K-divisibility of the K-functional and Calderón couples

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0. Introduction

Let f be a function in $L^p(\mathbf{R}^n)$ with L^p modulus of continuity $\omega(f, t)=\sup_{|h|\leqq t}\|f(x+h)-f(x)\|_{L^p}$ satisfying $\omega(f, t)\leqq\sum_{n=1}^{\infty}\psi_n(t)$ where $\psi_n(t)$ is a positive concave function of t for each n and $\sum_{n=1}^{\infty}\psi_n(1)<\infty$. Then f can be decomposed into a sum of functions in $L^p(\mathbf{R}^n)$, $f=\sum_{n=1}^{\infty}f_n$ such that, for each n, $\omega(f_n, t)\leqq\gamma\psi_n(t)$ for all $t>0$. (Here γ denotes a constant which does not depend on f.)

This result, which seems far from obvious, is presented as a brief comment in the remarkable note [2] of Ju. A. Brudnyĭ and N. Ja. Krugljak. It is only one of many consequences of their theorem on the property of "K-divisibility" of the Peetre K-functional (Theorem 1 below). Most of the other consequences studied thus far both by Brudnyĭ and Krugljak themselves [2, 3] and by others, notably Per Nilsson [13, 14], are formulated within the context of the theory of interpolation spaces. However, as the above result strongly suggests, it is to be expected that the advances in interpolation theory made possible by the work of Brudnyĭ and Krugljak will also have many further interesting new applications in various branches of analysis (cf. [13], Section 6.2).

In the present paper we present an alternative proof of the K-divisibility theorem and subsequently, with the help of the techniques developed in our proof, we show that all interpolation spaces with respect to a large class of compatible couples $\bar{A}=(A_0, A_1)$ have certain monotonicity properties with respect to the K-functional. Among the corollaries of our main theorem we obtain the result of Sparr [16] and its generalization due to Dmitriev [8] characterizing all interpolation spaces with respect to couples of L^p spaces. But now we can also give a weaker form of Dmitriev's theorem which holds for values of the exponents p for which the original version breaks down. Another corollary describes monotonicity conditions satisfied

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by all interpolation spaces with respect to the "Sobolev couple" $\overline{W}=(L^p, W^{1,p})$. The result here is best possible in a certain sense and answers a question posed in [6] (p. 135).

The paper is organized as follows. Section 1 contains definitions, terminology, and statements and discussion of the results to be proved in the following sections. We have amplified the discussion a little in order to refer the interested reader, who may not yet have access to [3], to other papers whose results, taken in conjunction with our work here, lead to proofs of some of the other main theorems of Brudnyĭ and Krugljak. Section 2 is devoted to some preliminary results concerning couples of weighted L^1 spaces. Section 3 contains our proof of the K-divisibility theorem. In Section 4 the proof is adapted to give our new result on monotonicity properties of interpolation spaces. Finally in Section 5 we briefly discuss the application of that result to the couple $\overline{W}$. We mention that one reason for our particular interest in this couple is the fact that its K-functional is equivalent to a rather concrete quantity, the L^p modulus of continuity, which appears in many other contexts. (Cf. [1], Chapter 6, and also the example with which we began here.)

It is a pleasure to acknowledge very helpful discussions and correspondence with Per Nilsson and Jaak Peetre. In particular we are indebted to Professor Peetre for pointing out that our results in Section 4 can be presented in the more general setting of relative interpolation spaces and thus yield the theorem of Dmitriev. We have incorporated his remark into our presentation here.

Remark 0.1. Weaker forms of the K-divisibility property can be established for couples of quasi Banach spaces or normed Abelian groups by methods not very different from those for Banach spaces. There is also a version in which the K-functional is replaced by the approximation functional. We refer to [3] and [13] for details. In fact Sparr also obtained his theorem in this more general setting, namely for couples of L^p spaces with p in the extended range $(0, \infty]$, under suitable restrictions on the measure space. (Cf. [6], Section 4 for a related result.) Our techniques here for obtaining Sparr's theorem via K-divisibility use the Hahn—Banach theorem and do not seem to be readily adaptable for $p<1$.

Remark 0.2. We shall present some converse results to our main theorem here in forthcoming papers with Per Nilsson [17, 18].

1. Notation, terminology and statements of results

We begin by recalling some notions from interpolation theory. For the most part we use the notation and terminology of [1] and will assume that the reader has some familiarity with the K- and J-functionals and their elementary properties ([1], Chapter 3).

Let $\bar{A}=(A_0, A_1)$ and $\bar{B}=(B_0, B_1)$ be compatible couples of Banach spaces. Let A and B be intermediate spaces with respect to $\bar{A}$, respectively $\bar{B}$. We say that A and B are *relative interpolation spaces* with respect to $\bar{A}$ and $\bar{B}$ if every linear operator T satisfying $T\colon \bar{A}\to\bar{B}$ maps A into B boundedly with

$$\|T\|_{A,B} \leqq C\max\left(\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\right)$$

for some constant C independent of T. If $C=1$ then we say that A and B are *exact* relative interpolation spaces. Analogously A and B are termed *relative K spaces* with respect to $\bar{A}$ and $\bar{B}$ if, for some positive constant C and for each $a\in A$, whenever the inequality $K(t, b; \bar{B})\leqq K(t, a; \bar{A})$ holds for all $t>0$ and some $b\in\Sigma(\bar{B})$ then $b\in B$ and $\|b\|_B\leqq C\|a\|_A$. Again A and B are *exact* relative K spaces if we can take $C=1$.

Clearly if A and B are relative K spaces then they must also be relative interpolation spaces. For many couples, $\bar{A}$ and $\bar{B}$ all pairs of relative interpolation spaces are characterized by this property. We say of such pairs of couples $\bar{A}$ and $\bar{B}$ that they are *relative Calderón couples.* (See e.g. [7], [8], [14].) Note that in all the above terminology we cannot change the order in which we write A and B or $\bar{A}$ and $\bar{B}$ without changing the meaning of the corresponding condition.

Much of the research on this topic has dealt with the case $\bar{A}=\bar{B}$ and in this context the above definitions reduce to saying that $\bar{A}$ is a *Calderón couple* if all interpolation spaces A with respect to $\bar{A}$ are *K spaces,* that is, if they each have the property that $a\in A$, $b\in\Sigma(\bar{A})$ and $K(t, b; \bar{A})\leqq K(t, a; \bar{A})$ for all $t>0$ imply that $b\in A$ with $\|b\|_A\leqq C\|a\|_A$. The couples which are known to be Calderón include couples of weighted L^p spaces [16] and all "reiterated" couples of the form $\bar{A}=(\bar{E}_{\theta_0,p_0}, \bar{E}_{\theta_1,p_1})$ where $\bar{E}$ is any other compatible Banach couple [6, 9]. For further examples, references and remarks see e.g. [4], [7] p. 2, [16], [14].

Turning our attention more specifically now to K spaces themselves we remark that a straightforward way of obtaining a K space A with respect to a couple $\bar{A}$ is to define the norm of A by

$$\|a\|_A = \|K(t, a; \bar{A})\|_\Phi$$

where Φ is a lattice norm defined on measurable functions on $(0, \infty)$. Spaces of this type, denoted for example by $A=\bar{A}_{\Phi;K}$, have been studied widely. Of course the much used spaces $\bar{A}_{\theta,p}=(A_0, A_1)_{\theta,p}$ are special examples of such spaces generated by taking Φ to be a suitable weighted L^p norm on $(0, \infty)$. Another important result of Brudnyĭ and Krugljak, which in fact can be proved using K-divisibility, states that for all couples $\bar{A}$ all K spaces A are of the form $\bar{A}_{\Phi;K}$ for some suitable Φ, to within isomorphism. (There are also related results concerning relative K spaces.) Thus, whenever $\bar{A}$ is a Calderón couple, all its interpolation spaces can be generated in this comparatively simple manner.

The K-divisibility theorem may be formulated as follows:

Theorem 1 ([2], Theorem 4). *Let $\bar{A}$ be a compatible couple of Banach spaces and let a be any element of $\Sigma(\bar{A})$. Suppose that $K(t, a; \bar{A}) \leqq \sum_{n=1}^{\infty} \psi_n(t)$ for all $t>0$, where each $\psi_n(t)$ is a positive concave function on $(0, \infty)$ and $\sum_{n=1}^{\infty} \psi_n(1) < \infty$. Then there exists a sequence of elements $a_n \in \Sigma(\bar{A})$ such that $a = \sum_{n=1}^{\infty} a_n$ (with the series converging in the norm of $\Sigma(\bar{A})$) and, for some constant γ,*

$$K(t, a_n; \bar{A}) \leqq \gamma\psi_n(t) \quad \textit{for each } n \textit{ and all} \quad t > 0.$$

One way of obtaining a proof of the above-mentioned result that all K spaces are of the form $\bar{A}_{\Phi;K}$ is to combine Theorem 1 above with Theorem 2.1 of [7] (p. 11). See also Section 4 of [14] for a treatment in a more general setting.

In [7] it was shown that many couples $\bar{A}$ have the following property which is related to K-divisibility and which may be considered as a refinement of the "fundamental lemma" ([1], p. 45).

(P) There exists a constant c such that each $a \in \Sigma_0(\bar{A})$ can be expressed as the sum $a = \sum_{\nu=-\infty}^{\infty} a_\nu$ of a series of terms $a_\nu \in \Delta(\bar{A})$ and for all $t>0$

$$\sum_{\nu=-\infty}^{\infty} \min(1, t/2^\nu) J(2^\nu, a_\nu; \bar{A}) \leqq cK(t, a; \bar{A}).$$

From the arguments to be given in Section 3 (see the proof of Theorem 4 or Remark 3.1) it will be apparent that all couples which are "mutually closed" ([7], p. 8) have property (P). This in turn, by application of Theorems 4.6, 4.7 and their corollaries in [7] leads to results very similar to those in [2] Section 7 concerning the equivalence of K spaces and interpolation spaces generated by the J-functional. These matters are also treated in more detail and greater generality in [13].

The alternative proof of Theorem 1 which we shall present here (in Section 3) is motivated to a considerable extent by the proof of Theorem 4.8 in [7]. Some similar ideas have been developed independently and used to study other aspects of real interpolation spaces by Svante Janson. (See [11], Example 0 and Theorem 16.) As in Theorem 4.8 of [7], we use the Sedaev—Semenov theorem that any compatible couple of weighted L^1 spaces $(L^1_{w_0}, L^1_{w_1})$ is a Calderón couple [15] (for alternative treatments cf. [5], p. 234 or [16]). However in the present context we need to extend the Sedaev—Semenov theorem to the case where each of the weight functions w_0 and w_1 may assume the value $+\infty$ on some set of positive measure. This is done in Section 2.

Brudnyĭ and Krugljak obtained that the constant γ in Theorem 1 satisfies $1 \leqq \gamma \leqq 14$. However our proof here shows that one can take $\gamma \leqq 8+\varepsilon$ for each $\varepsilon>0$. (This does not necessarily imply that $\gamma \leqq 8$ since our decomposition of a, $a = \sum_{n=1}^{\infty} a_n$, depends on the choice of ε.)

We now turn to the new results to be proved in Sections 4 and 5 of the present paper. In order to state them concisely it is convenient to introduce an "index" q for each couple $\bar{A}$ which in some sense measures how close $\bar{A}$ is to being a Calderón couple. The starting point for such a notion goes back to [5, 6] where we considered interpolation spaces which have properties somewhat weaker than that of being a K space. We may reformulate these properties in the more general "relative" setting as follows:

Let A and B be intermediate spaces with respect to the couples $\bar{A}$ and $\bar{B}$ respectively and let q be a positive number. We say that A and B are *relative L^q-K spaces* with respect to $\bar{A}$ and $\bar{B}$ if, for some fixed constant $C>0$, whenever $a\in A$, $b\in\Sigma(\bar{B})$ and

$$\left(\int_0^\infty [K(t, b; \bar{B})/K(t, a; \bar{A}]^q\, dt/t\right)^{1/q} \leqq 1$$

then $b\in B$ with $\|b\|_B\leqq C\|a\|_A$. Similarly A is an *L^q-K space* with respect to $\bar{A}$ if whenever $a\in A$, $b\in\Sigma(\bar{A})$ and

$$\left(\int_0^\infty [K(t, b; \bar{A})/K(t, a; \bar{A})]^q\, dt/t\right)^{1/q} \leqq 1$$

then $b\in A$ with $\|b\|_A\leqq C\|a\|_A$.

Of course the case $q=\infty$ in the above corresponds to the previous definitions of relative K spaces and K spaces. Note that for $q<\infty$, relative L^q-K spaces are not necessarily relative interpolation spaces (e.g. $A=\bar{A}_{\theta,\infty}$, $B=\bar{B}_{\theta,q}$, or, more drastically, $A=A_0\cap A_1$, $B=\{0\}$). However all relative interpolation spaces are necessarily relative L^1-K spaces for all choices of compatible couples $\bar{A}$ and $\bar{B}$. This is proved for the case $\bar{A}=\bar{B}$ in [5], but the proof for the general case is almost identical. In [5, 6] we also exhibited examples of couples $\bar{A}$ for which all interpolation spaces are L^q-K spaces for some q, $1<q<\infty$. This leads us to adopt the following terminology:

Definition. The couple $\bar{A}$ is of *Calderón type q* (or *C-type q*) if all interpolation spaces with respect to $\bar{A}$ are L^q-K spaces.

The couples $\bar{A}$ and $\bar{B}$ are of *relative Calderón type q* if all relative interpolation spaces with respect to $\bar{A}$ and $\bar{B}$ are relative L^q-K spaces.

Examples. Calderón couples of course have C-type ∞ and all couples have C-type 1. However, there exist couples (see e.g. [5], p. 223) which are not of C-type q for any $q>1$. The couple (L^1, C) (see [5], pp. 224—225) is not a Calderón couple but it is of C-type q for every $q<\infty$. Let $\bar{W}$ be the couple $(L^p, W^{1,p})$ for $1<p<\infty$, where $W^{1,p}$ denotes the usual Sobolev space of functions which, together with their (generalised) first derivatives, are in L^p, the underlying measure space being $\mathbf{R}^n$ or $\mathbf{T}^n$. Theorem 4 of [6] (p. 133) shows that $\bar{W}$ is of C-type p_* where $p_*=\min(p, 2)$, but this can be improved using our main theorem here (Theorem 2) to show that $\bar{W}$ is of C-type $p_{**}=2p/|p-2|$. This result is sharp in that $\bar{W}$ is not of C-type q for any $q>p_{**}$ since ([6], p. 135) the complex interpolation spaces $\bar{W}_{[\alpha]}$ are not L^q-K spaces for any $q>p_{**}$.

Remark 1.1. If $\bar{A}$ is of C-type q then $\bar{A}$ is of C-type r for all r, $0<r\leqq q$. This is an immediate consequence of the equivalence of the expressions

$$\left(\int_0^\infty [K(t, b; \bar{A})/K(t, a; \bar{A})]^q \, dt/t\right)^{1/q}$$

and

$$\left(\sum_{\nu=-\infty}^{\infty} [K(2^\nu, b; \bar{A})/K(2^\nu, a; \bar{A})]^q\right)^{1/q}$$

for any $a, b\in\Sigma(\bar{A})$ and $q\in(0, \infty]$. An analogous remark holds of course concerning the relative C-type of $\bar{A}$ and $\bar{B}$.

Our main theorem shows that couples $\bar{A}$ are of C-type q for appropriate values of q if they satisfy two properties which are relatively easy to verify in many cases. We now describe these properties.

Definition. Let A be a Banach space of (equivalence classes of) measurable functions on a measure space (X, Σ, μ) such that if $f\in A$ and $F\in\Sigma$ then $f\chi_F\in A$ with $\|f\chi_F\|_A\leqq\|f\|_A$. We shall say that A is *q-decomposable* for some $q>0$ if, whenever $f\in A$, $(F_n)_{n=1}^\infty$ is a sequence of disjoint sets in Σ and $(g_n)_{n=1}^\infty$ a sequence of disjointly supported elements in A and $\left(\sum_{n=1}^\infty (\|g_n\|_A/\|f\chi_{F_n}\|_A)^q\right)^{1/q}\leqq 1$, then it follows that $g=\sum_{n=1}^\infty g_n\in A$ with $\|g\|_A\leqq\|f\|_A$. Analogously A is *∞-decomposable* if the above condition holds with the sum replaced by the supremum in the usual way.

Finally if A and B are Banach spaces of measurable functions on possibly different measure spaces with $\|f\chi_F\|_A\leqq\|f\|_A$ as before, then we say that A and B are *relatively q-decomposable* if, for f and $(F_n)_{n=1}^\infty$ as above and $(g_n)_{n=1}^\infty$ disjointly supported functions in B, the estimate $\left(\sum_{n=1}^\infty (\|g_n\|_B/\|f\chi_{F_n}\|_A)^q\right)^{1/q}\leqq 1$ implies that $g=\sum_{n=1}^\infty g_n\in B$ with $\|g\|_B\leqq\|f\|_A$.

Examples. L^p is ∞-decomposable for all $p\in[1, \infty]$, L^q and L^p are relatively ∞-decomposable if $q\leqq p$. If $q>p$ then a simple application of Hölder's inequality shows that L^q and L^p are relatively $\dfrac{q}{q-p}$-decomposable. By related arguments, if A and B are Banach lattices and B is p-convex and A is q-concave ([12], p. 46), then A and B are relatively ∞-decomposable if $q\leqq p$ whereas, for $q>p$, A and B are relatively $\dfrac{q}{q-p}$-decomposable. Of course any pair of Banach lattices A and B are relatively 1-decomposable. The mixed norm space $L^p(l^2)$ is $\dfrac{2p}{|p-2|}$-decomposable. (See Section 5.)

Definition. Let $\bar{A}=(A_0, A_1)$ be a compatible couple of Banach spaces of (equivalence classes of) measurable functions on a measure space (X, Σ, μ). We say that $\bar{A}$ is a *Holmstedt couple* if, for each $a\in\Sigma(\bar{A})$, there exists a family of measurable sets

$(E_t)_{t>0}$ such that $E_s \subset E_t$ for $s \leqq t$ and such that, for a fixed constant $C = C(\bar{A})$ depending only on $\bar{A}$,

$$\|a\chi_{E_t}\|_{A_0} + t\|a\chi_{X \setminus E_t}\|_{A_1} \leqq CK(t, a; \bar{A})$$

for all $t>0$. (Note that the choice of sets E_t may depend on a.)

Examples. From Holmstedt's formula [10] $K(t, f; L^{p_0}, L^{p_1}) \sim \left(\int_0^{t^\alpha} f^*(s)^{p_0} ds\right)^{1/p_0} + t\left(\int_{t^\alpha}^\infty f^*(s)^{p_1} ds\right)^{1/p_1}$ it follows that (L^{p_0}, L^{p_1}) is a Holmstedt couple if the underlying measure space is non-atomic. The case when $p_1 = \infty$ follows a little less obviously from the K-functional formulae of Peetre and Krée ([1], p. 109). (For our purposes here it suffices to consider only non-atomic measure spaces or even simply $(0, \infty)$ equipped with Lebesgue measure, since all corollaries of Theorem 2 below concerning couples of rearrangement invariant spaces can be extended to the case of arbitrary σ-finite underlying measure spaces from the case of $(0, \infty)$ using the operators constructed by A. P. Calderón in [4], (p. 277, Lemma 2.) Using a "Stein—Weiss transformation" (see e.g. [5], p. 234) one can also deduce that $(L^{p_0}_{w_0}, L^{p_1}_{w_1})$ (weighted L^p spaces) is a Holmstedt couple, again under the assumption that the measure space is non atomic. From Lemmata 4.3 and 4.5 in [14] it follows that a large class of quasilinearizable couples are Holmstedt couples. See also [18].

For our final example let us suppose that A is a Banach space of measurable functions such that if $f \in A$ and g is measurable and $|g(x)| \leqq f(x)$ for a.e. x then $g \in A$ with $\|g\|_A \leqq \|f\|_A$. Let σ be a positive measurable function on the same measure space and define $A_\sigma = \{f \mid f\sigma \in A\}$ with norm $\|f\|_{A_\sigma} = \|f\sigma\|_A$. It is easy to check that, for the couple (A, A_σ),

$$K(t, f; A, A_\sigma) \leqq \|\chi_{\{t\sigma>1\}} f\|_A + t\|\chi_{\{t\sigma \leqq 1\}} f\|_{A_\sigma} \leqq 2K(t, f; A, A_\sigma).$$

This of course shows that (A, A_σ) is a Holmstedt couple. In particular if the underlying measure space is $\mathbf{R}^n \times \mathbf{N}$ and $A = L^p(l^2)$ is defined by

$$\|f(x, k)\|_A = \left(\int_{\mathbf{R}^n} \left(\sum_{k=0}^\infty |f(x, k)|^2\right)^{p/2} \mathrm{d}x\right)^{1/p}$$

and if $\sigma(x, k) = 2^k$ then (A, A_σ) is precisely the couple $\bar{L} = (L^p(l^2_0), L^p(l^2_1))$ studied, for example in [6], p. 133, in connection, with the couple $\bar{W} = (L^p, W^{1,p})$, $1 < p < \infty$.

We can now close this section with the statement of our main theorem. The preceding examples already indicate many of its applications. The proof is given in Section 4.

Theorem. 2. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be mutually closed Holmstedt couples. Let A_j and B_j be relatively q-decomposable for some q, $1 \leqq q \leqq \infty$, $j = 0, 1$. Then $\bar{A}$ and $\bar{B}$ are of relative Calderón type q.*

2. Positive concave functions and couples of weighted L^1 spaces

Let w_0 and w_1 be positive measurable weight functions on a measure space (X, Σ, μ). For our particular needs here we will need to permit each function w_j to assume the value $+\infty$ on a set, which we will denote by E_j, which may have positive measure.

For $j=0, 1$, let $P_j=L^1(w_j d\mu)$ denote the Banach space of measurable functions f on X satisfying

$$\|f\|_{P_j} = \int_X |f(x)| w_j(x)\, d\mu(x) < \infty.$$

Thus of course each $f\in P_j$ satisfies $f(x)=0$ for a.e. $x\in E_j$. The pair $\bar{P}=(P_0, P_1)=(L^1(w_0 d\mu), L^1(w_1 d\mu))$ is clearly a Banach couple. Each $f\in \Sigma(\bar{P})$ vanishes almost everywhere on $E_0\cap E_1$. The calculation of the K-functional for $\bar{P}$ is very easy and yields (very similarly to the much studied case where w_j are finite a.e.) that

$$K(t, f; \bar{P}) = \int_X |f(x)| \min(w_0(x), t w_1(x))\, d\mu(x)$$
$$= \int_{F_0} |f| w_0\, d\mu + \int_Y |f| \min(w_0, t w_1)\, d\mu + t\int_{F_1} |f| w_1\, d\mu$$

where $Y=X\setminus E_0\setminus E_1$, $F_0=E_1\setminus E_0$ and $F_1=E_0\setminus E_1$.

As is well known $K(t, f; \bar{P})$ is a positive concave function ([1], p. 39). Conversely, the following lemma and its corollary show that all positive concave functions on $(0, \infty)$ are of this form.

Lemma 1. *Let $\psi(t)$ be a positive concave function on $(0, \infty)$. Then there exist non negative constants α, β and a positive measure ν on $(0, \infty)$ such that for all $t>0$*

$$\psi(t) = \alpha + \beta t + \int_0^\infty \min(x, t)\, d\nu(x).$$

Proof. This is part of the proof of Lemma 5.4.3. of [1], p. 117. ∎

Corollary. *For each $\psi(t)$ as above there exists a measure space (X, Σ, μ), weight functions w_0, w_1 and a function $f(x)$ on X such that $\psi(t)=K(t, f; \bar{P})$ for all $t>0$.*

Proof. Given $\psi(t)$ let α, β and ν be as in Lemma 1 and let X be the union of $(0, \infty)$ with two additional points which we may conveniently denote by 0 and ∞. We define the measure μ by $\mu(\{0\})=\alpha$, $\mu(\{\infty\})=\beta$ and $\mu(E)=\nu(E)$ for all measurable $E\subset(0, \infty)$. Define w_0 and w_1 by

$$w_0(x) = x \quad x\in(0, \infty)$$
$$w_0(0) = 1, \quad w_0(\infty) = \infty$$
$$w_1(x) = 1 \quad x\in(0, \infty)$$
$$w_1(0) = \infty, \quad w_1(\infty) = 1.$$

Then clearly $F_0=\{0\}$, $F_1=\{\infty\}$ and if $f(x)=1$ for all $x\in X$ then the formulae for $\psi(x)$ and for $K(t,f;\bar{P})$ coincide. ∎

We now turn to the main result of this section.

Theorem 3. *Let* $\bar{P}=(L^1(w_0\,d\mu), L^1(w_1 d\mu))$ *be a Banach couple of weighted* L^1 *spaces on an arbitrary measure space* (X, Σ, μ). *Let* $f, g\in\Sigma(\bar{P})$ *such that for all* $t>0$

$$K(t,\, g;\, \bar{P}) \leqq K(t,\, f;\, \bar{P}). \tag{2.1}$$

Then for each $\varepsilon>0$ *there exists an operator* $T=T_\varepsilon$, $T\colon \bar{P}\to\bar{P}$ *with* $\|T|_{P_j\to P_j}\|\leqq 1+\varepsilon$ *for* $j=0, 1$ *such that* $Tf=g$.

Remark 2.1. This theorem was originally proved by Sedaev and Semenov subject to the condition that $w_j<\infty$ a.e. $(\mu(E_0)=\mu(E_1)=0)$. See [15]. It is of course equivalent to the statement that $\bar{P}$ is an "exact Calderón couple", i.e. all exact interpolation spaces with respect to $\bar{P}$ are exact K spaces.

Proof. We can suppose without loss of generality that the functions f and g are non negative. Since $\lim_{t\to 0} K(t, g)\leqq \lim_{t\to 0} K(t,f)$ and $\lim_{t\to\infty} K(t, g)/t\leqq \lim_{t\to\infty} K(t,f)/t$ we deduce that

$$\int_{F_j} g w_j\, d\mu \leqq \int_{F_j} f w_j\, d\mu, \quad j = 0, 1. \tag{2.2}$$

(Here and in the sequel we use the notation w_j, F_j and Y as defined at the beginning of this section.)

The main step of the proof will be to obtain an operator $W\colon \bar{P}\to\bar{P}$ with norm 1 on P_0 and on P_1 such that $(Wf)\chi_{F_j}=g\chi_{F_j}$, $j=0, 1$ and

$$K(t,\, g\chi_Y;\, \bar{P}) \leqq K\big(t,\, (Wf)\chi_Y;\, \bar{P}\big) \quad \text{for all} \quad t > 0. \tag{2.3}$$

Once W has been constructed the proof can be completed as follows: Since w_0 and w_1 are finite on Y the inequality (2.3) together with the Sedaev—Semenov theorem in its original form [15] implies that for each $\varepsilon>0$ there exists an operator $V=V_\varepsilon$ of norm less than $1+\varepsilon$ on P_0 and on P_1 which maps $(Wf)\chi_Y$ to $g\chi_Y$ and leaves functions supported on $F_0\cup F_1$ unchanged. The desired operator T is thus given by $T=VW$.

We shall define the operator W, acting on each $h\in\Sigma(\bar{P})$, by

$$W(h\chi_Y) = h\chi_Y \tag{2.4}$$

and

$$W(h\chi_{F_j}) = \int_{F_j} h w_j\, d\mu (g\chi_{F_j}+g_j)\Big/\int_{F_j} f w_j\, d\mu, \quad j = 0, 1 \tag{2.5}$$

provided $\int_{F_j} f w_j d\mu\neq 0$. If $\int_{F_j} f w_j d\mu=0$ then we take $W(h\chi_{F_j})=0$. (In this case (2.2) implies that $g(x)=0$ for a.e. $x\in F_j$.) The functions g_j in (2.5) are non negative and supported on Y. Their precise form will be given below. In particular they must satisfy the equations

$$\int_{F_j} gw_j\,d\mu+\int_Y g_jw_j\,d\mu = \int_{F_j} fw_j\,d\mu, \quad j = 0, 1 \tag{2.6}$$

from which it follows that W has norm 1 on P_0 and on P_1.

Define the numbers α_0 and α_1 by

$$\alpha_j = \int_{F_j} fw_j\,d\mu-\int_{F_j} gw_j\,d\mu \quad (j = 0, 1).$$

Then from (2.1) we have that for all $t>0$

$$\int_Y g\min(w_0, tw_1)\,d\mu \leqq \alpha_0+\int_Y f\min(w_0, tw_1)\,d\mu+t\alpha_1.$$

If $\alpha_j=0$ for either $j=0$ or $j=1$ we take $g_j=0$ and (2.6) is satisfied. Otherwise $\alpha_j>0$ by (2.2) and we can choose a positive number t_j such that

$$t_j^{-j}\int_Y g\min(w_0, t_jw_1)\,d\mu \leqq \alpha_j.$$

Let $\Omega_0=\{x\in Y|w_0(x)\leqq t_0w_1(x)\}$ and $\Omega_1=\{x\in Y|w_0(x)\geqq t_1w_1(x)\}$. If both α_0 and α_1 are strictly positive then we can choose t_0 and t_1 as above so that $t_0<t_1$ and consequently Ω_0 and Ω_1 are disjoint.

We can now define the functions g_j precisely for $j=0, 1$ also in the case $\alpha_j>0$ as follows:

Clearly $\int_{\Omega_j} gw_jd\mu\leqq\alpha_j$. If the integral is strictly positive then g_j is defined by $g_j=\lambda_j^{-1}g\chi_{\Omega_j}$ where $\lambda_j=\int_{\Omega_j} gw_jd\mu/\alpha_j\leqq 1$. Alternatively, if the integral vanishes so that $\lambda_j=0$ and $g(x)=0$ for a.e. $x\in\Omega_j$, we can take g_j to be any non negative function in P_j supported on Ω_j normalised so that here, as in the case $\lambda_j>0$,

$$\int_{\Omega_j} g_jw_j\,d\mu = \alpha_j.$$

Consequently (2.6) is satisfied in all cases.

A small problem may arise in the construction of g_j as above if $\mu(\Omega_j)=0$. This is readily overcome by enlarging the measure space appropriately. For example we may add an atom $\tilde{\Omega}_j$ and define $\mu(\tilde{\Omega}_j)=1$, $g(\tilde{\Omega}_j)=f(\tilde{\Omega}_j)=0$, $w_0(\tilde{\Omega}_j)=t_jw_1(\tilde{\Omega}_j)=1$ and $g_j(\tilde{\Omega}_j)=\alpha_j/w_j(\tilde{\Omega}_j)$.

Finally in order to check that (2.3) holds for $(Wf)\chi_Y=f\chi_Y+g_0+g_1$, we examine the function $\Phi(t)$ defined by

$$\begin{aligned}\Phi(t) &= K(t, (Wf)\chi_Y)-K(t, g\chi_Y) = K(t, f+g_0+g_1)-K(t, g)-\alpha_0-t\alpha_1\\ &= [K(t, f)-K(t, g)]+[K(t, g_0)-\alpha_0]+[K(t, g_1)-t\alpha_1].\end{aligned}$$

The first bracketed term is non negative by (2.1) for all t. The second bracketed term is zero for $t=t_0$ and so is non negative for all $t\geqq t_0$. The third bracketed term is zero for $t=t_1$ and so is non negative for all $t\leqq t_1$. Thus $\Phi(t)\geqq 0$ for all $t\in[t_0, t_1]$.

If $\alpha_0=0$ we see similarly that $\Phi(t)\geqq 0$ for all $t\in(0, t_1]$ and if $\alpha_1=0$ then $\Phi(t)\geqq 0$ for all $t\in[t_0, \infty)$. If $\alpha_0=\alpha_1=0$ then $\Phi(t)\geqq 0$ for all $t>0$.

If $\alpha_0>0$ then for each $t\leqq t_0$ we can write $\Phi(t)/t$ in the form

$$\Phi(t)/t = \int_Y (f+g_0+g_1-g)\min(w_0/t, w_1)\,d\mu$$

$$= \int_{\Omega_0}(f+(1-\lambda_0)g_0)\min(w_0/t, w_1)\,d\mu+\int_{Y\setminus\Omega_0}(f+g_1-g)\min(w_0/t, w_1)\,d\mu.$$

The first integral has a non negative integrand and so is clearly a non increasing function of t. The second integral is constant for all $t\leqq t_0$ since $\min(w_0/t, w_1)=w_1$ on $Y\setminus\Omega_0$. Since $\Phi(t_0)\geqq 0$ we deduce that $\Phi(t)\geqq 0$ on $(0, t_0]$.

Similarly, if $\alpha_1>0$ then for each $t\geqq t_1$ we have

$$\Phi(t) = \int_{\Omega_1}(f+(1-\lambda_1)g_1)\min(w_0, tw_1)\,d\mu+\int_{Y\setminus\Omega_1}(f+g_0-g)\,w_0\,d\mu.$$

We see that $\Phi(t)$ is non decreasing on $[t_1, \infty)$ and, since $\Phi(t_1)\geqq 0$, it follows that $\Phi(t)\geqq 0$ for all $t\geqq t_1$. Thus we have shown that $\Phi(t)\geqq 0$ for all $t>0$ which establishes (2.3) and completes the proof of the theorem. ∎

3. The proof of Theorem 1

Let $\bar{A}=(A_0, A_1)$ be a Banach couple. We recall that the Gagliardo completion of A_0 is the space $A_0+\infty\cdot A_1$ consisting of all elements $a\in\Sigma(\bar{A})$ such that

$$\|a\|_{A_0+\infty\cdot A_1} = \sup_{t>0} K(t, a; \bar{A}) < \infty$$

and similarly the Gagliardo completion of A_1 is the space $A_1+\infty\cdot A_0$ consisting of all elements $a\in\Sigma(\bar{A})$ such that

$$\|a\|_{A_1+\infty\cdot A_0} = \sup_{t>0} K(t, a; \bar{A})/t < \infty.$$

(See e.g. [7], p. 8 or [1], p. 34.)

Let $B_0=A_0+\infty\cdot A_1$ and $B_1=A_1+\infty\cdot A_0$. If $v\in B_0$ but $v\notin B_1$ then we use the notation $\|v\|_{B_1}=\infty$ and in particular for each $t\geqq 0$ we write

$$\min(\|v\|_{B_0}, t\|v\|_{B_1}) = \|v\|_{B_0}. \tag{3.1}$$

Similarly if $v\in B_1$ but $v\notin B_0$ then we write $\|v\|_{B_0}=\infty$ and

$$\min(\|v\|_{B_0}, t\|v\|_{B_1}) = t\|v\|_{B_1}. \tag{3.2}$$

Theorem 1 will follow readily from the following theorem.

Theorem 4. *Let $\bar{A}=(A_0, A_1)$ be an arbitrary Banach couple and let $\bar{B}=(B_0, B_1)$ where $B_0=A_0+\infty\cdot A_1$, $B_1=A_1+\infty\cdot A_0$. Let $a\in\Sigma(\bar{A})$. Then for each $\varepsilon>0$ there* *exists a sequence of elements* $(v_\nu)_{\nu=-\infty}^{\infty}$ *in* $\Sigma(\bar{A})$, *such that*

(3.3) $$v_\nu \in A_0 \cap A_1 \text{ for all but at most two values of } \nu,$$

(3.4) $$\sum_{\nu=-\infty}^{\infty} v_\nu = a$$

where the series converges with respect to the norm of $\Sigma(\bar{A})$, *and*

(3.5) $$\sum_{\nu=-\infty}^{\infty} \min(\|v_\nu\|_{B_0}, t\|v_\nu\|_{B_1}) \leqq 8(1+\varepsilon)K(t, a; \bar{A})$$

for all $t>0$. *If* $v_\nu \notin B_0 \cap B_1$ *then the* ν*-th term of this series is defined by* (3.1) *or* (3.2).

Corollary. *If* $\lim_{t\to 0} K(t, a; \bar{A})=0$ *and* $\lim_{t\to\infty} K(t, a; \bar{A})/t=0$ *then each* $v_\nu \in B_0 \cap B_1$ *and so* (3.4) *defines an admissible representation of a with respect to the couple* $\bar{B}$ (see [7], p. 6).

Proof. Fix a non zero element a of $\Sigma(\bar{A})$ and write $K(t)=K(t, a; \bar{A})$ for all $t>0$.

Define

$$n_0 = \lim_{t\to 0} K(t), \quad N_0 = \|a\|_{B_1} = \lim_{t\to 0} K(t)/t,$$

$$n_\infty = \lim_{t\to\infty} K(t)/t, \quad N_\infty = \|a\|_{B_0} = \lim_{t\to\infty} K(t).$$

(N_0 and N_∞ may of course be infinite.)

We now construct a non decreasing sequence $(t_\nu)_{\nu=-\infty}^{\infty}$ with $0 \leqq t_\nu \leqq \infty$ for each ν. It depends on a fixed parameter $r>1$. For our purposes in this section the optimal choice for r is $r=3$ but for later applications we prefer to obtain estimates for general r.

We choose $t_0=1$ and let t_1 be the smallest number such that both of the inequalities

$$K(t_1) \geqq rK(t_0) \quad \text{and} \quad rK(t_1)/t_1 \leqq K(t_0)/t_0$$

are satisfied. The continuity and monotonicity properties of $K(t)$ ensure that $t_1>t_0$ and that one of the above inequalities is in fact an equality. If there does not exist any t_1 in $(0, \infty)$ satisfying both of the above inequalities we define $t_1=\infty$ (and also $t_\nu=\infty$ for all $\nu>1$). We obtain t_2 from t_1 exactly as t_1 was obtained from t_0. In fact, inductively, for each $\nu>0$, t_ν is defined as the smallest finite positive number, if it exists, which satisfies both the inequalities.

(3.6) $$K(t_\nu) \geqq rK(t_{\nu-1})$$

(3.7) $$rK(t_\nu)/t_\nu \leqq K(t_{\nu-1})/t_{\nu-1}.$$

As before, if no finite value for t_ν satisfies (3.6) and (3.7) then we put $t_\nu=\infty$. Again, for $t_\nu<\infty$ one of (3.6) and (3.7) must be an equality. If $t_\nu=\infty$ then we also take $t_\mu=\infty$ for all $\mu>\nu$. We let ν_∞ denote the first (i.e. minimal) integer ν such that $t_\nu=\infty$. Alternatively, we define $\nu_\infty=+\infty$ if $t_\nu<\infty$ for all integers $\nu \geqq 0$.

We now define t_ν for $\nu<0$. We take t_{-1} to be the largest number satisfying both

$$K(t_0) \geqq rK(t_{-1}) \quad \text{and} \quad rK(t_0)/t_0 \leqq K(t_{-1})/t_{-1}$$

and similarly by induction, given t_ν we obtain $t_{\nu-1}$ as the largest number satisfying both (3.6) and (3.7). If no such number exists then we take $t_\nu=0$ and $t_\mu=0$ for all $\mu<\nu$. We let ν_0 be the first (i.e. maximal) integer ν such that $t_\nu=0$. Alternatively we write $\nu_0=-\infty$ if $t_\nu>0$ for all integers ν.

In summary the sequence $(t_\nu)_{\nu=-\infty}^{\infty}$ has the following properties:

(3.8) For each ν such that $\nu_0+1<\nu<\nu_\infty$, $t_{\nu-1}<t_\nu$ and the inequalities (3.6) and (3.7) hold, one of them being an equality.

(3.9) If $\nu_\infty=+\infty$ then $\lim_{\nu\to+\infty} t_\nu=+\infty$, and if $\nu_0=-\infty$ then $\lim_{\nu\to-\infty} t_\nu=0$.

(3.10) If $\nu_\infty<\infty$ then either

(3.10.1) $K(t)<rK(t_{\nu_\infty-1})$ for all $t>0$ implying that $n_\infty=0$ and $N_\infty<\infty$ $(a\in B_0)$ or

(3.10.2) $rK(t)/t>K(t_{\nu_\infty-1})/t_{\nu_\infty-1}$ for all $t>0$ implying that $n_\infty>0$ and $N_\infty=\infty$.

(3.11) If $\nu_0>-\infty$ then either

(3.11.1) $K(t_{\nu_0+1})<rK(t)$ for all $t>0$ implying that $n_0>0$ and $N_0=\infty$, or

(3.11.2) $rK(t_{\nu_0+1})/t_{\nu_0+1}>K(t)/t$ for all $t>0$ implying that $n_0=0$ and $N_0<\infty$ $(a\in B_1)$.

Having chosen $\varepsilon>0$ we now let $a=a_\nu+a'_\nu$ for each ν such that $\nu_0<\nu<\nu_\infty$, where $a_\nu\in A_0$, $a'_\nu\in A_1$ and

$$\|a_\nu\|_{A_0}+t_\nu\|a'_\nu\|_{A_1} \leqq (1+\varepsilon)K(t_\nu).$$

For each ν such that $\nu_0+1<\nu<\nu_\infty$ we define

$$v_\nu = a_\nu-a_{\nu-1} = a'_{\nu-1}-a'_\nu.$$

If $\nu=\nu_\infty<\infty$ then we define

$$v_{\nu_\infty} = a-a_{\nu_\infty-1} = a'_{\nu_\infty-1}$$

and

$$v_\nu = 0 \quad \text{for all} \quad \nu > \nu_\infty.$$

If $\nu=\nu_0+1>-\infty$ then we define

$$v_{\nu_0+1} = a-a'_{\nu_0+1} = a_{\nu_0+1}$$

and

$$v_\nu = 0 \quad \text{for all} \quad \nu \leqq \nu_0.$$

Note that $v_\nu\in A_0\cap A_1$ except possibly when $\nu=\nu_\infty$ or $\nu=\nu_0+1$. If $\nu_0+1<\nu<\nu_\infty$

(3.12) $$\|v_\nu\|_{A_0} \leqq \|a_\nu\|_{A_0}+\|a_{\nu-1}\|_{A_0} \leqq (1+\varepsilon)[K(t_\nu)+K(t_{\nu-1})]$$

and

$$\|v_\nu\|_{A_1} \leqq \|a'_\nu\|_{A_1} + \|a'_{\nu-1}\|_{A_1} \leqq (1+\varepsilon)[K(t_\nu)/t_\nu + K(t_{\nu-1})/t_{\nu-1}]. \tag{3.13}$$

If $\nu=\nu_\infty<\infty$ then

$$\|v_{\nu_\infty}\|_{A_1} = \|a'_{\nu_\infty-1}\|_{A_1} \leqq (1+\varepsilon)K(t_{\nu_\infty-1})/t_{\nu_\infty-1}. \tag{3.14}$$

In this case either (3.10.1) holds and so $a\in B_0$ with

(3.14.1)

$$\|v_{\nu_\infty}\|_{B_0} \leqq \|a\|_{B_0} + \|a_{\nu_\infty-1}\|_{B_0} \leqq rK(t_{\nu_\infty-1}) + \|a_{\nu_\infty-1}\|_{A_0} \leqq (r+1+\varepsilon)K(t_{\nu_\infty-1}),$$

or (3.10.2) holds and so, using (3.14),

$$\|v_{\nu_\infty}\|_{A_1} \leqq r(1+\varepsilon)K(t)/t \quad \text{for all} \quad t>0. \tag{3.14.2}$$

If $\nu=\nu_0+1>-\infty$ then

$$\|v_{\nu_0+1}\|_{A_0} = \|a_{\nu_0+1}\|_{A_0} \leqq (1+\varepsilon)K(t_{\nu_0+1}). \tag{3.15}$$

In this case either (3.11.1) holds and so, by (3.15),

$$\|v_{\nu_0+1}\|_{A_0} \leqq r(1+\varepsilon)K(t) \quad \text{for all} \quad t>0, \tag{3.15.1}$$

or (3.11.2) holds and so $a\in B_1$ and

$$\begin{aligned} \|v_{\nu_0+1}\|_{B_1} &\leqq \|a\|_{B_1} + \|a'_{\nu_0+1}\|_{B_1} \\ &\leqq rK(t_{\nu_0+1})/t_{\nu_0+1} + (1+\varepsilon)K(t_{\nu_0+1})/t_{\nu_0+1} \\ &= (r+1+\varepsilon)K(t_{\nu_0+1})/t_{\nu_0+1}. \end{aligned} \tag{3.15.2}$$

(Note that, under the hypotheses of the corollary, $n_0=n_\infty=0$ so that (3.10.1) and (3.11.2) hold. This implies that $v_\nu\in B_0\cap B_1$ for $\nu=\nu_\infty$, $\nu=\nu_0+1$ and indeed for all integers ν, as required.)

We now verify that $\Sigma_{\nu=-\infty}^{\infty} v_\nu=a$ where the series converges with respect to the norm of $\Sigma(\bar{A})$. This will follow from the two equalities:

$$\sum_{\nu\leqq 0} v_\nu = a_0, \quad \sum_{\nu\geqq 1} v_\nu = a'_0.$$

Indeed, if $\nu_0=-\infty$ then $n_0=0$ and $\lim_{\nu\to-\infty}\|a_\nu\|_{A_0}=0$. Thus $\Sigma_{\nu\leqq 0}v_\nu=a_0-\lim_{\nu\to-\infty}a_\nu=a_0$. However, if $\nu_0>-\infty$ then $\Sigma_{\nu\leqq 0}v_\nu=a_{\nu_0+1}+\sum_{\nu_0+2\leqq\nu\leqq 0}a_\nu-a_{\nu-1}=a_0$. By exactly analogous reasoning $\Sigma_{\nu\geqq 1}v_\nu=a'_0$.

It remains only to establish the estimate (3.5). We fix $t\in(0,\infty)$ and use the notation

$$m_\nu = \min(\|v_\nu\|_{B_0}, t\|v_\nu\|_{B_1}).$$

(Recall that m_ν may be defined by (3.1) or (3.2) when $\nu=\nu_0+1$ or $\nu=\nu_\infty$.)

We choose an integer ν_* such that $t_{\nu_*-1} \leqq t \leqq t_{\nu_*}$. We first consider the case when $\nu_0+1<\nu_*<\nu_\infty$. Thus by (3.8) we must have either

$$K(t_{\nu_*}) = rK(t_{\nu_*-1}) \tag{3.16}$$

or

$$rK(t_{\nu_*})/t_{\nu_*} = K(t_{\nu_*-1})/t_{\nu_*-1}. \tag{3.17}$$

If (3.16) holds then, by (3.12),

$$m_{\nu_*} \leqq (r+1)(1+\varepsilon)K(t_{\nu_*-1}).$$

If (3.17) holds then (3.13) can be applied to give

$$m_{\nu_*} \leqq t(1+\varepsilon)(r+1)K(t_{\nu_*})/t_{\nu_*}. \tag{3.18}$$

In both of these cases we deduce that

$$m_{\nu_*} \leqq (r+1)(1+\varepsilon)K(t). \tag{3.19}$$

For each ν such that $\nu_0+1<\nu<\nu_*$ we obtain from (3.12) and repeated applications of (3.6) (see (3.8)) that

$$m_\nu \leqq (1+\varepsilon)\left(1+\frac{1}{r}\right)K(t_\nu) \leqq \left(1+\frac{1}{r}\right)(1+\varepsilon)\left(\frac{1}{r}\right)^{\nu_*-1-\nu} K(t_{\nu_*-1}). \tag{3.20}$$

If $\nu_0>-\infty$ then for $\nu=\nu_0+1$ by (3.15) and (3.6)

$$m_{\nu_0+1} \leqq (1+\varepsilon)K(t_{\nu_0+1}) \leqq (1+\varepsilon)\left(\frac{1}{r}\right)^{\nu_*-1-(\nu_0+1)} K(t_{\nu_*-1}). \tag{3.21}$$

From (3.20) and (3.21) we deduce that for all $\nu<\nu_*$, whether or not $\nu_0>-\infty$,

$$m_\nu \leqq \left(1+\frac{1}{r}\right)(1+\varepsilon)\left(\frac{1}{r}\right)^{\nu_*-1-\nu} K(t).$$

Consequently

$$\sum_{\nu<\nu_*} m_\nu \leqq \left(1+\frac{1}{r}\right)(1+\varepsilon)K(t)\left[1\Big/\left(1-\frac{1}{r}\right)\right] = (1+\varepsilon)\left(\frac{r+1}{r-1}\right)K(t). \tag{3.22}$$

By an almost identical argument we can show that

$$\sum_{\nu>\nu_*} m_\nu \leqq (1+\varepsilon)\left(\frac{r+1}{r-1}\right)K(t). \tag{3.23}$$

We deduce (3.5) from (3.19), (3.22) and (3.23) and by choosing $r=3$.

To complete the proof of (3.5) we must consider the two cases when

$$\nu_* = \nu_0+1 \quad (0<t<t_{\nu_0+1}) \quad \text{or}$$

$$\nu_* = \nu_\infty \quad (t_{\nu_\infty-1}<t<\infty).$$

We shall treat the first of these, leaving the reader to construct the closely parallel argument which treats the second.

Since at least one of the estimates (3.15.1) or (3.15.2) must hold we deduce that

either $\quad m_{\nu_0+1} \leqq r(1+\varepsilon)K(t) \quad$ or $\quad m_{\nu_0+1} \leqq t(r+1+\varepsilon)K(t_{\nu_0+1})/t_{\nu_0+1}$
so that

$$m_{\nu_0+1} \leqq (r+1+\varepsilon)K(t).$$

For $\nu_0+1<\nu<\nu_\infty$ we obtain by (3.13) and repeated applications of (3.7) that

$$m_\nu \leqq t(1+\varepsilon)\left(1+\frac{1}{r}\right)K(t_{\nu-1})/t_{\nu-1} \leqq \left(1+\frac{1}{r}\right)(1+\varepsilon)t\left(\frac{1}{r}\right)^{\nu-1-\nu_0-1} K(t_{\nu_0+1})/t_{\nu_0+1}.$$

If $\nu_\infty<\infty$ then by (3.14) and (3.7)

$$m_{\nu_\infty} \leqq t(1+\varepsilon)K(t_{\nu_\infty-1})/t_{\nu_\infty-1} \leqq t(1+\varepsilon)\left(\frac{1}{r}\right)^{\nu_\infty-1-\nu_0-1} K(t_{\nu_0+1})/t_{\nu_0+1}.$$

Thus, regardless of whether $\nu_\infty<\infty$ or $\nu_\infty=\infty$,

$$\sum_{\nu>\nu_0+1} m_\nu \leqq \left(1+\frac{1}{r}\right)(1+\varepsilon)\,tK(t_{\nu_0+1})/t_{\nu_0+1}\sum_{\nu>\nu_0+1}\left(\frac{1}{r}\right)^{\nu-\nu_0-2}$$

$$\leqq (1+\varepsilon)\left(\frac{r+1}{r-1}\right)K(t).$$

Combining these estimates and again taking $r=3$ we obtain $\sum_{\nu=-\infty}^{\infty} m_\nu \leqq 6(1+\varepsilon)K(t)$ which of course implies (3.5).

This completes the proof of Theorem 4 and its corollary. ▮

We are finally ready to prove Theorem 1. At this stage our argument closely resembles the corresponding part of the proof of Theorem 4.8 of [7], p. 38.

For each positive concave function $\psi_n(t)$ we construct a measure μ_n and weight functions $w_0^n(s)$ and $w_1^n(s)$ on $[0, \infty]$ as in Lemma 1 and its corollary.

Take $X=[0, \infty]\times\mathbf{N}$ i.e. X is the union of disjoint copies I_n of $[0, \infty]$ for $n=1, 2, \ldots$. Let $w_j(s, n)=w_j^n(s)$, $j=0, 1$ for each $(s, n)\in X$ and let Σ be the σ-algebra of subsets E whose intersections with I_n are Borel subsets of $[0, \infty]$ for each n. For each such $E=\bigcup_{n=1}^{\infty}(E\cap I_n)\in\Sigma$ define $\mu(E)=\sum_{n=1}^{\infty}\mu_n(E\cap I_n)$. Then for the function f on X defined by $f(s, n)=1$ for all $(s, n)\in X$ we obtain that

$$K(t, f\chi_{I_n}; \bar{P}) = \psi_n(t) \quad \text{for each} \quad n\in\mathbf{N}$$

and

$$K(t, f; \bar{P}) = \sum_{n=1}^{\infty}\psi_n(t)$$

where $\bar{P}=(L^1(w_0\,d\mu), L^1(w_1\,d\mu))$ on (X, Σ, μ).

We recall that $a \in \Sigma(\bar{A})$ satisfies $K(t, a; \bar{A}) \leqq \sum_{n=1}^{\infty} \psi_n(t)$ and let $a = \sum_{\nu=-\infty}^{\infty} v_\nu$ be the decomposition of a obtained in Theorem 4. Let (X_0, μ) be the measure space of all integers ν for which $v_\nu \neq 0$, where here μ denotes counting measure. We define w_0 and w_1 on X_0 by $w_j(\nu) = \|v_\nu\|_{B_j}$. (Note that $w_j(\nu)$ may be infinite for some values of ν.)

If g is defined by $g(\nu) = 1$ for all $\nu \in X_0$ we obtain, taking $P_j = L^1(w_j d\mu)$ on the measure space which is the (disjoint) union of X and X_0, that

$$K(t, g; \bar{P}) = \sum_{\nu=-\infty}^{\infty} \min(\|v_\nu\|_{B_0}, t\|v_\nu\|_{B_1})$$
$$\leqq 8(1+\varepsilon) K(t, a; \bar{A}) \leqq 8(1+\varepsilon) K(t, f; \bar{P}).$$

Invoking Theorem 3 we obtain an operator T on $\bar{P}$ of norm less than $8(1+2\varepsilon)$ on P_0 and on P_1 such that $Tf = g$. We also have a "canonical" operator S from $\bar{P}$ to $\bar{B}$ (where $\bar{B} = (A_0 + \infty \cdot A_1, A_1 + \infty \cdot A_0)$) defined by

$$Sh = S(h\chi_{X_0}) = \sum_{\nu=-\infty}^{\infty} h(\nu) v_\nu.$$

Clearly $\|S|_{P_j \to B_j}\| = 1$ for $j = 0, 1$, and $Sg = \sum_{\nu=-\infty}^{\infty} v_\nu = a$. Now let $a_n = ST(f\chi_{I_n})$. Since the series $\sum_{n=1}^{\infty} f\chi_{I_n}$ converges to f in $\Sigma(\bar{P})$, $\sum_{n=1}^{\infty} a_n$ converges to $STf = a$ in $\Sigma(\bar{B}) = \Sigma(\bar{A})$ and $K(t, a_n; \bar{B}) = K(t, a_n; \bar{A}) \leqq 8(1+2\varepsilon) K(t, f\chi_{I_n}; \bar{P}) = 8(1+2\varepsilon) \cdot \cdot \psi_n(t)$. ∎

Remark 3.1. By a part of the above argument, for any $f \in \Sigma(\bar{P})$ and $a \in \Sigma(\bar{A})$ such that $K(t, a; \bar{A}) \leqq K(t, f; \bar{P})$ there exists a bounded operator $ST: \bar{P} \to \bar{B}$ such that $STf = a$. Thus $\bar{P}$ and $\bar{B}$ are relative Calderón couples and Theorem 4.4 of [3] shows that $\bar{B}$ has property (P) as claimed in Section 1.

4. The proof of Theorem 2

Suppose that $\bar{A}$ and $\bar{B}$ satisfy the hypotheses of the theorem (Section 1) and that a and b are arbitrary elements of $\Sigma(\bar{A})$ and $\Sigma(\bar{B})$ respectively which satisfy

$$\left(\int_0^\infty [K(t, b; \bar{B})/K(t, a; \bar{A})]^q \, dt/t\right)^{1/q} \leqq 1. \tag{4.1}$$

Clearly the proof of the theorem can be accomplished by constructing a linear operator T mapping A_j into B_j with norm bounded by a constant depending only on $\bar{A}$ and $\bar{B}$, $j = 0, 1$ such that $b = Ta$.

Our first step in the construction of T is to apply a variant of the proof of Theorem 4 in the previous section to $a \in \Sigma(\bar{A})$. For fixed $r > 1$ we obtain a sequence $(t_\nu)_{\nu=-\infty}^{\infty}$ of elements in $[0, \infty]$ satisfying (3.6) and (3.7) for $K(t) = K(t, a) = K(t, a; \bar{A})$. [In the sequel we shall frequently omit $\bar{A}$ or $\bar{B}$ in the notation for K-functionals when this will not introduce any ambiguity.] We define ν_0 and ν_∞ as in Section 3 and denote by Γ the set of all integers ν which satisfy $\nu_0+1<\nu<\nu_\infty$. Note that, depending on the behaviour of $K(t, a)$ and the choice of r, Γ may contain all the integers or be a strict or even empty subset of them.

Since $\bar{A}$ and $\bar{B}$ are Holmstedt couples, there exist measurable subsets E_{t_ν}, F_{t_ν} of the respective underlying measure spaces such that

$$K(t_\nu, a; \bar{A}) \leqq \|a\chi_{E_{t_\nu}}\|_{A_0}+t_\nu\|a(1-\chi_{E_{t_\nu}})\|_{A_1} \leqq C_1K(t_\nu, a; \bar{A}) \tag{4.2}$$

$$K(t_\nu, b; \bar{B}) \leqq \|b\chi_{F_{t_\nu}}\|_{B_0}+t_\nu\|b(1-\chi_{F_{t_\nu}})\|_{B_1} \leqq C_2K(t_\nu, b; \bar{B}) \tag{4.3}$$

for each integer ν, $\nu_0<\nu<\nu_\infty$ and for constants $C_1=C(\bar{A})$ and $C_2=C(\bar{B})$ depending only on $\bar{A}$ and $\bar{B}$ respectively. Thus we can write $a=a_\nu+a'_\nu$, $b=b_\nu+b'_\nu$ where $a_\nu=a\chi_{E_{t_\nu}}$, $b_\nu=b\chi_{F_{t_\nu}}$. As in Section 3, we let $v_\nu=a_\nu-a_{\nu-1}$, and analogously $w_\nu=b_\nu-b_{\nu-1}$ for all $\nu\in\Gamma$. If there are finite values of ν satisfying $\nu\geqq\nu_\infty$ or $\nu\leqq\nu_0+1$ then we define v_ν, and analogously w_ν, by the same formulae as used in Section 3. Since $E_{t_{\nu-1}}\subset E_{t_\nu}$ the sets G_ν defined by $G_\nu=E_{t_\nu}\setminus E_{t_{\nu-1}}$ are pairwise disjoint and $v_\nu=a\chi_{G_\nu}$ for $\nu_0+1<\nu<\nu_\infty$. The sets defined by $H_\nu=F_{t_\nu}\setminus F_{t_{\nu-1}}$ have a similar property and $w_\nu=b\chi_{H_\nu}$ for $\nu_0+1<\nu<\nu_\infty$. In order to be able to extend the notation $v_\nu=a\chi_{G_\nu}$ and $w_\nu=b\chi_{H_\nu}$ to any finite ν which may satisfy $\nu\leqq\nu_0+1$ or $\nu\geqq\nu_\infty$ we define $G_{\nu_0+1}=E_{t_{\nu_0+1}}$ and $H_{\nu_0+1}=F_{t_{\nu_0+1}}$ and let G_{ν_∞} be the complement of $E_{t_{\nu_\infty-1}}$ and H_{ν_∞} the complement of $F_{t_{\nu_\infty-1}}$ in the respective measure spaces. For $\nu\leqq\nu_0$ or $\nu\geqq\nu_\infty+1$, G_ν and H_ν are both taken to be empty sets. The sequences $(G_\nu)_{\nu=-\infty}^{\infty}$ and $(H_\nu)_{\nu=-\infty}^{\infty}$ are thus each pairwise disjoint also for this (possibly) extended range of ν.

Exactly as in Section 3 we have $\sum_{\nu=-\infty}^{\infty} v_\nu=a$ with convergence in $\Sigma(\bar{A})$. Furthermore, since (4.1) implies that $\sup_{t>0} K(t, b)/K(t, a)<\infty$, (cf. Remark 1.1 above or (4.10) below) we can similarly deduce that $\sum_{\nu=-\infty}^{\infty} w_\nu=b$ with convergence in $\Sigma(\bar{B})$.

Let $m_\nu=m_\nu(t)=\min(\|v_\nu\|_{A_0}, t\|v_\nu\|_{A_1})$ as in Section 3. Obvious and slight modifications of the estimates there show that for all $t>0$, if ν_* is defined by $t_{\nu_*-1}\leqq t\leqq t_{\nu_*}$ then

$$m_\nu \leqq C_1r^{1-|\nu-\nu_*|}K(t, a) \quad \text{for all } \nu. \tag{4.4}$$

(C_1 replaces $(1+\varepsilon)$ and the estimates (3.12) and (3.13) are simplified because here $\|v_\nu\|_{v_0}\leqq\|a_\nu\|_{A_0}$ and $\|v_\nu\|_{A_1}\leqq\|a'_{\nu-1}\|_{A_1}$.) Consequently $\sum_{\nu=-\infty}^{\infty} m_\nu\leqq C_1\big(r+2r/(r-1)\big)\cdot K(t, a)$.

Our next step is to define a "thickened" sequence $(u_\nu)_{\nu\in\Gamma}$ from the sequence $(v_\nu)_{\nu=-\infty}^{\infty}$ and obtain estimates from below for $K(t, u_\nu; \bar{A})$. Let β be a fixed positive integer chosen sufficiently large to ensure $C_3>0$ where C_3 is defined by

$$C_3 = [1-2C_1r^{-\beta}/(1-1/r)].$$

For each $\nu\in\Gamma$ define $u_\nu=\sum_{|n|\leqq\beta} v_{\nu+n}$. Then for each such ν

$$a = u_\nu+\sum_{|n|>\beta} v_{\nu+n}$$

so that

$$K(t, a) \leqq K(t, u_\nu)+\sum_{|n|>\beta} K(t, v_{\nu+n})$$

$$\leqq K(t, u_\nu)+\sum_{|n|>\beta} \min(\|v_{\nu+n}\|_{A_0}, t\|v_{\nu+n}\|_{A_1}).$$

Now, if we restrict t to the range $t_{\nu-1}\leqq t\leqq t_\nu$, the estimate (4.4) can be rewritten as

$$m_{\nu+n} \leqq C_1 r^{1-|n|}K(t, a) \quad \text{for all } n.$$

Consequently, for $t_{\nu-1}\leqq t\leqq t_\nu$,

$$K(t, u_\nu; \bar{A}) \geqq [1-2C_1 r^{-\beta}/(1-1/r)]K(t, a; \bar{A}) = C_3 K(t, a; \bar{A}). \tag{4.5}$$

The operator T which we require will be obtained as a sum of three operators $T=T_{-\infty}+T_\Gamma+T_\infty$ where $T_\Gamma=0$ if Γ is empty, $T_{-\infty}=0$ if $\nu_0=-\infty$ and $T_\infty=0$ if $\nu_\infty=\infty$. Furthermore, if Γ is non empty then $T_\Gamma a=\sum_{\nu\in\Gamma} w_\nu$, if ν_0 is finite then $T_{-\infty}a=w_{\nu_0+1}$ and if ν_∞ is finite then $T_\infty a=w_{\nu_\infty}$. Thus in all cases, whatever the behaviour of ν_0, ν_∞ and Γ, $Ta=\sum_{\nu=-\infty}^{\infty} w_\nu=b$. It remains to give precise constructions for each of the operators T_Γ, $T_{-\infty}$ and T_∞ and to show that each of them maps A_j boundedly into B_j for $j=0, 1$.

If Γ is non empty then for each $\nu\in\Gamma$ we define ε_ν by

$$\varepsilon_\nu = \begin{cases} 1 & \text{if } \ K(t_\nu, a) = rK(t_{\nu-1}, a) \\ 0 & \text{otherwise} \end{cases}$$

Note that if $\varepsilon_\nu=0$ then necessarily $K(t_{\nu-1}, a)/t_{\nu-1}=rK(t_\nu, a)/t_\nu$.

For each $\nu\in\Gamma$ let l_ν be a continuous linear functional on $\Sigma(\bar{A})$ such that $l_\nu(u_\nu)=K(t_{\nu-\varepsilon_\nu}, u_\nu; \bar{A})$ and $l_\nu(h)\leqq K(t_{\nu-\varepsilon_\nu}, h; \bar{A})$ for all $h\in\Sigma(\bar{A})$. (The existence of l_ν is of course guarenteed by the Hahn—Banach theorem.) The operator T_Γ is defined by

$$T_\Gamma h = \sum_{\nu\in\Gamma} l_\nu(h\chi_{\cup_{|n|\leqq\beta}G_{\nu+n}})w_\nu/K(t_{\nu-\varepsilon_\nu}, u_\nu)$$

for all $h\in\Sigma(\bar{A})$. Clearly $T_\Gamma a=\sum_{\nu\in\Gamma} w_\nu$.

In order to show that T_Γ maps A_j boundedly into B_j for $j=0, 1$ it is convenient to express T_Γ as a sum of operators $T_\Gamma=\sum_{|n|\leqq\beta} T_n$ where T_n is defined by $T_n h=\sum_{\nu\in\Gamma} l_\nu(h\chi_{G_{\nu+n}})w_\nu/K(t_{\nu-\varepsilon_\nu}, u_\nu)$ and to show that each T_n has the required boundedness properties. We observe that T_n maps disjointly supported "pieces" $h\chi_{G_{\nu+n}}$ of h to disjointly supported functions which are scalar multiples of w_ν. This means that we can use the fact that A_j and B_j are relatively q-decomposable, together with estimates for norms of the rank one operators $T_{n\nu}$ defined by

$$T_{n\nu}h = l_\nu(h\chi_{G_{\nu+n}})w_\nu/K(t_{\nu-\varepsilon_\nu}, u_\nu)$$

in order to estimate the norms of T_n. For $j=0, 1$ we have $\|T_{n\nu}h\|_{B_j}\leqq\|w_\nu\|_{B_j}\cdot$

$K(t_{\nu-\varepsilon_\nu}, h\chi_{G_{\nu+n}}; \bar{A})/K(t_{\nu-\varepsilon_\nu}, u_\nu; \bar{A})$. Using (4.3) we see that

$$\|w_\nu\|_{B_0} \leqq \|b\chi_{F_{t_\nu}}\|_{B_0} \leqq C_2 K(t_\nu, b; \bar{B})$$

and

$$\|w_\nu\|_{B_1} \leqq \|b(1-\chi_{F_{t_{\nu-1}}})\|_{B_1} \leqq C_2 t_{\nu-1}^{-1} K(t_{\nu-1}, b; \bar{B}).$$

Combining the last three inequalities with (4.5) we obtain that

$$\|T_{n\nu} h\|_{B_j} \leqq C_2 t_{\nu-j}^{-j} K(t_{\nu-j}, b) t_{\nu-\varepsilon_\nu}^{j} \|h\chi_{G_{\nu+n}}\|_{A_j} / C_3 K(t_{\nu-\varepsilon_\nu}, a).$$

The q-decomposability hypothesis now implies that

$$\|T_n h\|_{B_j} \leqq \frac{C_2}{C_3} \left(\sum\nolimits_{\nu\in\Gamma} \theta_\nu^q\right)^{1/q} \|h\|_{A_j},$$

where

$$\theta_\nu = t_{\nu-j}^{-j} K(t_{\nu-j}, b) / t_{\nu-\varepsilon_\nu}^{-j} K(t_{\nu-\varepsilon_\nu}, a).$$

For $j=0$, in view of the definition of ε_ν,

$$\theta_\nu \leqq r K(t_\nu, b)/K(t_\nu, a). \tag{4.7}$$

Similarly, for $j=1$,

$$\theta_\nu \leqq r K(t_{\nu-1}, b)/K(t_{\nu-1}, a). \tag{4.8}$$

We claim that

$$\left(\sum\nolimits_{\nu\in\Gamma} \theta_\nu^q\right)^{1/q} \leqq r^2/(\log r)^{1/q} \quad \text{for} \quad j=0 \quad \text{and} \quad j=1. \tag{4.9}$$

For $q=\infty$ we in fact obtain the "sharper" estimate $\sup_\nu \theta_\nu \leqq r$ as an immediate consequence of (4.1), (4.7) and (4.8). For $q<\infty$ we first note that, since the sequence $(t_\nu)_{\nu=-\infty}^{\infty}$ satisfies (3.6), it follows necessarily from the concavity of $K(t)$ that $t_\nu \geqq r t_{\nu-1}$ for all $\nu\in\Gamma$. Thus the intervals $I_\nu = (t_\nu/\sqrt{r}, t_\nu\sqrt{r})$ are disjoint. For all $t\in I_\nu$

$$K(t_\nu, a)/\sqrt{r} \leqq K(t, a) \leqq \sqrt{r}\, K(t_\nu, a)$$

and

$$K(t_\nu, b)/\sqrt{r} \leqq K(t, b) \leqq \sqrt{r}\, K(t_\nu, b).$$

Thus, for both $j=0$, $j=1$,

$$\sum\nolimits_{\nu\in\Gamma} \theta_\nu^q = r^q \sum\nolimits_{\nu\in\Gamma} [K(t_\nu, b)/K(t_\nu, a)]^q$$

$$= r^q \sum\nolimits_{\nu\in\Gamma} \int_{I_\nu} [K(t_\nu, b)/K(t_\nu, a)]^q \frac{dt}{t} \Big/ \int_{I_\nu} \frac{dt}{t}$$

$$\leqq r^{2q} \sum\nolimits_{\nu\in\Gamma} \int_{I_\nu} [K(t, b)/K(t, a)]^q \frac{dt}{t} \Big/ \log r$$

$$\leqq r^{2q} \int_0^\infty [K(t, b)/K(t, a)]^q \frac{dt}{t} \Big/ \log r \leqq r^{2q}/\log r$$

which establishes (4.9) and shows that T_r maps A_j into B_j with norm bounded by $C_2(2\beta+1)r/(C_3(\log r)^{1/q})$.

Before proceeding with the description of the operators $T_{-\infty}$ and T_∞ we note that by similar estimates to those above, for any $s>0$ and $r>1$,

$$K(s, b)/K(s, a) \leqq \left[(\log r)^{-1} \int_{s/\sqrt{r}}^{s\sqrt{r}} [rK(t, b)/K(t, a)]^q \, dt/t\right]^{1/q} \leqq r(\log r)^{-1/q}.$$

Choosing the optimal value $e^{1/q}$ for r we obtain that

$$\sup_{s>0} K(s, b)/K(s, a) \leqq (qe)^{1/q}. \tag{4.10}$$

If $\nu_0 > -\infty$ then we construct the operator $T_{-\infty}$ by taking $T_{-\infty}h = l_{-\infty}(h) w_{\nu_0+1}$. Here $l_{-\infty}$ is a linear functional satisfying $l_{-\infty}(a)=\sigma(a)$ and $|l_{-\infty}(h)| \leqq \sigma(h)$ for all $h \in \Sigma(\bar{A})$ where σ is a suitable seminorm on $\Sigma(\bar{A})$. The definition of σ depends on whether $K(t)=K(t, a)$ satisfies (3.11.1) or (3.11.2). In the former case we take $\sigma(h) = \limsup_{t\to 0} K(t, h; \bar{A})/K(t, a)$ and in the latter $\sigma(h) = K(t_{\nu_0+1}, h; \bar{A})/K(t_{\nu_0+1}, a)$. Clearly in both cases $T_{-\infty} a = w_{\nu_0+1}$.

If (3.11.1) holds and $h \in A_1$ then $T_{-\infty} h = 0$. If however $h \in A_0$, then using (3.11.1), (4.3) and (4.10) we have that

$$\begin{aligned} \|T_{-\infty} h\|_{B_0} &\leqq \|w_{\nu_0+1}\|_{B_0} \sigma(h) \\ &\leqq \|b\chi_{F_{t_{\nu_0+1}}}\|_{B_0} \|h\|_{A_0} / r^{-1} K(t_{\nu_0+1}, a) \\ &\leqq [rC_2 K(t_{\nu_0+1}, b)/K(t_{\nu_0+1}, a)] \|h\|_{A_0} \leqq rC_2 (qe)^{1/q} \|h\|_{A_0}. \end{aligned}$$

Alternatively if (3.11.2) holds and $h \in A_0$ then, similarly to the preceding estimates, we obtain that

$$\|T_{-\infty} h\|_{B_0} \leqq \|w_{\nu_0+1}\|_{B_0} \|h\|_{A_0} / K(t_{\nu_0+1}, a) \leqq C_2 (qe)^{1/q} \|h\|_{A_0}.$$

If however $h \in A_1$ then, since $w_{\nu_0+1} = b\chi_{H_{\nu_0+1}}$ and $\bar{B}$ is mutually closed,

$$\begin{aligned} \|T_{-\infty} h\|_{B_1} &\leqq \|b\|_{B_1} t_{\nu_0+1} \|h\|_{A_1} / K(t_{\nu_0+1}, a) \\ &= \sup_{t>0} \frac{K(t, b)}{t} t_{\nu_0+1} \|h\|_{A_1} / K(t_{\nu_0+1}, a) \leqq (qe)^{1/q} r \|h\|_{A_1}, \end{aligned}$$

by (4.10) and (3.11.2).

If $\nu_\infty < \infty$ we define T_∞ by $T_\infty h = l_\infty(h) w_{\nu_\infty}$, where the linear functional l_∞ is defined as above via a seminorm σ. Here we take $\sigma(h) = K(t_{\nu_\infty - 1}, h; \bar{A})/K(t_{\nu_\infty - 1}, a)$ if (3.10.1) holds, or alternatively $\sigma(h) = \limsup_{t\to\infty} K(t, h; \bar{A})/K(t, a)$ if (3.10.2) holds. The verification that T_∞ maps A_j boundedly into B_j for $j=0, 1$ is similar to that for $T_{-\infty}$ and is left to the reader.

This completes the proof of Theorem 2. ∎

5. The couples $\overline{W}=(L^p, W^{1,p})$ and $\overline{L}=(L^p(l_0^2), L^p(l_1^2))$

In this final section we show how Theorem 2 applies to the couples $\overline{W}$ and $\overline{L}$ discussed in Section 1, to show that they both have Calderón type $\frac{2p}{|p-2|}$ for $1<p<\infty$. As in [6], p. 133, we first point out that it is sufficient to show this for $\overline{L}$. The corresponding result for $\overline{W}$ follows from the fact that $\overline{W}$ is a retract of $\overline{L}$ ([1], Theorem 6.4.3 p. 151). $\overline{L}$ is mutually closed (as is every compatible couple of reflexive spaces) and Holmstedt, so it remains only to show that $L^p(l_\alpha^2)$ is $\frac{2p}{|p-2|}$-decomposable for $\alpha=0$ and 1.

To this end we let $f=f(x, k)$ and $g=g(x, k)$ be measurable functions on $X=\mathbf{R}^n\times\mathbf{N}$ and let $(F_n)_{n=0}^\infty$ and $(G_n)_{n=0}^\infty$ be two sequences of disjoint measurable subsets of X. Suppose further that $f\in L^p(l_\alpha^2)$, that g is supported on $\bigcup_{n=0}^\infty G_n$ and that $g\chi_{G^n}\in L^p(l_\alpha^2)$ with $\sum_{n=0}^\infty [\|g\chi_{G_n}\|_{L^p(l_\alpha^2)}/\|f\chi_{F_n}\|_{L^p(l_\alpha^2)}]^{2p/|p-2|}\leqq 1$. It will be convenient to denote the characteristic functions of F_n and G_n by $F_n(x, k)$ and $G_n(x, k)$ respectively.

Suppose first that $p>2$.

$$\|g\|_{L^p(l_\alpha^2)} = \|\Sigma_n gG_n\|_{L^p(l_\alpha^2)} = \|(\Sigma_k|\Sigma_n gG_n 2^{\alpha k}|^2)^{1/2}\|_{L^p}.$$

For each fixed (x, k) the numbers $g(x, k)\ G_n(x, k)2^{\alpha k}$ are non zero for at most one value of n. Therefore $|\Sigma_n gG_n 2^{\alpha k}|^2=\Sigma_n|gG_n 2^{\alpha k}|^2$ and

$$\|g\|_{L^p(l_\alpha^2)} = \|(\Sigma_n\Sigma_k|gG_n 2^{\alpha k}|^2)^{1/2}\|_{L^p}$$

$$\leqq \big(\Sigma_n\|(\Sigma_k|gG_n 2^{\alpha k}|^2)\|_{L^{p/2}}\big)^{1/2} = \big(\Sigma_n\|gG_n\|^2_{L^p(l_\alpha^2)}\big)^{1/2}$$

$$= \big(\Sigma_n\|fF_n\|^2_{L^p(l_\alpha^2)}(\|gG_n\|_{L^p(l_\alpha^2)}/\|fF_n\|_{L^p(l_\alpha^2)})^2\big)^{1/2}$$

$$\leqq \big(\Sigma_n\|fF_n\|^{2s}_{L^p(l_\alpha^2)}\big)^{1/2s}\big(\Sigma_n[\|gG_n\|_{L^p(l_\alpha^2)}/\|fF_n\|_{L^p(l_\alpha^2)}]^{2s'}\big)^{1/2s'}$$

by Hölder's inequality. If we choose $s=p/2$ then $s'=p/(p-2)$ and the series in the second factor is bounded by one and we have

$$\|g\|_{L^p(l_\alpha^2)} \leqq \big(\Sigma_n\|fF_n\|^p_{L^p(l_\alpha^2)}\big)^{1/p} = \Big(\Sigma_n\int(\Sigma_k|fF_n 2^{\alpha k}|^2)^{p/2}\,dx\Big)^{1/p}$$

$$= \Big(\int\Sigma_n(\Sigma_k|fF_n 2^{\alpha k}|^2)^{p/2}\,dx\Big)^{1/p} \leqq \Big(\int(\Sigma_n\Sigma_k|fF_n 2^{\alpha k}|^2)^{p/2}\,dx\Big)^{1/p}$$

$$= \Big(\int(\Sigma_k\Sigma_n F_n^2|f2^{\alpha k}|^2)^{p/2}\,dx\Big)^{1/p} \leqq \Big(\int(\Sigma_k|f2^{\alpha k}|^2)^{p/2}\,dx\Big)^{1/p} = \|f\|_{L^p(l_\alpha^2)}.$$

For $p<2$ the argument is rather similar. The first equality is exactly as before.

$$\|g\|_{L^p(l^2_\alpha)} = \|(\Sigma_n \Sigma_k |gG_n 2^{\alpha k}|^2)^{1/2}\|_{L^p} = \left(\int (\Sigma_n \Sigma_k |gG_n 2^{\alpha k}|^2)^{p/2}\, dx\right)^{1/p}$$

$$\leqq \left(\int \Sigma_n (\Sigma_k |gG_n 2^{\alpha k}|^2)^{p/2}\, dx\right)^{1/p} = \left(\Sigma_n \|gG_n\|^p_{L^p(l^2_\alpha)}\right)^{1/p}$$

$$\leqq (\Sigma_n \|fF_n\|^{ps}_{L^p(l^2_\alpha)})^{1/ps} (\Sigma_n [\|gG_n\|_{L^p(l^2_\alpha)}/\|fF_n\|_{L^p(l^2_\alpha)}]^{ps'})^{1/ps'}.$$

This time we take $s=2/p$ so $s'=2/(2-p)$ and again the second factor is bounded by 1.

$$\|g\|_{L^p(l^2_\alpha)} \leqq (\Sigma_n \|fF_n\|^2_{L^p(l^2_\alpha)})^{1/2} = \left(\Sigma_n \left(\int (\Sigma_k |fF_n 2^{\alpha k}|^2)^{p/2}\, dx\right)^{2/p}\right)^{1/2}$$

$$\leqq \left(\int \left[\Sigma_n ((\Sigma_k |fF_n 2^{\alpha k}|^2)^{p/2})^{2/p}\right]^{p/2} dx\right)^{1/p}$$

(by the integral form of Minkowski's inequality in $l^{2/p}$)

$$= \left(\int [\Sigma_n \Sigma_k |fF_n 2^{\alpha k}|^2]^{p/2}\, dx\right)^{1/p} = \left(\int [\Sigma_k (\Sigma_n F_n^2) |f 2^{\alpha k}|^2]^{p/2}\, dx\right)^{1/p}$$

$$\leqq \left(\int [\Sigma_k |f 2^{\alpha k}|^2]^{p/2}\, dx\right)^{1/p} = \|f\|_{L^p(l^2_\alpha)}.$$

The estimate $\|g\|_{L^p(l^2_\alpha)} \leqq \|f\|_{L^p(l^2_\alpha)}$ establishes the $\dfrac{2p}{|p-2|}$-decomposability of $L^p(l^2_\alpha)$.

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