Complex interpolation of quasi-Banach spaces with an A-convex containing space*

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1. Introduction

Complex interpolation of general couples (X_0, X_1) of quasi-Banach spaces has been considered by several authors [8], [11], [16]. The first approach was made by Rivière in his thesis [16] and recently it has been developed by Cwikel, Milman and Sagher in [8], where some new interpolation results have been obtained. See also [9] and [17].

These authors have used in the classical construction of Calderón (see [4] or [3]) the space $\mathscr{G}$ of all functions $f(z)=\sum_{k=1}^{N} f_k(z)x_k$, where $x_k \in X_0 \cap X_1$ and $f_k \in A(S, \mathbf{C})$, the class of scalar valued functions analytic on the strip $S=\{z\colon 0<\operatorname{Re} z<1\}$ and continuous and bounded on $\bar{S}$. As a result, a quasi-normed space of analytic functions is obtained in which the quasi-norm $\|\cdot\|_{\mathscr{F}}$ on $\mathscr{G}$ is the one introduced by Calderón [4], i.e.

$$\|f\|_{\mathscr{F}} = \max\{\sup_{t\in\mathbf{R}} \|f(it)\|_{X_0}, \sup_{t\in\mathbf{R}} \|f(1+it)\|_{X_1}\}.$$

When making the quotient to define the intermediate spaces, the following pathologies can occur for the quasi-seminorm $\|x\|_\theta=\inf\{\|f\|_{\mathscr{F}}\colon f(\theta)=x, f\in\mathscr{G}\}$ on $X_0\cap X_1$. First, the intermediate space is always the intersection and only the resulting quasi-seminorms defined on it vary. Moreover, these quasi-seminorms can be identically zero. If they are genuine quasi-norms, the intersection need not necessarily be complete. Its inclusion in the sum space may also fail to be continuous. Even if this latter problem does not arise, it is not clear whether the extension of the continuous inclusion to the completion is one to one. For information about these facts see [8], [14], [16], [18] and [19].

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Another approach can be found in a paper by Janson and Jones [11], where a third term $\sup_{z\in S}\|f(z)\|_{X_0+X_1}$ is added to define a space of vector valued functions as the completion of $\mathcal{G}$ with respect to the quasi-norm $\|f\|=\max\{\|f\|_{\mathcal{F}}, \sup_{z\in S}\|f(z)\|_{X_0+X_1}\}$. The complex interpolation methods introduced in that paper always yield quasi-Banach intermediate spaces and have been succesfully used to interpolate Hardy spaces and *BMO*. However, those methods give no convexity inequality in the interpolation theorem (in the terminology of Bergh—Löfström [3] they are not of exponent θ).

The purpose of this paper is to present a complex interpolation method based on complete spaces of analytic functions (hence yielding quasi-Banach interpolation spaces) and with a convexity inequality in the interpolation theorem. In doing so, we will restrict the category of general quasi-Banach pairs to some smaller category in which important examples remain included.

In the construction of the interpolation method, one has to deal with analytic functions with values in a quasi-Banach space. These functions are defined by the local existence of power series expansions and have many of the usual properties. For example, the power series expansion about a point z_0 converges uniformly on compact subsets of any disk D centered at z_0 and contained in the domain of the function. Information can be found in [12], [13] and [21].

We will denote by Δ and S the unit disk $|z|<1$ and the strip $0<\operatorname{Re} z<1$ respectively. $A(\Delta,\mathcal{U})$ and $A(S,\mathcal{U})$ will be the corresponding spaces of $\mathcal{U}$-valued bounded analytic functions that are continuous up to the boundary, for a given quasi-Banach space $\mathcal{U}$.

We will consider quasi-Banach spaces $\mathcal{U}$ which have an equivalent plurisubharmonic quasi-norm. Those spaces are called A-convex by Kalton [13], locally analytically pseudo-convex by Peetre [15] and locally holomorphic by Aleksandrov [1]. They are characterized by the existence of a constant C so that if $f\in A(\Delta,\mathcal{U})$ then

$$\|f(0)\| \leqq C \sup_{|z|=1}\|f(z)\|. \tag{1}$$

It is known that any quasi-Banach space $(X,\|\cdot\|)$ has an equivalent quasi-norm $|\cdot|$ so that, for certain r, $0<r\leqq 1$, $|\cdot|^r$ is subadditive. Such a quasi-norm is said to be an r-norm. Throughout the paper we will consider the case where all the quasi-norms are r-norms for some r, $0<r\leqq 1$. The general statements can be reduced to this case by equivalently re-quasi-norming the spaces with suitable r-norms (phrases like "with equal quasi-norms" would have then to be replaced by "with equivalent quasi-norms"). Similarly, let $(\mathcal{U},\|\cdot\|)$ be A-convex. The quasi-norm $\|\cdot\|$ satisfies the inequality (1). If we pick an equivalent r-norm $|\cdot|$, (1) still holds with another constant and $(\mathcal{U},|\cdot|)$ is an r-Banach space satisfying this weak maximum modulus principle. According to [13, Theorem 3.7], there exists an r-norm

equivalent to $|\cdot|$ which is plurisubharmonic. Throughout the paper, whenever an A-convex space $(\mathcal{U}, \|\cdot\|)$ appears, we will suppose that $\|\cdot\|$ is at the same time r-norm and plurisubharmonic, thus we will have $C=1$ in (1).

Also we shall make free use of the basic terminology of interpolation theory, like that of [3], and follow [13] for definitions and results about A-convex spaces.

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2. Interpolation pairs with an A-convex containing space

Let (X_0, X_1) be a compatible pair of quasi-Banach spaces. The pair (X_0, X_1) will be said to have an A-convex containing space if there is an A-convex space $\mathcal{U}$ so that

$$X_0+X_1 \subset \mathcal{U}$$

with continuous inclusion. We will denote by $\|\cdot\|_{\mathcal{U}}$ and $\|\cdot\|_j$ respectively the quasi-norms of $\mathcal{U}$ and X_j, $j=0, 1$.

We define the space $\mathcal{H}(\mathcal{U})=\mathcal{H}(X_0, X_1; \mathcal{U})$ to consist of all functions $f\in A(S, \mathcal{U})$ such that $f(j+it)\in X_j$, and $f(j+it)$ defines a bounded continuous function f_j from $\mathbf{R}$ to X_j, $j=0, 1$.

Since $\mathcal{U}$ is A-convex, the three lines theorem for functions with values in $\mathcal{U}$ can be proved from (1). Also, by [12, Theorem 6.3], $A(S,\mathcal{U})$ is complete with respect to the quasi-norm $\|f\|=\sup_{z\in S}\|f(z)\|_{\mathcal{U}}$. The argument given in [3] for the Banach space case works now to prove that

$$\|f\|_{\mathcal{H}(\mathcal{U})} = \max_{j=0,1} \sup_{t\in\mathbf{R}} \|f_j(t)\|_j$$

defines a complete quasi-norm on $\mathcal{H}(\mathcal{U})$, and we do not need a third term like in [11].

As in [11], we consider the $\|\cdot\|_{\mathcal{H}(\mathcal{U})}$-closure $\mathcal{F}(\mathcal{U})=\mathcal{F}(X_0, X_1; \mathcal{U})$ of

$$\mathcal{G} = \left\{\sum_{n=1}^N \varphi_n x_n\colon x_n\in X_0\cap X_1,\ \varphi_n\in A(S, \mathbf{C}),\ N\in\mathbf{N}\right\}.$$

For $0\leqq\theta\leqq1$, $(X_0, X_1)_{[\theta],\mathcal{U}}$ will denote the space of all $f(\theta)$, $f\in\mathcal{F}(\mathcal{U})$, with the complete quasi-norm

$$\|x\|_{[\theta],\mathcal{U}} = \inf\{\|f\|_{\mathcal{H}(\mathcal{U})}\colon f\in\mathcal{F}(\mathcal{U}),\ f(\theta) = x\}.$$

The same spaces and quasi-norms are obtained if we consider functions in $\mathcal{F}(\mathcal{U})$ vanishing at infinity (multiply by suitable scalar functions).

Thus the existence of an A-convex containing space $\mathcal{U}$ allows us to make the construction in a very similar way to that of Calderón [4] and guarantees that the technical difficulties mentioned in Section 1 do not arise. Also Proposition 5, which is a result on interpolation of Banach spaces and, as far as we know, appears here for the first time is proved following ideas from the A-convex case.

We shall see that, in general, $f \in \mathscr{F}(\mathscr{U})$ does not take its values in the sum X_0+X_1, and so $(X_0, X_1)_{[\theta], \mathscr{U}}$ need not to be contained there. However we will prove that the interpolation spaces are independent in a certain sense of the choice of $\mathscr{U}$.

3. The independence theorem

Let (X_0, X_1) be a compatible pair of quasi-Banach spaces and suppose that there exist two A-convex spaces $\mathscr{U}$ and $\mathscr{V}$ such that $X_0+X_1 \subset \mathscr{U}$ and also $X_0+X_1 \subset \mathscr{V}$ with continuous inclusions. Suppose further that both of the spaces $\mathscr{U}$ and $\mathscr{V}$ are contained with continuous inclusions in an arbitrary Hausdorff topological vector space $\mathscr{A}$.

Theorem 1. *The spaces $\mathscr{F}(\mathscr{U})$ and $\mathscr{F}(\mathscr{V})$ are equal as sets of $\mathscr{A}$-valued functions and their quasi-norms coincide. The spaces $(X_0, X_1)_{[\theta], \mathscr{U}}$ and $(X_0, X_1)_{[\theta], \mathscr{V}}$ are isometric and coincide as vector subspaces of $\mathscr{A}$.*

Proof. Clearly, we only have to prove the assertion about the spaces of analytic functions. If $f \in \mathscr{F}(\mathscr{U})$, pick $g_n \in \mathscr{G}$ so that $\|g_n - f\|_{\mathscr{H}(\mathscr{U})} \to 0$ as $n \to \infty$. The functions g_n converge X_j-uniformly to f over $j+i\mathbf{R}$, $j=0, 1$. By the A-convexity of $\mathscr{U}$, they also converge $\mathscr{U}$-uniformly to f over S. On the other hand, $\{g_n\}_n$ is also a Cauchy sequence in $\mathscr{H}(\mathscr{V})$, so there exists $f_1 \in \mathscr{F}(\mathscr{V})$ with $\|g_n - f_1\|_{\mathscr{H}(\mathscr{V})} \to 0$ as $n \to \infty$. Thus $g_n \to f_1$ as $n \to \infty$ X_j-uniformly over $j+i\mathbf{R}, j=0, 1$, and $\mathscr{V}$-uniformly over S. Since both $\mathscr{U}$ and $\mathscr{V}$ are continuously contained on $\mathscr{A}$, it follows that $f=f_1 \in \mathscr{F}(\mathscr{V})$. Thus $\mathscr{F}(\mathscr{U}) \subset \mathscr{F}(\mathscr{V})$ and, interchanging the roles of $\mathscr{U}$ and $\mathscr{V}$, the equality follows. ∎

In view of Theorem 1, notations like $(X_0, X_1)_{[\theta], \mathscr{U}}$ and $\mathscr{F}(\mathscr{U})$ can now be replaced by $(X_0, X_1)_{[\theta], \mathscr{A}}$ and $\mathscr{F}(\mathscr{A})$ to indicate that these spaces do not depend on the containing A-convex spaces $\mathscr{U}$ so that

$$X_0+X_1 \subset \mathscr{U} \subset \mathscr{A}.$$

However, in order to simplify the notation, we will denote by $(X_0, X_1)_{[\theta]}$ the interpolation spaces and by $\mathscr{F}$ or $\mathscr{F}(X_0, X_1)$ the corresponding spaces of analytic functions if there is no danger of confusion.

We will see later that we cannot dispense with the condition in Theorem 1 that both $\mathscr{U}$ and $\mathscr{V}$ must be contained in the same space $\mathscr{A}$.

Let $(X, \|\cdot\|)$ be a quasi-Banach space and, as in [13], denote

$$\|x\|_A = \inf\{\max_{|z|=1} \|\phi(z)\| : \phi \in A(\Delta, X), \phi(0) = x\}$$

the biggest plurisubharmonic quasi-seminorm on X such that $\|\cdot\|_A \leqq \|\cdot\|$.

If we have a continuous embedding $X \subset \mathscr{U}$, $\mathscr{U}$ being A-convex, it follows that

$$\|x\|_{\mathscr{U}} \leqq C\|x\|_A \tag{2}$$

for each $x \in X$ and consequently in this case $\|\cdot\|_A$ is a quasi-norm on X. The completion X_A of $(X, \|\cdot\|_A)$ is an A-convex space and X is continuously embedded in X_A.

If a quasi-Banach space has a separating dual, it has a Banach envelope. In analogy to this case, we call X_A the *A-convex envelope* of X. It follows that any continuous linear operator from X into an A-convex space factors through X_A. See the remarks after [13, Theorem 3.7].

It will be convenient to compute the interpolation spaces $(X, X)_{[\theta], X_A}$ when $0<\theta<1$.

Lemma 1. *If* $0<\theta<1$, *the relation*

$$(X, X)_{[\theta], X_A} = X_A$$

holds with equality of quasi-norms.

Proof. By the definition of the interpolation spaces, it follows that $(X, X)_{[\theta], X_A} \subset X_A$, and

$$\|x\|_A \leqq \|x\|_{[\theta], X_A}, \tag{3}$$

for each $x \in (X, X)_{[\theta], X_A}$.

Let $x \in X_A$. We consider $x_n \in X$, $n \in \mathbf{N}$, so that $\sum_{n=1}^{\infty} x_n = x$, the series being convergent in X_A, and

$$\sum_{n=1}^{\infty} \|x_n\|_A^r < \|x\|_A^r + \varepsilon.$$

Pick $f_n \in A(\Delta, X)$ such that $f_n(0) = x_n$ and

$$\|f_n\|_{\mathbf{T}}^r < \|x_n\|_A^r + \frac{\varepsilon}{2^n},$$

where

$$\|f_n\|_{\mathbf{T}} = \max_{|w|=1} \|f_n(w)\|.$$

Thus, the series $\sum_{n=1}^{\infty} f_n$ converges X-uniformly on $\mathbf{T}$ (and hence X_A-uniformly on Δ) to a function $f \in A(\Delta, X_A)$ because $\sum_{n=1}^{\infty} \|f_n\|_{\mathbf{T}}^r < +\infty$.

Let Φ be a conformal mapping between $\bar{S}$ and $\bar{\Delta} \setminus \{z_0, z_1\}$ where z_0 and z_1 are certain points in $\mathbf{T}$ and $\Phi(\theta) = 0$. The function $g(z) = f(\Phi(z))$ belongs to $\mathscr{F}(X_A)$.

To see this, consider $\delta>0$ and $N\geqq 1$ such that

$$\sum_{n=N+1}^{\infty} \|f_n\|_{\mathbf{T}}^r < \delta.$$

Denote by $\varphi_1=\sum_{n=1}^{N} f_n$. It follows that $\varphi_1\in A(\Delta, X)$. If we dilate a little, we will obtain a function φ_2, X-uniformly close to φ_1 over $\bar{\Delta}$, defined and analytic on a neighborhood of $\bar{\Delta}$. Hence the power series of φ_2 will converge X-uniformly over $\bar{\Delta}$. We take a partial sum of the power series of φ_2 X-uniformly close to φ_2 on $\bar{\Delta}$ and we obtain an X-valued function of finite rank φ_3 (a polynomial), X-uniformly close to φ_1 over $\bar{\Delta}$. Say $\|\varphi_1-\varphi_3\|_{\mathbf{T}}^r<\delta$. Then $\varphi_3\circ\Phi\in\mathscr{G}$ and $\|g-\varphi_3\circ\Phi\|_{\mathscr{H}(X_A)}^r<2\delta$.

Thus we have that $g\in\mathscr{F}(X_A)$ and $g(\theta)=x$, so $x\in(X, X)_{[\theta], X_A}$ and the equality $(X, X)_{[\theta], X_A}=X_A$ follows.

Moreover

$$\begin{aligned}\|x\|_{[\theta], X_A}^r &\leqq \|g\|_{\mathscr{H}(X_A)}^r \\ &= \|f\|_{\mathbf{T}}^r \\ &\leqq \sum_{n=1}^{\infty} \|f_n\|_{\mathbf{T}}^r \\ &< \|x\|_A^r+2\varepsilon.\end{aligned}$$

From this inequality, and from (3), the lemma follows. ∎

In the sequel we will need to use the fact that there exist quasi-Banach spaces which have separating duals but which are not A-convex. The following example of such a space was suggested to us by N. J. Kalton.

Consider $E=L^p/H^p$ $(0<p<1)$ and $f\in A(\Delta, E)$ vanishing on $\mathbf{T}$ but with $\|f(0)\|=1$. See [1] or [12, pp. 276 and 278]. As in the proof of Lemma 1, we can obtain $f_n\in A(\Delta, E)$ with finite rank and so that $\|f_n(0)\|=1$ and

$$\|f_n\|_{\mathbf{T}} \to 0,$$

as $n\to\infty$. Let X_n be the linear span of $f_n(\Delta)$ with the quasi-norm $\|\cdot\|$ of E, and define the space X to consist of all sequences $x=\{x_n\}_n$ with $x_n\in X_n$ and $\|x\|_X=\sum\|x_n\|<\infty$.

It is clear that X has a separating dual. However it is not A-convex, since the functions $F_m(z)=\{f_{mn}(z)\}_n$, with

$$f_{mn}=\begin{cases} f_n, & \text{if} \quad n=m, \\ 0, & \text{if} \quad n\neq m,\end{cases}$$

are in $A(\Delta, X)$ and satisfy $\|F_m(z)\|_X=\|f_m(z)\|$ for all $z\in\bar{\Delta}$.

The following proposition will enable us, among other things, to show that in some cases the space $(X_0, X_1)_{[\theta], \mathscr{A}}$ may depend on $\mathscr{A}$. If X is a quasi-Banach space so that for an A-convex space $\mathcal{U}$ the continuous inclusion $X\subset\mathcal{U}$ holds, (2) shows that the inclusion extends continuously to an operator

$$J\colon X_A \to \mathcal{U}.$$

Proposition 1. *Let X be a non A-convex space with separating dual. There exists a Banach space $\mathcal{U}$ such that $X\hookrightarrow\mathcal{U}$ but the continuous extension*

$$J\colon X_A \to \mathcal{U}$$

is not one to one.

Proof. Let $\|\cdot\|_1$ be the Banach envelope norm of the quasi-Banach space X. Thus $\|\cdot\|_1$ is the Minkowski functional of the convex envelope of the unit ball on X. Denote by X_1 the completion of the normed space $(X, \|\cdot\|_1)$. The Banach space X_1 is an A-convex containing space for X, hence the inclusion

$$(X, \|\cdot\|_A) \subset X_1$$

is continuous and extends to a continuous operator $J\colon X_A\to X_1$.

If $J(X_A)\subset X$, then J is not one to one and we have a counterexample, $\mathcal{U}=X_1$.

Suppose that $J(X_A)\not\subset X$. Pick $v=Jx_0\in J(X_A)\setminus X$, $x_0\in X_A\setminus X$ and consider $\mathcal{U}=X_1/[v]$. Let $\pi\colon X_1\to\mathcal{U}$ denote the quotient map, then the restriction of π to X is a continuous inclusion of X into the Banach space $\mathcal{U}$ and the continuous extension to X_A is $\pi\circ J$, which vanishes at $x_0\neq 0$. ∎

Now, to see, as mentioned above, that $f\in\mathcal{F}$ need not take its values in X_0+X_1 we simply choose $X_0=X_1=X$ with X and $\mathcal{U}$ as in Proposition 1. By Lemma 1, $(X_0, X_1)_{[\theta], X_A}$ is strictly larger that X_0+X_1.

4. Some properties of the spaces $(X_0, X_1)_{[\theta]}$

Let (X_0, X_1) be an interpolation pair with the containing A-convex space $\mathcal{U}$, and $0<\theta<1$. Let (Y_0, Y_1) be another interpolation pair with an A-convex containing space $\mathcal{V}$, and let

$$T\colon \mathcal{U} \to \mathcal{V}$$

be a linear continuous map which is of type (X_j, Y_j) with constant M_j $(j=0, 1)$.

Theorem 2. *The operator T is bounded from $(X_0, X_1)_{[\theta]}$ to $(Y_0, Y_1)_{[\theta]}$ with constant $M_0^{1-\theta}M_1^{\theta}$.*

Proof. See [3]. ∎

If the operator T is only defined in the sum X_0+X_1 and is of type (X_j, Y_j) with constant M_j, it extends continuously to

$$T_A\colon (X_0+X_1)_A \to \mathscr{V},$$

thus Theorem 2 applies and T_A maps $(X_0, X_1)_{[\theta], (X_0+X_1)_A}$ into $(Y_0, Y_1)_{[\theta]}$. This observation makes it interesting to compute the interpolation spaces with respect to the A-convex envelope of the sum space. In the section of applications we have done this for a class of vector valued L^p spaces.

If

$$Y \subset \mathscr{U} \subset \mathscr{A}$$

we denote by $Y_{\mathscr{A}}$ the interpolation space $(Y, Y)_{[\theta]}$. We note that $(Y, Y)_{[\theta]}$ are the same spaces if $0<\theta<1$. Also, by Proposition 1, it follows that there are examples where the natural map between Y_A and $Y_{\mathscr{A}}$, the extension to Y_A of the inclusion $Y\subset\mathscr{U}$, is not one to one (the fact that this extension maps Y_A into $Y_{\mathscr{A}}$ can be deduced from Theorem 2 and it can be proved directly that it maps Y_A onto $Y_{\mathscr{A}}$).

Proposition 2. *The intersection* $X_0\cap X_1$ *is dense in* $(X_0, X_1)_{[\theta]}$, $(X_0, X_1)_{[j]}=X_j^\circ$ *(the closure of* $X_0\cap X_1$ *in* X_j, $j=0, 1$*) and*

$$(X_0\cap X_1)_{\mathscr{A}} \subset (X_0, X_1)_{[\theta]} = (X_0^\circ, X_1^\circ)_{[\theta]} \subset (X_0^\circ+X_1^\circ)_{\mathscr{A}} \subset \mathscr{U}.$$

Proof. $X_0\cap X_1$ is dense in $(X_0, X_1)_{[\theta]}$ by construction. The equalities

$$X_{[j]} = X_j^\circ \quad \text{and} \quad (X_0, X_1)_{[\theta]} = (X_0^\circ, X_1^\circ)_{[\theta]}$$

can be proved as in the locally convex case. See [3].

Thus $\mathscr{U}$ is a containing space of (X_0°, X_1°) and

$$(X_0\cap X_1)_{\mathscr{A}} \subset (X_0, X_1)_{[\theta]} \subset (X_0^\circ+X_1^\circ)_{\mathscr{A}} \subset \mathscr{U}. \qquad \blacksquare$$

As mentioned in the previous section, if all the conditions of Theorem 1 are not fulfilled, we may have a situation similar to that described in [7] where the interpolation spaces $(X_0, X_1)_{[\theta], \mathscr{U}}$ can be essentially different for different choices of the containing space $\mathscr{U}$. Let us describe this phenomenon more precisely. We shall consider a particular couple (X_0, X_1) for which there exist A-convex spaces $\mathscr{U}$ and $\mathscr{V}$ both of which continuously contain X_0+X_1. Let T be any continuous linear map from $(X_0, X_1)_{[\theta], \mathscr{V}}$ into $(X_0, X_1)_{[\theta], \mathscr{U}}$ whose restriction to $X_0\cap X_1$ is the identity map. We shall see that T cannot be one to one. This example is obtained using the spaces X and $\mathscr{U}$ and the map J of Proposition 1, and setting $X_0=X_1=X$ and $\mathscr{V}=X_A$. The restrictions of the maps T and J to $X_0\cap X_1$ coincide, so by density and interpolation (Theorem 2 and Proposition 2) they also coincide on $X_A=(X_0, X_1)_{[\theta], \mathscr{V}}$.

We want to note that the following result of [16] holds true in our setting:

Proposition 3. *Let* φ_j $(j=0, 1)$ *be a pair of increasing functions on* $(0, +\infty)$ *such that* $\varphi_j(\exp t)$ *is convex, and let* $f\in\mathcal{F}$. *Then*

(a)
$$\log\|f(\theta)\|_{[\theta]} \leqq \int_{-\infty}^{\infty} \log\|f(it)\|_0 P_0(\theta, t)\,dt + \int_{-\infty}^{\infty} \log\|f(1+it)\|_1 P_1(\theta, t)\,dt,$$

(b)
$$\|f(\theta)\|_{[\theta]} \leqq \left\{\varphi_0^{-1}\left[\frac{1}{1-\theta}\int_{-\infty}^{\infty} \varphi_0(\|f(it)\|_0) P_0(\theta, t)\,dt\right]\right\}^{1-\theta} \cdot \left\{\varphi_1^{-1}\left[\frac{1}{\theta}\int_{-\infty}^{\infty} \varphi_1(\|f(1+it)\|_1) P_1(\theta, t)\,dt\right]\right\}^{\theta}.$$

Proof. For (a) see [3] or [4]. We sketch the proof of (b) given in [16] to provide an easy reference. The inequality in (a) can be re-written as

$$\|f(\theta)\|_{[\theta]} \leqq \left(\exp\frac{1}{1-\theta}\int_{-\infty}^{\infty} \log\|f(it)\|_0 P_0(\theta, t)\,dt\right)^{1-\theta} \cdot \left(\exp\frac{1}{\theta}\int_{-\infty}^{\infty} \log\|f(1+it)\|_1 P_1(\theta, t)\,dt\right)^{\theta}.$$

Now we observe that

$$\int_{-\infty}^{\infty} P_0(\theta, t)\,dt = 1-\theta \quad\text{and}\quad \int_{-\infty}^{\infty} P_1(\theta, t)\,dt = \theta.$$

Then, since $\varphi_j(\exp t)$ $(j=0, 1)$ are convex functions, we have

$$\varphi_0\left[\exp\left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} \log\|f(it)\|_0 P_0(\theta, t)\,dt\right)\right] \leqq \frac{1}{1-\theta}\int_{-\infty}^{\infty} \varphi_0(\|f(it)\|_0) P_0(\theta, t)\,dt,$$

and a corresponding inequality for φ_1. ∎

Finally we show a relation with the interpolation method given in [16] and [8]. If (X_0, X_1) is a general quasi-Banach couple, in [16] and [8] the quasi-seminorm $\|x\|_\theta$ is defined in $X_0\cap X_1$ by

$$\|x\|_\theta = \inf\{\|f\|_{\mathcal{F}}: f\in\mathcal{G}, f(\theta) = x\},$$

where the space $\mathcal{G}$ and the quasi-norm $\|\cdot\|_{\mathcal{F}}$ on it are the ones introduced in Section 1 as in the locally convex case. If (X_0, X_1) has a containing A-convex space $\mathcal{U}$, the relation $\|x\|_{\mathcal{U}}\leqq C\|x\|_\theta$ holds for some absolute constant C and all $x\in X_0\cap X_1$. Thus, in this case, $\|\cdot\|_\theta$ is a genuine quasi-norm. Moreover:

Proposition 4. *The inclusion*

$$(X_0 \cap X_1, \|\cdot\|_\theta) \hookrightarrow (X_0, X_1)_{[\theta]}$$

is continuous.

It is unknown to us whether the continuous extension to the completion is one to one, at least in the case where the containing space is taken to be $(X_0+X_1)_A$.

5. Examples

The following examples show a number of quasi-Banach couples (X_0, X_1) with a containing A-convex space.

Example 1. If X_0 is continuously contained in X_1 and X_1 is A-convex, then X_1 is a containing A-convex space. This is the case of the pairs (L^{p_1}, L^{p_0}) on a finite measure space, $(H^{p_1}(\Delta), H^{p_0}(\Delta))$ and (l^{p_0}, l^{p_1}), if $0<p_0<p_1<\infty$.

Example 2. (Hardy spaces on $\mathbf{R}^n$.) For the pair $(H^{p_0}(\mathbf{R}^n), H^{p_1}(\mathbf{R}^n))$, the sum can be imbedded into a Banach space because it has a separating dual.

To see this, take $\phi \in S(\mathbf{R}^n)$ with $\hat{\phi}(0) \neq 0$ and define

$$\phi_t(y) = t^{-n}\phi(t^{-1}y),$$

for $y \in \mathbf{R}^n$, $t>0$, and

$$u(x, t) = f * \phi_t(x).$$

It is well-known that

$$|u(x, t)| \leqq C(n)\, t^{-n/p} \|f\|_{H^p},$$

if $f \in H^p$, and the evaluations $f \mapsto u(x, t)$ are continuous on the sum space. We remark that when computing the interpolated spaces for this example, another containing space will be used.

Example 3. (L^p spaces on a σ-finite measure space (Ω, Σ, μ)). If $(Y, \|\cdot\|)$ is a quasi-Banach space, a function $f\colon \Omega \to Y$ is said to be measurable if it is the limit of an almost everywhere convergent sequence of measurable simple functions. If $0<p<\infty$, $L^p(Y)$ will be the completion of the simple measurable functions with respect to the quasi-norm

$$\|f\|_{L^p(Y)} = \left(\int_\Omega \|f(\omega)\|_Y^p \, d\mu(\omega)\right)^{1/p}.$$

If f is a measurable function, so is $\|f(\cdot)\|_Y$. It follows that $L^p(Y)$ (denoted by $\tilde{L}^p(Y)$ in [22]) consists of all measurable functions f such that $\|f\|_{L^p(Y)} < +\infty$. See [22] "korollar 4.(2)".

For every quasi-Banach pair (X_0, X_1) with an A-convex containing space $\mathscr{U}$, let us see that $(L^{p_0}(X_0), L^{p_1}(X_1))$ has an A-convex containing space too. Suppose for simplicity that the inclusion $X_0+X_1\subset\mathscr{U}$ is norm decreasing. Define $\varrho=\min\{p_0, p_1, 1\}$.

Let $\{B_n\}_n$ be an increasing sequence of measurable sets of finite measure such that $\Omega=\bigcup_{n=1}^{\infty}B_n$. For $f=f_0+f_1$ in $L^{p_0}(X_0)+L^{p_1}(X_1)$, we have $(j=0, 1)$

$$\|f_j\chi_{B_n}\|_{L^{\varrho}(\mathscr{U})} \leqq \mu(B_n)^{(1/\varrho)-(1/p_j)}\|f_j\|_{L^{p_j}(X_j)},$$

so that, for a certain constant $C(n)=C(n, p_0, p_1, \mathscr{U})>0$,

$$\|f\chi_{B_n}\|_{L^{\varrho}(\mathscr{U})} \leqq C(n)\|f\|_{L^{p_0}(X_0)+L^{p_1}(X_1)}.$$

Now we define the space $\mathscr{V}$ to consist of all measurable $\mathscr{U}$-valued functions g such that

$$\|g\|_{\mathscr{V}} = \sup_{n\geqq 1} C(n)^{-1}\|g\chi_{B_n}\|_{L^{\varrho}(\mathscr{U})} < +\infty.$$

We notice that $L^{\varrho}(\mathscr{U})$ is A-convex. In fact, if $\|\cdot\|_{\mathscr{U}}$ is plurisubharmonic and is an r-norm, then $\|\cdot\|_{\mathscr{U}}=\|\cdot\|_{\mathscr{U}_A}$. Then, by [13, Theorem 3.7], the functions $\log\|\cdot\|_{\mathscr{U}}$, and hence $\|\cdot\|_{\mathscr{U}}^{\varrho}$, are plurisubharmonic. Thus the following inequalities hold:

$$\begin{aligned}\|f\|_{L^{\varrho}(\mathscr{U})} &= \Big(\int_{\Omega}\|f(\omega)\|_{\mathscr{U}}^{\varrho}\,d\mu(\omega)\Big)^{1/\varrho}\\ &\leqq \Big(\int_{\Omega}\frac{1}{2\pi}\int_0^{2\pi}\|f(\omega)+e^{is}g(\omega)\|_{\mathscr{U}}^{\varrho}\,ds\,d\mu(\omega)\Big)^{1/\varrho}\\ &\leqq \frac{1}{2\pi}\int_0^{2\pi}\Big(\int_{\Omega}\|f(\omega)+e^{is}g(\omega)\|_{\mathscr{U}}^{\varrho}\,d\mu(\omega)\Big)^{1/\varrho}ds\\ &= \frac{1}{2\pi}\int_0^{2\pi}\|f+e^{is}g\|_{L^{\varrho}(\mathscr{U})}\,ds.\end{aligned}$$

After this observation, we prove that $\|\cdot\|_{\mathscr{V}}$ is plurisubharmonic. If $F\in A(\Delta, \mathscr{V})$, we have

$$\begin{aligned}\|F(0)\|_{\mathscr{V}} &= \sup_{n\geqq 1} C(n)^{-1}\|F(0)\chi_{B_n}\|_{L^{\varrho}(\mathscr{U})} \leqq \sup_{n\geqq 1} C(n)^{-1}\sup_{|w|=1}\|F(w)\chi_{B_n}\|_{L^{\varrho}(\mathscr{U})}\\ &= \sup_{|w|=1}\|F(w)\|_{\mathscr{V}}.\end{aligned}$$

Also, completeness of $\mathscr{V}$ follows easily from the completeness of $L^{\varrho}(\mathscr{U})$.

Thus $L^{p_0}(X_0)+L^{p_1}(X_1)$ is continuously contained in the A-convex space $\mathscr{V}$.

Example 4. (Tent spaces of Coifman, Meyer and Stein [6].) Let $0<p, q<\infty$. The tent space T_q^p is the set of all measurable functions f on $\mathbf{R}_+^{n+1}$ such that

$$\|f\|_{p,q} = \left[\int_{\mathbf{R}^n} \left(\int_{\Gamma(x)} |f(y,t)|^q \frac{dy\,dt}{t^{n+1}}\right)^{p/q} dx\right]^{1/p} < +\infty,$$

where $\Gamma(x)$ is the cone of all $(y,t)\in\mathbf{R}_+^{n+1}$ such that $y\in B(x,t)$.

If f is a function defined on $\mathbf{R}_+^{n+1}$, we define

$$Tf(x) = \chi_{\Gamma(x)} f$$

if $x\in\mathbf{R}^n$. If $f\in T_q^p$ it is clear that, for almost every $x\in\mathbf{R}^n$ $Tf(x)$ belongs to the L^q space on $\mathbf{R}_+^{n+1}$ with respect to the measure $dy\,dt/t^{n+1}$. We denote by L_*^q this space. Thus we have a function $Tf\colon \mathbf{R}^n\to L_*^q$.

Lemma 2. *The function Tf is measurable.*

Proof. The proof is straightforward and we leave the details to the reader. First one has to prove that measurability is preserved by a.e. limits, this can be done using Egorov's theorem. Now, approximating f by simple scalar functions σ defined on $\mathbf{R}_+^{n+1}$, it is enough to prove the measurability of $\sigma\chi_{\Gamma(\cdot)}$. Also, we can restrict ourselves to consider the functions $\chi_K\chi_{\Gamma(\cdot)}$, whenever $K\subset\mathbf{R}_+^{n+1}$ is compact. Now, if $I_1^r, \dots, I_{N_r}^r$ are the dyadic cubes of volume 2^{-rn} which are contained in $B(0,r)$, we consider $x_j^r\in I_j^r$ and define the simple L_*^q-valued function defined on $\mathbf{R}^n$

$$S_r(x) = \sum_{k=1}^{N_r} \chi_{I_k^r}(x)(\chi_K\chi_{\Gamma(x_k^r)}).$$

It follows that, for each $x\in\mathbf{R}^n$,

$$\|S_r(x)-\chi_K\chi_{\Gamma(x)}\|_{L_*^q}\to 0,$$

as $r\to\infty$. ∎

It is clear that the measurable function Tf is in $L^p(L_*^q)$. Here we understand L^p as the Lebesgue space on $\mathbf{R}^n$ with respect to the Lebesgue measure. Moreover, it is also clear that T is a one to one map from the space $\mathcal{M}$ of all complex measurable functions on $\mathbf{R}_+^{n+1}$ to the space of all functions defined on $\mathbf{R}^n$ with values in $\mathcal{M}$.

The operator T maps $T_{q_0}^{p_0}+T_{q_1}^{p_1}$ into $L^{p_0}(L_*^{q_0})+L^{p_1}(L_*^{q_1})$ and it is an isometry from $T_{q_j}^{p_j}$ into $L^{p_j}(L_*^{q_j})$, $0<p_j, q_j<\infty$, $(j=0,1)$. From Example 3, we have an A-convex containing space $\mathscr{V}$ for the pair $(L^{p_0}(L_*^{q_0}), L^{p_1}(L_*^{q_1}))$. We define $\mathscr{W}$ to be the set of all complex measurable functions f on $\mathbf{R}_+^{n+1}$ so that $Tf\in\mathscr{V}$ and we put in $\mathscr{W}$ the induced quasi-norm by $\mathscr{V}$. It follows by standard arguments that $\mathscr{W}$ is an A-convex quasi-Banach space which contains continuously the sum $T_{q_0}^{p_0}+T_{q_1}^{p_1}$.

Now, some interpolation spaces can be computed.

Theorem 3. *Let* $0<p_0, p_1<\infty$, $0\leq\theta\leq 1$, (X_0, X_1), $\mathscr{U}$ *and* $\mathscr{V}$ *be as in Example 3. Then*

$$\left(L^{p_0}(X_0), L^{p_1}(X_1)\right)_{[\theta],\mathscr{V}} = L^p\left((X_0, X_1)_{[\theta],\mathscr{U}}\right)$$

isometrically (here $1/p=(1-\theta)/p_0+\theta/p_1$*).*

It will be useful to have the following auxiliary result:

Lemma 3. *The space* $\mathscr{S}(X_0\cap X_1)$ *of all simple* $X_0\cap X_1$*-valued measurable functions is dense in* $L^{p_0}(X_0)\cap L^{p_1}(X_1)$.

Proof. Let B_n be an increasing sequence of measurable sets on Ω of finite measure with $\bigcup B_n=\Omega$. Let $f\in L^{p_0}(X_0)\cap L^{p_1}(X_1)$ and $\varepsilon>0$. The functions $f\chi_{B_n}$ approximate f in $L^{p_0}(X_0)\cap L^{p_1}(X_1)$ and so we only have to approximate $f\chi_{B_n}$ in $L^{p_0}(X_0)\cap L^{p_1}(X_1)$ by simple $X_0\cap X_1$-valued measurable functions.

Fix an integer n. The function $f\chi_{B_n}$ maps B_n into $X_0\cap X_1$ and is X_j-measurable, $(j=0, 1)$. The argument given in [4, pp. 171—172] works, without any change, to prove that $f\chi_{B_n}$ is $X_0\cap X_1$-measurable. Pick simple $X_0\cap X_1$-measurable functions S_m supported on B_n so that

$$\|S_m(x)-f(x)\|_{X_0\cap X_1} \to 0,$$

as $m\to\infty$, for almost each $x\in B_n$. Let $\delta>0$ be such that, whenever N is a measurable subset of B_n with $\mu(N)<\delta$ it follows that

$$\int_N \left(\|f(x)\|_0^{p_0}+\|f(x)\|_1^{p_1}\right) d\mu(x) < \varepsilon.$$

By Egorov's theorem, there is a measurable subset N_δ of B_n with $\mu(N_\delta)<\delta$ so that S_m converges $X_0\cap X_1$-uniformly to $f\chi_{B_n}$ outside N_δ. Define $\sigma_m=S_m\chi_{B_n\setminus N_\delta}$. The functions σ_m are in $\mathscr{S}(X_0\cap X_1)$ and, if $j=0, 1$,

$$\|\sigma_m-f\chi_{B_n}\|_{L^{p_j}(X_j)}^{p_j} < \varepsilon+\int_{B_n\setminus N_\delta} \|S_m(x)-f(x)\|_{X_0\cap X_1}^{p_j} d\mu(x).$$

By the uniform convergence of S_m over $B_n\setminus N_\delta$, the last integral can be made arbitrarily small and the lemma is proved. ∎

Proof of Theorem 3. The case where $\theta=0$ or $\theta=1$ follows from Proposition 2. We consider $0<\theta<1$.

(a) Let $f\in\left(L^{p_0}(X_0), L^{p_1}(X_1)\right)_{[\theta]}$ and $F\in\mathscr{F}\left(L^{p_0}(X_0), L^{p_1}(X_1)\right)$ with $F(\theta)=f$. We consider F_n of finite rank so that $\|F_n-F\|_{\mathscr{K}(\mathscr{V})}\to 0$ as $n\to\infty$. In view of Lemma 3 we can in fact assume that each F_n is of the form $\sum_{j=1}^N \phi_j x_j$, where $\phi_j\in A(S, \mathbf{C})$ and $x_j\in\mathscr{S}(X_0\cap X_1)$. For $x\in\Omega$, the functions $F_{n,x}(z)=F_n(z)(x)$ are in $\mathscr{F}(X_0, X_1)$.

We apply Proposition 3 to the functions $\varphi_j(t)=t^{p_j}$ $(j=0, 1)$ and we obtain

$$\|F_{n,x}(\theta)\|_{(X_0,X_1)_{[\theta]}} \leqq \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} \|F_{n,x}(it)\|_0^{p_0} P_0(\theta, t)\,dt\right)^{(1-\theta)/p_0} \cdot \left(\frac{1}{\theta}\int_{-\infty}^{\infty} \|F_{n,x}(1+it)\|_1^{p_1} P_1(\theta, t)\,dt\right)^{\theta/p_1}. \tag{2}$$

We observe that the right-hand side of the last inequality is a simple measurable function of x. Also $f_n=F_n(\theta)\in\mathscr{S}(X_0\cap X_1)$ are $(X_0, X_1)_{[\theta]}$-measurable. Hence following the usual steps, we obtain from (2) the estimates

$$\begin{aligned}
\|f_n\|^p_{L^p((X_0,X_1)_{[\theta]})} &\leqq \int_\Omega \left[\left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} \|F_{n,x}(it)\|_0^{p_0} P_0(\theta, t)\,dt\right)^{p(1-\theta)/p_0}\right. \\
&\quad \left.\cdot\left(\frac{1}{\theta}\int_{-\infty}^{\infty} \|F_{n,x}(1+it)\|_1^{p_1} P_1(\theta, t)\,dt\right)^{p\theta/p_1}\right] d\mu(x) \\
&\leqq \left(\int_\Omega \frac{1}{1-\theta}\int_{-\infty}^{\infty} \|F_{n,x}(it)\|_0^{p_0} P_0(\theta, t)\,dt\,d\mu(x)\right)^{p(1-\theta)/p_0} \\
&\quad \cdot\left(\int_\Omega \frac{1}{\theta}\int_{-\infty}^{\infty} \|F_{n,x}(1+it)\|_1^{p_1} P_1(\theta, t)\,dt\,d\mu(x)\right)^{p\theta/p_1} \\
&= \left(\frac{1}{1-\theta}\int_{-\infty}^{\infty} \|F_n(it)\|^{p_0}_{L^{p_0}(X_0)} P_0(\theta, t)\,dt\right)^{p(1-\theta)/p_0} \\
&\quad \cdot\left(\frac{1}{\theta}\int_{-\infty}^{\infty} \|F_n(1+it)\|^{p_1}_{L^{p_1}(X_1)} P_1(\theta, t)\,dt\right)^{p\theta/p_1} \\
&\leqq \|F_n\|^p_{\mathscr{H}(\mathscr{V})}.
\end{aligned}$$

It follows that $\|f_n-f_m\|_{L^p((X_0,X_1)_{[\theta]})}\leqq\|F_n-F_m\|_{\mathscr{H}(\mathscr{V})}\to 0$, as $n, m\to\infty$ and there is $g\in L^p((X_0, X_1)_{[\theta]})$ so that

$$\|f_n-g\|_{L^p((X_0,X_1)_{[\theta]})} \to 0.$$

Also $f_n\to f$ in $(L^{p_0}(X_0), L^{p_1}(X_1))_{[\theta]}$. Since this space and $L^p((X_0, X_1)_{[\theta]})$ are both continuously contained in $\mathscr{V}$, we conclude that $f=g\in L^p((X_0, X_1)_{[\theta]})$ and we obtain the inclusion

$$(L^{p_0}(X_0), L^{p_1}(X_1))_{[\theta]} \subset L^p((X_0, X_1)_{[\theta]}). \tag{4}$$

Furthermore:

$$\begin{aligned}
\|f\|_{L^p((X_0,X_1)_{[\theta]})} &= \lim_n \|f_n\|_{L^p((X_0,X_1)_{[\theta]})} \\
&\leqq \lim_n \|F_n\|_{\mathscr{H}(\mathscr{V})} \\
&= \|F\|_{\mathscr{H}(\mathscr{V})},
\end{aligned}$$

and the above inclusion has norm not exceeding 1.

(b) For the reverse inclusion, let S be a simple function of the form

$$S = \sum_k x_k \chi_{E_k},$$

where $x_k \in X_0 \cap X_1$ and E_k are pairwise disjoint measurable sets on Ω of finite measure. Let $\phi_k \in \mathcal{F}(X_0, X_1)$ be such that $\phi_k(\theta) = x_k$ and $\|\phi_k\|_{\mathcal{H}(\mathcal{U})}$ is close to $\|x_k\|_{[\theta]}$.

As in [3, Theorem 5.1.2], define

$$F(z) = \sum_k \psi_k(z) \phi_k(z) \chi_{E_k},$$

where $\psi_k \in A(S, \mathbf{C})$ is defined by

$$\psi_k(z) = \left(\frac{\|x_k\|_{[\theta]}}{\|S\|_{L^p((X_0, X_1)_{[\theta]})}} \right)^{p((1/p_1)-(1/p_0))(z-\theta)}.$$

Since $\phi_k \in \mathcal{F}(X_0, X_1)$, it is easy to see that $F \in \mathcal{F}\big(L^{p_0}(X_0), L^{p_1}(X_1)\big)$. Since $F(\theta) = S$, it follows that

$$\|S\|_{(L^{p_0}(X_0), L^{p_1}(X_1))_{[\theta]}} \leqq \|F\|_{\mathcal{H}(\mathcal{V})}.$$

Now

$$\|F(j+it)\|_{L^{p_j}(X_j)} = \Big(\int_\Omega \sum_k |\psi_k(j+it)|^{p_j} \|\phi_k(j+it)\|_j^{p_j} \chi_{E_k}(x)\, d\mu(x)\Big)^{1/p_j}$$

$$\leqq \Big(\int_\Omega \sum_k |\psi_k(j+it)|^{p_j} \|x_k\|_{[\theta]}^{p_j} \chi_{E_k}(x)\, d\mu(x)\Big)^{1/p_j} + \varepsilon$$

$$= \|S\|_{L^p((X_0, X_1)_{[\theta]})} + \varepsilon.$$

We have proved that

$$\|S\|_{(L^{p_0}(X_0), L^{p_1}(X_1))_{[\theta]}} \leqq \|S\|_{L^p((X_0, X_1)_{[\theta]})}, \tag{5}$$

for each simple $X_0 \cap X_1$-valued function S.

This shows that the quasi-norms of $L^p((X_0, X_1)_{[\theta]})$ and of $(L^{p_0}(X_0), L^{p_1}(X_1))_{[\theta]}$ concide in $\mathcal{S}(X_0 \cap X_1)$. Applying Lemma 3, the theorem follows. ∎

In the preceeding theorem, the A-convex containing space $\mathcal{U}$ for the pair (X_0, X_1) is arbitrary and the space $\mathcal{V}$ is the associated A-convex space given in Example 3, or any other A-convex containing space which is compatible with $\mathcal{V}$ in the sense of Theorem 1.

The paricular nature of $\mathcal{V}$ was used in part (a) of the proof when it was claimed that the space $L^p((X_0, X_1)_{[\theta]})$ was continuously contained in $\mathcal{V}$.

In part (b) the nature of $\mathcal{V}$ was used to claim that F was in

$$\mathcal{F}(L^{p_0}(X_0), L^{p_1}(X_1); \mathcal{V})$$

and to obtain the reverse inclusion of (4) as a consequence of (5).

We observe however that, if $X_0=X_1=\mathcal{U}=\mathbf{C}$, the above proof works for any A-convex containing space for the pair (L^{p_0}, L^{p_1}). In the scalar case $L^p\subset L^{p_0}+L^{p_1}\subset\mathcal{V}$ and the functions ϕ_k can be taken to be costant, $\phi_k=x_k$, then

$$F\in\mathcal{G}(L^{p_0}, L^{p_1}) \tag{6}$$

and the result follows for any A-convex containing space for $L^{p_0}+L^{p_1}$.

Moreover, if $\|\cdot\|_\theta$ denotes the quasi-norm introduced in $L^{p_0}\cap L^{p_1}$ by the method of Rivière, Cwikel, Milman and Sagher [16], [8], it follows from (6) that

$$\|f\|_{(L^{p_0}, L^{p_1})_{[\theta]}} = \|f\|_\theta = \|f\|_{L^p},$$

for each simple function f. By Lemma 3, the same is true for functions $f\in L^{p_0}\cap L^{p_1}$.

We identify the interpolation spaces of vector valued L^p spaces with respect to the A-convex envelope of the sum when the pair (X_0, X_1) and the space $\mathcal{U}$ satisfy the following property:

$$\|x\|_{[\theta],\mathcal{U}} = \|x\|_\theta, \tag{7}$$

for each $x\in X_0\cap X_1$.

Theorem 4. *If* (X_0, X_1) *and* $\mathcal{U}$ *satisfy* (7), *and* A *denotes the A-convex envelope of* $L^{p_0}(X_0)+L^{p_1}(X_1)$, *then the identity map on* $L^{p_0}(X_0)\cap L^{p_1}(X_1)$ *extends to an isometry from* $L^p((X_0, X_1)_{[\theta],\mathcal{U}})$ *onto* $(L^{p_0}(X_0), L^{p_1}(X_1))_{[\theta],A}$. *Here* $1/p=(1-\theta)/p_0+\theta/p_1$. *Moreover, the pair* $(L^{p_0}(X_0), L^{p_1}(X_1))$ *and the containing space* A *satisfy* (7).

Proof. Let $S\in\mathcal{S}(X_0\cap X_1)$ as in part (b) of the proof of Theorem 3. We can pick the functions ϕ_k appearing in that part of the proof in $\mathcal{G}(X_0, X_1)$. Then the function F is in $\mathcal{G}(L^{p_0}(X_0), L^{p_1}(X_1))$ and we obtain:

$$\|S\|_{(L^{p_0}(X_0), L^{p_1}(X_1))_{[\theta],A}} \leqq \|S\|_\theta \leqq \|S\|_{L^p((X_0, X_1)_{[\theta],\mathcal{U}})}, \tag{8}$$

and the inclusion

$$\mathcal{S}(X_0\cap X_1) \subset (L^{p_0}(X_0), L^{p_1}(X_1))_{[\theta],A}$$

is continuous with norm not exceeding one. The inclusion extends to a bounded operator

$$I\colon L^p((X_0, X_1)_{[\theta],\mathcal{U}}) \to (L^{p_0}(X_0), L^{p_1}(X_1))_{[\theta],A}$$

with norm no greater than one.

On the other hand, if $\mathcal{V}$ is the A-convex containing space given in Example 3, the inclusion

$$L^{p_0}(X_0)+L^{p_1}(X_1) \subset \mathcal{V}$$

extends to $K\colon A\to\mathcal{V}$ and, by Theorems 2 and 3,

$$K\colon (L^{p_0}(X_0), L^{p_1}(X_1))_{[\theta],A} \to L^p((X_0, X_1)_{[\theta],\mathcal{U}}),$$

with norm not exceeding one.

The relations

$$K \circ I = \mathrm{Id} \quad \text{and} \quad I \circ K = \mathrm{Id}$$

(where the identity operator is understood in the corresponding space) hold in $\mathcal{S}(X_0 \cap X_1)$. By Lemma 3, we conclude that the operator I is an isometry from $L^p((X_0, X_1)_{[\theta],\mathcal{U}})$ onto $(L^{p_0}(X_0), L^{p_1}(X_1))_{[\theta],A}$ that leaves fixed $\mathcal{S}(X_0 \cap X_1)$.

We note that

$$L^{p_0}(X_0) \cap L^{p_1}(X_1) \subset (L^{p_0}(X_0), L^{p_1}(X_1))_{[\theta],\mathcal{V}} = L^p((X_0, X_1)_{[\theta],\mathcal{U}}).$$

Again by Lemma 3, the intersection $L^{p_0}(X_0) \cap L^{p_1}(X_1)$ is fixed by I.

From (8), it follows that the quasi-norms of $L^p((X_0, X_1)_{[\theta],\mathcal{U}})$, $(L^{p_0}(X_0), L^{p_1}(X_1))_{[\theta],A}$ and $\|\cdot\|_\theta$ (the one introduced by Rivière, Cwikel, Milman and Sagher) coincide on $\mathcal{S}(X_0 \cap X_1)$ and therefore on $L^{p_0}(X_0) \cap L^{p_1}(X_1)$. ∎

Corollary. *Let (Y_0, Y_1) be an interpolation pair of quasi-Banach spaces with an A-convex containing space $\mathcal{W}$. Suppose that the pair (X_0, X_1) satisfies (7) for some A-convex containing space. Let A be a linear operator defined on $\mathcal{S}(X_0 \cap X_1)$, the space of all simple $X_0 \cap X_1$-measurable functions, with values in $Y_0 \cap Y_1$. If the operator A satisfies the estimates*

$$\|Af\|_{Y_j} \leqq M_j \|f\|_{L^{p_j}(X_j)},$$

for each $f \in \mathcal{S}(X_0 \cap X_1)$, $j=0, 1$, it follows.

$$\|Af\|_{[\theta]} \leqq M_0^{1-\theta} M_1^{\theta} \|f\|_{L^p((X_0, X_1)_{[\theta]})}.$$

Proof. We extend A to an operator defined on $Z = L^{p_0}(X_0^\circ) + L^{p_1}(X_1^\circ)$. We remark that $L^{p_j}(X_j^\circ)$ is the $L^{p_j}(X_j)$-closure of $\mathcal{S}(X_0 \cap X_1)$. We extend A to $A_A \colon Z_A \to \mathcal{W}$. We observe that if the pair (X_0, X_1) satisfies (7), by Proposition 2, the pair (X_0°, X_1°) also satisfies (7). The proof ends applying Theorem 2, Theorem 4 and Proposition 2. ∎

We consider now the A-convex containing space $\mathcal{W}$ introduced in Example 4. By definition, the operator T is an isometry from $\mathcal{W}$ into $\mathcal{V}$ and, by Theorems 2 and 3, it follows that

$$T((T_{q_0}^{p_0}, T_{q_1}^{p_1})_{[\theta]}) \subset L^p(L_*^q) \tag{9}$$

with $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. Thus we arrive to the following result:

Lemma 4. *If $0 < p_j, q_j < \infty$ $(j=0, 1)$ and $0 \leqq \theta \leqq 1$ then*

$$(T_{q_0}^{p_0}, T_{q_1}^{p_1})_{[\theta]} \subset T_q^p,$$

if $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$. The inclusion has norm $\leqq 1$.

Proof. The inclusion follows immediately from the definitions of the tent spaces, the operator T and (9).

The fact that the inclusion is norm decreasing follows by interpolation of T. ∎

From this lemma and using duality like in [6] we obtain a result that, in part, complements [6, Theorem 4 and Proposition 6], in which $q_0=q_1$, and [2, Theorem 6.5], in which $q_j=\infty$, $(j=0, 1)$.

Proposition 5. *If* $1<p_j, q_j<\infty$ $(j=0, 1)$, $0\leqq\theta\leqq 1$, $1/p=(1-\theta)/p_0+\theta/p_1$ *and* $1/q=(1-\theta)/q_0+\theta/q_1$ *then*

$$(T_{q_0}^{p_0}, T_{q_1}^{p_1})_{[\theta]} = T_q^p.$$

Finally, we observe that complex interpolation of H^p spaces $(0<p\leqq\infty)$ and BMO works exactly as in [5] and [11]. We remark that H^p is continuously contained in $\mathcal{S}'$ endowed with the weak topology relative to $\mathcal{S}$ [10, p. 174]. Also BMO is continuously contained in $\mathcal{S}'$ when normed by

$$\|f\|_{BMO} = \left|\int_{[0,1]^n} f(x)\,dx\right| + \|f\|_*,$$

where $\|\cdot\|_*$ is the usual BMO seminorm as in [10]. It is not necessary to make any new argument in the interpolation theorems. It is convenient to take as A-convex containing space $\mathcal{U}$ the linear span in $\mathcal{S}'$ of the closed convex hull B of the unit ball in the corresponding sum space.

This is a Banach space with respect to the Minkowski functional because B is a bounded, convex, balanced and complete subset of $\mathcal{S}'$. In fact, since $\mathcal{S}$ is a Fréchet space and B is bounded, convex, balanced and closed, B is weakly compact. See [20].

We point out that in the original proofs of [5] and [11] only functions of finite rank are considered. Thus, when reproducing these proofs in our context, the A-convex containing space $\mathcal{U}$ will not appear.

Also, we observe that the conditions on the analytic functions appearing in the interpolation theorem of [5] are satisfied by functions in $\mathcal{F}(H^{p_0}, H^{p_1}; \mathcal{U})$ vanishing at infinity. This can be seen using that $\mathcal{S}$ is a Montel space and so any weakly convergent sequence in $\mathcal{S}'$ is strongly convergent and thus uniformly convergent on compact sets of $\mathcal{S}$. See [20].

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