1^{i,j}=R^j\Gamma(W\Omega^\cdot)^i\Rightarrow H^*(A/W)$ is the first spectral sequence. By duality for $H^4(A/W)$ the pairing for $H^4(A/W)$ is perfect so $\operatorname{discr} N=2\,(\operatorname{length} E_\infty^{0,4}+\operatorname{length} E_\infty^{1,3})$. On the other hand, $H^3(W\mathcal{O})$ and $H^3(W\Omega^1)$ are torsion (by the slope conditions from above) and there is no semi-simple torsion in them because it would be dual to the semi-simple torsion of $H^2(W\Omega^3)$ resp. $H^2(W\Omega^4)$ and as $H^1(W\Omega^4)$ is torsion and by "survie du coeur" it would give rise to torsion in $H^5(A/W)$ resp. $H^6(A/W)$. Therefore $E_2^{1,3}=H^1(A^2(R^3\Gamma(W\Omega^\cdot)),d)$, so by (IV: Thm. 3.5 ii, 3.3.6) and snake lemma we get an exact sequence $E_2^{1,3}\to N^\vee/N\to E_2^{1,3^\vee}$.

Hence $\operatorname{discr} N\leqq 2\cdot\operatorname{length} E_2^{1,3}$ and combining we obtain $\operatorname{length} E_2^{1,3}-\operatorname{length} E_\infty^{1,3}\geqq\operatorname{length} E_\infty^{0,4}$. If we can show that $E_\infty^{0,4}\neq 0$ we conclude that $E_2^{1,3}\neq E_\infty^{1,3}$. However as $H^5(A/W)$ is without torsion and the Hodge to de Rham spectral sequence degenerates we see that $H^4(A/W)\to H^4_{DR}(A/k)$ and $H^4_{DR}(A/k)\to H^4(\mathcal{O})$ both are surjective and therefore their composite is non zero, but this composite factors through $E_\infty^{0,4}$.

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