On additive automorphic and rotation automorphic functions

R. Aulaskari and P. Lappan

1. Introduction

Let D denote the unit disc in the complex plane and let Γ be a Fuchsian group. A function $W(z)$ meromorphic in D is said to be *additive automorphic relative to the Fuchsian group* Γ if for each transformation $T\in\Gamma$ there exists a constant A_T such that $W(T(z))=W(z)+A_T$ for each $z\in D$. The numbers A_T are called *periods* of $W(z)$. A function $W(z)$ is said to be *additive automorphic* if it is additive automorphic relative to some non-trivial Fuchsian group. An analytic function $f(z)$ in D is said to be a *Bloch function* if there exists a constant B_f such that $(1-|z|^2)|f'(z)|\leqq B_f$ for each $z\in D$. A function $f(z)$ meromorphic in D is said to be a *normal function* if there exists a constant N_f such that

$$(1-|z|^2)|f'(z)|/(1+|f(z)|^2) \leqq N_f$$

for each $z\in D$.

Conditions under which an additive automorphic function is a normal function have been studied by Aulaskari [1], [2]. Pommerenke [6] has given an example of an additive automorphic function $W(z)$ such that $W(z)$ is not a Bloch function but $\iint_F |W'(z)|^2 dx\,dy<\infty$, where F denotes the fundamental region for Γ. In this note the main result is the following theorem.

Theorem 1. *There exists an additive automorphic function $W(z)$ relative to a Fuchsian group Γ such that $W(z)$ is not a normal function, $W(z)$ has only imaginary periods, and*

$$\iint_F |W'(z)|^2\, dx\, dy < \infty,$$

where F denotes the fundamental region of Γ.

This theorem will be proved in Section 2, by means of a modification of Pommerenke's method. In Section 3, some important consequences of this theorem will be given.

§ 2. Proof of Theorem 1

The proof of Theorem 1 consists of constructing an appropriate Riemann surface together with appropriate conformal mappings.

Let $\mathbf{N}$ denote the set of all positive integers, and let $\mathbf{N}^*$ denote the set of all non-zero integers. For each $n\in\mathbf{N}$, let

$$Q_n = \{z = x+iy\colon\ |x| < a_n,\ |y| \leqq b_n\},$$

where $a_n \geqq 3^n$, $a_{n+1} > a_n+3$, and $b_n > 0$. Also for each $n\in N$, let

$$T_n = \{z = x+iy\colon\ a_n-2 \leqq x \leqq a_n-1,\ y = 0\},$$

and for $n\in\mathbf{N}-\{1\}$ let

$$T_n' = \{z = x+iy\colon\ a_{n-1}-2 \leqq x \leqq a_{n-1}-1,\ y = 0\}.$$

Finally, for $n\in\mathbf{N}^*$, let

$$E_n = \{z = x+iy\colon\ |x| < a_{|n|},\ y = (\operatorname{sgn} n)\, b_{|n|}\},$$

where $\operatorname{sgn} n = n/|n|$, and let $Q = \bigcup_{n\in\mathbf{N}} Q_n$.

Let S be a covering surface for Q obtained by joining copies of the rectangles of Q_n as follows. Let $Q_n(s)$ denote a copy of Q_n, and let $T_n(s)$, $T_n'(s)$, $E_n(s)$ and $E_{-n}(s)$ denote the subsets of $Q_n(s)$ corresponding to T_n, T_n', E_n, and E_{-n}, respectively. For each $n\in\mathbf{N}$, let $Q_n(s)$ be slit along the line segment $T_n(s)$ and let $Q_{n+1}(s)$ be slit along the line segment $T_{n+1}'(s)$, and let the upper edge of the slit along $T_n(s)$ be joined to the lower edge of the slit along $T_{n+1}'(s)$ and also let the upper edge of the slit along $T_{n+1}'(s)$ be joined to the lower edge of the slit along $T_n(s)$, so that the two sheets $Q_n(s)$ and $Q_{n+1}(s)$, thus joined, have a local resemblence to the surface corresponding to the function $\sqrt{(z-a_n+1)(z-a_n+2)}$. Thus, $Q_n(s)\cup Q_{n+1}(s)$, joined as indicated, is a two-sheeted surface with points of ramification of order one at the points corresponding to $z=a_n-1$ and $z=a_n-2$. Let $S=\bigcup_{n\in\mathbf{N}} Q_n(s)$, where for each $n\in\mathbf{N}$, $Q_n(s)$ and $Q_{n+1}(s)$ are joined as described above, with no additional identification of points on S. For each $w\in Q_n(s)$, let $\pi_0(w)$ be the point of Q_n to which w corresponds. Thus S is a covering surface for the set Q with the projection map π_0. On the surface S, a point w is a point of ramification if and only if there exists $n\in N$ such that $w\in Q_n(S)$ and $\pi_0(w)\in\{a_n-1,\ a_n-2\}$.

Let γ be a free group which is generated by the elements $\sigma_1, \sigma_2, \sigma_3, \ldots, \sigma_n, \ldots$. Then each element $\sigma\in\gamma$ has the form

$$\sigma = \sigma_{n_1}^{k_1}\sigma_{n_2}^{k_2}\ldots\sigma_{n_p}^{k_p}$$

where each $n_j\in\mathbf{N}$ and each $k_j\in\mathbf{N}^*$. If we define $\sigma_{-k}=\sigma_k^{-1}$, we can take $n_j\in\mathbf{N}^*$ and $k_j\in\mathbf{N}$ in the representation of σ, and we will do this in what follows. For each $\sigma\in\gamma$, let S_σ denote a copy of S and if I denotes the identity element of γ we take

$S=S_I$. Thus if $\tilde{w}\in S_\sigma$, we may take $\tilde{w}=(w, \sigma)$, where w is the point of S corresponding to $\tilde{w}\in S_\sigma$. Let $R=\bigcup_{\sigma\in\gamma} S_\sigma$, where the only identification of points on the surfaces S_σ is given by $\tilde{w}=(w, \sigma)=\tilde{w}'=(\tilde{w}', \sigma')$ if and only if either (1) $w=w'$ and $\sigma=\sigma'$, or (2) there exists $n\in N^*$ such that $w\in E_n(S)$, $w'\subseteqq E_{-n}(S)$, $\sigma=\sigma_n\sigma'$, and $\pi_0(w)=\pi_0(w')+2i(\operatorname{sgn} n)b_{|n|}$. Define the projection mapping $\pi\colon R\to\mathbb{C}$ by

$$\pi(\tilde{w}) = \pi_0(w)+2i\sum_{j=1}^{q} k_j(\operatorname{sgn} n_j)b_{|n|}$$

where $\tilde{w}=(w, \sigma)$ with $\sigma=\sigma_{n_1}^{k_1}\sigma_{n_2}^{k_2}\dots\sigma_{n_q}^{k_q}$. It is easily verified that R is a simply connected open Riemann surface with projection mapping π such that the only points of ramification are those of the form $\tilde{w}=(w, \sigma)$, where w is a point of ramification of the surface S, that is, there exists $n\in N$ such that $w\in Q_n(S)$ and $\pi_0(w)\in\{a_n-2, a_n-1\}$.

Let f be a conformal mapping from the unit disc $D=\{z\colon |z|<1\}$ onto the simply connected Riemann surface R, and let $W(z)=(\pi\circ f)(z)$. Then $W(z)$ is an analytic function. For $\sigma\in\gamma$, define a mapping $\tilde{\sigma}\colon R\to R$ by $\tilde{\sigma}(\tilde{w})=(w, \sigma\tau)$ where $\tilde{w}=(w, \tau)$. Since $\pi\circ\tilde{\sigma}(\tilde{w})-\pi(\tilde{w})$ is a constant, we have that $\tilde{\sigma}$ is a conformal mapping of R onto itself and thus $T=f^{-1}\circ\tilde{\sigma}\circ f$ is a conformal mapping of D onto itself. Further, for any point $\tilde{w}\in R$, the orbit $\{\tilde{\sigma}(\tilde{w})\colon \sigma\in\gamma\}$ contains no limit point because at most two points of the orbit may belong to a single S_τ for a fixed $\tau\in\gamma$. Thus, the collection $\Gamma=\{f^{-1}\circ\tilde{\sigma}\circ f\colon \sigma\in\gamma\}$ is a Fuchsian group. The set

$$F = f^{-1}(S-\bigcup_{n\in\mathbf{N}} E_n(S))$$

is a fundamental set for the Fuchsian group Γ, that is, for each $z\in D$ there exists a unique $T\in\Gamma$ and $z'\in F$ such that $z=T(z')$. Since F is connected — it is the image of a connected set under the homeomorphism f^{-1} — we call F the fundamental region for Γ.

For $T\in\Gamma$, there exists $\sigma\in\gamma$ such that $T=f^{-1}\circ\tilde{\sigma}\circ f$, so that

$$\begin{aligned} W(T(z)) &= (\pi\circ f)(T(z)) = \pi((\tilde{\sigma}\circ f)(z)) \\ &= w(z)+2i\sum_{j=1}^{q} k_j(\operatorname{sgn} n_j)b_{|n_j|} \end{aligned}$$

where $\sigma=\sigma_{n_1}^{k_1}\sigma_{n_2}^{k_2}\dots\sigma_{n_q}^{k_q}$. Thus $W(z)$ is an additive automorphic function having only imaginary periods.

Let $Q_n(S_\sigma)=\{\tilde{w}=(w, \sigma)\colon w\in Q_n(S)\}$ and let

$$Q_n^* = \bigcup_{j\in N^*\cup\{0\}} Q_n(S_{\sigma_n^j}).$$

Then Q_n^* is a connected subset of R and

$$\pi(Q_n^*) = \{z = x+iy\colon |x| < a_n\}.$$

Further, the set of points of ramification in Q_n^* is a subset of the set

$$\{\tilde{w}\in Q_n^*\colon \operatorname{Re}(\pi(\tilde{w})) \geqq a_{n-1}-2\}.$$

For each $n \in N$, let $z_n \in D$ be such that $f(z_n) \in Q_n^*$ and $W(z_n)=0$. Then there exists a neighborhood U_n of z_n such that $W(z)$ maps U_n conformally onto the disc $\{t\colon |t|<a_{n-1}-2\}$. It follows that $(1-|z_n|^2)|W'(z_n)| \geqq a_{n-1}-2$ (see, for example, [5]). Since $a_n \to \infty$, we have that

$$\sup\{(1-|z|^2)|W'(z)|\colon z \in D, W(z)=0\} = \infty$$

and hence

$$\sup\{(1-|z|^2)|W'(z)|/(1+|W|z||^2)\colon z \in D\} = \infty.$$

Thus $W(z)$ is not a normal function. Therefore, the function $W(z)$ is an additive automorphic function which is not a normal function and all periods of $W(z)$ are imaginary. To complete the proof, we need only choose the sequences $\{a_n\}$ and $\{b_n\}$ such that $\sum_{n=1}^{\infty} a_n b_n < \infty$. Noting that $\iint_F |W'(z)|^2 dx\,dy$ gives the Euclidean area of the image of the fundamental set F under the function $W(z)$, counted according to multiplicity, we have that $\iint_F |W'(z)|^2 dx\,dy = 4 \sum_{n=1}^{\infty} a_n b_n < \infty$. This completes the proof.

§ 3. Consequences of Theorem 1

There are a number of important consequences which flow from Theorem 1 and its proof, and we now explore some of these.

Corallary 1. *There exists an additive automorphic function $W(z)$ having only imaginary periods such that $W(z)$ is not a normal function but $W(z)$ omits 3 values in the fundamental region F.*

Proof. For the function $W(z)$ in Theorem 1, the image of F under the function $W(z)$ has finite area, and so omits many values.

Remark 1. If $W(z)$ is an additive automorphic function such that $\iint_F |W'(z)|^2 dx\,dy < \infty$ and if appropriate restrictions are placed on F (or equivalently, on the group Γ), then $W(z)$ is a normal function (see [1, Theorems 3.7 and 7.2]). In the construction in the proof of Theorem 1, the fundamental region F does not satisfy these restrictions. In particular, the group Γ in the construction is infinitely generated and contains no parabolic transformations.

Definition 1. *We say that the harmonic function $u(z)$ is a normal function if*

$$\sup\{(1-|z|^2)|\operatorname{grad} u(z)|/(1+|u(z)|^2)\colon z \in D\} < \infty.$$

(Here, $|\operatorname{grad} u(z)|$ denotes the length of the gradient vector.)

This definition has appeared in [4].

Corollary 2. *There exists an additive automorphic function $W(z)$ such that $\iint_F |W'(z)|^2 dx\,dy < \infty$ and $u(z) = \mathrm{re}\,(W(z))$ is an automorphic harmonic function which is not a normal function but $\iint_F (u_x^2(z) + u_y^2(z))dx\,dy < \infty$.*

Proof. Let $W(z)$ be the function constructed in Theorem 1. Noting that $|W'(z)|^2 = u_x^2 + u_y^2$, we have that $|\mathrm{grad}\, u(z)| = |W'(z)|$ and hence

$$\sup \{(1-|z|^2)|\mathrm{grad}\, u(z)|:\ u(z) = 0\} = \infty.$$

Thus $u(z)$ is not a normal function. Since the periods of $W(z)$ are all imaginary, the function $u(z)$ is an automorphic function.

Remark 2. The function $u(z)$ in Corollary 2 can be written in the form $u(z) = u_1(z) - u_2(z)$, where both $u_1(z)$ and $u_2(z)$ are non-negative harmonic automorphic functions. To see this, let g be a conformal mapping from D onto the interior of F. Then the function $h = W \circ g$ has finite Dirichlet integral, so h is in the Hardy class H^1 and so $\mathrm{Re}\, h(t)$ can be written as a Poisson integral of its boundary values. Also, $\mathrm{Re}\, h(t) = V_1(t) - V_2(t)$, where $V_j(t)$ is the Poisson integral of the boundary values of $\frac{1}{2}(|\mathrm{Re}\, h(t)| - (-1)^j \mathrm{Re}\, h(t))$, $j = 1, 2$. Now let $u_j(z) = (V_j \circ g^{-1})(z)$ for $z \in F$. For $z \in \partial F$ and $T \in \Gamma$ such that $T(z) \in \partial F$ we have that $u_j(T(z)) = u_j(z)$. $j = 1, 2$. because $u(T(z)) = u(z)$. Thus $u_j(z)$ can be continued harmonically across ∂F to all of D so as to satisfy the relationship $u_j(T(z)) = u_j(z)$ for all $T \in \Gamma$, $z \in D$, $j = 1, 2$.

Definition 2. *We say that the meromorphic function $G(z)$ in D is a rotation automorphic function relative to a Fuchsian group Γ if for each $T \in \Gamma$ there exists a linear fractional transformation S_T, where S_T is a rotation of the Riemann sphere, such that $G(T(z)) = S_T(G(z))$ for each $z \in D$.*

Let $W(z)$ be an analytic additive automorphic function having only imaginary periods. Then $G(z) = e^{W(z)}$ is a rotation automorphic function (for which the rotations are all about the origin). If $W(z)$ is the specific function constructed in the proof of Theorem 1 and if $G(z) = e^{W(z)}$, then, since $G'(z) = W'(z)$ when $W(z) = 0$,

$$\sup \{(1-|z|^2)|G'(z)|:\ G(z) = 1,\ z \in D\} = \infty$$

and thus

$$\sup \{(1-|z|^2)|G'(z)|/(1+|G(z)|^2):\ z \in D\} = \infty.$$

Hence $G(z)$ is not a normal function in D. This conclusion has not made use of any condition on the sequence $\{b_n\}$, so we are free to choose the b_n's to suit our purposes. The image of $g^{-1}(Q_n(S))$ under the function $G(z)$ is contained in the sector of the circle with center at the origin, radius e^{a_n}, and central angle $2b_n$, so the area of $G(g^{-1}(Q_n(S)))$ is not more than $b_n e^{2a_n}$. Thus, if we require that the sequence $\{b_n\}$

be such that $\Sigma b_n e^{2a_n} < \infty$, we will have as a consequence that $\iint_F |G'(z)|^2 dx\, dy < \infty$. This proves the following result.

Theorem 2. *There exists a rotation automorphic function* $G(z)$ *such that* $G(z)$ *is not a normal function and*

$$\iint_F |G'(z)|^2 \, dx\, dy < \infty.$$

Corollary 3. *There exists a rotation automorphic function* $G(z)$ *such that* $G(z)$ *is not a normal function and*

$$\iint_F (|G'(z)|/(1+|G(z)|^2))^2 \, dx\, dy < \infty.$$

Corollary 3. follows from Theorem 2 immediately from the inequality

$$|G'(z)|/(1+|G(z)|^2) \leqq |G'(z)|$$

for each $z \in D$. However, a slightly stronger result can be obtained by a minor modification of the function $W(z)$ constructed in the proof of Theorem 1. In place of the sets Q_n we may use the sets $Q'_n = \{z = x+iy: |y| \leqq b_n\}$ and proceed to construct a Riemann surface R' following the construction as in the proof of Theorem 1. (The only other modification needed in this construction is to remove the restriction $|x| < a_n$ from the definition of the sets E_n and E_{-n}.) Let $W_1(z)$ denote the additive automorphic function constructed using the surface R'. Then $W_1(z)$ is not a normal function, since by the same argument as for $W(z)$ we have

$$\sup \{(1-|z|^2)|W_1'(z)|: W_1(z) = 0, \ z \in D\} = \infty.$$

But in the present construction, the fundamental region F does have parabolic vertices. In fact, the group Γ is generated by parabolic transformations. Taking $G_1(z) = \exp\{W_1(z)\}$, we note that the spherical image of Q'_n under the exponential function is $b_n/6$, so that the spherical image of the fundamental region F under the function $G_1(z)$ has area $\frac{1}{6}\Sigma b_n$. Thus, if we require that $\Sigma b_n < \infty$, then $G_1(z)$ is a non-normal rotation automorphic function for which the fundamental region F has infinitely many parabolic vertices and $\iint_F (|G_1'(z)|/(1+|G_1(z)|^2))^2 dx\, dy < \infty$.

Remark 3. The result of Corollary 2 (and the previous paragraph) contrasts with results of Aulaskari and Servaldi [3] under which, with some additional conditions on F the condition $\iint_F (|G'(z)|/(1+|G(z)|^2))^2 dx\, dy < \infty$ implies that a rotation automorphic function $G(z)$ is a normal function. Clearly, the fundamental regions for the constructions in this paper do not satisfy these additional conditions.

References

1. AULASKARI, R., On normal additive automorphic functions, *Ann. Acad. Sci. Fenn. Ser. A. I. Math. Dissertationes,* **23** (1978), 53 pp.
2. AULASKARI, R., On F_0-normality and additive automorphic functions of the first kind, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.* **5** (1980), 327—340.
3. AULASKARI, R. and SORVALI T., Rotation-automorphic functions near the boundary *Math. Scand* **49** (1981), 222—228.
4. LAPPAN, P., Some on results harmonic normal functions, *Math. Z.* **90** (1965), 155—159.
5. POMMERENKE, CH., On Bloch functions, *J. London Math. Soc.* (2) **2** (1970), 689—695.
6. POMMERENKE, CH., On inclusion relations for spaces of automorphic forms, *Lecture notes in Mathematics,* **505**, Springer-Verlag, 1976, 92—100.

Received April 6, 1982

R. Aulaskari
University of Joensuu
SF-80 101 Joensuu 10
Finland
and
P. Lappan
Michigan State University
East Lansing, Michigan 48824 USA