Curve singularities of finite Cohen—Macaulay type

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Let R be a reduced local Noetherian ring of dimension one, and assume that the integral closure $\tilde{R}$ is finitely generated as an R-module. The main theorem of [13] gives necessary and sufficient conditions in order that R have only finitely many non-isomorphic indecomposable finitely generated torsionfree modules. These necessary and sufficient conditions (listed in (1.2)) were introduced by Drozd and Roiter in [2] and shown to be equivalent to finite CM type for localizations of orders in algebraic number fields. (See also [8] and [5].) Since the non-zero finitely generated torsionfree modules are exactly the maximal Cohen—Macaulay modules, these rings are said to have *finite CM type*. In the geometric case, when R is the local ring of a singular point of an algebraic curve over a field k, these conditions impose stringent conditions on the singularity: Its multiplicity must be less than or equal to three, and when the multiplicity is three there is an additional condition, harder to describe in geometric terms. While this latter condition is easy to test for any specific singularity, it seems worthwhile to give an explicit classification, up to analytic isomorphism, of those singularities whose local rings have finite CM type.

In their 1985 paper [4] Greuel and Knörrer gave explicit equations for the plane curve singularities of finite CM type over an algebraically closed field of characteristic zero. The classification in positive characteristics (but still over an algebraically closed field) was obtained by Kiyek and Steinke in [99]. The classification given here (in § 5) is valid over arbitrary fields and includes space curves as well as plane curves.

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1. Pullbacks and Artinian pairs

Let R be a reduced local Noetherian ring of dimension one with finite normalization $\tilde{R} \neq R$. Let $\mathbf{c}$ be the conductor of R in $\tilde{R}$, and put $A = R/\mathbf{c}$, $B = \tilde{R}/\mathbf{c}$. Then R is represented as a pullback:

$$\begin{array}{ccc} R & \rightarrow & \tilde{R} \\ \downarrow & & \downarrow \\ A & \rightarrow & B \end{array} \tag{1.1}$$

The bottom line of the pullback is an *Artinian pair*, that is, a module-finite extension $A \rightarrow B$ of Artinian rings. The symbols $A \rightarrow B$ and (A, B) will be used interchangeably. Artinian pairs (A_i, B_i) are said to be isomorphic if there is a ring isomorphism f: $B_1 \rightarrow B_2$ carrying A_1 onto A_2. If the A_i are k-algebras, we define k-isomorphism by requiring f to be k-linear. By an (A, B)-module, we mean pair (V, W), where W is a finitely generated projective B-module and V is an A-submodule of W satisfying $BV = W$. A morphism from (V_1, W_1) to (V_2, W_2) is simply a B-linear homomorphism $W_1 \rightarrow W_2$ carrying V_1 into V_2. The (A, B)-modules form an additive category, and we have the notions of direct-sum decompositions and indecomposables. The Artinian pair (A, B) has *finite representation type* provided there are, up to isomorphism, only finitely many indecomposable (A, B)-modules.

1.2. Proposition. *Let R, A, B be as above, and let $\mathbf{M}$ be the maximal ideal of R. The following are equivalent:*

(a) *R has finite CM type.*

(b) *The $\mathbf{M}$-adic completion $\hat{R}$ has finite CM type.*

(c) *The Artinian pair (A, B) has finite representation type.*

These conditions imply

(F1) *B is generated by 3 elements as an A-module; and*

(F2) $(\mathbf{m}B + A)/A$ *is cyclic as an A-module, where* $\mathbf{m} = \mathbf{M}/\mathbf{c}$ *is the maximal ideal of A.*

If (F1) *and* (F2) *hold and the residue field(s) of B are separable over the residue field of A, then R has finite CM type.*

Proof. (a) and (c) are equivalent, by [13, 1.9]. The completion $\hat{R}$ satisfies all the standing hypotheses on R, and its associated Artinian pair is the same as that of R. (See [15, A.1].) Therefore (b) and (c) are equivalent. The remaining implications are (2.1) and (3.1) of [15].

In view of the equivalence of (a) and (b), we will usually restrict our attention to complete, reduced, local rings of dimension one. (The integral closure of such a ring is automatically finitely generated. [M, § 31].) Our classification theorem covers only the equicharacteristic case. Furthermore, we will usually need the separability condition imposed in the last assertion of (1.2).

Most of the work involved in our classification will take place in the bottom line of the pullback (1.1). In § 3 and § 4 we will classify the Artinian pairs coming from rings R of finite CM type, and in § 5 we will lift the classification to the rings R. The following theorem from [15] shows that the Artinian pair carries all the information we need to recover R:

1.3. Theorem. *Let R be a one-dimensional, reduced, complete, equicharacteristic local ring with coefficient field k. Assume that the residue field(s) of $\tilde{R}$ are separable extensions of k. Then the ring R is determined, up to k-isomorphism, by the k-isomorphism class of the Artinian pair (A, B) in diagram* (1.1).

Proof. If $R=\tilde{R}$ then $R\cong k[[X]]$ by Cohen's structure theorem for complete local rings. If $R\neq\tilde{R}$, we appeal [15, A.2]. (In the statement of [15, A.2], k is assumed to be perfect, but the proof goes through under the alternate assumption that the residue fields of $\tilde{R}$ are separable algebraic extensions of k.

2. Notation and lemmas

Let R be a reduced, local ring of dimension one, with module-finite integral closure $\tilde{R}$. The *multiplicity* $\mu=\mu(R)$ can be defined to be the number of generators required for $\tilde{R}$ as an R-module. Of course, this is the same as the number $\mu(A, B)$ of generators required for B as an A-module (where A and B are as in (1.1)). Also, μ is the exact bound on the number of generators required for ideals of R. (See, for example, [3].) If $\mu(R)=1$, then R is a discrete valuation ring, so of course R has finite CM type. On the other hand, if $\mu\geqq 4$, then R can never have finite CM type, by (F1). Therefore we assume from now on that $\mu=2$ or 3. Suppose, now, that R satisfies the hypotheses of (1.3). Since $\mu<4$, $\tilde{R}$ has *at most* one residue field K properly extending k. If no residue field of $\tilde{R}$ extends k properly, put $K=k$. We let $d=d(R)=[K:k]$ in either case. Let $s=s(R)$ denote the number of minimal prime ideals of R. (Then $s\leqq\mu\leqq 3$.) Since $\tilde{R}$ is a direct product of discrete valuation domains, it follows that $s(R)$ is the number of maximal ideals of $\tilde{R}$. Write $\tilde{R}=\prod_{i=1}^{s}\tilde{R}_i$ with $\tilde{R}_i$ local, and number them so that $\tilde{R}_1$ has residue field K. By [11, Theorem 91] and the separability of K/k, $\tilde{R}_1$ has a unique coefficient field containing k; we denote this coefficient field by K. By Cohen's structure theorem $\tilde{R}_1\cong K[[X]]$, and the other $\tilde{R}_i$ (if there are any others) are isomorphic to $k[[X]]$.

Now we consider the Artinian pair associated to R. We know B is the direct product of local rings B_i, and each B_i is of the form $F[[X]]/(X^n)$, where $F=K$ if $i=1$ and $F=k$ otherwise. We will usually write $F[x]$, $x^n=0$ for the truncated power series ring $F[[X]]/(X^n)$. We know A is a local k-subalgebra of B, and k is a coefficient field of A. Also, A contains no non-zero ideal of B.

Conversely, every Artinian pair satisfying these conditions comes from a complete local ring R of the sort we have been discussing. To see this, take $\tilde{R}=\prod_{i=1}^{s}\tilde{R}_i$, where $\tilde{R}_1\cong K[[X]]$ and $\tilde{R}_i\cong k[[X]]$ for $i\geqq 2$; and *define* R by the pullback diagram (1.1). See [14, 3.1] for the details.

From (1.2), the separability of K/k, and the fact that $\mu\leqq 3$, we know that R has finite *CM* type if and only if (A, B) satisfies (F2). In § 3 and § 4, where we will classify the Artinian pairs coming from rings R of finite *CM* type, the following (somewhat redundant) assumptions and notation will be in effect:

(2.1) K/k is a separable extension of degree $d\leqq 3$.

(2.2) A is a local k-subalgebra of the finite-dimensional k-algebra B.

(2.3) k is a coefficient field of A.

(2.4) $B=\prod_{i=1}^{s}B_i$, where $s\leqq 3$ and each B_i is local.

(2.5) Each B_i is of the form $F[x]$, $x^n=0$, where $F=K$ if $i=1$, and $F=k$ if $i>1$.

(2.6) No non-zero ideal of B is contained in A.

(2.7) $\mu=\mu(A, B)=\dim_k(B/\mathbf{m}B)\leqq 3$, where $\mathbf{m}$ is the maximal ideal of A.

Having fixed a separable extension K/k of degree $\leqq 3$, we will refer to an Artinian pair $A\to B$ satisfying (2.2)—(2.7) as a *special* Artinian pair. We record the following useful identity:

(2.8) $s+d+\dim_k(\mathbf{J}/\mathbf{m}B)=\mu+1$, where $\mathbf{J}$ is the radical of B.

2.9. Lemma. *Let $A\to B$ be the special Artinian pair for the ring R.*

(2.9.1) $\dim_k B\geqq 2\dim_k A$, *with equality if and only if R is Gorenstein.*

(2.9.2) *If* (F2) *is satisfied, then* $\dim_k B\leqq 2\dim_k A+1$.

Proof. The first assertion is a special case of [6, Korollar 3.7]. To prove (2.9.2), let $e=\dim_k((\mathbf{m}B+A)/A)=\dim_k(\mathbf{m}B/\mathbf{m})$. Then $\dim_k(B/A)=\mu+e-1$. By (F2), $\mathbf{m}B/\mathbf{m}$ is a cyclic A-module, and since it is unfaithful we have $e<\dim_k A$. Thus $\dim_k B/A\leqq\dim_k A+\mu-2\leqq\dim_k A+1$.

This lemma will be used repeatedly in the following way: We will use (F2) to force various specific elements of B to be in A. Once we get A to contain a subalgebra A_0 of dimension at least half that of B, we'll know that $A=A_0$. We will apply (F2) in the following equivalent form:

(F2) $\dim_k(\mathbf{m}B+A)/(\mathbf{m}^2B+A)\leqq 1$.

Notice that $\dim_k(\mathbf{m}B+A)=\dim_k(\mathbf{m}B)+1$. The dimension of $\mathbf{m}B$ will be usually be clear from (2.8).

Most of the algebras we encounter will have to be manipulated into the form specified by our classification. We will use the following simple lemma:

2.10. Lemma. *Let F be a field, and let $\Gamma=F[t]$, $t^n=0$. Let α be any unit of Γ. Then there is a unique F-automorphism of Γ taking t to αt.* (Warning: The inverse automorphism does not necessarily take t to $\alpha^{-1}t$.)

We end this preliminary section with a simple proof of the fact (implicit in [1]) that double points (i.e. $\mu=2$) have finite CM type. The advantage of a direct proof (independent of (1.2)) is that no separability condition is needed. We drop temporarily our standing assumptions on R.

2.11. Theorem. *Let R be a reduced local ring of dimension one. If the integral closure $\tilde{R}$ is generated by 2 elements as an R-module, then R has finite CM type.*

Proof. By [10, 2.1], every indecomposable torsionfree R-module is isomorphic to an ideal of R. If I is a non-zero ideal of R, then I is a faithful ideal of $R/(0:I)$, and the latter ring enjoys all the standing hypotheses on R, by [10, 1.2]. Since R has at most 2 minimal primes, there are, by [10, 1.2], at most 3 possibilities for the annihilators $(0:I)$ as I ranges over all non-zero ideals. Therefore it will suffice to prove that R has only finitely many isomorphism classes of faithful ideals.

Each faithful ideal I is an invertible ideal of its endomorphism ring $\varrho(I)$, by [10, 2.1], and the latter is a ring between R and $\tilde{R}$. Therefore $\varrho(I)$ is semilocal, whence I and $\varrho(I)$ are isomorphic as $\varrho(I)$-modules, hence as R-modules. We now appeal to [10, 2.2], which implies that there are only finitely many rings between R and $\tilde{R}$.

3. The geometric case: $K=k$

In this section we will classify the special Artinian pairs of finite representation type, under the additional assumption that there is no residue field growth, that is, $d=1$. We will present each k-algebra A in a form that will make the classification of the corresponding rings R in § 5 essentially transparent, even though this will sometimes involve what appears to be an unnatural choice of generators.

In the next three theorems some of the Artinian pairs are listed with a parenthetical assumption on the characteristic of the ground field k. The pairs listed *without* such an assumption occur in *all* characteristics. Thus, for example, if char $k\neq2$, the special Artinian pairs with $\mu=2$ and $s=1$ are classified by (3.1.2); while if char $k=2$ the classification includes (3.1.2) and (3.1.3). In all three theorems the classification is irredundant; e.g., in characteristic 2, no Artinian pair in in (3.1.2) is isomorphic to a pair in (3.1.3).

We begin with the case $\mu=2$. By (1.2) and (2.11), finite representation type is automatic when $\mu\leqq 2$, so our computations amount to a classification of double points. This classification is well-known for algebraically closed fields of characteristic $\neq 2$.

Notation. Throughout this chapter and the next, when working with a truncated power series ring $F[X]$, $x^n=0$, we will use the notation h_m for an unspecified element of order at least m. Thus, for example, x^2+h_4 represents an element of the form $x^2+ax^4+bx^5+\dots$.

3.1. Theorem. *Suppose* $\mu=2$ *and* $d=1$. *If* $s>1$, *then* $A\to B$ *is k-isomorphic to*

(3.1.1) $k[(t,u)]\to k[t]\times k[u]$, $t^n=u^n=0$, $n\geqq 1$.

If $s=1$, *then* $A\to B$ *is k-isomorphic to*

(3.1.2) $k[t^2]\to k[t]$, $t^{2n}=0$, $n\geqq 1$; *or*

(char k=2) (3.1.3) $k[t^2+\alpha t^r]\to k[t]$, $t^{2n}=0$, $n\geqq 2$, r *an odd integer*, $1<r<n$, $\alpha\in k^*:=k-\{0\}$.

Moreover, the integer r *in* (3.1.3) *is uniquely determined by the pair* $A\to B$, *and* α *is uniquely determined modulo the group* k^{*r-2} *of* $(r-2)^{\text{th}}$ *powers in* k^*. *Conversely, each of these pairs is a special Artinian pair with* $\mu=2$.

Proof. If $s>1$, we can write $B=k[t]\times k[u]$, $t^m=u^n=0$, with $1\leqq m\leqq n$. From (2.8) we see that $\mathbf{m}B=tk[t]\times uk[u]$. If $t\neq 0$, it follows that $\mathbf{m}B$ is not contained in $(t^2k[t]\times k[u])\cup(k[t]\times u^2k[u])$. In any case (even if $t=0$). It follows that $\mathbf{m}$ contains an element of the form $f=(t+h_2, au+h_2)$, with $a\neq 0$. By (2.10) we may assume $(t,u)\in A$. An application of (2.9) now shows that $m=n$ and $A=k[(t,u)]$, as desired.

Suppose $s=1$. We note that $\dim_k B=2\dim_k A$. (This follows from the proof of (2.9) or from its statement and the fact [1] that R is Gorenstein if $\mu(R)=2$.) Then $B=k[t]$, $t^{2n}=0$, for some $n\geqq 1$; and by (2.8) we have $\mathbf{m}B=t^2B$. Therefore $\mathbf{m}$ cannot contain an element with non-zero linear term, but $\mathbf{m}$ *must* contain an element of the form $f=t^2+h_3$. If $\operatorname{char} k\neq 2$, write $f=t^2\beta$, and use Hensel's lemma (or, more to the point, [14, 2.3]) to find $\alpha\in B$ with $\alpha^2=\beta$. By (2.10) there is a k-automorphism φ of B taking t^2 to f. Then, on replacing A by $\varphi^{-1}(A)$, we may assume $t^2\in A$. Now $\dim_k k[t^2]=n=A$, so $A=k[t^2]$, and we have (3.1.2).

Now suppose k has characteristic 2, and assume $A\to B$ is not isomorphic to the pair (3.1.2). Choose, in $\mathbf{m}$, an element $f=t^2+\alpha t^r+h_{r+1}$, with $\alpha\neq 0$ and r ($<2n$ by our assumption) as large as possible. Then r is odd, for otherwise we could replace f by $f-f^{r/2}$, thereby increasing r.

Our goal is to eliminate h_{r+1}. If $h_{r+1}\neq 0$, write $f=t^2+\alpha t^r+\beta t^q+h_{q+1}$, with $r<q<2n$. It will suffice, by induction, to replace $A\to B$ by an isomorphic pair for which $\mathbf{m}$ contains an element of the form $f=t^2+\alpha t^r+h_{q+1}$. If q is even, we have the element $f+\beta f^{q/2}$; so assume q is odd. Put $u=q-r+1$, and consider the automorphism $\varphi\colon t\to t+(\beta/\alpha)t^u$. Let $g=\varphi(f)=t^2+\alpha t^r+(\beta^2/\alpha^2)t^{2u}+h_{q+1}$. We want to eliminate the term of even degree without disturbing the rest.

More generally, suppose $g=t^2+\alpha t^r+e_v+h_{q+1}$, where "$e_v$" denotes a sum of terms of even degrees greater than or equal to $2v$. If $v\geqq q-r+1$, we will see that $g^v=t^{2v}+e_{v+1}+h_{q+1}$. Assuming this for the moment, we can replace g by $g+\gamma g^v$ for a suitable constant γ, getting an element $g'=t^2+\alpha t^r+e_{v+1}+h_{q+1}$. Repeating this process if necessary, we eventually get the element we want.

Now $g^v=t^{2v}+h_{q+1}+\sum p_{ij}$, where p_{ij} is a scalar multiple of $t^{2(v-i-j)}(e_v)^j t^{ri}$, and the sum is extended over pairs (i,j) satisfying $0\leqq i\leqq v$, $0\leqq j\leqq v-i$, $(i,j)\neq(0,0)$. If $i=0$ and $j\geqq 1$, p_{ij} is a sum of terms of even degrees greater than or equal to $2(v-j)+2vj=2v+2j(v-1)>2v$. Therefore it will suffice to show that the remaining p_{ij} (for $i\geqq 1$ and $j\geqq 0$) have order at least $q+1$. But this is clear, since $2(v-i-j)+2vj+ri$ is increasing in i and j and is greater than q when $i=1$ and $j=0$.

Now we will prove the uniqueness assertion and the fact that none of the pairs in (3.1.3) is isomorphic to the pair of (3.1.2). Let $B=k[t]$, $t^{2n}=0$; and consider the subrings $A_1=k[t^2+\alpha t^r]$ and $A_2=k[t^2+\beta t^q]$, where $\alpha\neq 0$, r and q are odd and $1<r\leqq q<2n$. We will show that $(A_1,B)\cong(A_2,B)$ if and only if $r=q$, $\beta\neq 0$ and β/α is an $(r-2)^{\text{th}}$ power.

If the pairs are isomorphic there is an automorphism $\varphi\colon t\to at+h_2$, $a\neq 0$, carrying $f:=t^2+\alpha t^r$ into $k[g]$, where $g=t^2+\beta t^q$. Now $\varphi(f)=a^2t^2+\alpha a^r t^r+e_2+h_{r+1}$, where e_2 denotes even stuff of degree at least 4. Write $\varphi(f)=xg+yg^2+\dots$, and note that $x=a^2$. Also, since $\alpha a^r t^r$ is the first odd-degree term of $\varphi(f)$, it follows that $q=r$ and $x\beta=\alpha a^r$. Therefore $\beta/\alpha=a^{r-2}$. Conversely, if $r=q$ and $\beta/\alpha=a^{r-2}\neq 0$, the automorphism $\varphi\colon t\to at$ yields an isomorphism from (A_1,B) onto (A_2,B).

The proof of the last statement is left to the reader.

The next three theorems deal with the case $\mu=3$ (but still $d=1$). Of course $s(R)\leqq 3$, and we treat the three possibilities for s separately. In each case, we omit the straightforward verification that the pairs listed are special Artinian pairs with the specified parameters.

3.2. Theorem. *Up to k-isomorphism, the special Artinian pairs satisfying* (F2), *and with* $\mu=3$, $s=1$ *and* $d=1$ *are exactly the following:*

	(3.2.1)	k	$\to k[t],\ t^3=0.$
	(3.2.2)	$k[t^3]$	$\to k[t],\ t^5=0.$
(char k = 3)	(3.2.3)	$k[t^3+t^4]$	$\to k[t],\ t^5=0.$
	(3.2.4)	$k[t^3,t^4]$	$\to k[t],\ t^6=0.$

(char k = 2)	(3.2.5)	$k[t^3, t^4+t^5] \to k[t]$,	$t^6=0$.
(char k = 3)	(3.2.6)	$k[t^3+\beta t^5, t^4] \to k[t]$,	$t^6=0$.
	(3.2.7)	$k[t^3, t^5] \to k[t]$,	$t^8=0$.
(char k = 3)	(3.2.8)	$k[t^3+t^4, t^5] \to k[t]$,	$t^8=0$.
(char k = 3)	(3.2.9)	$k[t^3+\gamma t^7, t^5] \to k[t]$,	$t^8=0$.
(char k = 5)	(3.2.10)	$k[t^3, t^5+\delta t^7] \to k[t]$,	$t^8=0$.

Moreover, the constants β, γ and δ in (3.2.6), (3.2.9) *and* (3.2.10) *are uniquely determined modulo k^{*2}, k^{*4} and k^{*2}, respectively.*

Proof. We know that any such pair $A \to B$ has $B=k[t]$, $t^n=0$, for some $n \geqq 1$. If $A=k$ then $n=\mu=3$, and we get (3.2.1).

Assume now that A contains k properly, that is, $\mathbf{m} \neq 0$. By (2.8) $\mathbf{m}B=t^3k[t]$, so $\mathbf{m}$ contains a non-zero element $f=t^3+h_4$. Then $n \geqq 5$, since otherwise Bf would be a non-zero ideal of A, contradicting (2.6).

Suppose $n=5$. Then $A=k[f]$ by (2.9). If char $k \neq 3$ we use (2.10) to eliminate h_4 and get (3.2.2). If char $k=3$ and $h_4 \neq 0$, write $f=t^3+\alpha t^4$, $\alpha \neq 0$. The automorphism φ: $t \to \alpha t$ carries $k[t^3+t^4]$ onto $k[f]$, and we obtain the pair of (3.2.3). It is easy to check that the pairs $(k[t^3], B)$ and $(k[t^3+t^4], B)$ are not isomorphic in characteristic 3.

Supposing next that $n \geqq 6$, we see that $n \neq 7$ by (2.6), and A contains $k[f]$ properly by (2.9.2). Select a non-zero element $g=t^e+h_{e+1}$, with $e>3$ and minimal. Then (F2) implies that e is either 4 or 5. Moreover, these possibilities force n to be 6 or 8, respectively, by (2.6); and $A=k[f,g]$ in either case by (2.9).

Assume $e=4$ (and $n=6$), and write $g=t^4+\alpha t^5$. If char $k \neq 3$, we may assume $f=t^3$. If also char $k \neq 2$, the automorphism $t \to t+(1/4)\alpha t^2+(3/16)\alpha^2 t^3$ carries t^3 to $t^3+(3/4)\alpha g$ and t^4 to g, both of which are in $k[t^3, g]$. Therefore, if char $k \neq 2, 3$, (A, B) is isomorphic to the pair in (3.2.4). If k has characteristic 2 and $\alpha \neq 0$, the automorphism $t \to \alpha t$ carries $k[t^3, t^4+t^5]$ to $k[t^3, t^4+\alpha t^5]$, and we obtain (3.2.5).

Now, still assuming $e=4$ and $n=6$, we suppose k has characteristic 3. The automorphism $t \to t-\alpha t^2$ carries $g=t^4+\alpha t^5$ to t^4 and carries f another element of the form t^3+h_4. Thus we may assume $A=k[t^3+\beta t^5, t^4]$, and if $\beta \neq 0$ we have (3.2.6). It is easy to check that β is uniquely determined modulo k^{*2}. Conversely, if $a \in k^*$, the automorphism $t \to at$ takes $k[t^3+\beta t^5, t^4]$ to $k[t^3+a^2\beta t^5, t^4]$.

Next we assume $e=5$ and $n=8$. If char $k \neq 3$, we may assume $A=k[t^3, g]$, where $g=t^5+\delta t^7$. If k has characteristic 5, we have (3.2.7) or (3.2.10). In the latter case, we leave to the reader the easy proof of uniqueness of δ modulo k^{*2}. If, on the other hand, char $k\neq 3, 5$, we note that the automorphism $\varphi\colon t\to t+(1/5)\delta t^3+(4/25)\delta^2 t^5$ takes t^3 to $t^3+(3/5)\beta g$ and t^5 to g. Then φ^{-1} carries A to $k[t^3, t^5]$, and we have (3.2.7).

Finally, we examine the case $e=5$, $n=8$ and char $k=3$. Write $g=t^5(1+h_1)$, and use Hensel's lemma or [14, 2.3] to solve $\varepsilon^5=1+h_1$ in B. Using the inverse of the automorphism $t\to\varepsilon t$, we may assume $g=t^5$, and now $A=k[f, t^5]$, where $f=t^3+h_4$. We can clearly eliminate the terms of degree 5 and 6 in f, leaving $f=t^3+\mu t^4+\gamma t^7$. If $\mu\neq 0$, the automorphism $t\to\mu t+\gamma t^4$ takes $k[t^3+t^4, t^5]$ to $k[f, t^5]$, and we get (3.2.8). If $\mu=0$ we have either (3.2.7) or (3.2.9). The proofs of uniqueness of γ modulo k^{*4} and of irredundancy in the classification are left to the reader.

3.3. Theorem. *Up to k-isomorphism, the special Artinian pairs satisfying* (F2), *and with* $\mu=3$, $s=2$ *and* $d=1$ *are exactly the following:*

(3.3.1) $\quad k[(0, u^2)] \to k\times k[u]$,
$u^{2n}=0, \quad n\geqq 1.$

(char k = 2) (3.3.2) $\quad k[(0, u^2+\alpha u^r)] \to k\times k[u]$,
$u^{2n}=0$, r *odd*, $1<r<2n$, $n\geqq 1$, $\alpha\in k^*$.

(3.3.3) $\quad k[(0, u^2), (t, u^{2n+1})] \to k[t]\times k[u]$,
$t^2=u^{2n+2}=0, \quad n\geqq 1.$

(char k = 2) (3.3.4) $\quad k[(0, u^2+\alpha u^r), (t, u^{2n+1})] \to k[t]\times k[u]$,
$t^2=u^{2n+2}=0$, r *odd*, $1<r<2n+2$, $n\geqq 1$, $\alpha\in k^*$.

(3.3.5) $\quad k[(t, u^2), (0, u^3)] \to k[t]\times k[u]$,
$t^3=u^5=0.$

(char k = 3) (3.3.6) $\quad k[(t, u^2), (0, u^3+u^4) \to k[t]\times k[u]$,
$t^3=u^5=0.$

Moreover, the integer r *in* (3.3.2) *and* (3.3.4) *is uniquely determined by the isomorphism class of the pair* $A\to B$, *and the constant* α *is uniquely determined modulo* k^{*r-2}.

Proof. Let (A, B) be a special Artinian pair satisfying (F2) and the listed parameters μ, s, d. Then $B=k[t]\times k[u]$, $t^p=u^q=0$. By (2.8), $\mathbf{J}/\mathbf{m}B$ is one-dimensional, so we may assume $\mathbf{m}B=tk[t]\times u^2k[u]$, and of course $u\neq 0$ (that is, $q\geqq 2$). The arguments used in the case of (3.1.1) show that we may assume A contains $f:=(t, u^2+h_3)$. Furthermore, if char $k\neq 2$, we can take $h_3=0$.

If $p\leqq 2$, it follows from (2.6) that q is even. Moreover, since $t^2=0$, the analysis we did in the proof of (3.1) shows that in characteristic 2 we may assume either $f=(t, u^2)$ or $f=(t, u^2+\alpha t^r)$, where r is odd, $1<r<q$ and $\alpha \in k^*$.

Suppose first that $p=1$, that is, $t=0$. Then $\dim_k B=q+1$, while $\dim_k k[f]=q/2$. By (2.9), $A=k[f]$, and we have (3.3.1) or (3.3.2).

If $p=2$, the conductor condition (2.6) forces $q \geqq 4$ (one of the requirements of (3.3.3) and (3.3.4)). Condition (F2) implies that A has to contain $k[f]$ properly. On the other hand, $k[f]$ already has dimension $q/2$, just one less than $(\dim_k B)/2$; so by (2.9) $A/k[f]$ is one-dimensional. Choose $g \in A-k[f]$. By subtracting an appropriate element of $k[f]$, we may assume $g=(at, bu^e+h_{e+1})$, where e is odd and $b \neq 0$. On multiplying by $b^{-1}f^{(q-e-1)/2}$, we can take $g=(at, u^{q-1})$, and now $a \neq 0$ by (2.6). (Of course, this means e was equal to $q-1$ at the outset.) Now $A=k[f, g]$, and it is harmless to replace f by $f-a^{-1}g$. The new f looks like $(0, u^2-a^{-1}u^{q-1})$ or (only in characteristic 2) $(0, u^2+\alpha u^r+a^{-1}u^{q-1})$, with $\alpha \neq 0$.

In the former case, if $\operatorname{char} k \neq 2$, we can eliminate the u^{q-1} term from f via the automorphism $t \to t$, $u \to u+(1/2)a^{-1}u^{q-2}$. Since this automorphism fixes both t and u^{q-1}, g is still in A, and now $f=(0, u^2)$. In the latter case, if $r<q-1$, we follow exactly the same steps as in the proof of (3.1) to eliminate the term of degree $q-1$. We now have $f=(0, u^2)$ or $f=(0, u^2+\alpha u^r)$, and (after replacing at by t) $g=(t, u^{q-1})$. We have arrived at (3.3.3) or (3.3.4) with $n=q/2-1$.

Finally, assume $p \geqq 3$. Our goal is (3.3.5) or (3.3.6), so we must show that $p=3$ and $q=5$. We have $f=(t, u^2+h_3) \in A$, and it follows easily from (2.6) that either $q=2p-1$ or $q=2m$ with $m \geqq 2p$. Also, (F2) implies that $\mathfrak{m}^2B+A$ contains 2 linearly independent elements in the span of $\{(t, 0), (0, u^2), (0, u^3)\}$ and hence a non-zero element of the form $(0, au^2+bu^3)$. Therefore A contains an element of the form $g=(h_2, au^2+bu^3+h_4)$, where either a or b is non-zero.

Suppose first that $a \neq 0$. Then one checks that $1, f, \ldots, f^{p-1}, g, \ldots, g^e$ are linearly independent, where e is the greatest integer strictly less than $q/2$. Then $p+q=\dim_k B \geqq 2 \dim_k A \geqq 2(p+e)=2p+2(e+1)-2 \geqq 2p+q-2$, which contradicts our assumption that $p \geqq 3$. Therefore $a=0$, and we may assume that $g=(h_2, u^3+h_4)$. If q were greater than 5, one could use the fact that $q \geqq 2p-1$ to choose suitable exponents a and b to get $f^a g^b=(0, u^{q-1})$, which would in turn contradict (2.6). Therefore $p=3$ and $q=5$.

Now we have $A=k[f, g]$, where $f=(t, u^2+h_3)$, $g=(ct^2, u^3+du^4)$, and $t^3=u^5=0$. After replacing g by $b-cf^2$, we can assume $c=0$. If $\operatorname{char} k \neq 3$, we proceed as follows: We use the automorphism $t \to t$, $u \to u-(d/3)u^2$ to make $d=0$. Next, we use a multiple of g to bring f to the form $f=(t, u^2+xu^4)$. By (2.9) $A=k[f-xf^2, g]=k[(t-xt^2, u^2), (0, u^3)]$. The automorphism $t \to t+xt^2$, $u \to u$ puts A in the form prescribed by (3.3.5). Finally, if $\operatorname{char} k=3$ we can eliminate the h_3 term in f, getting $A=k[(t, u^2), (0, u^3+du^4)]$. If $d=0$ we have (3.3.5), while if $d \neq 0$ we use the automorphism $t \to d^{-2}t$, $u \to d^{-1}u$ to transform (A, B) to the form of (3.3.6).

3.4. Theorem. *Up to k-isomorphism, the special Artinian pairs satisfying* (F2) *and with* $s=3$ *are exactly the following:*

(3.4.1) $k[(0,u,v)] \to k\times k[u]\times k[v], \quad u^n=v^n=0, \quad n\geqq 1.$

(3.4.2) $k[(0,u,v),\ (t,0,v^n)] \to k[t]\times k[u]\times k[v], \quad t^2=0,$

$$u^{n+1}=v^{n+1}=0, \quad n\geqq 1.$$

Proof. Write $B=k[t]\times k[u]\times k[v]$, $t^p=u^q=v^r=0$, with $1\leqq p\leqq q\leqq r$. Then $\mathbf{m}B=tk[t]\times uk[u]\times vk[v]$, by (2.8). Using (2.6), one checks easily that $q=r$. If $p=q=1$, we get the case $n=1$ of (3.4.1); while if $p=1<q$ we see that $\mathbf{m}$ is not contained in $B(0,u,v^2)\cup B(0,u^2,v)$. In this case we use (2.10) and (2.9) to put (A,B) in the form of (3.4.1).

Suppose $p\geqq 2$. Assume first that $\mathbf{m}$ contains an element with non-zero linear term in each variable. By (2.10) this element can be assumed to be (t,u,v). Using (2.9), we deduce that the codimension of $k[(t,u,v)]$ in A is the greatest integer $\leqq p/2$. A little computation shows that A contains an element of the form $g=(t+h_2,0,av^e+h_{e+1})$, where $a\neq 0$ and $0<e<q$. Since $g,\dots,g^{p-1}$ are linearly independent modulo $k[(t,u,v)]$, our observation on the codimension shows that $p=2$, whence $A/k[(t,u,v)]$ is one-dimensional. Therefore $e=q-1$, and $A=k[(t,u,v),\ (t,0,av^{q-1})]$.

If $q\geqq 3$, the transformation $t\to a^{q-2}t$, $u\to au$, $v\to av-a^2v^{q-1}$ takes

$$k[(0,u,v),\ (t,0,v^{q-1})]$$

to A, and we have (3.4.2) with $n=q-1$. Next, suppose $p=q=2$, so that $A=k[(t,u,v),\ (t,0,av)]$. If $a=1$, then A contains $B(0,u,0)$, contradicting (2.6). Thus $a\neq 1$, and we can use the automorphism $t\to at$, $u\to(1-a)u$, $v\to v$ to transform A to the algebra $k[(0,u,v),\ (t,0,v)]$ of (3.4.2).

Now assume that $\mathbf{m}$ does *not* contain an element with order 1 in each variable. Then $\mathbf{m}$ is contained in the union of three proper subspaces, and it follows that k is the 2-element field. We may assume that $\mathbf{m}$ contains an element $f:=(t,u,h_2)$, and again we see that the codimension of $k[f]$ in B is the greatest integer less than or equal to $p/2$. Now A also contains (up to an automorphism of $k[v]$) an element of the form $g=(t+h_2,h_2,v)$. Since $g,\dots,g^{q-1}$ are linearly independent modulo $k[f]$, it follows that $p=q=2$. Now $A=k[f,g]=k[(t,u,0),\ (t,0,v)]$, which, up to a permutation of the variables, is the case $n=1$ of (3.4.2).

The case-by-case application of (2.10) may lead one to conjecture that higher degree terms can always be eliminated if the ground field has characteristic 0 or $p\gg 0$. Indeed, a general theorem to this effect would have saved a lot of tedious repetition. Unfortunately, no such theorem seems possible, as shown by the following example:

3.5. Example. The following two special Artinian pairs are k-isomorphic if and only if $\operatorname{char} k=3$:

$$A_1 := k[t^3, t^7] \to B := k[t], \quad t^{12} = 0.$$

$$A_2 := k[t^3, t^7+t^8] \to B.$$

Consequently, the rings $R_1=k[[T^3, T^7]]$ and $R_2=k[[T^3, T^7+T^8]]$ are isomorphic if and only if $\operatorname{char} k=3$.

Proof. Suppose φ is a k-automorphism of B such that $\varphi(A_1)=A_2$. Then $\varphi(t)=xt$ for some unit x of B, say, $x=a+bt+h_2$, with $a\neq 0$. Then $t^3x^3\in A_2$, and it follows that $3b=0$. On the other hand, the fact that $t^7x^7\in A_2$ forces $7b=a$. Thus k has characteristic 3. Conversely, if $\operatorname{char}(k)=3$, the map φ: $t\to t+t^2$ takes t^3 to $t^3+t^6\in A_2$ and t^7 to $(t^7+t^8)(1-t^3)\in A_2$.

To prove the last statement, it is enough, by (1.3), to show that (A_i, B) is the Artinian pair associated to R_i; and this is really just a matter of computing the conductor. The conductor of R_1 in its normalization $k[[T]]$ is clearly $T^{12}k[[T]]$. For R_2 we reason as follows: Given any $n\geqq 12$, we can find $F_n\in R_2$ of the form T^n+H_{n+1}, where H has order at least $n+1$. Then $T^n=F_n+a_1F_{n+1}+a_2F_2+\ldots\in R_2$ for suitably chosen constants a_i.

Notice that (F2) fails for these Artinian pairs.

4. Residue field growth

In this section, K/k is a separable field extension of degree $d=2$ or 3. We begin with the case $d=2$, $\mu=2$.

4.1. Theorem. *Up to k-isomorphism, the special Artinian pairs satisfying* (F2) *and with* $d=2$, $\mu=2$ *are the following:*

(4.1.1) $k[t] \to K[t], \quad t^n = 0, \quad n \geqq 1.$

Proof. Clearly $s=1$, so $B=K[t]$, $t^n=0$, for some $n\geqq 1$. Also, $\mathbf{J}=\mathbf{m}B$, and by (2.10) we may assume $t\in A$. Then $A=k[t]$ by (2.9).

Next we tackle the most difficult case of all: $d=2$ and $\mu=3$.

4.2. Lemma. *If* $d=2$ *and* $\mu=3$, *then* $s=2$ *and* $\mathbf{J}=\mathbf{m}B$.

Proof. From (2.8) we have $s=2-\dim_k \mathbf{J}/\mathbf{m}B$. If s were equal to 1, $\mathbf{J}/\mathbf{m}B$ would be a vector space over K, and its k-dimension would be even, an obvious contradiction.

4.3. Theorem. *Let* $[K:k]=2$, *and let* ξ *be an arbitrary but fixed element of* $K-k$. *Up to k-isomorphism, the special Artinian pairs for* K/k *satisfying* (F2) *and with* $\mu=3$ *are exactly the following:*

(4.3.1) $k[(t,0)] \to K[t]\times k, \quad t^n=0, \quad n\geqq 1.$

(4.3.2) $k[(t,u),(\xi t^n,u)] \to K[t]\times k[u], \quad t^{n+1}=u^2=0, \quad n\geqq 1.$

Proof. By the lemma $B=K[t]\times k[u]$, $t^p=u^q=0$, and $\mathbf{m}B=\mathbf{J}$. Using (2.10), we may assume $(t,u)\in\mathbf{m}$. Now $p\geqq q$, for otherwise $B(t,u)^{q-1}\in A$, contradicting (2.6). If $q=1$, (2.9) forces A to equal $k[(t,u)]$; and we have the family (4.3.1) with $n=q$.

Suppose now that $q\geqq 2$, and count dimensions: $\mathbf{m}B+A$ has k-dimension $2p+q-2$, while the dimension of $\mathbf{m}^2B+k[(t,u)]$ is $2p+q-4$. By (F2) A contains an element $f=(at+h_2, bu+h_2)$ in which $a\neq b$. Then A also contains $g=:f-b(t,u)-h_2=(c_1t+\ldots+c_{p-1}t^{p-1},0)$, with $c_1\neq 0$. Now $g,g^2,\ldots,g^{q-1}$ are linearly independent modulo $k[(t,u)]$, so the k-dimension of A is at least $p+q-1$. On the other hand, $\dim_k B=2p+q$. It follows from (2.9) that $q=2$ and A has $\{(t,u)^i\colon 0\leqq i\leqq p-1\}\cup\{g\}$ as a k-basis.

I claim that not all c_i are in k. For otherwise we could subtract k-multiples of the powers of (t,u) from g, getting $(0,u+h_2)\in A$ and contradicting (2.6). Letting r be the first subscript i for which $c_i\notin k$, we see that A contains $h:=(c_1t^{p-r}+\ldots+c_rt^{p-1},0)=(t,u)^{p-1-r}g$.

Suppose for the moment that $r\leqq p-2$. Then $p-r\geqq 2$, and we can subtract k-multiples of powers of (t,u) from h, getting $(c_rt^{p-1},0)\in A$. Now $u^{p-1}=0$ because $p-1\geqq r+1\geqq 2=q$. Therefore A contains $(t^{p-1},0)=(t,u)^{p-1}$. But then $A\supseteq B(t^{p-1},0)$, violating (2.6). This contradiction shows that $r=p-1$.

We now know that $c_i\in k$ for $i<p-1$, but $c_{p-1}\notin k$. Therefore A contains an element of the form $(\alpha t^{p-1},u)$, where $\alpha\in K-k$. (In fact, $\alpha=c_1+1$ if $p=2$ and $\alpha=-c_{p-1}/c_1$ if $p>2$.) Also, A is the k-span of this element and the $(t,u)^i$, $0\leqq i\leqq p-1$. More generally, for each $\beta\in K-k$, let A_β be the k-span of $(\beta t^{p-1},u)$ and the $(t,u)^i$. Then (A_β,B) is easily seen to be a special Artinian pair satisfying (F2). Moreover, when $\beta=\xi$ we get the Artinian pair in (4.3.2) with $n=p-1$. We just need to check that, for fixed p, the isomorphism class of (A_β,B) does not depend on the choice of $\beta\in K-k$.

Suppose $p=2$, and let $\beta,\gamma\in K-k$. We seek $\delta\in K$, $d\in k$, both non-zero, such that the $(K\times k)$-automorphism $t\to\delta t$, $u\to du$ carries A_β onto A_γ. In other words, we want $(\delta-d)/(\gamma-1)\in k$ and $(\beta\delta-1)/(\gamma-1)\in k$. The latter condition can be satisfied by letting δ be either $1/\beta$ or γ/β. One can then solve for d by writing $\delta=d\cdot 1+e\cdot(\gamma-1)$ with $d,e\in k$. The two choices for δ can't *both* force $d=0$; for otherwise γ would be in k.

If $p>2$, it's easier since now $u^{p-1}=0$: Given $\beta, \gamma \in K-k$, write $\beta=d\gamma+e$, with $d, e \in k$. The $(K\times k)$-automorphism $t\to t+\gamma(d-1)t^{p-1}$, $u\to du$ carries A_β onto A_γ.

The final case $[K:k]=3$ is much easier to handle:

4.4. Theorem. *Let K/k be a separable field extension of degree 3, and let ξ be an arbitrary but fixed element of $K-k$. Up to k-isomorphism, the special Artinian pairs for K/k satisfying* (F2) *are exactly the following:*

(4.4.1) $k \to K$.

(4.4.2) $k[t, \xi t]\to K[t]$, $t^2=0$.

Proof. Clearly the listed pairs are special and satisfy (F2). Conversely, let (A, B) be a special pair satisfying (F2). From (2.8) we have $B=K[t]$, $t^p=0$, for some $p\geqq 1$; and $\mathbf{m}B=tK[t]$. If $p=1$ we get the pair (4.4.1); so suppose $p>1$. An application of (2.10) allows us to assume that $t\in A$, and then (F2) implies that A contains an element $f=\alpha t+h_2$, with $\alpha\in K-k$. Then A contains $t^2, \alpha t^2+h_3$ and $\alpha^2 t^2+h_3$; and if $p>2$ we would have $A\supseteq Bt^{p-1}$, a contradiction. This shows that $t^2=0$, and A is the 3-dimensional k-algebra spanned by 1, t and αt. For $\beta\in K-k$, let A_β be the k-span of 1, t and βt. The pair in (4.4.2) is (A_ξ, B), and we have to show that (A_α, B) and (A_ξ, B) are k-isomorphic.

Since $[K:k]=3$ there exist $a, b, c, d\in k$, not all zero, such that $a\alpha+b\alpha\xi= c+d\xi$. Put $\varrho=a+b\xi$, and observe that $t\to\varrho t$ is an automorphism of B carrying A_α onto A_ξ. Thus, the k-isomorphism class of the pair (A_ξ, B) is independent of the choice of the element $\xi\in K-k$.

5. The classification

Let R be a one-dimensional, reduced, complete, equicharacteristic local ring, and let k be a coefficient field. We will classify, up to k-isomorphism, the rings R of finite CM type, assuming only that the residue field(s) of the normalization $\tilde{R}$ are separable over k. (This assumption is imposed partly as a technical convenience, but primarily so that (1.2) can be applied. It is unknown whether (1.2) is valid without the separability hypothesis.) We begin with the rings having no residue field growth. We list all the rings in parametric form, as subrings of $k[[T]]$, $k[[T]]\times k[[U]]$, or $k[[T]]\times k[[U]]\times k[[V]]$. In some cases, we also represent the rings by equations, as homomorphic images of $k[[X, Y]]$ or $k[[X, Y, Z]]$. I have tried to find the defining equations in characteristic 2 for the rings B_n' and M_n' in the next theorem but have succeeded only for small values of n. The equations in these examples appear to have an interesting combinatorial structure that deserves further investigation. On the other hand, a smarter choice of generators in the parametric forms may result in simpler equations. For example, it should be possible to eliminate the term X^3Y in the equation for E'.

We denote the ideal generated by f and g by $\langle f, g\rangle$. Parentheses, e.g., (f, g) are used for elements of direct products. The parameter n runs through all positive integers.

5.1. Theorem. *Let R be a one-dimensional, reduced, complete, equicharacteristic local ring, and let k be a coefficient field. Assume that the residue field(s) of the normal-*

Table 1

Curve singularities of finite CM type with no residue field growth

	$A = k[[T]]$
	$B_n = k[[T^2, T^{2n+1}]]$
(char k = 2)	$B'_n = k[[T^2+\alpha T^{2r+1}, T^{2n+3}]], \quad r \leq n$
	$C_n = k[[(T, U), (0, U^n)]] = k[[X, Y]]/Y\langle Y-X^n\rangle$
	$E = k[[T^3, T^4]]$
(char k = 2)	$E' = k[[T^3, T^4+T^5]]$
	$= k[[X, Y]]/\langle Y^3+X^3Y+X^4+X^5\rangle$
(char k = 3)	$E'' = k[[T^3+\gamma T^5, T^4]]$
	$= k[[X, Y]]/\langle \gamma^4 Y^5+\gamma^2 Y^4+Y^3+\gamma X^2 Y^2-X^4\rangle$
	$F = k[[T^3, T^5]]$
(char k = 3)	$F' = k[[T^3+T^4, T^5]]$
	$= k[[X, Y]]/\langle Y^4+Y^3-XY^3-X^2Y^2-X^5\rangle$
(char k = 3)	$F'' = k[[T^3+\beta T^7, T^5]]$
	$= k[[X, Y]]/\langle \beta^5 Y^7+\beta^2 XY^4+Y^3-\beta X^3Y^2-X^5\rangle$
(char k = 5)	$F''' = k[[T^3, T^5+\gamma T^7]]$
	$= k[[X, Y]]/\langle Y^3+2\gamma X^4Y-X^5-\gamma^3X^7\rangle$
	$G = k[[T^3, T^4, T^5]]$
	$H = k[[T^3, T^5, T^7]]$
(char k = 3)	$H' = k[[T^3+T^4, T^5, T^7]]$
	$L = k[[(T, U^2), (0, U^3)]] = k[[X, Y]]/\langle Y^3-X^2\rangle$
(char k = 3)	$L' = k[[(T, U^2), (0, U^3+U^4)]]$
	$= k[[X, Y]]/Y\langle Y^2+X^2Y-X^3+X^4\rangle$
	$M_n = k[[(0, U^2), (T, U^{2n+1})]]$
	$= k[[X, Y]]/X\langle Y^2-X^{2n+1}\rangle$
(char k = 2)	$M'_n = k[[(0, U^2+\alpha U^{2r+1}), (T, U^{2n+1})]], \quad r \leq n$
	$N_n = k[[(0, U^2), (0, U^{2n+1}), (T, 0)]]$
	$= k[[X, Y, Z]]/(\langle X, Y\rangle \cap \langle Y^2-X^{2n+1}, Z\rangle)$
(char k = 2)	$N'_n = k[[(0, U^2+\alpha U^{2r+1}), (0, U^{2n+1}), (T, 0)]], \quad r \leq n-1$
	$P_n = k[[(0, U, V), (T, 0, V^n)]]$
	$= k[[X, Y]]/XY\langle Y-X^n\rangle$
	$Q_n = k[[(0, U, V), (0, 0, V^n), (T, 0, 0)]]$
	$= k[[X, Y, Z]]/(\langle X, Y\rangle \cap \langle Y, Z\rangle \cap \langle Y-X^n, Z\rangle)$

ization $\tilde{R}$ are of degree one over k. Then R has finite CM type if and only if R is k-isomorphic to one of the rings listed in table 1. In this table the parameters n and r are positive integers uniquely determined by the the isomorphism class of R, and α, β and γ are non-zero elements of k, unique up to congruence modulo k^{*2r-1}, k^{*4} and k^{*2}, respectively.

Proof. By (1.2) and (1.3), it is enough to check, using table 2, that the list of Artinian pairs corresponding to these rings agrees with the union of the lists in Theorems 3.1—3.4. Computing the Artinian pairs for the rings in the theorem is straightforward, and the details are left to the reader. For the exceptional rings in small characteristic, where high powers of the variables are not polynomial functions of the generators, the method used in the second paragraph of the proof of (3.5) can be used to compute the conductor. Notice that the odd integer r of (3.1) and (3.3) has been replaced by $2r+1$, and that in some cases the parameter n has been shifted by 1, so that in the tables it takes on all positive integer values.

Table 2 lists the Artinian pair (A, B), as well as other possibly useful information, for each of the rings in table 1. The number "char." gives the characteristic

Table 2

Artinian pairs associated to the curve singularities of finite CM type with no residue field growth

char.	R	Art. pair	$\mu(R)$	$s(R)$	$\dim_k A$	$\dim_k B$	emb.	ind.
	A	$0\to 0$	1	1	0	0	1	0
	B_n	(3.1.2)	2	1	n	$2n$	2	n
2	B'_n	(3.1.3)	2	1	n	$2n$	2	n
	C_n	(3.1.1)	2	2	n	$2n$	2	n
	E	(3.2.4)	3	1	3	6	2	2
2	E'	(3.2.5)	3	1	3	6	2	2
3	E''	(3.2.6)	3	1	3	6	2	2
	F	(3.2.7)	3	1	4	8	2	3
3	F'	(3.2.8)	3	1	4	8	2	3
3	F''	(3.2.9)	3	1	4	8	2	3
5	F'''	(3.2.10)	3	1	4	8	2	3
	G	(3.2.1)	3	1	1	3	3	1
	H	(3.2.2)	3	1	2	5	3	2
3	H'	(3.2.3)	3	1	2	5	3	2
	L	(3.3.5)	3	2	4	8	2	3
3	L'	(3.3.6)	3	2	4	8	2	3
	M_n	(3.3.3)	3	2	$n+2$	$2n+4$	2	$n+1$
2	M'_n	(3.3.4)	3	2	$n+2$	$2n+4$	2	$n+1$
	N_n	(3.3.1)	3	2	n	$2n+1$	3	n
2	N'_n	(3.3.2)	3	2	n	$2n+1$	3	n
	P_n	(3.4.2)	3	3	$n+2$	$2n+4$	2	$n+1$
	Q_n	(3.4.1)	3	3	n	$2n+1$	3	n

(2, 3 or 5) in which the exceptional rings occur. (The rings with no entry in the "char." column occur in all characteristics.) The number "emb." is the embedding dimension of R, that is, the number of generators required for the maximal ideal $\mathbf{M}$ of R; and "ind." is the index of nilpotency of the maximal ideal $\mathbf{m}$ of A.

Before moving on to the case of residue field growth, we discuss the geometry of the curves of finite CM type with more than one analytic branch ($s \geqq 2$). To keep the discussion from getting too technical, we restrict to characteristic 0 or $p > 3$. The equational descriptions represent R as the completion of the local ring at the origin of an algebraic curve. The double points ($\mu = 2$) are the rings C_n. These correspond to the union of two smooth curves meeting at the origin, transversally if $n = 1$, tangentially if $n > 1$.

The triple points with two branches are L, M_n and N_n. L corresponds to the union of the cusp $y^2 = x^3$ and the x-axis. (In fact, the x-axis can be replaced by any smooth curve tangent to the x-axis at the origin.) M_n and N_n correspond to the union of the (possibly higher order) cusp $y^2 = x^{2n+1}$ and a smooth curve whose tangent line at the origin is distinct from the x-axis. For M_n, the tangent line of the smooth component lies in the xy-plane, and for N_n it does not.

Finally, P_n and Q_n come from curves with three analytic branches. All three branches must be smooth, and they have either two (if $n > 1$) or three (if $n = 1$) distinct tangent lines at the origin. (Thus, for example, $y(y - x^2)(y + x^2) = 0$ defines a curve singularity of infinite CM type, since all three branches have the same tangent line.) The curves for P_n are planar, that is, the tangent space at the origin is two-dimensional; with Q_n the tangent space is three-dimensional.

The singularities for which every indecomposable CM module is an ideal have been studied by Bass [1], Nazarova and Roiter [12] and Greither [3], and were finally classified by Haefner and Levy in [HL]. In our context, these rings are the double points B_n and C_n, and just two of the triple points: P_1 and Q_1.

Now we suppose $\tilde{R}$ has a residue field K properly extending k, and we keep the notation established in the beginning of § 2. Thus K is the coefficient field of the first component of $\tilde{R}$, and $d(R) = [K:k]$ is either 2 or 3. For a given separable extension K/k, we will list all the rings R of finite CM type. Of course the k-isomorphism class of R depends on the extension K/k, so there will in general be continuous families of non-isomorphic k-algebras of finite CM type. For each K, let ξ be a fixed element of $K - k$. In the classification below, the k-isomorphism class of the ring R does *not* depend on the choice of ξ.

We will use symbols such as $A2_n$ to denote a family of rings R with $d(R) = 2$. Similarly, $D3$ denotes a ring R with $d(R) = 3$.

5.2. Theorem. *Let R be a one-dimensional, reduced, complete, equicharacteristic local ring, and let k be a coefficient field. Assume that the normalization $\tilde{R}$ has a res-*

Table 3

Curve singularities of finite CM type with residue field growth

$A2_n = k[[T, \xi T^n]] = k[[T]] + T^n K[[T]]$ $(d = 2)$

$B2_n = k[[(T, 0), (\xi T^n, 0), (0, U)]]$

$\quad = k[[(T, U]] + (T^n K[[T]] \times U k[[U]])$ $(d = 2)$

$C2_n = k[[(T, U), (\xi T^n, U), (0, U^2)]]$

$\quad = k[[(T, U)]] + k(\xi T^n, U) + (T^{n+1} K[[T]] \times U^2 k[[U]])$ $(d = 2)$

$D3 = k[[T, \xi T, \xi^2 T]] = k + TK[[T]]$ $(d = 3)$

$E3 = k[[T, \xi T]] = k + (k + k\xi) T + T^2 K[[T]]$ $(d = 3)$

idue field K properly extending k and that K/k is separable. Then R has finite CM type if and only if $d := [K:k] \leqq 3$ and R is k-isomorphic to one of the rings in table 3. The parameters n range over the positive integers. The parenthetical number d is the degree of the extension K/k, and ξ is an arbitrary element of $K - k$. The isomorphism class of R does not depend on the choice of ξ.

As before, the proof amounts to computing the Artinian pair for each ring and comparing these pairs with those listed in § 4.

A remark on notation: In the second description of $C2_n$, $k[[(T, U)]]$ is the power series ring in the *single* variable (T, U) and is regarded as a subring of $K[[T]] \times k[[U]]$. Also, $k(\xi T^n, U)$ denotes the k-linear span of $(\xi T^n, U)$.

Table 4

Artinian pairs associated to the curve singularities of finite CM type with residue field growth

R	Art. pair	$d(R)$	$\mu(R)$	$s(R)$	$\dim_k A$	$\dim_k B$	emb.	ind.
$A2_n$	(4.1.1)	2	2	1	n	$2n$	2	n
$B2_n$	(4.3.1)	2	3	2	n	$2n+1$	3	n
$C2_n$	(4.3.2)	2	3	2	$n+2$	$2n+4$	3	$n+1$
$D3$	(4.4.1)	3	3	1	1	3	3	1
$E3$	(4.4.2)	3	3	1	3	6	2	2

In table 4, we list the Artinian pairs (A, B) associated to the rings in table 3. Again, "emb." is the embedding dimension of R, and "ind." is the least i such that $\mathbf{m}^i = 0$, where $\mathbf{m}$ is the maximal ideal of A.

Perhaps it is worth giving explicit equations for the singularities of finite CM type when $k = \mathbf{R}$, the field of real numbers. We assume $K = \mathbf{C}$, since the case $K = \mathbf{R}$ is described adequately in Theorem 5.1. The only possibilities are $A2_n$, $B2_n$ and $C2_n$. Taking $\xi = \sqrt{-1}$ in Theorem 5.2, we easily deduce the following:

5.3. Corollary. *Let R be as in theorem 5.2, with $k=\mathbf{R}$, the real number field. Then R has finite CM type if and only if R is one of the following (for some $n \geqq 1$):*

$$\mathbf{R}[[X,Y]]/\langle X^{2n}+Y^2\rangle$$
$$\mathbf{R}[[X,Y,Z]]/\langle X^{2n}+Y^2,\ XZ\rangle$$
$$\mathbf{R}[[X,Y,Z]]/\langle X^{2n}+Y^2-Z^n-Z,\quad (X^2-Z)Z\rangle$$

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