TAIWANESE JOURNAL OF MATHEMATICS
Vol. 23, No. 6, pp. 1479–1510, December 2019
DOI: 10.11650/tjm/190109

Nonlocal Elliptic Systems Involving Critical Sobolev-Hardy Exponents and Concave-convex Nonlinearities

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Abstract. In this paper, a system of fractional elliptic equation is investigated, which involving fractional critical Sobolev-Hardy exponent and concave-convex terms. By means of variational methods and analytic techniques, the existence and multiplicity of positive solutions to the system is established.

1. Introduction

In this work, we study the following nonlocal elliptic system:

$$\begin{cases} (-\Delta)^{\alpha/2}u - \gamma \dfrac{u}{|x|^\alpha} = \lambda \dfrac{|u|^{q-2}u}{|x|^s} + \dfrac{2\eta}{\eta+\theta} \dfrac{|u|^{\eta-2}u|v|^\theta}{|x|^t} & \text{in } \Omega, \\ (-\Delta)^{\alpha/2}v - \gamma \dfrac{v}{|x|^\alpha} = \mu \dfrac{|v|^{q-2}v}{|x|^s} + \dfrac{2\theta}{\eta+\theta} \dfrac{|u|^{\eta}|v|^{\theta-2}v}{|x|^t} & \text{in } \Omega, \\ u = v = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ containing 0 in its interior, $0 < \alpha < 2$, $N > \alpha$, $0 \le s, t < \alpha$, $\lambda, \mu > 0$, $0 \le \gamma < \gamma_H$, $1 \le q < 2$ and $\eta, \theta > 1$ satisfy $\eta + \theta = 2_\alpha^*(t)$, $2_\alpha^*(t) = 2(N-t)/(N-\alpha)$ is the so-called fractional critical Sobolev-Hardy exponent.

Let $0 < \alpha < 2$, the fractional Laplacian in $\mathbb{R}^N$ taking the form

$$(-\Delta)^{\alpha/2}u(x) = c(N,\alpha)\, \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+\alpha}}\, dy, \quad \forall\, x \in \mathbb{R}^N,$$

where $c(N,\alpha)$ is the following normalization constant:

$$c(N,\alpha) = \left(\int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+\alpha}} \right)^{-1} \quad \text{with } \zeta = (\zeta_1, \zeta') \in \mathbb{R}^1 \times \mathbb{R}^{N-1},$$

and P.V. stands for the Cauchy principle value. Precisely, setting

$$\text{P.V.} \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+\alpha}}\, dy = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+\alpha}}\, dy,$$

Received November 5, 2018; Accepted January 31, 2019.
Communicated by Eiji Yanagida.
2010 *Mathematics Subject Classification*. 35J50, 47G20, 35B65.
Key words and phrases. fractional Laplacian, Hardy potential, multiple positive solutions, fractional critical Sobolev-Hardy exponent, concave-convex nonlinearities.
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we can write

$$(-\Delta)^{\alpha/2}u(x) = c(N,\alpha)\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+\alpha}}\,dy.$$

Here the value $(-\Delta)^{\alpha/2}u(x)$ depends not only on the value of u near x, but also on the value of u on the whole $\mathbb{R}^N$. Hence, we say that $(-\Delta)^{\alpha/2}$ is a non-local operator. See [3, 7, 23] and references therein for the basics on the fractional Laplacian.

In this paper, we work on the bounded domain $\Omega \subset \mathbb{R}^N$ and defined the space $X_0^{\alpha/2}(\Omega)$ as

$$X_0^{\alpha/2}(\Omega) = \left\{u \in H^{\alpha/2}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\right\},$$

which generalizes to the space introduced in [22]. As in [24], we define the following scalar product on $X_0^{\alpha/2}(\Omega)$,

$$\langle u,v\rangle_{X_0^{\alpha/2}(\Omega)} = c(N,\alpha)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+\alpha}}\,dxdy,$$

which induces the following norm

$$\|u\|_{X_0^{\alpha/2}(\Omega)} = \left(c(N,\alpha)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\,dxdy\right)^{1/2}.$$

Problem (1.1) is related to the following fractional Sobolev-Hardy inequality

$$C\left(\int_\Omega \frac{|u|^p}{|x|^t}\,dx\right)^{2/p} \le c(N,\alpha)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\,dxdy, \quad \forall\, u \in X_0^{\alpha/2}(\Omega),$$

where $2 \le p \le 2_\alpha^*(t)$, $0 \le t < \alpha$, and the constant $C > 0$. If $t = \alpha$ and $p = 2$, the above inequality becomes the well-known fractional Hardy inequality

$$\gamma_H \int_\Omega \frac{|u|^2}{|x|^\alpha}\,dx \le c(N,\alpha)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\,dxdy, \tag{1.2}$$

where $\gamma_H = 2^\alpha\Gamma^2(\frac{N+\alpha}{4})/\Gamma^2(\frac{N-\alpha}{4})$ is the best fractional Hardy constant on $\mathbb{R}^N$. Note that γ_H converges to the classical Hardy constant $\left(\frac{N-2}{2}\right)^2$ as $\alpha \to 2$. Using the fractional Hardy inequality (1.2), we employ the following norm on $X_0^{\alpha/2}(\Omega)$:

$$\|u\|_\gamma = \left(c(N,\alpha)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\,dxdy - \gamma\int_\Omega \frac{|u|^2}{|x|^\alpha}\,dx\right)^{1/2},$$

which is equivalent to the usual norm $\|\cdot\|_{X_0^{\alpha/2}(\Omega)}$ on $X_0^{\alpha/2}(\Omega)$.

By the fractional Hardy inequality and Sobolev-Hardy inequality, for $\gamma < \gamma_H$, $0 \le t < \alpha$ and $2 \le p \le 2_\alpha^*(t)$, we can define the best fractional Sobolev-Hardy constant:

$$\Lambda_{\alpha,t,p} := \inf_{u \in X_0^{\alpha/2}(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\,dxdy - \gamma\int_\Omega \frac{|u|^2}{|x|^\alpha}\,dx}{\left(\int_\Omega \frac{|u|^p}{|x|^t}\,dx\right)^{2/p}}, \tag{1.3}$$

and $\Lambda_{\alpha,t,2^*_\alpha(t)}$ is achieved for $\Omega = \mathbb{R}^N$ by a function u (see [14]).

The existence and multiplicity of nontrivial solutions for nonlinear elliptic equations with singular potentials and fractional Laplace operator has been recently studied by several authors. We refer, e.g., in bounded domains to [2,8,9,24,28,30], and for the whole space to [13,14,29] and the references therein. For example, Shakerian in [24] studied the following singular nonlocal elliptic problems via the the variational methods and Nehari manifold decomposition techniques:

$$\begin{cases} (-\Delta)^{\alpha/2}u - \gamma\dfrac{u}{|x|^\alpha} = \lambda f(x)|u|^{q-2}u + g(x)\dfrac{|u|^{p-2}u}{|x|^s} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$ is a bonded domain, $0 < \alpha < 2$, $N > \alpha$, $0 \le s < \alpha$, $\lambda > 0$, $0 \le \gamma < \gamma_H$, $1 < q < 2 < r \le 2^*_\alpha(s)$, f and g are possibly change sign in $\overline{\Omega}$. The authors prove that problem (1.4) has at least two positive solutions for λ sufficiently small. In [2], Barrios, Medina and Peral studied the subcritical case of (1.4) and proved that there exists $\lambda_* > 0$ such that the problem has at least two solutions for all $0 < \lambda < \lambda_*$ when $f(x) = g(x) \equiv 1$, $t = 0$ and $\gamma < \gamma_H$, and discussed the existence and multiplicity of solutions depending on the value p.

It should be mentioned that, for $\lambda, \mu > 0$, $2 \le r \le 2^*_s$ and $2 \le q \le 2^*_\alpha(s)$, the following nonlocal elliptic problems with Dirichlet boundary condition

$$\begin{cases} (-\Delta)^{\alpha/2}u = \lambda|u|^{r-2}u + \mu\dfrac{|u|^{q-2}u}{|x|^s} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

has been studied in [30] by Yang and Yu, and the existence results about positive solutions were obtained. In addition, by a Pohozaev-type identity, they verify that, as $q = 2$, $s = \alpha$ and $r = 2^*_\alpha(0)$ in (1.5), if Ω is a start-shaped domain with respect to the origin, the problem has no nontrivial solutions. In [28], Wang, Yang and Zhou using the variational methods, proved the existence of infinity many solutions for the following singular nonlocal elliptic equation

$$\begin{cases} (-\Delta)^{\alpha/2} - \gamma\dfrac{u}{|x|^\alpha} = |u|^{2^*_\alpha(0)-2}u + au & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when $\Omega \subset \mathbb{R}^N$ $(N > 2)$ is an open bounded domain with smooth boundary and $0 \in \Omega$, $a > 0$ and $2^*_\alpha = 2N/(N-\alpha)$ is the fractional critical Sobolev exponent. Moreover, in [9], Fall and Felli considered the unique continuation property and local asymptotic of solutions for the fractional elliptic problems with Hardy potential. We point out that the homogeneous Dirichlet datum in (1.1) is given in $\mathbb{R}^N \setminus \Omega$ and not simply on $\partial\Omega$, which consistently with the nonlocal character of the operator $(-\Delta)^{\alpha/2}$.

The main difficulty of studying (1.1) is that the equation with fractional critical Sobolev-Hardy exponent. By very technical and complicated analysis, for $0 \leq \gamma < \gamma_H$ and $0 < t < \alpha$, Ghoussoub-Shakerian [14] and Ghoussoub et al. [13] consider the following limiting problem

$$\begin{cases} (-\Delta)^{\alpha/2} u - \gamma \dfrac{u}{|x|^\alpha} = \dfrac{u^{2^*_\alpha(t)-1}}{|x|^t} & \text{in } \mathbb{R}^N, \\ u \geq 0, \quad u \not\equiv 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1.6}$$

and proved that the problem (1.6) has positive, radially symmetric, radially decreasing ground states, and which approaches zero as $|x| \to \infty$. In addition, the ground states solution u satisfying $u \in C^1(\mathbb{R}^N \setminus \{0\})$ and

$$\lim_{|x|\to 0} |x|^{\beta_-(\gamma)} u(x) = \lambda_0 \quad \text{and} \quad \lim_{|x|\to\infty} |x|^{\beta_+(\gamma)} u(x) = \lambda_\infty,$$

where $\lambda_0, \lambda_\infty > 0$ and $\beta_-(\gamma)$, $\beta_+(\gamma)$ are zero of the function

$$\Psi_{N,\alpha}(\beta) = 2^\alpha \frac{\Gamma(\frac{N-\beta}{2})\Gamma(\frac{\alpha+\beta}{2})}{\Gamma(\frac{N-\alpha-\beta}{2})\Gamma(\frac{\beta}{2})} - \gamma, \quad \forall\, \beta \geq 0,\ 0 \leq \gamma < \gamma_H$$

and satisfy

$$0 \leq \beta_-(\gamma) < \frac{N-\alpha}{2} < \beta_+(\gamma) \leq N - \alpha.$$

In particular, there exists $C_1, C_2 > 0$ such that

$$\frac{C_1}{|x|^{\beta_-(\gamma)} + |x|^{\beta_+(\gamma)}} \leq u(x) \leq \frac{C_2}{|x|^{\beta_-(\gamma)} + |x|^{\beta_+(\gamma)}}, \quad \forall\, x \in \mathbb{R}^N \setminus \{0\}.$$

Unlike the case of the classical Laplacian operator, no explicit formula is known for this ground states solution $u(x)$, but all ground states must be the form

$$U_\varepsilon(x) = \varepsilon^{-(N-\alpha)/2} u(x/\varepsilon), \quad \forall\, \varepsilon > 0,$$

which satisfies

$$\begin{aligned} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|U_\varepsilon(x) - U_\varepsilon(y)|^2}{|x-y|^{N+\alpha}}\, dxdy - \gamma \int_{\mathbb{R}^N} \frac{|U_\varepsilon|^2}{|x|^\alpha}\, dx &= \int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{2^*_\alpha(t)}}{|x|^t}\, dx \\ &= (\Lambda_{\alpha,t,2^*_\alpha(t)})^{(N-t)/(\alpha-t)}. \end{aligned} \tag{1.7}$$

Now, we define the space $W = X_0^{\alpha/2}(\Omega) \times X_0^{\alpha/2}(\Omega)$ with the norm

$$\|(u,v)\|_W^2 = \|u\|_\gamma^2 + \|v\|_\gamma^2.$$

For any $0 \le \gamma < \gamma_H$, $0 \le t < \alpha$ and $\eta, \theta > 1$ satisfy $\eta + \theta = 2^*_\alpha(t)$, we denote the best constant:

$$S_{\alpha,\eta,\theta} := \inf_{(u,v)\in W\setminus\{(0,0)\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\big(\frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\,dxdy + \frac{|v(x)-v(y)|^2}{|x-y|^{N+\alpha}}\big)\,dxdy - \gamma\int_\Omega \frac{|u|^2+|v|^2}{|x|^\alpha}\,dx}{\big(\int_\Omega \frac{|u|^\eta|v|^\theta}{|x|^t}\,dx\big)^{2/2^*_\alpha(t)}}. \tag{1.8}$$

We will establish a relationship between $\Lambda_{\alpha,t,2^*_\alpha(t)}$ and $S_{\alpha,\eta,\theta}$ as follows:

Theorem 1.1. *For the constants $\Lambda_{\alpha,t,2^*_\alpha(t)}$ and $S_{\alpha,\eta,\theta}$ introduced in* (1.3) *and* (1.8)*, it holds*

$$S_{\alpha,\eta,\theta} = \left(\left(\frac{\eta}{\theta}\right)^{\theta/(\eta+\theta)} + \left(\frac{\theta}{\eta}\right)^{\eta/(\eta+\theta)}\right)\Lambda_{\alpha,t,2^*_\alpha(t)}.$$

Also, in recent years, several authors have used the variational methods and Nehari manifold to solve semilinear elliptic equation, quasilinear elliptic equation and fractional Laplacian problem, see [1, 4, 5, 10, 11, 16–21, 26, 27] and references therein. This paper is devoted to the existence and multiplicity of positive solutions for the nonlocal elliptic systems with fractional critical Soboev-Hardy exponents and concave-convex terms.

To the best of our knowledge, the existence and multiplicity of solutions to nonlocal elliptic system (1.1) has not ever been studied by variational methods. Our proofs are based on variational methods. Let us point out that although the idea was used before for other problems, the adaptation to the procedure to our problem is not trivial at all. Because the appearance of nonlocal term and critical growth, we must reconsider this problem and need more delicate estimates. Moreover, since no explicit formula is known for the ground states of limiting problem (1.6), we will get around the difficulty by working with certain asymptotic estimates for this solution at zero and infinity.

To formulate the main result, we introduce

$$\begin{aligned}\Lambda_0 = {}& \left(\frac{2-q}{2(2^*_\alpha(t)-q)}\right)^{\frac{2}{2^*_\alpha(t)-2}} \left(\Lambda_{\alpha,t,2^*_\alpha(t)}\right)^{\frac{2^*_\alpha(t)}{2^*_\alpha(t)-2}} \left(\frac{2^*_\alpha(t)-q}{2^*_\alpha(t)-2}\right)^{-\frac{2}{2-q}} \\ & \times \left(\frac{N-s}{N\omega_N R_0^{N-s}}\right)^{\frac{2(2^*_\alpha(s)-q)}{2^*_\alpha(s)(2-q)}} \left(\Lambda_{\alpha,s,2^*_\alpha(s)}\right)^{\frac{q}{2-q}},\end{aligned} \tag{1.9}$$

where R_0 is a positive constant such that $\Omega \subset B_{R_0}(0)$ and $\omega_N = \frac{2\pi^{N/2}}{N\Gamma(N/2)}$ is the volume of the unit ball in $\mathbb{R}^N$.

For $C > 0$, set

$$\mathfrak{D}_C = \left\{(\lambda,\mu) \in \mathbb{R}^2_+ : 0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < C\right\}.$$

We are now ready to state our main results.

Theorem 1.2. *Assume that* $0 < \alpha < 2$, $N > \alpha$, $0 \le \gamma < \gamma_H$, $0 \le s, t < \alpha$, $1 \le q < 2$ *and* $\eta, \theta > 1$ *with* $\eta + \theta = 2^*_\alpha(t)$. *Then we have the following results.*

(1) *If* $(\lambda, \mu) \in \mathfrak{D}_{\Lambda_0}$, *the problem* (1.1) *has at least one positive solution in* W.

(2) *There exists* $0 < \Lambda^* < \Lambda_0$ *such that* (1.1) *has at least two positive solutions in* W *for all* $(\lambda, \mu) \in \mathfrak{D}_{\Lambda^*}$.

We prove Theorem 1.2 by critical point theory. However, the energy functional $I_{\lambda,\mu}$ does not satisfy Palais-Smale ((PS) in short) condition due to the lack of compactness of the embedding $X_0^{\alpha/2}(\Omega) \hookrightarrow L^{2^*_\alpha(t)}(\Omega, |x|^{-t}\,dx)$, so the standard variational argument is not applicable directly. In order to construct suitable Palais-Smale compact sequences, we need to locate the energy range where $I_{\lambda,\mu}$ satisfies Palais-Smale condition. By Nehari manifold methods and analytic techniques, the existence and multiplicity of positive solutions to the problem is established. The conclusion are new for the elliptic system with critical Sobolev-Hardy exponent and fractional Laplacian operator.

This paper is organized as follows. In Section 2, we introduce the variational setting of the problem and present some estimates about the ground states of problem (1.6), and complete proofs of Theorem 1.1. In Section 3, we give some properties about the Nehari manifold and fibering maps. In Sections 4 and 5, we investigate the Palais-Smale condition and prove the existence of solutions to some related local minimization problems. Finally, the proof of Theorem 1.2 is given in Section 6.

In the end of this section, we fix some notations that will be used in the sequel.

- $W := X_0^{\alpha/2}(\Omega) \times X_0^{\alpha/2}(\Omega)$ is equipped with the norm $\|(u,v)\|_W^2 = \|u\|_\gamma^2 + \|v\|_\gamma^2$.
- $O(\varepsilon^t)$ denotes $|O(\varepsilon^t)|/\varepsilon^t \le C$ and $o(\varepsilon^t)$ means $|o(\varepsilon^t)|/\varepsilon^t \to 0$ as $\varepsilon \to 0$ for $t \ge 0$.
- $L^q(\Omega, |x|^{-s}\,dx)$ denotes the usual weighted $L^q(\Omega)$ space with the weight $|x|^{-s}$.
- $o_n(1)$ denotes $o_n(1) \to 0$ as $n \to \infty$.
- C, C_i, c will denote various positive constants which may vary from line to line.

2. Some preliminary facts

In this section, we collect some preliminary facts in order to establish the functional setting.

First of all, let us introduce the standard properties of $(-\Delta)^{\alpha/2}$ on the whole $\mathbb{R}^N$ for future use in this paper. For $0 < \alpha < 2$, the operator $(-\Delta)^{\alpha/2}$ taking the form

$$(-\Delta)^{\alpha/2} u(x) = c(N,\alpha)\,\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+\alpha}}\,dy, \tag{2.1}$$

where P.V. stands for the Cauchy principle value. It can be evaluated as

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{u(x)-u(y)}{|x-y|^{N+\alpha}}\,dy.$$

Here the value $(-\Delta)^{\alpha/2}u(x)$ depends not only on the value of u near x, but also on the value of u on the whole $\mathbb{R}^N$. Hence, we say that $(-\Delta)^{\alpha/2}$ is a non-local operator.

The singular integral given in (2.1) can be written as a weighted second-order differential quotient as follows (see [7, Lemma 3.2]):

$$(-\Delta)^{\alpha/2}u=-\frac{c(N,\alpha)}{2}\int_{\mathbb{R}^N}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+\alpha}}\,dy,\quad x\in\mathbb{R}^N.$$

For u in $\mathcal{S}$, the Schwartz space of rapidly decreasing C^∞ functions in $\mathbb{R}^N$, the fractional Laplacian in (2.1) can be equivalently defined by the Fourier transform:

$$\widehat{(-\Delta)^{\alpha/2}}u(\xi)=|\xi|^\alpha\widehat{u}(\xi),$$

where $\widehat{u}=\mathcal{F}(u)$ is the Fourier transform of u, i.e.,

$$\mathcal{F}(u)(\xi)=\int_{\mathbb{R}^N}e^{-2\pi i x\cdot\xi}u(x)\,dx.$$

By Proposition 3.6 in [7], for all $u\in H_0^{\alpha/2}(\mathbb{R}^N)$, the following relation holds:

$$\int_{\mathbb{R}^N}|2\pi\xi|^\alpha|\mathcal{F}u(\xi)|^2\,d\xi=\frac{c(N,\alpha)}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\,dxdy,$$

where the fractional Sobolev space $H_0^{\alpha/2}(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|^2_{H_0^{\alpha/2}(\mathbb{R}^N)}=\int_{\mathbb{R}^N}|2\pi\xi|^\alpha|\mathcal{F}u(\xi)|^2\,d\xi=\left(\int_{\mathbb{R}^N}|(-\Delta)^{\alpha/4}u|^2\,dx\right)^{1/2}.$$

We start with the fractional Sobolev inequality [6], which asserts that for $N>\alpha$ and $\alpha\in(0,2)$, there exists a constant $S(N,\alpha)>0$ such that

$$\|u\|^2_{2^*_\alpha(0)}\le S(N,\alpha)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\,dxdy,\quad\forall\,u\in H_0^{\alpha/2}(\mathbb{R}^N),$$

where $2^*_\alpha(0)=2^*_\alpha:=2N/(N-\alpha)$ is the so-called fractional critical Sobolev exponent. Another important inequality is the fractional Hardy inequality [15], which states that under the same conditions on N and α, there exists a constant $\gamma_H>0$ such that

$$\gamma_H\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^\alpha}\,dx\le c(N,\alpha)\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\,dxdy,\quad\forall\,u\in H_0^{\alpha/2}(\mathbb{R}^N),$$

where $\gamma_H = 2^\alpha \frac{\Gamma^2(\frac{N+\alpha}{4})}{\Gamma^2(\frac{N-\alpha}{4})}$ is the best constant in the above inequality on $\mathbb{R}^N$. Note that γ_H converges to the best classical Hardy constant $\big(\frac{N-2}{2}\big)^2$ as $\alpha \to 2$. Throughout this paper, without loss of generality, we may assume that $c(N,\alpha) \equiv 1$.

By interpolating these inequalities via Hölder's inequalities, one gets the following fractional Sobolev-Hardy inequality, see [14].

Lemma 2.1. *Assume that* $0 < \alpha < 2$, $0 \le s < \alpha < N$, $\gamma < \gamma_H$ *and* $2 \le p \le 2^*_\alpha(t)$. *Then, there exists a positive constant* C*, such that for all* $u \in H_0^{\alpha/2}(\mathbb{R}^N)$,

$$C\left(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^s}\,dx\right)^{2/p} \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\,dxdy - \gamma \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^\alpha}\,dx.$$

Remark 2.2. Under the same conditions on α, s and p, for any $u \in X_0^{\alpha/2}(\Omega)$, there exists a constant $C > 0$ such that

$$C\left(\int_{\Omega} \frac{|u|^p}{|x|^s}\,dx\right)^{2/p} \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\,dxdy - \gamma \int_{\Omega} \frac{|u|^2}{|x|^\alpha}\,dx, \tag{2.2}$$

as long as $\gamma < \gamma_H$.

From (2.2), we can define the following best critical Sobolev-Hardy constants:

$$\Lambda_{\alpha,t,2^*_\alpha(t)} := \inf_{u \in X_0^{\alpha/2}(\Omega)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}}\,dxdy - \gamma\int_\Omega \frac{|u|^2}{|x|^\alpha}\,dx}{\big(\int_\Omega \frac{|u|^{2^*_\alpha(t)}}{|x|^t}\,dx\big)^{2/2^*_\alpha(t)}}$$

and

$$S_{\alpha,\eta,\theta} := \inf_{(u,v)\in W\setminus\{(0,0)\}} \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \big(\frac{|u(x)-u(y)|^2}{|x-y|^{N+\alpha}} + \frac{|v(x)-v(y)|^2}{|x-y|^{N+\alpha}}\big)\,dxdy - \gamma\int_\Omega \frac{|u|^2+|v|^2}{|x|^\alpha}\,dx}{\big(\int_\Omega \frac{|u|^\eta|v|^\theta}{|x|^t}\,dx\big)^{2/(\eta+\theta)}}, \tag{2.3}$$

where $\eta, \theta > 1$ and $\eta + \theta = 2^*_\alpha(t)$. The relationship between $\Lambda_{\alpha,t,2^*_\alpha(t)}$ and $S_{\alpha,\eta,\theta}$ as follows (see Theorem 1.1):

$$S_{\alpha,\eta,\theta} = \left(\left(\frac{\eta}{\theta}\right)^{\theta/(\eta+\theta)} + \left(\frac{\theta}{\eta}\right)^{\eta/(\eta+\theta)}\right)\Lambda_{\alpha,t,2^*_\alpha(t)}.$$

Proof of Theorem 1.1. Let $\{w_n\} \subset X_0^{\alpha/2}(\Omega)$ be a minimizing sequence for $\Lambda_{\alpha,t,2^*_\alpha(t)}$. Let $l_1, l_2 > 0$ to be chosen later and consider the sequences $u_n = l_1 w_n, v_n = l_2 w_n \in X_0^{\alpha/2}(\Omega)$. By (2.3), we have

$$\frac{(l_1^2 + l_2^2)\big(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|w_n(x)-w_n(y)|^2}{|x-y|^{N+\alpha}}\,dxdy - \gamma\int_\Omega \frac{|w_n|^2}{|x|^\alpha}\,dx\big)}{(l_1^\eta l_2^\theta)^{2/(\eta+\theta)}\big(\int_\Omega \frac{|w_n|^{2^*_\alpha(t)}}{|x|^t}\,dx\big)^{2/2^*_\alpha(t)}} \ge S_{\alpha,\eta,\theta}. \tag{2.4}$$

Defining $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ by setting $g(x) = x^{2\theta/(\eta+\theta)} + x^{-2\eta/(\eta+\theta)}$, we have

$$g\left(\frac{l_1}{l_2}\right) = \frac{l_1^2 + l_2^2}{l_2^{2\theta/(\eta+\theta)} l_1^{2\eta/(\eta+\theta)}} = \frac{l_1^2 + l_2^2}{(l_1^\eta l_2^\theta)^{2/(\eta+\theta)}}$$

and

$$\min_{x>0} g(x) = g\left(\sqrt{\frac{\eta}{\theta}}\right) = \left(\frac{\eta}{\theta}\right)^{\theta/(\eta+\theta)} + \left(\frac{\theta}{\eta}\right)^{\eta/(\eta+\theta)}.$$

Choosing l_1, l_2 in the inequality (2.4) such that $l_1/l_2 = \sqrt{\eta/\theta}$ and letting $n \to \infty$ yields

$$\left(\left(\frac{\eta}{\theta}\right)^{\theta/(\eta+\theta)} + \left(\frac{\theta}{\eta}\right)^{\eta/(\eta+\theta)}\right)\Lambda_{\alpha,t,2_\alpha^*(t)} \geq S_{\alpha,\eta,\theta}. \tag{2.5}$$

On the other hand, let $\{(u_{1,n}, u_{2,n})\} \subset W\setminus\{(0,0)\}$ be a minimizing sequence for $S_{\alpha,\eta,\theta}$. Set $h_n = s_n u_{2,n}$ for some $s_n > 0$ and satisfies

$$\int_\Omega \frac{|u_{1,n}|^{2_\alpha^*(t)}}{|x|^t}\,dx = \int_\Omega \frac{|h_n|^{2_\alpha^*(t)}}{|x|^t}\,dx. \tag{2.6}$$

Then Young's inequality yields

$$\begin{aligned}\int_\Omega \frac{|u_{1,n}|^\eta |h_n|^\theta}{|x|^t}\,dx &\leq \frac{\eta}{\eta+\theta}\int_\Omega \frac{|u_{1,n}|^{2_\alpha^*(t)}}{|x|^t}\,dx + \frac{\theta}{\eta+\theta}\int_\Omega \frac{|h_n|^{2_\alpha^*(t)}}{|x|^t}\,dx\\ &= \frac{\eta+\theta}{\eta+\theta}\int_\Omega \frac{|u_{1,n}|^{2_\alpha^*(t)}}{|x|^t}\,dx = \int_\Omega \frac{|u_{1,n}|^{2_\alpha^*(t)}}{|x|^t}\,dx.\end{aligned} \tag{2.7}$$

Moreover, from (2.6) and (2.7), we can estimate

$$\begin{aligned}&\frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\big(\frac{|u_{1,n}(x)-u_{1,n}(y)|^2}{|x-y|^{N+\alpha}} + \frac{|u_{2,n}(x)-u_{2,n}(y)|^2}{|x-y|^{N+\alpha}}\big)\,dxdy - \gamma\int_\Omega \frac{|u_{1,n}|^2+|u_{2,n}|^2}{|x|^\alpha}\,dx}{\big(\int_\Omega \frac{|u_{1,n}|^\eta|u_{2,n}|^\theta}{|x|^t}\,dx\big)^{2/(\eta+\theta)}}\\
&= \frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\big(\frac{|u_{1,n}(x)-u_{1,n}(y)|^2}{|x-y|^{N+\alpha}} + \frac{|u_{2,n}(x)-u_{2,n}(y)|^2}{|x-y|^{N+\alpha}}\big)\,dxdy - \gamma\int_\Omega \frac{|u_{1,n}|^2+|u_{2,n}|^2}{|x|^\alpha}\,dx\big)}{s_n^{-2\theta/(\eta+\theta)}\big(\int_\Omega \frac{|u_{1,n}|^\eta|h_n|^\theta}{|x|^t}\,dx\big)^{2/(\eta+\theta)}}\\
&\geq s_n^{2\theta/(\eta+\theta)}\frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_{1,n}(x)-u_{1,n}(y)|^2}{|x-y|^{N+\alpha}}\,dxdy - \gamma\int_\Omega\frac{|u_{1,n}|^2}{|x|^\alpha}\,dx}{\big(\int_\Omega\frac{|u_{1,n}|^{\eta+\theta}}{|x|^t}\,dx\big)^{2/(\eta+\theta)}}\\
&\quad + s_n^{2\theta/(\eta+\theta)-2}\frac{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|h_n(x)-h_n(y)|^2}{|x-y|^{N+\alpha}}\,dxdy - \gamma\int_\Omega\frac{|h_n|^2}{|x|^\alpha}\,dx}{\big(\int_\Omega\frac{|h_n|^{\eta+\theta}}{|x|^t}\,dx\big)^{2/(\eta+\theta)}}\\
&\geq s_n^{2\theta/(\eta+\theta)}\Lambda_{\alpha,t,2_\alpha^*(t)} + s_n^{-2\eta/(\eta+\theta)}\Lambda_{\alpha,t,2_\alpha^*(t)}\\
&= \Lambda_{\alpha,t,2_\alpha^*(t)}\big(s_n^{2\theta/(\eta+\theta)} + s_n^{-2\eta/(\eta+\theta)}\big)\\
&= \Lambda_{\alpha,t,2_\alpha^*(t)}g(s_n)\\
&\geq \Lambda_{\alpha,t,2_\alpha^*(t)}g\big(\sqrt{\eta/\theta}\big)\\
&= \left(\left(\frac{\eta}{\theta}\right)^{n/(\eta+\theta)} + \left(\frac{\theta}{\eta}\right)^{\eta/(\eta+\theta)}\right)\Lambda_{\alpha,t,2_\alpha^*(t)}.\end{aligned} \tag{2.8}$$

Thus, (2.8) and the definition of $S_{\alpha,\eta,\theta}$ yield that

$$\left(\left(\frac{\eta}{\theta}\right)^{\theta/(\eta+\theta)}+\left(\frac{\theta}{\eta}\right)^{\eta/(\eta+\theta)}\right)\Lambda_{\alpha,t,2_\alpha^*(t)}\le S_{\alpha,\eta,\theta}. \tag{2.9}$$

Hence, from (2.5) and (2.9), the proof of Theorem 1.1 is completed. □

In the end of this section, we consider the following limiting problem

$$\begin{cases}(-\Delta)^{\alpha/2}u-\gamma\dfrac{u}{|x|^\alpha}=\dfrac{u^{2_\alpha^*(t)-1}}{|x|^t} & \text{in } \mathbb{R}^N,\\ u\ge 0,\quad u\not\equiv 0 & \text{in } \mathbb{R}^N.\end{cases} \tag{2.10}$$

From [13, 14], we known that the problem (2.10) has positive radial ground states

$$U_\varepsilon(x)=\varepsilon^{-(N-\alpha)/2}u_\gamma(x/\varepsilon),\quad \forall\,\varepsilon>0,$$

where u_γ is the unique radial solution of the limiting problem with $u_\gamma(x)\in C^1(\mathbb{R}^N\setminus\{0\})$. Furthermore, u_γ have the following properties:

$$\lim_{|x|\to 0}|x|^{\beta_-(\gamma)}u_\gamma(x)=\lambda_0,\quad \lim_{|x|\to\infty}|x|^{\beta_+(\gamma)}u_\gamma(x)=\lambda_\infty, \tag{2.11}$$

where $\lambda_0,\lambda_\infty>0$ and $\beta_-(\gamma)$, $\beta_+(\gamma)$ are zero of the function $\Psi_{N,\alpha}$.

Choose $\rho>0$ small enough such that $B_\rho(0)\subset\Omega$, $\varphi\in C_0^\infty(\Omega)$, $0\le\varphi(x)\le 1$, $\varphi(x)=1$ for $|x|<\rho/2$ and $\varphi(x)=0$ for $|x|\ge\rho$. Set $u_\varepsilon(x)=\varphi(x)U_\varepsilon(x)$. Then, we get the following results:

Proposition 2.3. *Assume that $0<\alpha<2$, $N>\alpha$, $0\le\gamma<\gamma_H$, $0\le s,t<\alpha$ and $1\le q<2_\alpha^*(s)$. Then, as $\varepsilon\to 0$, we have the following estimates:*

$$\|u_\varepsilon\|_\gamma^2=\left(\Lambda_{\alpha,t,2_\alpha^*(t)}\right)^{(N-t)/(\alpha-t)}+O(\varepsilon^{2\beta_+(\gamma)+\alpha-N}), \tag{2.12}$$

$$\int_\Omega\frac{|u_\varepsilon|^{2_\alpha^*(t)}}{|x|^t}\,dx=\left(\Lambda_{\alpha,t,2_\alpha^*(t)}\right)^{(N-t)/(\alpha-t)}+O(\varepsilon^{2_\alpha^*(t)\beta_+(\gamma)+t-N}) \tag{2.13}$$

and

$$\int_\Omega\frac{|u_\varepsilon|^q}{|x|^s}\,dx=\begin{cases}C\varepsilon^{N-s-q(N-\alpha)/2} & \text{if } q>(N-s)/\beta_+(\gamma),\\ C\varepsilon^{N-s-q(N-\alpha)/2)}|\ln\varepsilon| & \text{if } q=(N-s)/\beta_+(\gamma),\\ C\varepsilon^{q(\beta_+(\gamma)-(N-\alpha)/2)} & \text{if } q<(N-s)/\beta_+(\gamma),\end{cases} \tag{2.14}$$

where the number $\beta_+(\gamma)$ is a solution of $\Psi_{N,\alpha}(\beta)=0$ in $((N-\alpha)/2,N-\alpha]$.

Proof. For (2.12), we can compute

(2.15)

$$
\begin{aligned}
\|u_\varepsilon\|_\gamma^2 &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+\alpha}}\,dxdy - \gamma\int_\Omega \frac{|u_\varepsilon|^2}{|x|^\alpha}\,dx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi(x)U_\varepsilon(x)-\varphi(y)U_\varepsilon(y)|^2}{|x-y|^{N+\alpha}}\,dxdy - \gamma\int_\Omega \frac{|\varphi U_\varepsilon|^2}{|x|^\alpha}\,dx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi^2(x)U_\varepsilon^2(x)-2\varphi(x)U_\varepsilon(x)\varphi(y)U_\varepsilon(y)+\varphi^2(y)U_\varepsilon^2(y)}{|x-y|^{N+\alpha}}\,dxdy - \gamma\int_\Omega \frac{|\varphi U_\varepsilon|^2}{|x|^\alpha}\,dx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi^2(x)U_\varepsilon^2(x)-\varphi^2(x)U_\varepsilon(x)U_\varepsilon(y)+\varphi^2(y)U_\varepsilon^2(y)-\varphi^2(y)U_\varepsilon(y)U_\varepsilon(x)}{|x-y|^{N+\alpha}}\,dxdy \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi^2(x)U_\varepsilon(x)U_\varepsilon(y)+\varphi^2(y)U_\varepsilon(x)U_\varepsilon(y)-2\varphi(x)\varphi(y)U_\varepsilon(x)U_\varepsilon(y)}{|x-y|^{N+\alpha}}\,dxdy \\
&\quad - \gamma\int_{\mathbb{R}^N} \frac{|\varphi U_\varepsilon|^2}{|x|^\alpha}\,dx + \gamma\int_{\mathbb{R}^N\setminus\Omega} \frac{|\varphi U_\varepsilon|^2}{|x|^\alpha}\,dx \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(U_\varepsilon(x)-U_\varepsilon(y))(\varphi^2(x)U_\varepsilon(x)-\varphi^2(y)U_\varepsilon(y))}{|x-y|^{N+\alpha}}\,dxdy - \gamma\int_{\mathbb{R}^N} \frac{|\varphi U_\varepsilon|^2}{|x|^\alpha}\,dx \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\varphi^2(x)U_\varepsilon(x)U_\varepsilon(y)+\varphi^2(y)U_\varepsilon(x)U_\varepsilon(y)-2\varphi(x)\varphi(y)U_\varepsilon(x)U_\varepsilon(y)}{|x-y|^{N+\alpha}}\,dxdy \\
&\quad + \gamma\int_{\mathbb{R}^N\setminus\Omega} \frac{|\varphi U_\varepsilon|^2}{|x|^\alpha}\,dx \\
&= \int_{\mathbb{R}^N} \frac{\varphi^2(x)|U_\varepsilon|^{2_\alpha^*(t)}}{|x|^t}\,dx + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+\alpha}}U_\varepsilon(x)U_\varepsilon(y)\,dxdy \\
&\quad + \gamma\int_{\mathbb{R}^N\setminus\Omega} \frac{|\varphi U_\varepsilon|^2}{|x|^\alpha}\,dx \\
&= \int_{\mathbb{R}^N} \frac{|U_\varepsilon|^{2_\alpha^*(t)}}{|x|^t}\,dx + \int_{\mathbb{R}^N} \frac{(\varphi^2(x)-1)|U_\varepsilon|^{2_\alpha^*(t)}}{|x|^t}\,dx + \gamma\int_{\mathbb{R}^N\setminus\Omega} \frac{|\varphi U_\varepsilon|^2}{|x|^\alpha}\,dx \\
&\quad + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+\alpha}}U_\varepsilon(x)U_\varepsilon(y)\,dxdy.
\end{aligned}
$$

Using (2.11), it is easy to check that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} \frac{(\varphi^2(x)-1)|U_\varepsilon(x)|^{2_\alpha^*(t)}}{|x|^t}\,dx\right| \\
&\le \int_{\mathbb{R}^N\setminus B_{\rho/2}(0)} \frac{|U_\varepsilon(x)|^{2_\alpha^*(t)}}{|x|^t}\,dx = \varepsilon^{-(N-t)}\int_{\mathbb{R}^N\setminus B_{\rho/2}(0)} \frac{|u_\gamma(x/\varepsilon)|^{2_\alpha^*(t)}}{|x|^t}\,dx \qquad (2.16)\\
&= \int_{\mathbb{R}^N\setminus B_{\rho/(2\varepsilon)}(0)} \frac{|u_\gamma(x)|^{2_\alpha^*(t)}}{|x|^t}\,dx = O(\varepsilon^{\frac{2_\alpha^*(t)}{2}(2\beta_+(\gamma)-N+\alpha)}) = O(\varepsilon^{2\beta_+(\gamma)-N+\alpha}).
\end{aligned}
$$

In the same way, we have

$$
\int_{\mathbb{R}^N\setminus\Omega} \frac{|U_\varepsilon(x)|^2}{|x|^\alpha}\,dx = \varepsilon^{-(N-\alpha)}\int_{\mathbb{R}^N\setminus\Omega} \frac{|u_\gamma(x/\varepsilon)|^2}{|x|^\alpha}\,dx
$$

$$= O(\varepsilon^{2\beta_+(\gamma)-(N-\alpha)}) \int_{\mathbb{R}^N\setminus\Omega} \frac{1}{|x|^{2\beta_+(\gamma)+\alpha}}\, dx$$
$$= O(\varepsilon^{2\beta_+(\gamma)-(N-\alpha)}) \int_{\rho}^{+\infty} \frac{1}{r^{2\beta_+(\gamma)+\alpha-N+1}}\, dr.$$

Since $(N-\alpha)/2 < \beta_+(\gamma) \le N-\alpha$, we get $2\beta_+(\gamma)+\alpha-N+1 > 1$. Thus

$$\gamma \int_{\mathbb{R}^N\setminus\Omega} \frac{|\varphi(x)U_\varepsilon(x)|^2}{|x|^\alpha}\, dx = O(\varepsilon^{2\beta_+(\gamma)-N+\alpha}). \tag{2.17}$$

Moreover,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+\alpha}} U_\varepsilon(x)U_\varepsilon(y)\, dxdy \\
&= \varepsilon^{2\beta_+(\gamma)-N+\alpha} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+\alpha}} \frac{U_\varepsilon(x)}{\varepsilon^{(2\beta_+(\gamma)-N+\alpha)/2}} \frac{U_\varepsilon(y)}{\varepsilon^{(2\beta_+(\gamma)-N+\alpha)/2}}\, dxdy \\
&= \varepsilon^{2\beta_+(\gamma)-N+\alpha} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+\alpha}} \frac{u_\gamma(x/\varepsilon)}{\varepsilon^{\beta_+(\gamma)}} \frac{u_\gamma(y/\varepsilon)}{\varepsilon^{\beta_+(\gamma)}}\, dxdy.
\end{aligned} \tag{2.18}$$

It follows from Ground State Representation [12], Legesgue's convergence theorem yields that there exists $C>0$ such that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+\alpha}} \frac{u_\gamma(x/\varepsilon)}{\varepsilon^{\beta_+(\gamma)}} \frac{u_\gamma(y/\varepsilon)}{\varepsilon^{\beta_+(\gamma)}}\, dxdy \\
&= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{(\varphi(x)-\varphi(y))^2}{|x-y|^{N+\alpha}} \frac{1}{|x|^{\beta_+(\gamma)}} \frac{1}{|y|^{\beta_+(\gamma)}}\, dxdy \\
&= \|\varphi\|^2_{H_0^{\alpha/2,\beta_+(\gamma)}(\mathbb{R}^N)} \le C.
\end{aligned} \tag{2.19}$$

So, (2.15), (2.16), (2.17), (2.18), (2.19) and (1.7) imply that

$$\begin{aligned}
\|u_\varepsilon\|_\gamma^2 &= \int_{\mathbb{R}^N} \frac{|U_\varepsilon(x)|^{2_\alpha^*(t)}}{|x|^t}\, dx + O(\varepsilon^{2\beta_+(\gamma)-N+\alpha}) \\
&= \left(\Lambda_{\alpha,t,2_\alpha^*(t)}\right)^{(N-t)/(\alpha-t)} + O(\varepsilon^{2\beta_+(\gamma)+\alpha-N}).
\end{aligned}$$

In order to get (2.13), we first compute

$$\begin{aligned}
\int_\Omega \frac{|u_\varepsilon(x)|^{2_\alpha^*(t)}}{|x|^t}\, dx &= \int_\Omega \frac{|\varphi(x)U_\varepsilon(x)|^{2_\alpha^*(t)}}{|x|^t}\, dx \\
&= \int_\Omega \frac{|U_\varepsilon(x)|^{2_\alpha^*(t)}}{|x|^t}\, dx + \int_\Omega \frac{(\varphi^{2_\alpha^*(t)}(x)-1)|U_\varepsilon(x)|^{2_\alpha^*(t)}}{|x|^t}\, dx \\
&= \int_{\mathbb{R}^N} \frac{|U_\varepsilon(x)|^{2_\alpha^*(t)}}{|x|^t}\, dx - \int_{\mathbb{R}^N\setminus\Omega} \frac{|U_\varepsilon(x)|^{2_\alpha^*(t)}}{|x|^t}\, dx \\
&\quad + \int_\Omega \frac{(\varphi^{2_\alpha^*(t)}(x)-1)|U_\varepsilon(x)|^{2_\alpha^*(t)}}{|x|^t}\, dx \\
&= \int_{\mathbb{R}^N} \frac{|U_\varepsilon(x)|^{2_\alpha^*(t)}}{|x|^t}\, dx - (\mathrm{I}) + (\mathrm{II}).
\end{aligned} \tag{2.20}$$

Now we estimate last two terms in (2.20). For (I),

$$\begin{aligned}\int_{\mathbb{R}^N\setminus\Omega}\frac{|U_\varepsilon(x)|^{2^*_\alpha(t)}}{|x|^t}\,dx &= \int_{\mathbb{R}^N\setminus\Omega}\frac{|\varepsilon^{-(N-\alpha)/2}u_\gamma(x/\varepsilon)|^{2^*_\alpha(t)}}{|x|^t}\,dx\\ &= \varepsilon^{-(N-t)}\int_{\mathbb{R}^N\setminus\Omega}\frac{|u_\gamma(x/\varepsilon)|^{2^*_\alpha(t)}}{|x|^t}\,dx\\ &= \varepsilon^{-(N-t)}\int_{\mathbb{R}^N\setminus\Omega}\frac{O(|x/\varepsilon|^{-\beta_+(\gamma)\cdot 2^*_\alpha(t)})}{|x|^t}\,dx\\ &= \varepsilon^{-N+t+\beta_+(\gamma)2^*_\alpha(t)}\int_{\mathbb{R}^N\setminus\Omega}O(|x|^{-\beta_+(\gamma)\cdot 2^*_\alpha(t)-t})\,dx\\ &= \varepsilon^{-N+t+\beta_+(\gamma)2^*_\alpha(t)}\int_\rho^\infty O\left(\frac{1}{r^{\beta_+(\gamma)\cdot 2^*_\alpha(t)+t-N+1}}\right)dr.\end{aligned}$$

Since $(N-\alpha)/2<\beta_+(\gamma)<N-\alpha$, we have $\beta_+(\gamma)2^*_\alpha(t)-N+1+t>1$, which implies that there exists $C>0$ such that $\int_\rho^\infty \frac{1}{r^{\beta_+(\gamma)\cdot 2^*_\alpha(t)+t-N+1}}\,dr\le C$. Then

$$\int_{\mathbb{R}^N\setminus\Omega}\frac{|U_\varepsilon(x)|^{2^*_\alpha(t)}}{|x|^t}\,dx = O(\varepsilon^{\beta_+(\gamma)\cdot 2^*_\alpha(t)-N+t}). \tag{2.21}$$

For (II),

$$\begin{aligned}&\left|\int_\Omega\frac{(\varphi^{2^*_\alpha(t)}(x)-1)|U_\varepsilon(x)|^{2^*_\alpha(t)}}{|x|^t}\,dx\right|\\ &= \varepsilon^{-(N-t)}\int_\Omega\frac{(1-\varphi^{2^*_\alpha(t)}(x))|u_\gamma(x/\varepsilon)|^{2^*_\alpha(t)}}{|x|^t}\,dx\\ &\le \varepsilon^{-(N-t)}\int_{\Omega\setminus B_{\rho/2}(0)}\frac{|u_\gamma(x/\varepsilon)|^{2^*_\alpha(t)}}{|x|^t}\,dx\\ &= \varepsilon^{-(N-t)}\int_{\Omega\setminus B_{\rho/2}(0)}\frac{O(|x/\varepsilon|^{-\beta_+(\gamma)\cdot 2^*_\alpha(t)})}{|x|^t}\,dx\\ &= \varepsilon^{\beta_+(\gamma)\cdot 2^*_\alpha(t)-(N-t)}\int_{\Omega\setminus B_{\rho/2}(0)}\frac{O(|x/\varepsilon|^{-\beta_+(\gamma)\cdot 2^*_\alpha(t)})}{|x|^t}\,dx\\ &= C\varepsilon^{\beta_+(\gamma)\cdot 2^*_\alpha(t)-(N-t)}\int_{\rho/2}^\infty O\left(\frac{1}{r^{\beta_+(\gamma)\cdot 2^*_\alpha(t)+t-N+1}}\right)dr\\ &= O(\varepsilon^{\beta_+(\gamma)\cdot 2^*_\alpha(t)-(N-t)}).\end{aligned} \tag{2.22}$$

Therefore, from (2.20), (2.21), (2.22) and (1.7), we get

$$\begin{aligned}\int_\Omega\frac{|u_\varepsilon|^{2^*_\alpha(t)}}{|x|^t}\,dx &= \int_{\mathbb{R}^N}\frac{|U_\varepsilon(x)|^{2^*_\alpha(t)}}{|x|^t}\,dx + O(\varepsilon^{\beta_+(\gamma)\cdot 2^*_\alpha(t)-N+t})\\ &= \left(\Lambda_{\alpha,t,2^*_\alpha(t)}\right)^{(N-t)/(\alpha-t)} + O(\varepsilon^{\beta_+(\gamma)\cdot 2^*_\alpha(t)-N+t}).\end{aligned}$$

This completes the proof of (2.13).

Finally, we compute (2.14). For all $1 \le q < 2^*_\alpha(s)$, as $\varepsilon \to 0$,

$$
\begin{aligned}
\int_\Omega \frac{|u_\varepsilon(x)|^q}{|x|^s}\,dx &= \int_\Omega \frac{|\varphi(x)U_\varepsilon(x)|^q}{|x|^s}\,dx \\
&= \int_\Omega \frac{|\varphi(x)\varepsilon^{-(N-\alpha)/2}u_\gamma(x/\varepsilon)|^q}{|x|^s}\,dx \\
&\ge \varepsilon^{-q(N-\alpha)/2}\int_{B_{\rho/2}(0)} \frac{|u_\gamma(x/\varepsilon)|^q}{|x|^s}\,dx \\
&= \varepsilon^{-q(N-\alpha)/2+N-s}\int_{B_{\rho/(2\varepsilon)}(0)} \frac{O(|x|^{-\beta_+(\gamma)\cdot q})}{|x|^s}\,dx \\
&\ge \varepsilon^{-q(N-\alpha)/2+N-s}\int_{\rho_0}^{\rho/(2\varepsilon)} O\left(\frac{1}{r^{\beta_+(\gamma)\cdot q+s-N+1}}\right)\,dr,
\end{aligned}
\tag{2.23}
$$

where the constant $\rho_0 > 0$ small enough.

(i) If $\beta_+(\gamma)\cdot q + s - N = 0$, it is not difficult to calculate that

$$
\int_{\rho_0}^{\rho/(2\varepsilon)} \frac{1}{r^{\beta_+(\gamma)\cdot q+s-N+1}}\,dr = \int_{\rho_0}^{\rho/(2\varepsilon)} \frac{1}{r}\,dr = C\ln|\varepsilon|.
\tag{2.24}
$$

So, (2.23) and (2.24) yield that

$$
\int_\Omega \frac{|u_\varepsilon|^q}{|x|^s}\,dx \ge C\varepsilon^{N-s-q(N-\alpha)/2}\ln|\varepsilon|.
\tag{2.25}
$$

(ii) If $\beta_+(\gamma)\cdot q + s - N < 0$, it follows that $\beta_+(\gamma)\cdot q + s - N + 1 < 1$ and

$$
\int_{\rho_0}^{\rho/(2\varepsilon)} \frac{1}{r^{\beta_+(\gamma)\cdot q+s-N+1}}\,dr = \int_{\rho_0}^{\rho/(2\varepsilon)} r^{N-s-\beta_+(\gamma)\cdot q-1}\,dr = C\varepsilon^{-(N-s-\beta_+(\gamma)\cdot q)}.
\tag{2.26}
$$

Then, inserting (2.26) into (2.23), we obtain

$$
\int_\Omega \frac{|u_\varepsilon|^q}{|x|^s}\,dx \ge C\varepsilon^{N-s-q(N-\alpha)/2-N+s+\beta_+(\gamma)\cdot q} = C\varepsilon^{q(\beta_+(\gamma)-(N-\alpha)/2)}.
\tag{2.27}
$$

(iii) If $\beta_+(\gamma)\cdot q + s - N > 0$, we have $\beta_+(\gamma)\cdot q + s - N + 1 > 1$, then there exists $C > 0$ such that

$$
\left|\int_{\rho_0}^{\rho/(2\varepsilon)} \frac{1}{r^{\beta_+(\gamma)\cdot q+s-N+1}}\,dr\right| \le C.
\tag{2.28}
$$

Therefore, by (2.28) and (2.23),

$$
\int_\Omega \frac{|u_\varepsilon|^q}{|x|^s}\,dx \ge C\varepsilon^{N-s-q(N-\alpha)/2}.
\tag{2.29}
$$

Thus, (2.25), (2.27) and (2.29) imply (2.14). $\square$

3. The Nehari manifold

The corresponding energy functional of (1.1) is defined by

$$I_{\lambda,\mu}(u,v)=\frac{1}{2}\|(u,v)\|_W^2-\frac{2}{\eta+\theta}\int_\Omega\frac{|u|^\eta|v|^\theta}{|x|^t}\,dx-\frac{1}{q}\left(\lambda\int_\Omega\frac{|u|^q}{|x|^s}\,dx+\mu\int_\Omega\frac{|v|^q}{|x|^s}\,dx\right).$$

Denote

$$Q_{\lambda,\mu}(u,v)=\lambda\int_\Omega\frac{|u|^q}{|x|^s}\,dx+\mu\int_\Omega\frac{|v|^q}{|x|^s}\,dx.$$

By the Hölder and Sobolev-Hardy inequalities, for all $u\in X_0^{\alpha/2}(\Omega)$, we get

$$\tag{3.1}\begin{aligned}\int_\Omega\frac{|u|^q}{|x|^s}\,dx&=\int_\Omega\frac{|u|^q}{|x|^{\frac{q}{2_\alpha^*(s)}s}}\cdot\frac{1}{|x|^{(1-\frac{q}{2_\alpha^*(s)})s}}\,dx\\&\le\left(\int_{B_{R_0}(0)}\frac{1}{|x|^s}\,dx\right)^{(2_\alpha^*(s)-q)/2_\alpha^*(s)}\left(\int_\Omega\frac{|u|^{2_\alpha^*(s)}}{|x|^s}\,dx\right)^{q/2_\alpha^*(s)}\\&\le\left(N\omega_N\int_0^{R_0}r^{N-s-1}\,dr\right)^{(2_\alpha^*(s)-q)/2_\alpha^*(s)}\left(\Lambda_{\alpha,s,2_\alpha^*(s)}\right)^{-q/2}\|u\|_\gamma^q\\&=\left(\frac{N\omega_NR_0^{N-s}}{N-s}\right)^{(2_\alpha^*(s)-q)/2_\alpha^*(s)}\left(\Lambda_{\alpha,s,2_\alpha^*(s)}\right)^{-q/2}\|u\|_\gamma^q,\end{aligned}$$

where R_0 is the positive number such that $\Omega\subset B_{R_0}(0):=\{x\in\mathbb{R}^N:|x|<R_0\}$, and $\omega_N=\frac{2\pi^{N/2}}{N\Gamma(N/2)}$ is the volume of the unit ball in $\mathbb{R}^N$. Then, from the Sobolev inequality, Hölder inequality and (3.1), we get that

$$\tag{3.2}\begin{aligned}Q_{\lambda,\mu}(u,v)\le&\left(\frac{N\omega_NR_0^{N-s}}{N-s}\right)^{(2_\alpha^*(s)-q)/2_\alpha^*(s)}\left(\lambda^{2/(2-q)}+\mu^{2/(2-q)}\right)^{(2-q)/2}\\&\times\left(\Lambda_{\alpha,s,2_\alpha^*(s)}\right)^{-q/2}\|(u,v)\|_W^q.\end{aligned}$$

Definition 3.1. A pair of functions $(u,v)\in W$ is said to be a weak solution of (1.1), if for every $(\phi,\psi)\in W$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+\alpha}}\,dxdy-\gamma\int_\Omega\frac{u\phi}{|x|^\alpha}\,dx\\&\quad+\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+\alpha}}\,dxdy-\gamma\int_\Omega\frac{v\psi}{|x|^\alpha}\,dx\\&=\lambda\int_\Omega\frac{|u|^{q-2}u\phi}{|x|^s}\,dx+\mu\int_\Omega\frac{|v|^{q-2}v\psi}{|x|^s}\,dx\\&\quad+\frac{2\eta}{\eta+\theta}\int_\Omega\frac{|u|^{\eta-2}u\phi|v|^\theta}{|x|^t}\,dx+\frac{2\theta}{\eta+\theta}\int_\Omega\frac{|u|^\eta|v|^{\theta-2}v\psi}{|x|^t}\,dx.\end{aligned}$$

Since the energy functional $I_{\lambda,\mu}$ is not bounded from below on W, it is useful to consider the functional on the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} = \{(u,v) \in W \setminus \{(0,0)\} : \langle I'_{\lambda,\mu}(u,v), (u,v)\rangle = 0\}.$$

Thus, $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$\|(u,v)\|_W^2 - 2\int_\Omega \frac{|u|^\eta |v|^\theta}{|x|^t}\, dx - Q_{\lambda,\mu}(u,v) = 0. \tag{3.3}$$

The Nehari manifold $\mathcal{N}_{\lambda,\mu}$ is closely to the fibering maps $m\colon \tau \mapsto I_{\lambda,\mu}(\tau(u,v))$ given by

$$m(\tau) = \frac{\tau^2}{2}\|(u,v)\|_W^2 - \frac{2\tau^{2^*_\alpha(t)}}{2^*_\alpha(t)} \int_\Omega \frac{|u|^\eta |v|^\theta}{|x|^t}\, dx - \frac{\tau^q}{q} Q_{\lambda,\mu}(u,v).$$

Notice that

$$\begin{aligned} m'(\tau) &= \tau\|(u,v)\|_W^2 - 2\tau^{2^*_\alpha(t)-1} \int_\Omega \frac{|u|^\eta |v|^\theta}{|x|^t}\, dx - \tau^{q-1} Q_{\lambda,\mu}(u,v), \\ m''(\tau) &= \|(u,v)\|_W^2 - 2(2^*_\alpha(t)-1)\tau^{2^*_\alpha(t)-2} \int_\Omega \frac{|u|^\eta |v|^\theta}{|x|^t}\, dx - (q-1)\tau^{q-2} Q_{\lambda,\mu}(u,v). \end{aligned}$$

It is clear that $m'(\tau) = 0$ if and only if $(\tau u, \tau v) \in \mathcal{N}_{\lambda,\mu}$. Hence, $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if $m'(1) = 0$.

Now introduce the functional

$$\Phi_{\lambda,\mu}(u,v) := \langle I'_{\lambda,\mu}(u,v), (u,v)\rangle.$$

We see that $\Phi_{\lambda,\mu} \in C^1(W,\mathbb{R})$, $\mathcal{N}_{\lambda,\mu} = \Phi^{-1}_{\lambda,\mu}(0) \setminus \{(0,0)\}$, and for all $(u,v) \in \mathcal{N}_{\lambda,\mu}$, we get

$$\begin{aligned} \langle \Phi'_{\lambda,\mu}(u,v), (u,v)\rangle &= 2\|(u,v)\|_W^2 - qQ_{\lambda,\mu}(u,v) - 22^*_\alpha(t) \int_\Omega \frac{|u|^\eta |v|^\theta}{|x|^t}\, dx \\ &= (2-q)\|(u,v)\|_W^2 - 2(2^*_\alpha(t)-q) \int_\Omega \frac{|u|^\eta |v|^\theta}{|x|^t}\, dx \qquad (3.4) \\ &= (2^*_\alpha(t)-q) Q_{\lambda,\mu}(u,v) - (2^*_\alpha(t)-2)\|(u,v)\|_W^2. \qquad (3.5) \end{aligned}$$

Following the methods used in [31], we split $\mathcal{N}_{\lambda,\mu}$ into three parts:

$$\begin{aligned} \mathcal{N}^+_{\lambda,\mu} &= \{(u,v) \in \mathcal{N}_{\lambda,\mu} : \langle \Phi'_{\lambda,\mu}(u,v), (u,v)\rangle > 0\}, \\ \mathcal{N}^0_{\lambda,\mu} &= \{(u,v) \in \mathcal{N}_{\lambda,\mu} : \langle \Phi'_{\lambda,\mu}(u,v), (u,v)\rangle = 0\}, \\ \mathcal{N}^-_{\lambda,\mu} &= \{(u,v) \in \mathcal{N}_{\lambda,\mu} : \langle \Phi'_{\lambda,\mu}(u,v), (u,v)\rangle < 0\}. \end{aligned}$$

In order to understand the Nehari manifold and fibering maps, let us consider the function $f_{u,v}\colon \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f_{u,v}(\tau) = \tau^{2-q}\|(u,v)\|_W^2 - 2\tau^{2^*_\alpha(t)-q} \int_\Omega \frac{|u|^\eta |v|^\theta}{|x|^t}\, dx.$$

Thus,

$$f'_{u,v}(\tau) = (2-q)\tau^{1-q}\|(u,v)\|_W^2 - 2(2^*_\alpha(t)-q)\tau^{2^*_\alpha(t)-q-1}\int_\Omega \frac{|u|^\eta|v|^\theta}{|x|^t}\,dx.$$

We can see that if $(\tau u, \tau v) \in \mathcal{N}_{\lambda,\mu}$, then

$$\tau^{q-1} f'_{u,v}(\tau) = m''(\tau).$$

Hence, $(\tau u, \tau v) \in \mathcal{N}^+_{\lambda,\mu}$ (or $(\tau u, \tau v) \in \mathcal{N}^-_{\lambda,\mu}$) if and only if $f'_{u,v}(\tau) > 0$ (or $f'_{u,v}(\tau) < 0$). Moreover, it is clear that,

$$m'(\tau) = \tau^{q-1}(f_{u,v}(\tau) - Q_{\lambda,\mu}(u,v))$$

and for $\tau > 0$,

$$\tau(u,v) \in \mathcal{N}_{\lambda,\mu} \iff m'(\tau) = 0 \iff f_{u,v}(\tau) = Q_{\lambda,\mu}(u,v). \tag{3.6}$$

Lemma 3.2. *Assume that $(u,v) \in W \setminus \{(0,0)\}$. Then the function $f_{u,v}$ satisfies the following properties:*

(i) *$f_{u,v}$ has a unique critical point at*

$$\tau_{\max} = \left(\frac{(2-q)\|(u,v)\|_W^2}{2(2^*_\alpha(t)-q)\int_\Omega \frac{|u|^\eta|v|^\theta}{|x|^t}\,dx}\right)^{1/(2^*_\alpha(t)-2)};$$

(ii) *$f_{u,v}$ is strictly increasing on $(0,\tau_{\max})$, and is strictly decreasing on $(\tau_{\max},+\infty)$;*

(iii) *$\lim_{\tau\to 0^+} f_{u,v}(\tau) = 0$ and $\lim_{\tau\to+\infty} f_{u,v}(\tau) = -\infty$.*

Proof. This follows from a direct computation. □

From (3.6), we have seen that $m'(\tau) = 0$ if and only if there exist λ and $\mu > 0$ such that the following condition holds:

$$f_{u,v}(\tau_{max}) \geq Q_{\lambda,\mu}(u,v).$$

By a direct calculation,

$$\begin{aligned}
f_{u,v}(\tau_{\max}) &= \frac{\left(\|(u,v)\|_W^2\right)^{\frac{2^*_\alpha(t)-q}{2^*_\alpha(t)-2}}}{\left(\int_\Omega \frac{|u|^\eta|v|^\theta}{|x|^t}\,dx\right)^{\frac{2-q}{2^*_\alpha(t)-2}}}\left(\left(\frac{2-q}{2(2^*_\alpha(t)-q)}\right)^{\frac{2-q}{2^*_\alpha(t)-2}} - 2\left(\frac{2-q}{2(2^*_\alpha(t)-q)}\right)^{\frac{2^*_\alpha(t)-q}{2^*_\alpha(t)-2}}\right) \\
&\geq \|(u,v)\|_W^q\left(\left(\frac{2-q}{2(2^*_\alpha(t)-q)}\right)^{\frac{2-q}{2^*_\alpha(t)-2}} - 2\left(\frac{2-q}{2(2^*_\alpha(t)-q)}\right)^{\frac{2^*_\alpha(t)-q}{2^*_\alpha(t)-2}}\right) S_{\alpha,\eta,\theta}^{\frac{2^*_\alpha(t)(2-q)}{2(2^*_\alpha(t)-2)}} \\
&= \|(u,v)\|_W^q \left(\frac{2-q}{2(2^*_\alpha(t)-q)}\right)^{\frac{2-q}{2^*_\alpha(t)-2}} \left(\frac{2^*_\alpha(t)-2}{2^*_\alpha(t)-q}\right) S_{\alpha,\eta,\theta}^{\frac{2^*_\alpha(t)(2-q)}{2(2^*_\alpha(t)-2)}}.
\end{aligned}$$

And from (1.9) and (3.2), for all $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_0}$, we have

$$
\begin{aligned}
0 &< Q_{\lambda,\mu}(u,v) \\
&\le \left(\frac{N\omega_N R_0^{N-s}}{N-s}\right)^{(2_\alpha^*(s)-q)/2_\alpha^*(s)} \left(\lambda^{2/(2-q)}+\mu^{2/(2-q)}\right)^{(2-q)/2} \\
&\quad \times \left(\Lambda_{\alpha,s,2_\alpha^*(s)}\right)^{-q/2}\|(u,v)\|_W^q \\
&< \|(u,v)\|_W^q \left(\frac{2-q}{2(2_\alpha^*(t)-q)}\right)^{\frac{2-q}{2_\alpha^*(t)-2}} \left(\frac{2_\alpha^*(t)-2}{2_\alpha^*(t)-q}\right) S_{\alpha,\eta,\theta}^{\frac{2_\alpha^*(t)(2-q)}{2(2_\alpha^*(t)-2)}} \\
&\le f_{u,v}(\tau_{\max}).
\end{aligned}
\tag{3.7}
$$

Then, Lemma 3.2 and (3.7) deduce the following result.

Lemma 3.3. *Let $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_0}$. Then, for every $(u,v) \in W \setminus \{(0,0)\}$ with $Q_{\lambda,\mu}(u,v) > 0$, there exist unique $\tau_1 := \tau_1(u,v)$ and $\tau_2 := \tau_2(u,v) > 0$ such that*

(1) $0 < \tau_1 < \tau_{\max} < \tau_2$;

(2) $(\tau_1 u, \tau_1 v) \in \mathcal{N}_{\lambda,\mu}^+$ *and* $(\tau_2 u, \tau_2 v) \in \mathcal{N}_{\lambda,\mu}^-$;

(3) $I_{\lambda,\mu}(\tau_1 u, \tau_1 v) = \inf_{\tau\in[0,\tau_{\max}]} I_{\lambda,\mu}(\tau u, \tau v)$, $I_{\lambda,\mu}(\tau_2 u, \tau_2 v) = \sup_{\tau\in[0,\infty)} I_{\lambda,\mu}(\tau u, \tau v)$.

Proof. Let $(u,v) \in W \setminus \{(0,0)\}$ with $Q_{\lambda,\mu}(u,v) > 0$. By (3.7) and Lemma 3.2, there exist unique τ_1 and τ_2 with $0 < \tau_1 < \tau_{\max} < \tau_2$ such that

$$f_{u,v}(\tau_1) = f_{u,v}(\tau_2) = Q_{\lambda,\mu}(u,v), \quad f'_{u,v}(\tau_1) > 0 > f'_{u,v}(\tau_2).$$

This and (3.6) imply $m'(\tau_1) = m'(\tau_2) = 0$ and $m''(\tau_1) > 0 > m''(\tau_2)$. So, the fibering map $m(t)$ has a local minimum at τ_1 and a local maximum at τ_2 such that $(\tau_1 u, \tau_1 v) \in \mathcal{N}_{\lambda,\mu}^+$ and $(\tau_2 u, \tau_2 v) \in \mathcal{N}_{\lambda,\mu}^-$.

Since $m(\tau) = I_{\lambda,\mu}(\tau u, \tau v)$, we have $I_{\lambda,\mu}(\tau_1 u, \tau_1 v) \le I_{\lambda,\mu}(\tau u, \tau v) \le I_{\lambda,\mu}(\tau_2 u, \tau_2 v)$ for all $\tau \in [\tau_1, \tau_2]$, and $I_{\lambda,\mu}(\tau_1 u, \tau_1 v) \le I_{\lambda,\mu}(\tau u, \tau v)$ for all $\tau \in [0, \tau_1]$. Thus,

$$I_{\lambda,\mu}(\tau_1 u, \tau_1 v) = \inf_{\tau\in[0,\tau_{\max}]} I_{\lambda,\mu}(\tau u, \tau v), \quad I_{\lambda,\mu}(\tau_2 u, \tau_2 v) = \sup_{\tau\in[0,+\infty)} I_{\lambda,\mu}(\tau u, \tau v).$$

This completes the proof. □

Lemma 3.4. *If (u_0, v_0) is a local minimizer for $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ and $(u_0, v_0) \notin \mathcal{N}_{\lambda,\mu}^0$, then $I'_{\lambda,\mu}(u_0, v_0) = 0$.*

Proof. Let (u_0, v_0) be a local minimizer of $I_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$. Then, there exists a neighborhood $\mathcal{D}$ of (u_0, v_0) such that (u_0, v_0) is a nontrivial pair solution of the optimization problem

$$I_{\lambda,\mu}(u_0, v_0) = \min_{(u,v)\in\mathcal{D}\cap\mathcal{N}_{\lambda,\mu}} I_{\lambda,\mu}(u,v) = \min_{\substack{(u,v)\in\mathcal{D}\setminus\{(0,0)\} \\ \langle I'_{\lambda,\mu}(u,v),(u,v)\rangle = 0}} I_{\lambda,\mu}(u,v).$$

Hence by the theory of Lagrange multiplies, there exists $\kappa \in \mathbb{R}$ such that $I'_{\lambda,\mu}(u_0, v_0) = \kappa \Phi'_{\lambda,\mu}(u_0, v_0)$, which implies that

$$\langle I'_{\lambda,\mu}(u_0, v_0), (u_0, v_0)\rangle = \kappa \langle \Phi'_{\lambda,\mu}(u_0, v_0), (u_0, v_0)\rangle. \tag{3.8}$$

Since $(u_0, v_0) \notin \mathcal{N}^0_{\lambda,\mu}$, we have $\langle \Phi'_{\lambda,\mu}(u_0, v_0), (u_0, v_0)\rangle \neq 0$ and so by (3.8), $\kappa = 0$. This completes the proof. □

Lemma 3.5. *There exists a positive number $\Lambda_0 > 0$ such that if $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_0$, then $\mathcal{N}^0_{\lambda,\mu} = \emptyset$.*

Proof. Assume by contradiction that there exist λ and $\mu > 0$ with $0 < \lambda^{2/(2-q)} + \mu^{2/(2-q)} < \Lambda_0$ and such that $\mathcal{N}^0_{\lambda,\mu} \neq \emptyset$.

Let $(u, v) \in \mathcal{N}^0_{\lambda,\mu}$, by (3.4) and (3.5), we get

$$\|(u, v)\|^2_W = \frac{2(2^*_\alpha(t) - q)}{2 - q} \int_\Omega \frac{|u|^\eta |v|^\theta}{|x|^t}\, dx, \quad \|(u, v)\|^2_W = \frac{2^*_\alpha(t) - q}{2^*_\alpha(t) - 2} Q_{\lambda,\mu}(u, v). \tag{3.9}$$

By (3.1) and Young inequality, it follows that

$$\begin{aligned} \|(u, v)\|^2_W &= \frac{2(2^*_\alpha(t) - q)}{2 - q} \int_\Omega \frac{|u|^\eta |v|^\theta}{|x|^t}\, dx \\ &\le \frac{2(2^*_\alpha(t) - q)}{2 - q} \left(\frac{\eta}{\eta + \theta} \int_\Omega \frac{|u|^{\eta+\theta}}{|x|^t}\, dx + \frac{\theta}{\eta + \theta} \int_\Omega \frac{|v|^{\eta+\theta}}{|x|^t}\, dx \right) \\ &\le \frac{2(2^*_\alpha(t) - q)}{2 - q} \left(\Lambda_{\alpha,t,2^*_\alpha(t)}\right)^{-2^*_\alpha(t)/2} \|(u, v)\|^{2^*_\alpha(t)}_W, \end{aligned} \tag{3.10}$$

which leads to

$$\|(u, v)\|_W \ge \left(\frac{2 - q}{2(2^*_\alpha(t) - q)} \left(\Lambda_{\alpha,t,2^*_\alpha(t)}\right)^{2^*_\alpha(t)/2} \right)^{1/(2^*_\alpha(t)-2)}. \tag{3.11}$$

On the other hand, from (3.2) and (3.9),

$$\begin{aligned} \|(u, v)\|^2_W &= \frac{2^*_\alpha(t) - q}{2^*_\alpha(t) - 2} Q_{\lambda,\mu}(u, v) \\ &\le \frac{2^*_\alpha(t) - q}{2^*_\alpha(t) - 2} \left(\frac{N\omega_N R_0^{N-s}}{N - s} \right)^{(2^*_\alpha(s)-q)/2^*_\alpha(s)} \left(\lambda^{2/(2-q)} + \mu^{\frac{2}{2-q}}\right)^{(2-q)/2} \\ &\quad \times \left(\Lambda_{\alpha,s,2^*_\alpha(s)}\right)^{-q/2} \|(u, v)\|^q_W, \end{aligned}$$

it follows that

$$\begin{aligned} \|(u, v)\|_W \le{}& \left(\frac{2^*_\alpha(t) - q}{2^*_\alpha(t) - 2} \right)^{1/(2-q)} \left(\frac{N\omega_N R_0^{N-s}}{N - s} \right)^{\frac{2^*_\alpha(s)-q}{2^*_\alpha(s)(2-q)}} \\ &\times \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right)^{1/2} \left(\Lambda_{\alpha,s,2^*_\alpha(s)}\right)^{-q/[2(2-q)]}. \end{aligned} \tag{3.12}$$

Thus, (3.11) and (3.12) lead to the inequality

$$\lambda^{2/(2-q)} + \mu^{2/(2-q)} \geq \Lambda_0.$$

This is a contradiction. □

Lemma 3.6. *The energy functional $I_{\lambda,\mu}$ is bounded below and coercive on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Let $(u,v) \in \mathcal{N}_{\lambda,\mu}$, then by (3.2) and (3.3), we have

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= \frac{1}{2}\|(u,v)\|_W^2 - \frac{2}{2_\alpha^*(t)} \int_\Omega \frac{|u|^\eta |v|^\theta}{|x|^t}\,dx - \frac{1}{q} Q_{\lambda,\mu}(u,v) \\ &= \left(\frac{1}{2} - \frac{1}{2_\alpha^*(t)}\right) \|(u,v)\|_W^2 - \left(\frac{1}{q} - \frac{1}{2_\alpha^*(t)}\right) Q_{\lambda,\mu}(u,v) \\ &\geq \frac{2_\alpha^*(t)-2}{2\cdot 2_\alpha^*(t)} \|(u,v)\|_W^2 - \frac{2_\alpha^*(t)-q}{q\cdot 2_\alpha^*(t)} C_0 \|(u,v)\|_W^q, \end{aligned}$$

where

$$C_0 := \left(\frac{N\omega_N R_0^{N-s}}{N-s}\right)^{(2_\alpha^*(s)-q)/2_\alpha^*(s)} \left(\lambda^{2/(2-q)} + \mu^{2/(2-q)}\right)^{(2-q)/2} \left(\Lambda_{\alpha,s,2_\alpha^*(s)}\right)^{-q/2} > 0.$$

Since $1 \leq q < 2$, the functional $I_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}$. □

By Lemmas 3.5 and 3.6, for each $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_0}$, we can consider the following local minimizer problem on $\mathcal{N}_{\lambda,\mu}$, $\mathcal{N}_{\lambda,\mu}^+$ and $\mathcal{N}_{\lambda,\mu}^-$:

$$\begin{aligned} c_{\lambda,\mu} &= \inf\{I_{\lambda,\mu}(u,v) : (u,v) \in \mathcal{N}_{\lambda,\mu}\}, \\ c_{\lambda,\mu}^- &= \inf\{I_{\lambda,\mu}(u,v) : (u,v) \in \mathcal{N}_{\lambda,\mu}^-\}, \\ c_{\lambda,\mu}^+ &= \inf\{I_{\lambda,\mu}(u,v) : (u,v) \in \mathcal{N}_{\lambda,\mu}^+\}. \end{aligned}$$

We have the following properties about the Nehari manifold $\mathcal{N}_{\lambda,\mu}$.

Lemma 3.7. *The following facts hold:*

(i) *If $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_0}$, then $c_{\lambda,\mu} \leq c_{\lambda,\mu}^+ < 0$.*

(ii) *There exist $c_0, \Lambda_* > 0$ such that $c_{\lambda,\mu}^- \geq c_0$ for all $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_*}$, where*

$$\begin{aligned} \Lambda_* := &\left(\frac{q(2_\alpha^*(t)-2)}{2(2_\alpha^*(t)-q)}\right)^{\frac{2}{2-q}} \left(\frac{2-q}{2(2_\alpha^*(t)-q)}\right)^{\frac{2}{2_\alpha^*(t)-2}} \left(\Lambda_{\alpha,t,2_\alpha^*(t)}\right)^{\frac{2_\alpha^*(t)}{2_\alpha^*(t)-2}} \\ &\times \left(\frac{N\omega_N R_0^{N-s}}{N-s}\right)^{\frac{2(q-2_\alpha^*(s))}{2_\alpha^*(s)\,(2-q)}} \left(\Lambda_{\alpha,s,2_\alpha^*(s)}\right)^{\frac{q}{2-q}}. \end{aligned}$$

Proof. (i) First, from the definitions of $c_{\lambda,\mu}$ and $c^+_{\lambda,\mu}$, it is easy to deduce that $c_{\lambda,\mu} \le c^+_{\lambda,\mu}$. Moreover, for $(u,v) \in \mathcal{N}^+_{\lambda,\mu}$, we get

$$\frac{2-q}{2(2^*_\alpha(t)-q)}\|(u,v)\|^2_W > \int_\Omega \frac{|u|^\eta|v|^\theta}{|x|^t}\,dx,$$

and so

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= \left(\frac{1}{2}-\frac{1}{q}\right)\|(u,v)\|^2_W - 2\left(\frac{1}{2^*_\alpha(t)}-\frac{1}{q}\right)\int_\Omega \frac{|u|^\eta|v|^\theta}{|x|^t}\,dx \\ &\le \left[\left(\frac{1}{2}-\frac{1}{q}\right)+\left(\frac{1}{q}-\frac{1}{2^*_\alpha(t)}\right)\frac{2-q}{2^*_\alpha(t)-q}\right]\|(u,v)\|^2_W \\ &= -\frac{(2-q)(2^*_\alpha(t)-2)}{2q2^*_\alpha(t)}\|(u,v)\|^2_W \\ &< 0. \end{aligned}$$

This implies $c_{\lambda,\mu} \le c^+_{\lambda,\mu} < 0$ and completes the proof of (i).

(ii) Let $(u,v) \in \mathcal{N}^-_{\lambda,\mu}$. By (3.4) and (3.10),

$$\frac{2-q}{2(2^*_\alpha(t)-q)}\|(u,v)\|^2_W < \int_\Omega \frac{|u|^\eta|v|^\theta}{|x|^t}\,dx \le \left(\Lambda_{\alpha,t,2^*_\alpha(t)}\right)^{-2^*_\alpha(t)/2}\|(u,v)\|^{2^*_\alpha(t)}_W.$$

Hence, we have

$$\|(u,v)\|_W > \left(\frac{2-q}{2(2^*_\alpha(t)-q)}\right)^{1/(2^*_\alpha(t)-2)}\left(\Lambda_{\alpha,t,2^*_\alpha(t)}\right)^{\frac{2^*_\alpha(t)}{2(2^*_\alpha(t)-2)}}. \tag{3.13}$$

From (3.2) and (3.13), we infer that

$$\begin{aligned} I_{\lambda,\mu}(u,v) &= \Big(\frac{1}{2}-\frac{1}{2^*_\alpha(t)}\Big)\|(u,v)\|^2_W - \Big(\frac{1}{q}-\frac{1}{2^*_\alpha(t)}\Big)Q_{\lambda,\mu}(u,v) \\ &\ge \Big(\frac{1}{2}-\frac{1}{2^*_\alpha(t)}\Big)\|(u,v)\|^2_W - \Big(\frac{2^*_\alpha(t)-q}{q2^*_\alpha(t)}\Big)\Big(\frac{N\omega_N R_0^{N-s}}{N-s}\Big)^{(2^*_\alpha(s)-q)/2^*_\alpha(s)} \\ &\qquad\quad \times \big(\lambda^{2/(2-q)}+\mu^{2/(2-q)}\big)^{(2-q)/2}\big(\Lambda_{\alpha,s,2^*_\alpha(s)}\big)^{-q/2}\|(u,v)\|^q_W \\ &= \|(u,v)\|^q_W\bigg[\Big(\frac{1}{2}-\frac{1}{2^*_\alpha(t)}\Big)\|(u,v)\|^{2-q}_W - \Big(\frac{2^*_\alpha(t)-q}{q2^*_\alpha(t)}\Big)\Big(\frac{N\omega_N R_0^{N-s}}{N-s}\Big)^{(2^*_\alpha(s)-q)/2^*_\alpha(s)} \\ &\qquad\quad \times \big(\lambda^{2/(2-q)}+\mu^{2/(2-q)}\big)^{(2-q)/2}\big(\Lambda_{\alpha,s,2^*_\alpha(s)}\big)^{-q/2}\bigg] \\ &> \|(u,v)\|^q_W\bigg[\frac{2^*_\alpha(t)-2}{22^*_\alpha(t)}\Big(\frac{2-q}{2(2^*_\alpha(t)-q)}\Big)^{\frac{2-q}{2^*_\alpha(t)-2}}\big(\Lambda_{\alpha,t,2^*_\alpha(t)}\big)^{\frac{2^*_\alpha(t)(2-q)}{2(2^*_\alpha(t)-2)}} \\ &\qquad\quad - \Big(\frac{2^*_\alpha(t)-q}{q2^*_\alpha(t)}\Big)\Big(\frac{N\omega_N R_0^{N-s}}{N-s}\Big)^{(2^*_\alpha(s)-q)/2^*_\alpha(s)} \\ &\qquad\quad \times \big(\lambda^{2/(2-q)}+\mu^{2/(2-q)}\big)^{(2-q)/2}\big(\Lambda_{\alpha,s,2^*_\alpha(s)}\big)^{-q/2}\bigg]. \end{aligned}$$

Then, if

$$\frac{2^*_\alpha(t)-2}{22^*_\alpha(t)}\left(\frac{2-q}{2(2^*_\alpha(t)-q)}\right)^{\frac{2-q}{2^*_\alpha(t)-2}}\left(\Lambda_{\alpha,t,2^*_\alpha(t)}\right)^{\frac{2^*_\alpha(t)(2-q)}{2(2^*_\alpha(t)-2)}}$$
$$>\left(\frac{2^*_\alpha(t)-q}{q2^*_\alpha(t)}\right)\left(\frac{N\omega_N R_0^{N-s}}{N-s}\right)^{(2^*_\alpha(s)-q)/2^*_\alpha(s)}\left(\lambda^{2/(2-q)}+\mu^{2/(2-q)}\right)^{(2-q)/2}\left(\Lambda_{\alpha,s,2^*_\alpha(s)}\right)^{-q/2},$$

that is,

$$\lambda^{2/(2-q)}+\mu^{2/(2-q)}<\Lambda_*:=\left(\frac{q(2^*_\alpha(t)-2)}{2(2^*_\alpha(t)-q)}\right)^{\frac{2}{2-q}}\left(\frac{2-q}{2(2^*_\alpha(t)-q)}\right)^{\frac{2}{2^*_\alpha(t)-2}}$$
$$\times\left(\Lambda_{\alpha,t,2^*_\alpha(t)}\right)^{\frac{2^*_\alpha(t)}{2^*_\alpha(t)-2}}\left(\frac{N-s}{N\omega_N R_0^{N-s}}\right)^{\frac{2(q-2^*_\alpha(s))}{2^*_\alpha(s)\,(2-q)}}\left(\Lambda_{\alpha,s,2^*_\alpha(s)}\right)^{\frac{q}{2-q}},$$

we can get

$$I_{\lambda,\mu}(u,v)\geq c_0=c(q,t,s,\alpha,N)>0,\quad \forall\,(u,v)\in\mathcal{N}^-_{\lambda,\mu}.$$

Hence, for all $(\lambda,\mu)\in\mathfrak{D}_{\Lambda_*}$, we can prove (ii). $\square$

Remark 3.8. We can easily deduce that $\Lambda_0/\Lambda_*=(2/q)^{2/(2-q)}$ and $\Lambda_*<\Lambda_0$.

4. Appropriate Palais-Smale sequence

Definition 4.1. Let $c\in\mathbb{R}$, W be a Banach space and $I_{\lambda,\mu}\in C^1(W,\mathbb{R})$. $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ is a $(\mathrm{PS})_c$ sequence in W for $I_{\lambda,\mu}$ if $I_{\lambda,\mu}(u_n,v_n)=c+o_n(1)$ and $I'_{\lambda,\mu}(u_n,v_n)=o_n(1)$ as $n\to\infty$. We say that $I_{\lambda,\mu}$ satisfies the $(\mathrm{PS})_c$ condition if any $(\mathrm{PS})_c$ sequence $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ for $I_{\lambda,\mu}$ admits a convergent subsequence.

Proposition 4.2. *Assume that $N>\alpha$, $0\leq\gamma<\gamma_H$, $0\leq s,t<\alpha$, $1\leq q<2$ and $\eta+\theta=2^*_\alpha(t)$. If $\{(u_n,v_n)\}_{n\in\mathbb{N}}\subset W$ is a $(\mathrm{PS})_c$ sequence for $I_{\lambda,\mu}$ with c satisfying $0<c<\frac{\alpha-t}{N-t}\left(\frac{S_{\alpha,\eta,\theta}}{2}\right)^{(N-t)/(\alpha-t)}$, then there exists a subsequence of $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ converging weakly to a nontrivial solution of* (1.1).

Proof. Suppose that $\{(u_n,v_n)\}_{n\in\mathbb{N}}\subset W$ satisfies $I_{\lambda,\mu}(u_n,v_n)\to c$ and $I'_\lambda(u_n,v_n)\to 0$ with $c\in(0,\frac{\alpha-t}{N-t}(\frac{S_{\alpha,\eta,\theta}}{2})^{(N-t)/(\alpha-t)})$ as $n\to\infty$. Since $\{(u_n,v_n)\}_{n\in\mathbb{N}}$ is bounded in W, there is a subsequence, still denoted by $\{(u_n,v_n)\}$, and $(u_0,v_0)\in W$ such that $(u_n,v_n)\rightharpoonup(u_0,v_0)$ weakly in W. Therefore, by Sobolev embedding theorem, as $n\to\infty$,

$$\begin{aligned}
&u_n\rightharpoonup u_0,\quad v_n\rightharpoonup v_0\quad \text{weakly in } L^{2^*_\alpha(t)}(\Omega,|x|^{-t}\,dx),\\
&u_n\to u_0,\quad v_n\to v_0\quad \text{strongly in } L^q(\Omega,|x|^{-s}\,dx),\ \forall\,1\leq q<2^*_\alpha(s),\\
&u_n(x)\to u_0(x),\quad v_n(x)\to v_0(x)\quad \text{a.e. in } \Omega.
\end{aligned}\tag{4.1}$$

Hence, from (4.1), it is easy to see that $I'_{\lambda,\mu}(u_0, v_0) = 0$ and

$$Q_{\lambda,\mu}(u_n, v_n) = Q_{\lambda,\mu}(u_0, v_0) + o_n(1). \tag{4.2}$$

Now, we claim that $(u_0, v_0) \neq (0, 0)$. Arguing by contradiction, we assume $(u_0, v_0) \equiv (0, 0)$. By (4.1) and (4.2), as $n \to \infty$,

$$\begin{aligned} o_n(1) &= \langle I'_{\lambda,\mu}(u_n, v_n), (u_n, v_n)\rangle \\ &= \|(u_n, v_n)\|_W^2 - 2\int_\Omega \frac{|u_n|^\eta |v_n|^\theta}{|x|^t}\,dx - Q_{\lambda,\mu}(u_n, v_n) \\ &= \|(u_n, v_n)\|_W^2 - 2\int_\Omega \frac{|u_n|^\eta |v_n|^\theta}{|x|^t}\,dx - Q_{\lambda,\mu}(u_0, v_0) + o_n(1) \\ &= \|(u_n, v_n)\|_W^2 - 2\int_\Omega \frac{|u_n|^\eta |v_n|^\theta}{|x|^t}\,dx, \end{aligned}$$

which implies

$$2\int_\Omega \frac{|u_n|^\eta |v_n|^\theta}{|x|^t}\,dx \to l, \quad \|(u_n, v_n)\|_W^2 \to l \geq 0 \quad \text{as } n \to \infty. \tag{4.3}$$

Then, from the definition of $\Lambda_{\alpha,t,2^*_\alpha(t)}$ and (4.3), we have

$$l = \lim_{n\to\infty} \|(u_n, v_n)\|_W^2 \geq S_{\alpha,\eta,\theta} \lim_{n\to\infty} \left(\int_\Omega \frac{|u_n|^\eta |v_n|^\theta}{|x|^t}\,dx\right)^{2/2^*_\alpha(t)} = S_{\alpha,\eta,\theta}\left(\frac{l}{2}\right)^{2/2^*_\alpha(t)}.$$

This implies

$$\text{either} \quad l = 0 \quad \text{or} \quad l \geq 2\left(\frac{S_{\alpha,\eta,\theta}}{2}\right)^{2^*_\alpha(t)/(2^*_\alpha(t)-2)} = 2\left(\frac{S_{\alpha,\eta,\theta}}{2}\right)^{(N-t)/(\alpha-t)}.$$

If $l = 0$, then by (4.2) and (4.3), we get

$$c + o_n(1) = I_{\lambda,\mu}(u_n, v_n) = \frac{1}{2}l - \frac{1}{2^*_\alpha(t)}l - Q_{\lambda,\mu}(u_0, v_0) = 0,$$

which contradicts $c > 0$. Thus we conclude that $l \geq 2\big(\frac{S_{\alpha,\eta,\theta}}{2}\big)^{(N-t)/(\alpha-t)}$. Hence

$$\begin{aligned} c + o_n(1) &= I_{\lambda,\mu}(u_n, v_n) \\ &= \frac{1}{2}\|(u_n, v_n)\|_\gamma^2 - \frac{2}{2^*_\alpha(t)}\int_\Omega \frac{|u_n|^\eta |v_n|^\theta}{|x|^t}\,dx - Q_{\lambda,\mu}(u_0, v_0) \\ &= \left(\frac{1}{2} - \frac{1}{2^*_\alpha(t)}\right) l + o_n(1) \\ &\geq \frac{\alpha - t}{N - t}\left(\frac{S_{\alpha,\eta,\theta}}{2}\right)^{(N-t)/(\alpha-t)}. \end{aligned}$$

This contradicts the assumption on c. Therefore (u_0, v_0) is a nontrivial solution of (1.1) and completes the proof of Proposition 4.2. $\square$

Moreover, using the Ekeland variational principle, we get the following result, the proof is similar to [24, Proposition 3.8] and the details are omitted.

Proposition 4.3. *Assume that $N > \alpha$, $0 \le \gamma < \gamma_H$, $0 \le s,t < \alpha$ and $1 \le q < 2$. The following facts hold:*

(i) *Let $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_0}$. Then there is a $(\mathrm{PS})_{c_{\lambda,\mu}}$-sequence $\{(u_n,v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ for $I_{\lambda,\mu}$.*

(ii) *Let $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_*}$. Then there is a $(\mathrm{PS})_{c^-_{\lambda,\mu}}$-sequence $\{(u_n,v_v)\} \subset \mathcal{N}^-_{\lambda,\mu}$ for $I_{\lambda,\mu}$.*

5. Local minimization problem

In this section, we establish the existence of a local minimizer for $I_{\lambda,\mu}$ in $\mathcal{N}^+_{\lambda,\mu}$.

Theorem 5.1. *Assume that $N > \alpha$, $0 \le \gamma < \gamma_H$, $0 \le s,t < \alpha$ and $1 \le q < 2$. If $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_0}$, then $I_{\lambda,\mu}$ has a local minimizer (u_1,v_1) in $\mathcal{N}^+_{\lambda,\mu}$ and satisfies the following:*

(i) *$I_{\lambda,\mu}(u_1,v_1) = c_{\lambda,\mu} = c^+_{\lambda,\mu}$;*

(ii) *(u_1,v_1) is a positive solution of (1.1).*

Proof. By Proposition 4.3, there exists a minimizing sequence $\{(u_n,v_n)\}$ for $I_{\lambda,\mu}$ in $\mathcal{N}_{\lambda,\mu}$ such that, as $n \to \infty$,

$$I_{\lambda,\mu}(u_n,v_n) = c_{\lambda,\mu} + o_n(1) \quad \text{and} \quad I'_{\lambda,\mu}(u_n,v_n) = o_n(1). \tag{5.1}$$

By Lemma 3.6, we see that $I_{\lambda,\mu}$ is coercive on $\mathcal{N}_{\lambda,\mu}$, and $\{(u_n,v_n)\}$ is bounded in W. Then there exist a subsequence, still denoted by $\{(u_n,v_n)\}$, and $(u_1,v_1) \in W$ such that, as $n \to \infty$,

$$\begin{aligned}
&u_n \rightharpoonup u_1, \quad v_n \rightharpoonup v_1 \quad \text{weakly in } X_0^{\alpha/2}(\Omega),\\
&u_n \rightharpoonup u_1, \quad v_n \rightharpoonup v_1 \quad \text{weakly in } L^{2^*_\alpha(t)}(\Omega,|x|^{-t}),\\
&u_n \to u_1, \quad v_n \to v_1 \quad \text{strongly in } L^q(\Omega,|x|^{-s}),\ \forall\, 1 \le q < 2^*_\alpha(s),\\
&u_n(x) \to u_1(x), \quad v_n(x) \to v_1(x) \quad \text{a.e. in } \Omega.
\end{aligned} \tag{5.2}$$

This implies that

$$Q_{\lambda,\mu}(u_n,v_n) = Q_{\lambda,\mu}(u_1,v_1) + o_n(1). \tag{5.3}$$

We now claim that (u_1,v_1) is a nontrivial solution of (1.1). From (5.1), (5.2) and (5.3), it is easy to verify that (u_1,v_1) is a weak solution of (1.1). Moreover, from $(u_n,v_n) \in \mathcal{N}_{\lambda,\mu}$, we get

$$Q_{\lambda,\mu}(u_n,v_n) = \frac{q(2^*_\alpha(t)-2)}{2(2^*_\alpha(t)-q)}\|(u_n,v_n)\|^2_W - \frac{q \cdot 2^*_\alpha(t)}{2^*_\alpha(t)-q} I_{\lambda,\mu}(u_n,v_n). \tag{5.4}$$

Let $n \to \infty$ in (5.4), by (5.3) and Lemma 3.7(i), we have

$$Q_{\lambda,\mu}(u_1, v_1) \geq -\frac{q \cdot 2^*_\alpha(t)}{2^*_\alpha(t) - q} c_{\lambda,\mu} > 0.$$

Thus, $(u_1, v_1) \in \mathcal{N}_{\lambda,\mu}$ is a nontrivial weak solution of (1.1).

Next, we show that $(u_n, v_n) \to (u_1, v_1)$ strongly in W and $I_{\lambda,\mu}(u_1, v_1) = c_{\lambda,\mu}$. From the fact $(u_1, v_1) \in \mathcal{N}_{\lambda,\mu}$, and the Fatou's lemma it follows that

$$\begin{aligned} c_{\lambda,\mu} &\leq I_{\lambda,\mu}(u_1, v_1) \\ &= \frac{2^*_\alpha(t) - 2}{2 \cdot 2^*_\alpha(t)} \|(u_1, v_1)\|^2_W - \frac{2^*_\alpha(t) - q}{q \cdot 2^*_\alpha(t)} Q_{\lambda,\mu}(u_1, v_1) \\ &\leq \lim_{n\to\infty} \left[\frac{2^*_\alpha(t) - 2}{2 \cdot 2^*_\alpha(t)} \|(u_n, v_n)\|^2_W - \frac{2^*_\alpha(t) - q}{q \cdot 2^*_\alpha(t)} Q_{\lambda,\mu}(u_n, v_n) \right] \\ &= \lim_{n\to\infty} I_{\lambda,\mu}(u_n, v_n) \\ &= c_{\lambda,\mu}, \end{aligned}$$

which implies that $c_{\lambda,\mu} = I_{\lambda,\mu}(u_1, v_1)$ and $\lim_{n\to\infty} \|(u_n, v_n)\|^2_W = \|(u_1, v_1)\|^2_W$. A standard argument shows that $(u_n, v_n) \to (u_1, v_1)$ strongly in W.

Finally, we claim that $(u_1, v_1) \in \mathcal{N}^+_{\lambda,\mu}$. Assume by contradiction that $(u_1, v_1) \in \mathcal{N}^-_{\lambda,\mu}$. Then by Lemma 3.3, there exist unique τ_1^+ and $\tau_1^- > 0$ such that $(\tau_1^+ u_1, \tau_1^+ v_1) \in \mathcal{N}^+_{\lambda,\mu}$, $(\tau_1^- u_1, \tau_1^- v_1) \in \mathcal{N}^-_{\lambda,\mu}$ and $\tau_1^+ < \tau_1^- = 1$. Since

$$\frac{d}{d\tau} I_{\lambda,\mu}(\tau_1^+ u_1, \tau_1^+ v_1) = 0, \quad \frac{d^2}{d\tau^2} I_{\lambda,\mu}(\tau_1^+ u_1, \tau_1^+ v_1) > 0,$$

there exists $\tau_1^+ < \tau_1^* < \tau_1^-$ such that $I_{\lambda,\mu}(\tau_1^+ u_1, \tau_1^+ v_1) < I_{\lambda,\mu}(\tau_1^* u_1, \tau_1^* v_1)$. By Lemma 3.3, we have

$$I_{\lambda,\mu}(\tau_1^+ u_1, \tau_1^+ v_1) < I_{\lambda,\mu}(\tau_1^* u_1, \tau_1^* v_1) \leq I_{\lambda,\mu}(\tau_1^- u_1, \tau_1^- v_1) < I_{\lambda,\mu}(u_1, v_1),$$

a contradiction. Moreover, from $I_{\lambda,\mu}(u_1, v_1) = I_{\lambda,\mu}(|u_1|, |v_1|)$, $(|u_1|, |v_1|) \in \mathcal{N}_{\lambda,\mu}$ and Lemma 3.4 we may assume that (u_1, v_1) is a nontrivial nonnegative solution of (1.1) in W. By the strong maximum principle [25, Proposition 2.2.8], it follows that $u_1, v_1 > 0$ in Ω. Hence, (u_1, v_1) is a positive solution for (1.1). □

Remark 5.2. From Lemma 3.7(i) and (3.2), for this nontrivial solution (u_1, v_1), we obtain

$$\begin{aligned} 0 > c_{\lambda,\mu} &= I_{\lambda,\mu}(u_1, v_1) \\ &= \left(\frac{1}{2} - \frac{1}{2^*_\alpha(t)} \right) \|(u_1, v_1)\|^2_W - \left(\frac{1}{q} - \frac{1}{2^*_\alpha(t)} \right) Q_{\lambda,\mu}(u_1, v_1) \\ &\geq -\frac{2^*_\alpha(t) - q}{q \cdot 2^*_\alpha(t)} Q_{\lambda,\mu}(u_1, v_1) \end{aligned}$$

$$\geq -\frac{2^*_\alpha(t)-q}{q\cdot 2^*_\alpha(t)}\left(\frac{N\omega_N R_0^{N-s}}{N-s}\right)^{(2^*_\alpha(s)-q)/2^*_\alpha(s)}\left(\lambda^{2/(2-q)}+\mu^{2/(2-q)}\right)^{(2-q)/2}$$
$$\times\left(\Lambda_{\alpha,s,2^*_\alpha(s)}\right)^{-q/2}\|(u_1,v_1)\|_W^q.$$

This implies that $I_{\lambda,\mu}(u_1,v_1)\to 0$ as $\lambda\to 0^+$ and $\mu\to 0^+$.

Now, we will establish the existence of the second positive solution of (1.1) on $\mathcal{N}^-_{\lambda,\mu}$.

Lemma 5.3. *Assume that* $0<\alpha<2$, $0\leq\gamma<\gamma_H$, $0\leq s,t<\alpha<N$, $1\leq q<2$ *and* $\eta,\theta>1$ *with* $\eta+\theta=2^*_\alpha(t)$. *Then there exist* $(u,v)\in W\setminus\{(0,0)\}$ *and* $\Lambda_1>0$ *such that for* $(\lambda,\mu)\in\mathfrak{D}_{\Lambda_1}$,

$$\sup_{\tau\geq 0} I_{\lambda,\mu}(\tau u,\tau v)<\frac{\alpha-t}{N-t}\left(\frac{S_{\alpha,\eta,\theta}}{2}\right)^{(N-t)/(\alpha-t)}. \tag{5.5}$$

In particular, $c^-_{\lambda,\mu}<\frac{\alpha-t}{N-t}\left(\frac{S_{\alpha,\eta,\theta}}{2}\right)^{(N-t)/(\alpha-t)}$ *for all* $(\lambda,\mu)\in\mathfrak{D}_{\Lambda_1}$.

Proof. Let $\rho>0$ be small enough such that $B_\rho(0)\subset\Omega$. Taking a cut-off function $\varphi\in C_0^\infty(\Omega)$ satisfies $0\leq\varphi(x)\leq 1$, $\varphi(x)=1$ for $|x|<\rho/2$ and $\varphi(x)=0$ for $|x|\geq\rho$. Set $U_\varepsilon(x)=\varepsilon^{-(N-\alpha)/2}u_\gamma(x/\varepsilon)$ and $u_\varepsilon(x)=\varphi(x)U_\varepsilon(x)$, where $u_\gamma(x)$ is a ground state solution of problem (2.10). Then, we consider the functions

$$\begin{aligned}
g(\tau)&=I_{\lambda,\mu}(\tau\sqrt{\eta}\,u_\varepsilon,\tau\sqrt{\theta}\,u_\varepsilon)\\
&=\frac{\tau^2}{2}\|(\sqrt{\eta}\,u_\varepsilon,\sqrt{\theta}\,u_\varepsilon)\|_W^2-\frac{\tau^q}{q}Q_{\lambda,\mu}(\sqrt{\eta}\,u_\varepsilon,\sqrt{\theta}\,u_\varepsilon)\\
&\quad-\frac{2\tau^{2^*_\alpha(t)}}{2^*_\alpha(t)}\int_\Omega\frac{|\sqrt{\eta}\,u_\varepsilon|^\eta|\sqrt{\theta}\,u_\varepsilon|^\theta}{|x|^t}\,dx
\end{aligned}$$

and

$$h(\tau)=\frac{\tau^2}{2}\|(\sqrt{\eta}\,u_\varepsilon,\sqrt{\theta}\,u_\varepsilon)\|_W^2-\frac{2\tau^{2^*_\alpha(t)}}{2^*_\alpha(t)}\int_\Omega\frac{|\sqrt{\eta}\,u_\varepsilon|^\eta|\sqrt{\theta}\,u_\varepsilon|^\theta}{|x|^t}\,dx.$$

By the fact that

$$\begin{aligned}
\sup_{\tau\geq 0}\left(\frac{\tau^2}{2}B_1-\frac{2\tau^{2^*_\alpha(t)}}{2^*_\alpha(t)}B_2\right)&=\frac{B_1}{2}\left(\frac{B_1}{2B_2}\right)^{\frac{2}{2^*_\alpha(t)-2}}-\frac{B_2}{2^*_\alpha(t)}\left(\frac{B_1}{2B_2}\right)^{\frac{2^*_\alpha(t)}{2^*_\alpha(t)-2}}\\
&=\frac{\alpha-t}{2(N-t)}\frac{B_1^{\frac{2^*_\alpha(t)}{2^*_\alpha(t)-2}}}{(2B_2)^{\frac{2}{2^*_\alpha(t)-2}}}=\frac{\alpha-t}{2(N-t)}\left(\frac{B_1}{(2B_2)^{\frac{2}{2^*_\alpha(t)}}}\right)^{\frac{2^*_\alpha(t)}{2^*_\alpha(t)-2}}\\
&=\frac{\alpha-t}{2(N-t)}\left(\frac{B_1}{(2B_2)^{\frac{2}{2^*_\alpha(t)}}}\right)^{(N-t)/(\alpha-t)},\quad\text{where } B_1,B_2>0
\end{aligned}$$

and Proposition 2.3, we can get that

$$
\begin{aligned}
\sup_{\tau\geq 0} h(\tau) &\leq \frac{\alpha-t}{2(N-t)}\left(\frac{(\eta+\theta)\|u_\varepsilon\|_\gamma^2}{\left(2\eta^{\frac{\eta}{2}}\theta^{\frac{\theta}{2}}\int_\Omega \frac{|u_\varepsilon|^{2^*_\alpha(t)}}{|x|^t}\,dx\right)^{\frac{2}{2^*_\alpha(t)}}}\right)^{(N-t)/(\alpha-t)} \\
&= \frac{\alpha-t}{2(N-t)}\left(\frac{1}{2}\right)^{\frac{N-\alpha}{\alpha-t}}\left(\left(\frac{\eta}{\theta}\right)^{\theta/(\eta+\theta)}+\left(\frac{\theta}{\eta}\right)^{\theta/(\eta+\theta)}\right)^{(N-t)/(\alpha-t)} \\
&\quad\times\left(\frac{\left(\Lambda_{\alpha,t,2^*_\alpha(t)}\right)^{(N-t)/(\alpha-t)}+O(\varepsilon^{2\beta_+(\gamma)+\alpha-N})}{\left((\Lambda_{\alpha,t,2^*_\alpha(t)})^{(N-t)/(\alpha-t)}+O(\varepsilon^{2^*_\alpha(t)\beta_+(\gamma)+t-N})\right)^{\frac{2}{2^*_\alpha(t)}}}\right)^{(N-t)/(\alpha-t)} \\
&= \frac{\alpha-t}{N-t}\left(\frac{1}{2}\right)^{\frac{N-\alpha}{\alpha-t}+1}\left(\left(\frac{\eta}{\theta}\right)^{\theta/(\eta+\theta)}+\left(\frac{\theta}{\eta}\right)^{\theta/(\eta+\theta)}\right)^{(N-t)/(\alpha-t)} \\
&\quad\times\left(\Lambda_{\alpha,t,2^*_\alpha(t)}\right)^{(N-t)/(\alpha-t)}+O(\varepsilon^{2\beta_+(\gamma)+\alpha-N}) \\
&= \frac{\alpha-t}{N-t}\left[\frac{1}{2}\left(\left(\frac{\eta}{\theta}\right)^{\theta/(\eta+\theta)}+\left(\frac{\theta}{\eta}\right)^{\theta/(\eta+\theta)}\right)\left(\Lambda_{\alpha,t,2^*_\alpha(t)}\right)\right]^{(N-t)/(\alpha-t)} \\
&\quad+O(\varepsilon^{2\beta_+(\gamma)+\alpha-N}) \\
&= \frac{\alpha-t}{N-t}\left(\frac{S_{\alpha,\eta,\theta}}{2}\right)^{(N-t)/(\alpha-t)}+O(\varepsilon^{2\beta_+(\gamma)+\alpha-N}).
\end{aligned}
\tag{5.6}
$$

On the other hand, using the definitions of g and u_ε, we get

$$
g(\tau)=I_{\lambda,\mu}(\tau\sqrt{\eta}\,u_\varepsilon,\tau\sqrt{\theta}\,u_\varepsilon)\leq\frac{\tau^2}{2}\|(\sqrt{\eta}\,u_\varepsilon,\sqrt{\theta}\,u_\varepsilon)\|_W^2\quad\text{for all }\lambda,\mu>0\text{ and }\tau\geq 0.
$$

Combining this with (2.12), let $\varepsilon\in(0,1)$, then there exists $\tau_0\in(0,1)$ independent of $\varepsilon>0$ such that

$$
\sup_{\tau\in[0,\tau_0]} g(\tau)<\frac{\alpha-t}{N-t}\left(\frac{S_{\alpha,\eta,\theta}}{2}\right)^{(N-t)/(\alpha-t)}. \tag{5.7}
$$

Hence, by (5.6), for all $1\leq q<2$ we can choose $\Lambda_1>0$ so small that, for all $(\lambda,\mu)\in\mathfrak{D}_{\Lambda_1}$, we get

$$
\begin{aligned}
\sup_{\tau\geq\tau_0} g(\tau) &= \sup_{\tau\geq\tau_0}\left(h(\tau)-\frac{\tau^q}{q}Q_{\lambda,\mu}(\sqrt{\eta}\,u_\varepsilon,\sqrt{\theta}\,u_\varepsilon)\right) \\
&\leq \frac{\alpha-t}{N-t}\left(\frac{S_{\alpha,\eta,\theta}}{2}\right)^{(N-t)/(\alpha-t)}+O(\varepsilon^{2\beta_+(\gamma)+\alpha-N}) \\
&\quad-\frac{\tau_0^q}{q}\left(\lambda\eta^{q/2}+\mu\theta^{q/2}\right)\int_\Omega\frac{|u_\varepsilon|^q}{|x|^s}\,dx.
\end{aligned}
\tag{5.8}
$$

Now we need to distinguish two cases:

Case 1*:* $1\leq q<(N-s)/\beta_+(\gamma)$. By (2.14) one can get

$$
\int_\Omega\frac{|u_\varepsilon|^q}{|x|^s}\,dx\geq C\varepsilon^{q(\beta_+(\gamma)-(N-\alpha)/2)}.
$$

Moreover, from $\beta_+(\gamma) > (N-\alpha)/2$, we get

$$2\beta_+(\gamma) + \alpha - N - q\left(\beta_+(\gamma) - \frac{N-\alpha}{2}\right) = (2-q)\left(\beta_+(\gamma) - \frac{N-\alpha}{2}\right) > 0.$$

Then

$$2\beta_+(\gamma) + \alpha - N > q\left(\beta_+(\gamma) - \frac{N-\alpha}{2}\right).$$

Combining this with (5.7) and (5.8), for $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_1}$, we can choose ε small enough such that

$$\sup_{\tau\ge 0} g(\tau) < \frac{\alpha-t}{N-t}\left(\frac{S_{\alpha,\eta,\theta}}{2}\right)^{(N-t)/(\alpha-t)}.$$

Case 2*:* $(N-s)/\beta_+(\gamma) \le q < 2$. By (2.14) we have that

$$\int_\Omega \frac{|u_\varepsilon|^q}{|x|^s}\,dx \ge \begin{cases} C\varepsilon^{N-s-q(N-\alpha)/2} & \text{if } q > (N-s)/\beta_+(\gamma), \\ C\varepsilon^{N-s-q(N-\alpha)/2}|\ln\varepsilon| & \text{if } q = (N-s)/\beta_+(\gamma). \end{cases}$$

Moreover, it follows from $\beta_+(\gamma) > (N-\alpha)/2$ and $(N-s)/\beta_+(\gamma) \le q$ that

$$2\beta_+(\gamma) + \alpha - N > q\left(\beta_+(\gamma) - \frac{N-\alpha}{2}\right) \ge N - s - \frac{q(N-\alpha)}{2}. \tag{5.9}$$

Therefore, from (5.7), (5.8) and (5.9), we get that for $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_1}$, we can choose $\varepsilon > 0$ small enough such that

$$\sup_{\tau\ge 0} g(\tau) < \frac{\alpha-t}{N-t}\left(\frac{S_{\alpha,\eta,\theta}}{2}\right)^{(N-t)/(\alpha-t)}.$$

From Cases 1 and 2, (5.5) holds by taking $(u,v) = (\sqrt{\eta}\,u_\varepsilon, \sqrt{\theta}\,u_\varepsilon)$.

From Lemma 3.5, the definition of $c^-_{\lambda,\mu}$ and (5.5), for all $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_1}$, we obtain that there exists $\tau^- > 0$ such that $(\tau^-\sqrt{\eta}\,u_\varepsilon, \tau^-\sqrt{\theta}\,u_\varepsilon) \in \mathcal{N}^-_{\lambda,\mu}$ and

$$\begin{aligned} c^-_{\lambda,\mu} &\le I_{\lambda,\mu}(\tau^-\sqrt{\eta}\,u_\varepsilon, \tau^-\sqrt{\theta}\,u_\varepsilon) \\ &\le \sup_{\tau\ge 0} I_{\lambda,\mu}(\tau\sqrt{\eta}\,u_\varepsilon, \tau\sqrt{\theta}\,u_\varepsilon)) < \frac{\alpha-t}{N-t}\left(\frac{S_{\alpha,\eta,\theta}}{2}\right)^{(N-t)/(\alpha-t)}. \end{aligned}$$

The proof is thus complete. □

Now, we establish the existence of a local minimum of $I_{\lambda,\mu}$ on $\mathcal{N}^-_{\lambda,\mu}$.

Theorem 5.4. *Assume that $N > \alpha$, $0 \le \gamma < \gamma_H$, $0 \le s,t < \alpha$, $\eta,\theta > 1$ with $\eta+\theta = 2^*_\alpha(t)$ and $1 \le q < 2$. Let $\Lambda^* = \min\{\Lambda_0, \Lambda_*, \Lambda_1\}$. Then $I_{\lambda,\mu}$ has a minimizer (u_2,v_2) in $\mathcal{N}^-_{\lambda,\mu}$ for all $(\lambda,\mu) \in \mathfrak{D}_{\Lambda^*}$, and satisfying*

(i) $I_{\lambda,\mu}(u_2,v_2) = c^-_{\lambda,\mu}$*;*

(ii) (u_2, v_2) *is a positive solution of* (1.1).

Proof. By Proposition 4.3(ii), there exists a $(\mathrm{PS})_{c^-_{\lambda,\mu}}$-sequence $\{(u_n, v_n)\} \subset \mathcal{N}^-_{\lambda,\mu}$ for $I_{\lambda,\mu}$ for all $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_*}$. By Lemmas 3.4 and 5.3, Proposition 4.2, for all $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_1}$, $I_{\lambda,\mu}$ satisfies the $(\mathrm{PS})_{c^-_{\lambda,\mu}}$ condition for all $c^-_{\lambda,\mu} \in \big(0, \frac{\alpha-t}{N-t}(\frac{S_{\alpha,\eta,\theta}}{2})^{(N-t)/(\alpha-t)}\big)$. Then, for all $(\lambda,\mu) \in \mathfrak{D}_{\Lambda^*}$, there exist a subsequence still denoted by $\{(u_n, v_n)\}$ and $(u_2, v_2) \in W \setminus \{(0,0)\}$ such that $(u_n, v_n) \rightharpoonup (u_2, v_2)$ weakly in W. Arguing as in the proof of Theorem 5.1, for $(\lambda,\mu) \in \mathfrak{D}_{\Lambda^*}$, we obtain $(u_n, v_n) \to (u_2, v_2)$ strongly in W and (u_2, v_2) is a positive solution of (1.1).

Finally, we prove that $(u_2, v_2) \in \mathcal{N}^-_{\lambda,\mu}$. Arguing by contradiction, we assume $(u_2, v_2) \in \mathcal{N}^+_{\lambda,\mu}$. Since $\mathcal{N}^-_{\lambda,\mu}$ is closed in W, we have $\|(u_2, v_2)\|_W < \liminf_{n\to\infty} \|(u_n, v_n)\|_W$. Moreover, by Lemma 3.5, there exists a unique τ_2^- such that $(\tau_2^- u_2, \tau_2^- v_2) \in \mathcal{N}^-_{\lambda,\mu}$. This and $(u_n, v_n) \in \mathcal{N}^-_{\lambda,\mu}$ deduce that

$$c^-_{\lambda,\mu} \le I_{\lambda,\mu}(\tau_2^- u_2, \tau_2^- v_2) < \lim_{n\to\infty} I_{\lambda,\mu}(\tau_2^- u_n, \tau_2^- v_n) \le \lim_{n\to\infty} I_{\lambda,\mu}(u_n, v_n) = c^-_{\lambda,\mu}.$$

So, $(u_2, v_2) \in \mathcal{N}^-_{\lambda,\mu}$. This completes the proof of Theorem 5.4. □

6. Proof of Theorem 1.2

By Theorem 5.1, the system (1.1) has a positive solution $(u_1, v_1) \in \mathcal{N}^+_{\lambda,\mu}$ for all $(\lambda,\mu) \in \mathfrak{D}_{\Lambda_0}$. This gives the proof of Theorem 1.2(i).

By Theorem 5.4, for all $(\lambda,\mu) \in \mathfrak{D}_{\Lambda^*}$, the system (1.1) admits a positive solution $(u_2, v_2) \in \mathcal{N}^-_{\lambda,\mu}$. Since $\mathcal{N}^+_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \emptyset$, $\mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu} = \mathcal{N}_{\lambda,\mu}$ and $\Lambda^* < \Lambda_0$, we can get that (u_1, v_1), (u_2, v_2) are distinct positive solutions of the system (1.1), and complete the proof of Theorem 1.2(ii).

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