On the "symmetric commutant" — canonical decomposition of families of Hilbert space operators

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1. Preliminaries

$L(H)$ denotes the algebra of all linear bounded operators in a complex Hilbert space H. WOT (SOT, respectively) denotes the weak (strong, respectively) operator topology in $L(H)$. I_H or I stands for the identity in H. $P\in L(H)$ is a *projection* if $P^2=P^*=P$. The adjectives "largest" and "maximal" used with "projection" will concern the usual partial ordering of projections in $L(H)$. $U\in L(H)$ is a *partial isometry* if U^*U, UU^* are projections. If $P, Q\in L(H)$ are projections, then $P\vee Q$ denotes the projection onto the closure of $\{Px+Qy: x, y\in H\}$. If $\mathcal{S}\subset L(H)$, then $\mathcal{S}'$, $W^*(\mathcal{S})$, $A(\mathcal{S})$ stand for the commutant of $\mathcal{S}$, for the von Neumann algebra generated by $\mathcal{S}$ and for the WOT-closed algebra generated by $\mathcal{S}$ and I, respectively. $\mathcal{S}^*$ denotes the set $\{S^*: S\in\mathcal{S}\}$. $\mathcal{S}$ is *symmetric* if $\mathcal{S}=\mathcal{S}^*$, i.e. if $S\in\mathcal{S}$ implies $S^*\in\mathcal{S}$. If $P\in\mathcal{S}'$ is a projection, then $\mathcal{S}_P=\{S|PH: S\in\mathcal{S}\}$ denotes the part of $\mathcal{S}$ in PH. If $\mathcal{S}=\{S_1, \ldots, S_k\}$, then $A(S_1, \ldots, S_k)=A(\mathcal{S})$. For two projections P, Q in a von Neumann algebra $\mathcal{R}\subset L(H)$ the symbol $P\sim Q \pmod{\mathcal{R}}$ means that there is a partial isometry $U\in\mathcal{R}$ such that $U^*U=P$, $UU^*=Q$.

Consider a family of operators $\mathcal{S}\subset L(H)$. The question we are going to answer in the present paper is: Does there exist the largest projection $P_0\in\mathcal{S}'$ such that $(\mathcal{S}_{P_0})'$ is symmetric? If this question has positive answer, then the decomposition $H=P_0H\oplus(I-P_0)H$ will be called the (SC)-*canonical decomposition of* $\mathcal{S}$. (SC) replaces the expression "symmetric commutant". Let

$$\mathcal{P}=\{P\in\mathcal{S}': P \text{ is a projection, } (\mathcal{S}_P)' \text{ is symmetric}\}.$$

Each element of $\mathcal{P}$ will be called an (SC)-*projection for* $\mathcal{S}$. Now the above question reads: Does there exist the largest (SC)-projection for $\mathcal{S}$?

To explain, where this problem comes from, let us notice that an operator $T\in L(H)$ is normal if and only if $\{T\}'$ is symmetric. This is a consequence of Fuglede's theorem ([7], Corollary 1.18). Therefore, if $T\in L(H)$, then the (SC)-

canonical decomposition of $\{T\}$ (or of $A(T)$) amounts to the canonical decomposition of T onto normal and completely non-normal parts, which is well-known. If $\mathcal{S}$ is an arbitrary family of operators in $L(H)$, then the methods developed in [9], which will be also used further, allow to find the largest projection $P_1 \in \mathcal{S}'$ such that $\mathcal{S}_{P_1}$ consists of normal operators. $\mathcal{S}_{P_1}$ will be called the *normal part of* $\mathcal{S}$. In case of commutative $\mathcal{S}$ this P_1 coincides with the largest (SC)-projection for $\mathcal{S}$. Indeed, by Fuglede's theorem, a commutative family $\mathcal{S} \subset L(H)$ consists of normal operators if and only if $\mathcal{S}'$ is symmetric. Therefore

(1.1) *Each commutative family of operators has the* (SC)-*canonical decomposition.*

It is clear now that the question, whether a family $\mathcal{S} \subset L(H)$ has the (SC)-canonical decomposition, has its origin in the problem of decomposition of an operator onto normal and completely non-normal parts.

If P_1 is the largest projection in $\mathcal{S}'$ such that $\mathcal{S}_{P_1}$ consists of normal operators and if the largest (SC)-projection P_0 for $\mathcal{S}$ exists, then $P_1 \leqq P_0$. The following non-commutative example shows that P_1 can be strictly less than P_0. The operator algebra in this example is a non-symmetric algebra with symmetric commutant.

(1.2) *Example.* Let P be the projection of $H = C^2 \oplus C$ onto C. Define $T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A = A(T, Q)$. After a matrix computation we see that $A' = \{T\}' \cap \{Q\}' = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} : a, b \in C \right\}$. The only projections in A' are $O, I, P, I-P$. The projection P_1 that determines the normal part of A equals P and the largest (SC)-projection for A is I, because A' is symmetric. A is neither commutative ($TQ \neq QT$) nor symmetric ($T^* \notin A$).

Next example illustrates similar properties as (1.2) does, but it is sharper than (1.2). Namely, the algebra A in (1.3) has no normal part, whereas A' is symmetric. This shows that there is no dependence between the canonical decomposition on normal and completely non-normal parts and the (SC)-canonical decomposition in non-commutative cases.

(1.3) *Example.* In C^2 define $T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Put $S = T \oplus T$, $Q = P \oplus P$ in C^4. The algebra $A = A(S, Q)$ is neither symmetric nor commutative and its commutant has the form:

$$A' = \{S\}' \cap \{Q\}' = \left\{ \begin{pmatrix} a_1 & 0 & a_2 & 0 \\ 0 & a_1 & 0 & a_2 \\ a_3 & 0 & a_4 & 0 \\ 0 & a_3 & 0 & a_4 \end{pmatrix} : a_1, \ldots, a_4 \in C \right\}.$$

Let us observe that to simplify computing of A' it is enough to determine $\{T\}'$, $\{P\}'$ and then to apply Lemma 7.4 of [7]. A' is symmetric. A has no normal part.

Lemma 7.4. of [7] allows to prove the following property:

(1.4) *If* $\mathcal{S}\subset L(H)$ *has symmetric commutant, then the commutant of* $\mathcal{S}^{(n)}=\{\underbrace{S\oplus\ldots\oplus S}_{n \text{ times}}: S\in\mathcal{S}\}$ *is symmetric.*

It is also easy to check that

(1.5) *If* $\mathcal{S}'$ *is symmetric, then* $(\mathcal{S}^*)'$ *is.*

To finish this section we would like to mention that the other reason to investigate the (SC)-canonical decomposition more closely appeared after reading papers [3], [5], [6], [8], where similar problems for reductive operators and reductive operator algebras were studied. Here we shall present a more systematic approach to the (SC)-canonical decomposition problem to realize exactly, what is essential and important for such a decomposition to exist. In § 2 we construct the (SC)-canonical decomposition and we establish necessary and sufficient conditions for the existence of this decomposition. In § 3 we provide examples of operator algebras without the (SC)-canonical decomposition and we discuss the (SC)-maximal decomposition. § 4 deals with some sufficient conditions for the existence of the (SC)-canonical decomposition, which are easier to check than the general condition established in § 2. Consequences of one of these conditions called (N) for the (SC)-canonical decomposition are studied. Several examples are provided.

2. Constructing the decomposition

Let us fix a family $\mathcal{S}\subset L(H)$. As in § 1, let $\mathcal{P}$ be the family of all (SC)-projections for $\mathcal{S}$. The canonical decomposition method established in [9] will be the basis of our construction of the (SC)-canonical decomposition.

(2.1) *Suppose we are able to prove:*

(i) $\mathcal{P}$ *is hereditary, i.e. if* $P\in\mathcal{P}$, $Q\in\mathcal{S}'$ *is a projection,* $Q\leqq P$, *then* $Q\in\mathcal{P}$,

(ii) *If* $P\in\mathcal{P}$, $Q\in\mathcal{S}'$ *is a projection,* $Q\sim P$ $(\mathrm{mod}\, W^*(\mathcal{S})')$, *then* $Q\in\mathcal{P}$,

(iii) *If* $P, Q\in\mathcal{P}$, $PQ=0$, *then* $P+Q\in\mathcal{P}$,

(iv) *If* $P_n\in\mathcal{P}$ *is a net that converges to a projection* P *in* SOT *and* $P_n\leqq P$ *for each* n, *then* $P\in\mathcal{P}$.

Then $\mathcal{S}$ *has the* (SC)*-canonical decomposition. The largest* (SC)*-projection* P_0 *for* $\mathcal{S}$ *belongs to* $W^*(\mathcal{S})$ *and* $\mathcal{S}_{I-P_0}$ *has no non-zero* (SC)*-projection.*

Proof: Conditions (i), (ii), (iii) imply that $P \vee Q \in \mathscr{P}$ whenever $P, Q \in \mathscr{P}$, by [9], Proposition 1. Thus $\mathscr{P}$ is a directed family of projections. By Vigier's lemma ([11], p. 7), $P_0 = \text{LUB}\, \mathscr{P}$ is the SOT-limit of $\mathscr{P}$. Now (iv) implies that $P_0 \in \mathscr{P}$, what means exactly that $\mathscr{S}$ has the (SC)-canonical decomposition. By [9], Theorem 5, $P_0 \in W^*(\mathscr{S})$. $\mathscr{S}_{I-P_0}$ has no non-zero (SC)-projection by (i) and (A) of [9]. Q. e. d.

Now we shall check the conditions (i), (ii), (iv). The following technical property, whose proof is straightforward, will be used:

(2.2) *Suppose* $P \in \mathscr{S}'$ *is a projection. Then:*

(a) $T \in (\mathscr{S}_P)'$ *if and only if* $TP \in \mathscr{S}'$, *for each* $T \in L(PH)$,

(b) $PTP \in \mathscr{S}'$ *if and only if* $PT|PH \in (\mathscr{S}_P)'$, *for each* $T \in L(H)$.

Checking (i). Take $P \in \mathscr{P}$, a projection $Q \in \mathscr{S}'$ and $T \in (\mathscr{S}_Q)'$. By (2.2), $PTQP = TQ \in \mathscr{S}'$ and $PTQ|PH \in (\mathscr{S}_P)'$. Since P is an (SC)-projection for $\mathscr{S}$, $PT^*Q|PH = (PTQ|PH)^* \in (\mathscr{S}_P)'$. Again by (2.2), $T^*Q \in \mathscr{S}'$. Hence $T^* \in (\mathscr{S}_Q)'$. Thus $Q \in \mathscr{P}$.

Checking (ii). Take $P \in \mathscr{P}$ and a projection $Q \in \mathscr{S}'$ such that $Q \sim P$ (mod $W^*(\mathscr{S})'$). Notice that a projection commutes with $\mathscr{S}$ if and only if it commutes with $W^*(\mathscr{S})$. Let $U \in W^*(\mathscr{S})'$ be a partial isometry such that $U^*U = Q$, $UU^* = P$. Take $T \in (\mathscr{S}_Q)'$ and define $X = UTQU^*$. Observe that $X = UQTQU^* = UU^*UTU^*UU^* = PUTU^*P$. Thus $X = PXP$. Since $T \in (\mathscr{S}_Q)'$, $U \in W^*(\mathscr{S})'$, it follows by (2.2) that $X \in \mathscr{S}'$. Put $S = X|PH$. Then $X = SP$. By (2.2), $S \in (\mathscr{S}_P)'$. Since P is an (SC)-projection for $\mathscr{S}$, $S^* \in (\mathscr{S}_P)'$. Again by (2.2), $X^* = S^*P \in \mathscr{S}'$. Hence $T^*Q = QT^*QQ = U^*UT^*QU^*U = U^*X^*U \in \mathscr{S}'$. Now $T^* \in (\mathscr{S}_Q)'$, by (2.2). Thus $Q \in \mathscr{P}$.

Checking (iv). Let $P_n \in \mathscr{P}$ be a net that SOT-converges to a projection P and such that $P_n \leqq P$ for each n. Since the multiplication in $L(H)$ is separately SOT-continuous, $P \in \mathscr{S}'$. Take $T \in (\mathscr{S}_P)'$. By (2.2), $STP = TPS$, $S \in \mathscr{S}$. Now $SP_nTP_n = P_nSTPP_n = P_nTPSP_n = P_nTP_nS$, for each $S \in \mathscr{S}$, because P_n, $P \in \mathscr{S}'$ and $P_n \leqq P$. Each P_n is an (SC)-projection. Thus $P_nT^*P_n \in \mathscr{S}'$ for each n, by (2.2). Passing to the SOT-limit in the equality $P_nT^*P_nS = SP_nT^*P_n$, $S \in \mathscr{S}$ (the net P_n is norm-bounded), we get $T^*P \in \mathscr{S}'$. Again by (2.2), $T^* \in (\mathscr{S}_P)'$. Hence $P \in \mathscr{P}$.

We have just proved that the conditions (i), (ii), (iv) hold without any additional assumptions on $\mathscr{S}$. The following necessary and sufficient conditions for the existence of the (SC)-canonical decomposition are natural consequences of the above considerations:

(2.3) **Theorem.** *The following conditions are equivalent:*

(a) $\mathscr{S}$ *has the* (SC)-*canonical decomposition,*

(b) $\mathcal{S}$ *satisfies* (iii),

(c) *For all* $P, Q\in\mathcal{P}$, $PQ=0$ *and for each* $B=PBQ$ *if* $B\in\mathcal{S}'$, *then* $B^*\in\mathcal{S}'$.

Proof: (a)⇒(b). Let P_0 be the largest (SC)-projection for $\mathcal{S}$. If $P, Q\in\mathcal{P}$, then $P, Q\leqq P_0$. Hence $P\vee Q\leqq P_0$. By (i), $P\vee Q\in\mathcal{P}$. In particular, $P+Q=P\vee Q$ if $PQ=0$.

(b)⇒(a) follows from (2.1).

(b)⇒(c). Take P, Q, B as in (c). Then $B|(P+Q)H\in(\mathcal{S}_{P+Q})'$, by (2.2). Since $P+Q\in\mathcal{P}$, it follows that $B^*|(P+Q)H=(B|(P+Q)H)^*$ belongs to $(\mathcal{S}_{P+Q})'$. Again by (2.2), $B^*\in\mathcal{S}'$.

(c)⇒(b). Take $P, Q\in\mathcal{P}$, $PQ=0$. Let $T\in(\mathcal{S}_{P+Q})'$. Then $T(P+Q)\in\mathcal{S}'$. Now $T(P+Q)=(P+Q)T(P+Q)=PTP+PTQ+QTP+QTQ$. Since P, Q are (SC)-projections for $\mathcal{S}$, PT^*P, $QT^*Q\in\mathcal{S}'$, by (2.2). Thus $T^*(P+Q)=(P+Q)T^*(P+Q)\in\mathcal{S}'$, by (c). Again by (2.2), $T^*\in(\mathcal{S}_{P+Q})'$. Q. e. d.

The condition (c), although concerns nilpotents $\big((PBQ)^2=0$ if $PQ=0\big)$ and looks simpler than (iii), is not much easier to check, when one wants to apply it. Some easier applicable conditions will be discussed in § 4.

3. (SC)-canonical decomposition does not hold in general

First we present an example to justify the previous statement.

(3.1) *Example.* In C^2 define $T=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$, $P=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$. Put $S=T\oplus 0$, $Q=P\oplus 0$ in $H=C^2\oplus C$. Define $A=A(S, Q)$. Let E be the projection of H onto the first C^2. Then $A_E=A(T, P)$, $A_{I-E}=CI_2$ where I_2 denotes the identity in C. A matrix computation shows that $A(T, P)'$ consists of scalar multiples of the identity in C^2. Thus $E, I-E$ are two (SC)-projections for A. The form of the commutant of A:

$$A'=\left\{\begin{pmatrix}a_1 & 0 & 0\\ 0 & a_1 & 0\\ 0 & a_2 & a_3\end{pmatrix}: a_1, a_2, a_3\in C\right\}$$

shows that A' is not symmetric, whence the condition (iii) is not satisfied. Thus, by (2.3), A does not have the (SC)-canonical decomposition. Moreover, the only projections in A' are $O, I, E, I-E$. Thus $E, I-E$ are two maximal (SC)-projections for A.

A comment is now in order. Since we know that the (SC)-canonical decomposition does not exist in general, it is interesting to know, whether some kind of orthogonal decomposition does occur at all in connection with the property (SC). The decomposition we are going to say a few words about is very often of a big significance. Namely, suppose we are given a family $\mathcal{T}\subset L(H)$ of operators and a property (W) concerning this family. Let $\mathcal{F}=\{F\in\mathcal{T}'\colon F$ is a projection, $\mathcal{T}_F$ has the property $(W)\}$. Suppose that $\mathcal{F}$ has maximal elements. If they are mutually orthogonal, then H can be written as $H=\oplus\{FH\colon F$ is a maximal element of $\mathcal{F}\}\oplus H_1$. Let us call this decomposition the *(W)-maximal decomposition of $\mathcal{T}$*. It is clear that the (W)-canonical decomposition of $\mathcal{T}$ (which means that there is the largest projection in $\mathcal{F}$) is much stronger than the (W)-maximal one. But there are properties (W) and families of operators $\mathcal{T}$ without the (W)-canonical decomposition that still have the (W)-maximal decomposition. An example can be found in [10].

It occurs that even the (SC)-maximal decomposition does not exist in general and the reason is, that there may exist two maximal (SC)-projections which are not orthogonal each other.

(3.2) **Proposition.** *Let $\mathcal{S}$ be a family of operators in $L(H)$. There exist maximal (SC)-projections for $\mathcal{S}$ but they need not be mutually orthogonal.*

Proof: Let $\mathcal{P}$ be the family of all (SC)-projections for $\mathcal{S}$. Let $\mathcal{C}$ be a linearly ordered subset of $\mathcal{P}$. $\mathcal{C}$ is a directed family of projections. By Vigier's lemma ([11], p. 7), $C=LUB\,\mathcal{C}$ is the SOT-limit of $\mathcal{C}$. Thus $C\in\mathcal{P}$, because (iv) is satisfied. Applying the Kuratowski—Zorn lemma we see that there exist maximal (SC)-projections. To show that maximal (SC)-projections need not be mutually orthogonal, consider two operators $T=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $P=\begin{pmatrix}0&0\\0&1\end{pmatrix}$ in C^2. Define $S=T\oplus I_2\oplus T^*$, $Q=P\oplus 0\oplus P$ in $H=H_1\oplus H_2\oplus H_3$, where $H_1=H_3=C^2$, $H_2=C$. Let E_1, E_2 be the projection of H onto $H_1\oplus H_2$, $H_2\oplus H_3$, respectively. Put $A=A(S, Q)$. E_1 is an (SC)-projection for A, as it is shown in (1.3). Computations similar to those of (1.3) show that E_2 is an (SC)-projection for A. The commutant of A has the form:

$$A'=\left\{\begin{pmatrix}a_1&0&0&a_2&0\\0&a_1&0&0&0\\0&0&a_3&0&0\\0&0&0&a_4&0\\0&a_5&0&0&a_4\end{pmatrix}\colon a_1,\dots,a_5\in C\right\}.$$

It is clear that A' is not symmetric. The only projections E in A' such that $E\geqq E_i$ are I or E_i $(i=1, 2)$. Thus E_1, E_2 are maximal (SC)-projections for A. They are not orthogonal each other. Q. e. d.

4. Sufficient conditions for the (SC)-canonical decomposition to exist

We have already observed in (1.1) that the commutativity of a family $\mathcal{S}\subset L(H)$ guarantees that $\mathcal{S}$ has the (SC)-canonical decomposition. Another result in this direction has been found by R. L. Moore. Namely, the main result of [6] states that each reductive algebra of operators has the (SC)-canonical decomposition. To begin with let us say a few words about the reductivity. Usually it is said that an operator algebra is reductive if it contains the identity, it is WOT-closed and each subspace invariant under it reduces it ([7], p. 167). But we shall say that a family $\mathcal{S}\subset L(H)$ is *reductive* if each subspace invariant for $\mathcal{S}$ reduces $\mathcal{S}$. We shall make no assumption about neither algebraic nor topological properties of $\mathcal{S}$. As far as the definitions are concerned, it really does not make any difference, because invariant and reducing subspaces are the same for $\mathcal{S}$ and for $A(\mathcal{S})$. But troubles with the correctness of expressions may arise once one passes to parts of $\mathcal{S}$ in reducing subspaces. To be precise suppose that $P\in\mathcal{S}'$ is a projection. If $\mathcal{S}$ is reductive (in our sense), then $\mathcal{S}_P$ is ([7], Lemma 9.1). But one cannot say the same if the WOT-closedness is included into the definition of the reductivity, because it is not clear, why $\mathcal{S}_P$ has to be WOT-closed when $\mathcal{S}$ is. Therefore one has to be careful at this point. Looking through the proof of R. L. Moore's results ([6], Theorems 2, 3) it becomes clear that the WOT-closedness of algebras he deals with is never used and his results remain true for reductive families of operators in our sense). Moreover, requiring a reductive algebra to be WOT-closed makes the use of Theorem 1 of [6] in the proof of the main Theorem 2 of [6] not completely justified, by the above reasons.

In this section we present two conditions under which the (SC)-canonical decomposition holds. They are considerably weaker than the reductivity, as we shall show in some examples. Moreover, we are able to provide examples of operator algebras that satisfy our conditions, whereas it is difficult to find non-symmetric examples of reductive algebras at all (cf. [4]), mainly because of the close connection between the reductive algebra problem ([7], p. 167) and the invariant subspace problem (cf. [2]).

Let us introduce now our conditions. Suppose $\mathcal{S}\subset L(H)$ is a family of operators.

(4.1) We say that $\mathcal{S}$ satisfies the *idempotent condition* (Id) if for each idempotent $E\in L(H)$ if $E\in\mathcal{S}'$, then $E^*\in\mathcal{S}'$.

Define $\mathcal{N}=\{P\in\mathcal{S}'\colon P$ is a projection such that for each $B=PB(I-P)$ if $B\in\mathcal{S}'$, then $B^*\in\mathcal{S}'\}$.

(4.2) We say that $\mathcal{S}$ satisfies the *nilpotent condition* (N) if each projection $P\in\mathcal{S}'$ belongs to $\mathcal{N}$. Notice that operators $B=PB(I-P)$ are nilpotents.

(4.3) **Theorem.** *Let $\mathscr{S}\subset L(H)$ be a family of operators. The following implications hold: $\mathscr{S}$ is reductive $\Rightarrow\mathscr{S}$ satisfies* (Id)$\Rightarrow\mathscr{S}$ *satisfies* (N)$\Rightarrow\mathscr{S}$ *has the* (SC)-*canonical decomposition. Neither of these implications can be reversed.*

Proof: Suppose that $\mathscr{S}$ is reductive. Let $E\in\mathscr{S}'$ be an idempotent. Then $I-E$ is also an idempotent in $\mathscr{S}'$. Thus $\ker E$, $\ker(I-E)$ are invariant for $\mathscr{S}$. Since $\mathscr{S}$ is reductive, $\operatorname{ran} E^*$, $\operatorname{ran}(I-E^*)$ are invariant for $\mathscr{S}$. It is easy to check that an idempotent $F\in L(H)$ commutes with an operator $Y\in L(H)$ if and only if $\operatorname{ran} F$, $\ker F=\operatorname{ran}(I-F)$ are invariant for Y. Therefore $E^*\in\mathscr{S}'$. Thus $\mathscr{S}$ satisfies (Id).

Suppose now that $\mathscr{S}$ satisfies (Id). Take a projection $P\in\mathscr{S}'$ and a nilpotent $B=PB(I-P)\in\mathscr{S}'$. Then $P-B$ is an idempotent in $\mathscr{S}'$. By (Id), $P-B^*\in\mathscr{S}'$. Thus $B^*\in\mathscr{S}'$. Therefore (N) is satisfied.

Finally suppose that $\mathscr{S}$ satisfies (N). Take $P, Q\in\mathscr{S}'$, $PQ=0$ and a nilpotent $B=PBQ\in\mathscr{S}'$. Then $B=PB(I-P)$. By (N), $B^*\in\mathscr{S}'$. Hence $\mathscr{S}$ has the (SC)-canonical decomposition, by (2.3).

Up to now we have proved all the desired implications. Next we shall present four examples to show that these implications cannot be reversed.

(4.4) *Example.* Let S denote the unilateral shift of multiplicity one in a separable Hilbert space H. Since $A(S)=A(S)'$ consists of analytic Toeplitz operators, up to the unitary equivalence (cf. [7], Theorem 3.4), the only idempotents in $A(S)'$ are 0 and I. Thus (Id) is satisfied. But $A(S)$ is not reductive.

This example was commutative and irreducible. Another example of an algebra that satisfies (Id), but is not reductive is the algebra $A=A(T, Q)$ of Example (1.2), which is non-commutative and has reducing subspaces. A is not reductive for general reasons ([7], Corollary 9.12). Clearly, A satisfies (Id).

Next example shows that (N) does not imply (Id). This example illustrates also many patologies that can occur in the (SC)-canonical decomposition. The algebra A below is non-commutative, has no normal part and the largest (SC)-projection for A is different from $0, I$.

(4.5) *Example.* In C^4 consider three operators

$$T=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&0\\0&0&1&1\end{pmatrix},\quad Q=\begin{pmatrix}0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad P=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Define $A=A(T, Q, P)$. A is not commutative ($TQ\neq QT$). After determining the form of the commutant of A

$$A' = \{T\}' \cap \{Q\}' \cap \{P\}' = \left\{ \begin{pmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \\ 0 & 0 & a_3 & a_2+a_3 \end{pmatrix} : a_1, a_2, a_3 \in C \right\}$$

it is plain that the only projections in A' are $0, I, P, I-P$. The only operators $B=PB(I-P)$ or $B=(I-P)BP$ are 0. Thus (N) is satisfied. The operator T^2 is an idempotent that belongs to A', but $T^{2*} \notin A'$. Thus A does not satisfy (Id). P is the largest (SC)-projection for A. A has no normal part. Notice also that the largest (SC)-projection P for A belongs to A''. As we shall see in (4.7), this is a consequence of the condition (N).

Now we show that the existence of the (SC)-canonical decomposition does not imply the condition (N).

(4.6) *Example.* In $H=C\oplus C^2$ define $T=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ and consider the commutative algebra $A=A(T)$. By (1.1), A has the (SC)-canonical decomposition. The projection P of H onto C is the largest (SC)-projection for A. Take $B=\begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $B=PB(I-P)$ and $B\in A'$. But $BT^* \neq T^*B$. Thus A does not satisfy (N).

Now Theorem (4.3) is completely proved. Q. e. d.

Finally we would like to see, what additional informations does the condition (N) give concerning the (SC)-canonical decomposition. We shall prove the following theorem:

(4.7) **Theorem.** *Suppose a family* $\mathcal{S} \subset L(H)$ *satisfies the condition* (N). *Then the largest* (SC)*-projection* P_0 *for* $\mathcal{S}$ *belongs to* $\mathcal{S}''$. *Moreover, both* $\mathcal{S}_{P_0}, \mathcal{S}_{I-P_0}$ *satisfy* (N) *and each subspace reducing* $\mathcal{S}_{I-P_0}$ *is hyperreducing for* $\mathcal{S}_{I-P_0}$, *i.e., reduces* $(\mathcal{S}_{I-P_0})'$.

Similar properties of the (SC)-canonical decomposition have been proved in [3], [5], [7] for reductive operators and reductive operator algebras. These results are particular case of Theorem (4.7), by Theorem (4.3).

Before we prove (4.7) several preliminary propositions are needed.

(4.8) **Proposition.** *Let* $\mathcal{S}$ *be a family of operators in* $L(H)$. *Suppose* $T, T^* \in \mathcal{S}'$. *Let* P_1, P_2 *denote the projections onto* $\overline{\operatorname{ran} T}, \overline{\operatorname{ran} T^*}$, *respectively. If* P_1 *is an* (SC)*-projection for* $\mathcal{S}$, *then* P_2 *is.*

Proof: It is clear that $P_1, P_2 \in \mathcal{S}'$. Let U be the partial isometry of the polar decomposition of T. It is known that $U \in W^*(T)$ and that $U^*U = P_2$, $UU^* = P_1$ (cf. [1]). Thus $P_1 \sim P_2 \pmod{W^*(T)}$. Since $T, T^* \in \mathcal{S}'$ and since $\mathcal{S}'$ is WOT-closed, $W^*(T) \subset \mathcal{S}'$. Hence $P_1 \sim P_2 \pmod{W^*(\mathcal{S})'}$. Now Proposition follows from (ii) of § 2. Q. e. d.

(4.9) **Proposition.** *Suppose* $P \in \mathcal{N}$ *(as defined above). Let* $B = PB(I-P)$ *and suppose* $B \in \mathcal{S}'$. *Then the projection onto* $\overline{\operatorname{ran} B}$ *is an* (SC)*-projection for* $\mathcal{S}$.

The proof of this proposition is exactly the same as the proof of Theorem 1, (2) of [6]. One has to realize only that not reductivity, but the property (N) is essentially used in that proof.

(4.10) **Proposition.** *Suppose a family of operators* $\mathcal{S} \subset L(H)$ *satisfies* (N). *If* $P \in \mathcal{S}'$ *is a projection, then* $\mathcal{S}_P$ *satisfies* (N).

Proof: Take a projection $Q \in (\mathcal{S}_P)'$. By (2.2), $QP \in \mathcal{S}'$. It is clear that QP is a projection. Take $B = QB(I_{PH} - Q) \in (\mathcal{S}_P)'$. By (2.2), $BP \in \mathcal{S}'$. Moreover, $BP = QB(I_{PH} - Q)P = QP(BP)(I - QP)$, because $\operatorname{ran} B \subset \operatorname{ran} Q \subset \operatorname{ran} P$. (N) implies $B^*P = (BP)^* \in \mathcal{S}'$. Again by (2.2), $B^* \in (\mathcal{S}_P)'$. Q. e. d.

Proof of Theorem (4.7): Let P_0 be the largest (SC)-projection for $\mathcal{S}$ which does exist, by (4.3). Take $T \in \mathcal{S}'$. Since $\mathcal{S}$ satisfies (N), $I - P_0 \in \mathcal{N}$. By (4.9), the projection E_1 onto $\overline{\operatorname{ran}(I-P_0)TP_0}$ is an (SC)-projection for $\mathcal{S}$. Hence $E_1 \leqq P_0$. On the other hand, $E_1 \leqq I - P_0$. Thus $E_1 = 0$ and $P_0TP_0 = TP_0$. Let now $B = P_0T(I-P_0)$. By arguments similar to the above ones, the projection onto $\overline{\operatorname{ran} B}$ is an (SC)-projection for $\mathcal{S}$. By (4.8), the projection E_2 onto $\overline{\operatorname{ran} B^*}$ is an (SC)-projection for $\mathcal{S}$. Thus $E_2 \leqq P_0$. But $\operatorname{ran} B^* \subset (I-P_0)H$. Thus $E_2 = B^* = 0$, whence $B = 0$. We have just proved $TP_0 = P_0T$. Thus $P_0 \in \mathcal{S}''$, because T was arbitrary in $\mathcal{S}'$. It follows from (4.10) that both $\mathcal{S}_{P_0}$, $\mathcal{S}_{I-P_0}$ satisfy (N). To prove the last statement take a projection $Q \in (\mathcal{S}_{I-P_0})'$ and an operator $T \in (\mathcal{S}_{I-P_0})'$. Since $\mathcal{S}_{I-P_0}$ satisfies (N), the projection onto $\overline{\operatorname{ran} QT(I_{P_0H} - Q)}$ is an (SC)-projection for $\mathcal{S}_{I-P_0}$, by (4.9). Now (2.1) implies that this projection equals zero. Thus $QTQ = QT$. Considering the projection onto $\overline{\operatorname{ran}(I_{P_0H} - Q)TQ}$ we prove similarly that $QTQ = TQ$. Thus $Q \in (\mathcal{S}_{I-P_0})''$. Q. e. d.

We would like to close this paper with the following simple observation:

(4.11) *Suppose* $\mathcal{S} \subset L(H)$ *is a family of operators. The largest* (SC)*-projection for* $\mathcal{S}$ *equals* 0 *or* I *in each of the following cases:*
(a) *the only projections in* $\mathcal{S}'$ *are* $0, I$,
(b) $\dim H = 2$.

Proof: (a) is clear. The largest (SC)-projection for $\mathcal{S}$ equals I or 0 depending on whether $\mathcal{S}'$ is symmetric or it is not. To prove (b) it is enough to consider the case when there is a projection $P\in\mathcal{S}'$, $P\neq 0$, $P\neq I$. Let e_1, e_2 be an orthonormal basis of H such that e_1, e_2 span $PH, (I-P)H$, respectively. Each $T\in\mathcal{S}$ has the form $T=\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}$ $a, b\in C$, in this basis. If each $T\in\mathcal{S}$ is a scalar multiple of I, then $\mathcal{S}'=L(H)$. Therefore we may assume that there is $T=\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}\in$ with $a\neq b$. Then $P=(a-b)^{-1}(T-bI)\in\mathcal{S}$. Hence $\mathcal{S}'\subset\{P\}'$. It is clear that $\{P\}'=D(e_1, e_2)=$ the symmetric algebra of all diagonal operators with respect to the basis e_1, e_2 of H. The proof will be finished once we realize that *if f_1, f_2 is an orthonormal basis of H, then the only proper subalgebra of $D(f_1, f_2)$ is the algebra of all scalar multiples of I.* To see this take an algebra $A\subset D(f_1, f_2)$ that contains $S=\begin{pmatrix} a & 0 \\ 0 & b\end{pmatrix}$ with $a\neq b$. If $c, d\in C$, then there is a polynomial p such that $p(a)=c, p(b)=d$. Thus $p(S)=\begin{pmatrix} c & 0 \\ 0 & d\end{pmatrix}$, whence $D(f_1, f_2)\subset A(S)\subset A$.

Since $\mathcal{S}'$ is a subalgebra of $D(e_1, e_2)=\{P\}'$, the proof is finished. Q. e. d.

Now (4.11), (b) together with Example (3.1) show that 2 is the biggest dimension of a Hilbert space H such that each family of operators in $L(H)$ has the (SC)-canonical decomposition.

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