WHITEHEAD AND RUSSELL'S PRINCIPIA MATHEMATICA

Principia Mathematica. By Alfred North Whitehead and Bertrand Russell, Volume I. Second Edition. Cambridge, University Press, 1925. xlvi+674 pp.

The second edition of Volume I of Whitehead and Russell's *Principia Mathematica* leaves the first edition intact (except for minor changes), but adds some new sections. The new sections—an introduction and three appendices—are chiefly devoted to a restatement of the logical theories of the authors in the light of the reduction in the number of primitives in the *Principia* by Sheffer and Nicod and in the light of the views of Wittgenstein (expressed in his remarkable *Tractatus Logico-Philosophicus*) on propositions and propositional functions.

The authors announce in the new Introduction the main improvements which they find necessary to make in their logic. These improvements I may list as follows: the dropping of the distinction between "real" and "apparent" variables; the dropping of the primitive idea "assertion of a propositional function"; the reading, on all occasions, of "$\vdash \cdot fp$" as "$\vdash \cdot (p) \cdot fp$"; the dropping of the primitive proposition *1·11. The authors, also, apparently abandon their "axiom of reducibility," because "clearly it is not the sort of axiom with which we can rest content," though, because of this abandonment, "there is, so far as we can discover, no way by which our present primitive propositions can be made adequate to Dedekindian and well-ordered relations." I have not included in the above list of changes the Sheffer-Nicod simplification of the primitives of the *Principia*. This simplification, elegant as it is from a certain point of view, is not a logical necessity. The general logical make-up of the old *Principia* is not affected by the revised edition. The authors begin with a set of primitives in the logic of "elementary" propositions, whose independence (and consistency) are left unproved because "the ordinary methods of proving independence are not applicable, without reserve, to fundamentals," and from these primitives, together with primitives introduced later, they aim to derive, in a definitely restricted way, the rest of logic and all mathematics.

When one considers the caliber of our authors and the fact that the *Principia* has occupied a prominent place on mathematical shelves for fourteen years, one wonders that the book has influenced mathematics so little. Of course, a partial explanation lies in the magnitude and structure of the *Principia*. Volume I, which is only one of four royal octavo volumes, which deals only with mathematical logic and matters introductory to the theory of cardinals, contains over 600 pages (over 700 in the new edition), has over 400 different symbols (some of them representing extremely subtile ideas), and has thousands of propositions, wholly written and demonstrated in symbols and linked together in an unbreakable chain. But the chief reason for the aloof attitude of mathematicians toward the *Primcipia* seems to me to lie in the fact that the authors have admitted into the book concepts and principles based on considerations not sufficiently convincing—concepts and principles based on views opposed to those forced on mathematicians by the work of Peano, Pieri, Hilbert, Veblen, Huntington. Thus our authors have admitted into the first edition the primitive idea "assertion of an elementary function," the notion "all" as distinct from "any," the notion "function of functions" in an "intensional" sense. Thus we have an involved theory of types of propositions, with its demand that there be an infinite number of propositional logics, one for each type, instead of one logic for all propositions. Thus, also, we have the view that "the theory of propositions necessarily precedes the theory of classes" and the view that for the primitives underlying the propositional logic "the recognized methods of proving independence are not applicable."

I can perhaps make more clear the attitude toward the *Principia* of the mathematician interested in logic by making some analysis, from the latter's point of view, of the primitives for elementary propositions found in the first edition. The analysis will also hold in substance for the primitives of the revised edition. Among the primitives for elementary propositions are found the following undefined ideas: "elementary proposition," denoted by $p, q, r, \cdots$; "negation" of p, denoted by "$\sim p$"; "assertion" of p, denoted by "$\vdash p$"; "disjunction" of p and q, symbolized by "$p \vee q$". Our authors define "$p \supset q$" to mean "$\sim p \vee q$", and then give a list of primitive propositions of which the following two are types: (1) "Anything implied by a true elementary proposition is true," (2) "$\vdash : p \vee p . \supset . p$," where in (2) the dots stand for parentheses. Proposition (1), our authors observe, cannot be expressed in symbols. If we take this remark literally, then (1) says nothing about the indefinables of the system, and hence cannot be a proposition belonging to the system. Let us take it that our authors mean to say that (1) cannot be expressed wholly in symbols, that (1) means, say: (1′) "Given '$\vdash p$' and '$\vdash . p \supset q$', then '$\vdash q$'." The primitive propositions under discussion then consist of types (1′) and (2). Now what can we say of (2)? Evidently, since *all* the symbols are left undefined, *it says nothing of* p, any more than "$a+b$" says anything about a and b. Then (2) is merely a *function* of $\vdash, p, \vee$, and $\sim$; it is *not* a proposition. Our authors presumably mean to use "$\vdash$" and "$\supset$" as symbols of assertion; but in that case these symbols will have to be *outside* the system, in accordance with Russell's own very sound "vicious-circle principle." A proposition in our logic, then, if it is to say anything about the undefined symbols, must be of type (1′). This means that not only must any *non*-logical mathematical system use the notions and principles of general logic, but also that the logic of propositions itself must have a general logic, an unsymbolized logic, underlying it. This the authors of the *Principia* seem to overlook or to ignore when they give us propositions of type (2) above and when they regard the logic of propositions as more "fundamental" than the logic of classes.

But, it may be objected, our authors nevertheless obtain from their primitives all the facts in the theory of elementary propositions. The answer is: Not all. Nowhere among the primitive propositions or among the theorems derived from them is there found a proposition to the effect that every elementary proposition has one and only one of the truth-values *truth* and *falsity*. A proposition such as *2·11. "$\vdash, p \vee \sim p$" may *seem* to give this fact, but in reality it does not, as is readily seen when *2·11 is written in the partly classic notation "$p \vee \sim p = 1$." We need an existence proposition to establish our proposition, and our authors' theory of types could not admit existence notions in the theory of elementary propositions!

But how about non-existence propositions derived from our primitives? If propositions of type (2) are not really propositions, how account for the host of truths that are actually obtained from such propositions? The answer here is: The authors improperly read into the symbols of their systems ideas which properly belong *outside* the system. They do this with regard to the symbols "$\vdash$" and "$\supset$". The symbol "$\vdash p$" of the *Principia* is the same as Boole's or Schröder's "$p=1$" ("the truth-value of p is truth"), in which p and 1 are symbols of the system, but "$=$" is not. It is by means of that part of the symbol "$\vdash$" which belongs outside the system that our authors make a proposition out of the sequence of symbols (2) above. Again, "$p \supset q$" is, by definition, nothing more than "$\sim p \vee q$." It should therefore be read: "not-p or q." It must not be read, as the *Principia* does: "p implies q" nor "If p is true then q is true." Each of the last two forms is a *statement about* p and q; the first is not.

All this is not quibbling. If the distinctions made be disregarded, serious errors may result. Such an error Schröder makes when he "shows" that the duality principle for classes breaks down in the logic of propositions. And such an error our authors commit when they say (p. 121) that the analog for classes of the proposition *4·78. $\vdash :. p \supset q . \vee . p \supset r : \equiv : p . \supset q \vee r$ is false. "Put p = English people, q = men, r = women. Then p is contained in q or r, but is not contained in q and is not contained in r." The fact is that the analog for classes of *4·78 (or of any other proposition in the logic of elementary propositions) strictly holds, as may be seen by reading "$\vdash p$" as "p is the universal class 1," and by writing for "$p \supset q$" and "$p \equiv q$" only the expressions given them by definition. The error made by our authors consists in taking as the class analog of "$p \supset q$" the *statement* "p is contained in q," when the true analogue is the *function* "not-p or q." The statement "p is contained in q" is the analog of "$\vdash . \sim p \vee q$" (or "$\sim p \vee q = 1$"). Similar considerations hold for the symbol "$\equiv$".

The essence of the matter is this. The logic of propositions is simply a two-element logic of classes, an algebra of truth-values 0, 1, as I have shown elsewhere (in a paper forthcoming in the TRANSACTIONS OF THIS SOCIETY). Also, the symbols of this logic, *together with other symbols*, may be used as a *language*, a symbol language, in which all mathematical systems can be expressed, *including the propositional logic itself* (Cf. Peano's *Formulaire de Mathématiques*). As a mathematical system, the logic of propositions is amenable to the postulational treatment applicable to any other branch of mathematics. As a language, this logic has *all* its symbols *outside* the system which it expresses. This distinction between the propositional logic as a mathematical system and as a language must be made, if serious errors are to be avoided; this distinction the *Principia* does not make.

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