J. DIFFERENTIAL GEOMETRY
35 (1992) 511-519

COHOMOLOGY OF SCHUBERT SUBVARIETIES OF GL_n/P

E. AKYILDIZ, A. LASCOUX & P. PRAGACZ

Dedicated to Professor I. M. Gelfand on his seventy-fifth birthday

Abstract

Let GL_n be the group of $n \times n$ invertible complex matrices, and P a parabolic subgroup of GL_n. In this paper we give a geometric description of the cohomology ring of a Schubert subvariety Y of GL_n/P. Our main result (Theorem 3.1) states that the coordinate ring $A(Y \cap Z)$ of the scheme-theoretic intersection of Y and the zero scheme Z of the vector field V associated to a principal regular nilpotent element $\mathfrak{n}$ of $\mathfrak{gl}_n$ is isomorphic to the cohomology algebra $H^*(Y; \mathbf{C})$ of Y. This theorem was conjectured for any reductive algebraic group G in [4], and it was proved for the Grassmannian manifolds in [2]. We were recently informed that Professor D. H. Peterson has just proved that GL_n is exactly the algebraic group G where the cohomology ring of any Schubert subvariety Y of the space G/B is isomorphic to $A(Y \cap Z)$. Here B stands for a Borel subgroup of G. It is also interesting to note that the cohomology ring of the union of two Schubert subvarieties in GL_n/P may not admit such a description. This result is due to Professor J. B. Carrell.

0. Introduction

Let X be a nonlinear complex projective variety having the following properties:

(A) there exists an algebraic vector field V with exactly one zero x_0, and

(B) there exists an algebraic $\mathbf{C}^*$-action on X

$$\lambda: \mathbf{C}^* \times X \to X \qquad ((t, x) \to \lambda(t) \cdot x),$$

such that $d\lambda(t) \cdot V = t^p V$ for some $p > 0$ and for all t in $\mathbf{C}^*$, where $d\lambda(t)$ is the associated tangent action of $\lambda(t)$ on vector fields.

Let Z be the zero scheme of the vector field V, and let Y be any V- and $\mathbf{C}^*$-invariant subvariety of X. It follows from property (B) that Z is a $\mathbf{C}^*$-invariant subscheme of X. Thus, the coordinate ring $A(Z)$

Received July 12, 1989. The first author was partially supported by King Fahd University Petroleum and Minerals (KFUPM) Project MS/Action 2/100.

(respectively $A(Y \cap Z)$) of Z (respectively $Y \cap Z$) has a natural graded algebra structure induced from the $\mathbf{C}^*$-action λ. Here, $Y \cap Z$ stands for the scheme-theoretic intersection of Y and Z. Throughout the rest of the paper the rings $A(Z)$ and $A(Y \cap Z)$ will be regarded as graded algebras with the gradation above, and $H^*(W; \mathbf{C})$ will denote the cohomology ring of the variety W with coefficients in the field of complex numbers $\mathbf{C}$. The following theorem is proved in [4], [5].

Theorem. *There exists a graded algebra isomorphism*

$$\psi: A(Z) \to H^*(X; \mathbf{C})$$

which induces a graded algebra homomorphism

$$\overline{\psi}: A(Y \cap Z) \to H^*(Y; \mathbf{C})$$

commuting with the natural maps

$$A(Z) \to A(Y \cap Z) \quad \text{and} \quad H^*(X; \mathbf{C}) \to H^*(Y; \mathbf{C}).$$

For any parabolic subgroup P of a complex reductive algebraic group G, the space G/P has the properties (A) and (B). Moreover any Schubert subvariety $Y = \overline{B\sigma P}$ of G/P is V- and $\mathbf{C}^*$-invariant. Thus, by the Theorem we have a surjective graded algebra homomorphism

$$\overline{\psi}: A(Y \cap Z) \to H^*(Y; \mathbf{C}).$$

Definition. The cohomology ring of the Schubert variety Y is said to have a nilpotent description if $\overline{\psi}$ is an isomorphism. It is known that the cohomology ring of any Schubert subvariety Y of the Grassmann manifold $G_{k,n}$ has a nilpotent description [2]. In this paper, we generalize this result to any Schubert subvariety of the partial flag manifold GL_n/P. The paper is organized as follows. In §1, we begin with the preliminaries. In §2, we investigate a certain ideal in the cohomology ring of GL_n/B associated with a Schubert subvariety $Y = \overline{B\sigma B}$ of GL_n/B. This is done by finding a relation between the functions P_σ constructed by Bernstein, Gelfand, and Gelfand in [6] (independently by Demazure in [7]), and the Plücker coordinates. In §3, we first prove that if the cohomology ring of any Schubert subvariety of the space G/B has a nilpotent description, then so does the cohomology ring of any Schubert subvariety of G/P. Here P is a parabolic subgroup of a complex reductive linear algebraic group G which contains the Borel subgroup B of G. Then we finally prove that the cohomology rings of the Schubert subvarieties of GL_n/P have nilpotent descriptions.

1. Preliminaries

Let GL_n be the group of $n \times n$ invertible complex matrices, B the group of upper triangular matrices in GL_n, W the symmetric group in $1, 2, \cdots, n$, and $l(\tau)$ the length of $\tau \in W$. Let $R = \mathbf{C}[x_1, \cdots, x_n]$ be the polynomial algebra with the usual grading, and IR the ideal of R generated by the elementary symmetric polynomials in $x_1, \cdots, x_n$. W acts on R by permuting the variables. We denote this action by $\sigma \cdot f(x_1, \cdots, x_n) = f(x_{\sigma_1}, \cdots, x_{\sigma_n})$, $\sigma = (\sigma_1, \cdots, \sigma_n) \in W$. Let (i, j) denote the transposition of W obtained by changing i with j. We recall the following facts from [6], [7] (see also [10] for a more combinatorial approach). For any $1 \leq i < j \leq n$, the polynomial $f-(i, j)\cdot f$ is divisible by $x_i - x_j$. Thus, the operator

$$\partial_{(i,j)}: R \to R, \qquad \partial_{(i,j)}(f) = \frac{f-(i,j)\cdot f}{x_i - x_j},$$

is well defined.

Let $i_1, \cdots, i_r$ be integers in $\{1, \cdots, n\}$, and let $\omega = (i_1, i_1+1) \cdots (i_r, i_r+1)$ be any element of W. Then the following hold:

(a) If $l(\omega) \neq r$, then $\partial_{(i_1, i_1+1)} \cdots \partial_{(i_r, i_r+1)} = 0$.

(b) If $l(\omega) = r$, then the operator $\partial_{(i_1, i_1+1)} \cdots \partial_{(i_r, i_r+1)}$ depends only on ω and not on the representation in the form $\omega = (i_1, i_1+1) \cdots (i_r, i_r+1)$.

In case (b) we put $\partial_\omega = \partial_{(i_1, i_1+1)} \cdots \partial_{(i_r, i_r+1)}$. We note that the operator $\partial_\omega: R \to R$ preserves the ideal IR, and thus it induces an operator $\overline{\partial}_\omega: R/IR \to R/IR$ of homogeneous degree $-l(\omega)$. Let ω_0 be the permutation $(n, n-1, \cdots, 1)$ in W, and let $P_{\omega_0} = (\prod_{1 \leq i<j \leq n}(x_i - x_j))/n!$ mod (IR). For each ω in W, let $P_\omega = \overline{\partial}_{\omega\omega_0}(P_{\omega_0})$, and let $[X_\tau]$ denote the cycle class of the Schubert variety $X_\tau = \overline{B\tau B}$ in $H_*(GL_n/B; \mathbf{C})$. The following theorem is proved in [6], [7].

Theorem 1.1. *There exists a graded algebra isomorphism $\beta: R/IR \to H^*(GL_n/B; \mathbf{C})$ such that $\beta(P_\omega) = \mathscr{P}([X_{\omega_0\omega}])$ for any ω in W, where $\mathscr{P}$ stands for the Poincaré duality map*

$$\mathscr{P}: H_*(GL_n/B; \mathbf{C}) \to H^*(GL_n/B; \mathbf{C}).$$

We shall now discuss the nilpotent case $A(Z)$ for the space GL_n/B. Let U be the group of all lower triangular unipotent matrices in GL_n, and let $z_{i,j}$, $1 \leq j < i \leq n$, be the coordinate functions $z_{i,j}(x) = x_{i,j}$, $x \in U$. Let $\mathfrak{n}$ be the regular nilpotent $n \times n$ matrix, which is in the Jordan form, and let V be the vector field on GL_n/B induced from the one-parameter subgroup $\exp(t\mathfrak{n})$ of GL_n. V has a unique zero $x_0 = B$, and satisfies property (B) [1]. The coordinate ring $A(Z)$ of the zero scheme Z of V in the affine neighborhood U of x_0 has been computed in [2], and the following description has been obtained. Consider the grading on the polynomial algebra $A(U) = \mathbf{C}[z_{i,j}: 1 \leq j < i \leq n]$ determined by taking $\deg z_{i,j} = i - j$. Then $A(Z)$ is isomorphic, as a graded algebra, to $A(U)/I(Z)$, where $I(Z)$ is the ideal of $A(U)$ generated by the homogeneous elements

$$z_{i+1,j} - z_{i,j-1} + z_{i,j}(z_{j,j-1} - z_{j+1,j}),$$

where we take $z_{k,r} = 0$ if $k > n$, or $r < 1$, or $r > k$.

Let I_k, $k = 1, 2, \cdots, n-1$, denote the set of sequences of integers $(i_1, \cdots, i_k)$ such that $1 \leq i_1 < i_2 < \cdots < i_i \leq n$, and let W_k be the set of all permutations $(\mu_1, \cdots, \mu_n)$ in W such that $(\mu_1, \cdots, \mu_k) \in I_k$ and $(\mu_{k+1}, \cdots, \mu_n) \in I_{n-k}$. For any $(i_1, \cdots, i_k)$ in I_k there exists a unique permutation in the form $(i_1, \cdots, i_k, i_{k+1}, \cdots, i_n)$ in W_k. We denote this permutation by $\sigma(i_1, \cdots, i_k)$. For $(i_1, \cdots, i_k)$ in I_k, let $[i_1, \cdots, k_k]$ denote the function in $A(Z)$ which is induced from the Plücker coordinate $\det[z_{i_m,j}]$, $1 \leq m, j \leq k$.

Here and throughout the rest of the paper, we put $z_{k,r} = 0$ if $k > n$, or $r > k$, or $r < 1$. The following theorem is proved in [2].

Theorem 1.2. *The homomorphism $\varphi: R \to A(U)$ determined by $\varphi(x_i) = z_{i+1,i} - z_{i,i-1}$, $i = 1, \cdots, n$, induces a graded algebra isomorphism $\overline{\varphi}: R/IR \to A(Z)$. Moreover for any $(i_1, \cdots, i_k)$ in I_k we have*

$$\overline{\varphi}(P_{\sigma(i_1,\cdots,i_k)}) = [i_1, \cdots, i_k].$$

2. A certain ideal associated with a Schubert variety in the cohomology of GL_n/B

We keep the notation of §1, and moreover, for a given sequence of distinct integers $(j_1, \cdots, j_k)$, $(j_1, \cdots, j_k)^<$ (respectively $(j_1, \cdots, j_k)^>$) denotes the sequence $(j_{\tau_1}, \cdots, j_{\tau_k})$, where $j_{\tau_1} < \cdots < j_{\tau_k}$ (respectively, $j_{\tau_1} > \cdots > j_{\tau_k}$) for some permutation $\tau = (\tau_1, \cdots, \tau_k)$ of $\{1, 2, \cdots, k\}$. We recall the following well-known formula, which is due to Monk [11] (see also [6], [7], and [10]).

Theorem 2.1. *Let $\mu = (\mu_1, \cdots, \mu_n)$ be a permutation in W, and let $k = 1, 2, \cdots, n-1$. Then the identity*

$$P_\mu x_k = \sum \operatorname{sgn}(j-k) P_{\mu(j,k)}$$

holds in R/IR, where the sum is over all $j \neq k$ such that $l(\mu(j,k)) = l(\mu) + 1$.

For $k = 1, 2, \cdots, n-1$, let $\mathfrak{p}_k \colon W \to W_k$ denote the projection map

$$\begin{aligned}\mathfrak{p}_k(\mu_1, \cdots, \mu_n) &= \sigma((\mu_1, \cdots, \mu_k)^<) \\ &= ((\mu_1, \cdots, \mu_k)^<, (\mu_{k+1}, \cdots, \mu_n)^<).\end{aligned}$$

We note that the Bruhat ordering $\leq$ on W ($\tau \leq \mu$ if and only if $B\tau B \subseteq \overline{B\mu B}$ in GL_n/B) induces an ordering on W_k, which we will also denote by $\leq$. Recall that for $\mu = (\mu_1, \cdots, \mu_n)$ and $\nu = (\nu_1, \cdots, \nu_n)$ in W_k, $\mu \leq \nu$ (in W_k) if and only if $\mu_i \leq \nu_i$ for $i = 1, \cdots, k$.

Lemma 2.1. *Let $\mu = (\mu_1, \cdots, \mu_n)$ be a permutation in W which satisfies $\mu_1 > \cdots > \mu_k$ and $\mu_{k+1} > \cdots > \mu_n$. Then we have the following equality in R/IR,*

$$P_\mu = P_{\mathfrak{p}_k(\mu)} x_1^{k-1} x_2^{k-2} \cdots x_{k-1} x_{k+1}^{n-k-1} x_{k+2}^{n-k-2} \cdots x_{n-1} + \sum m_\tau P_\tau,$$

where the sum is over all τ in W such that $\mathfrak{p}_k(\mu) < \mathfrak{p}_k(\tau)$ in W_k.

Proof. By using Monk's formula for the successive multiplications

$$\begin{gathered}P_{\mathfrak{p}_k(\mu)} x_1, (P_{\mathfrak{p}_k(\mu)} x_1) x_1, \cdots, (P_{\mathfrak{p}_k(\mu)} x_1^{k-2}) x_1, \\ (P_{\mathfrak{p}_k(\mu)} x_1^{k-1}) x_2, \cdots, (P_{\mathfrak{p}_k(\mu)} x_1^{k-1} x_2^{k-2}), \cdots, \\ P_{\mathfrak{p}_k(\mu)} x_1^{k-1} x_2^{k-2} \cdots x_{k-1},\end{gathered}$$

it is not difficult to see that at each stage of the multiplication there appears in the sum only one P_ζ with $\mathfrak{p}_k(\zeta) = \mathfrak{p}_k(\mu)$, and all the remaining P_ν satisfy $\mathfrak{p}_k(\mu) < \mathfrak{p}_k(\nu)$. (Note that we start with the permutation $\mathfrak{p}_k(\mu)$, where the first k elements appear in ascending order.) Thus we get an expression in the form

$$P_{\mathfrak{p}_k(\mu)} x_1^{k-1} x_2^{k-2} \cdots x_{k-1} = P_{(\mu_1, \cdots, \mu_k, \mu_n, \cdots, \mu_{k+1})} + \sum m_\xi P_\xi,$$

where $m_\xi \in Z$, and the sum is over all ξ in W such that $\mathfrak{p}_k(\mu) < \mathfrak{p}_k(\xi)$. We repeat this process, multiplying $P_{\mathfrak{p}_k(\mu)} x_1^{k-1} x_2^{k-2} \cdots x_{k-1}$ first by x_{k+1}, then by $x_{k+1}^2, \cdots$, then by $x_{k+1}^{n-k-1}, \cdots$, and finally by x_{n-1}. It is clear that by arguing as above we obtain the claim.

Lemma 2.2. *For any permutation $\mu = (\mu_1, \cdots, \mu_n)$ in W, and $k = 1, 2, \cdots, n-1$, the equality*

$$P_\mu = fP_{\mathfrak{p}_k(\mu)} + \sum m_\tau P_\tau$$

holds in R/IR, where the sum is over all τ in W such that $\mathfrak{p}_k(\mu) < \mathfrak{p}_k(\tau)$ in W_k.

Proof. It follows from Lemma 2.1 that

$$P_{((\mu_1, \cdots, \mu_k)^>, (\mu_{k+1}, \cdots, \mu_n)^>)} = P_{\mathfrak{p}_k(\mu)} g + \sum m_\xi P_\xi,$$

where $g = x_1^{k-1} x_2^{k-2} \cdots x_{k-1} x_{k+1}^{n-k-1} \cdots x_{n-1}$. Since the operator $\partial_{(i,i+1)}$ has the property that

$$\partial_{(i,i+1)}(P_{(\xi_1, \cdots, \xi_n)}) = \begin{cases} P_{(\xi_1, \cdots, \xi_{i+1}, \xi_i, \cdots, \xi_n)} & \text{if } \xi_i > \xi_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

we can pass from $P_{((\mu_1, \cdots, \mu_k)^>, (\mu_{k+1}, \cdots, \mu_n)^>)}$ to P_μ by using the operators $\partial_{(i,i+1)}$ in an appropriate way. We note that in doing this we need to use only those $\partial_{(i,i+1)}$ where $i \neq k$. On the other hand for $i \neq k$ we have

(a) $\partial_{(i,i+1)}(P_{\mathfrak{p}_k(\mu)} g) = P_{\mathfrak{p}_k(\mu)} \partial_{(i,i+1)}(g)$, because $P_{\mathfrak{p}_k(\mu)}$ is a symmetric polynomial $x_1, \cdots, x_k$, and does not depend on the remaining variables $x_{k+1}, \cdots, x_n$.

(b) $\tilde{\mathfrak{p}}_k(\partial_{(i,i+1)}(P_\xi)) = \tilde{\mathfrak{p}}_k(P_\xi)$, where $\tilde{\mathfrak{p}}_k$ stands for the function $\tilde{\mathfrak{p}}_k(P_\tau) = P_{\mathfrak{p}_k(\tau)}$ for $\tau \in W$. Thus the assertion follows. q.e.d.

For a given permutation μ in W, let J_μ be the ideal of R/IR generated by P_σ, $\sigma \nleq \mu$, and let $\mathscr{G} = \bigcup_{k=1}^{n-1} W_k$ denote the set of the so-called Grassmannian permutations of $\{1, 2, \cdots, n\}$.

Theorem 2.2. *For any permutation μ in W, J_μ is the ideal generated by P_τ, where $\tau \nleq \mu$, and τ is in $\mathscr{G}$.*

Proof. The assertion is true for $\mu = \omega_0 = (n, n-1, \cdots, 1)$. For every permutation $\mu \neq \omega_0$ there exists a permutation ν and $k \in \{1, \cdots, n\}$ such that $\mu = \nu(k, k+1)$ and $l(\nu) = l(\mu) + 1$. Thus, it is sufficient to prove the following implication: If the assertion is true for ν, then it is true for μ. Let $\mathscr{J}(\mu)$ be the set of all permutations σ such that $\sigma \nleq \mu$. It suffices to show that for every $\omega \in \mathscr{J}(\mu) - \mathscr{J}(\nu)$ the polynomial P_ω belongs to the ideal J_μ. This is true for $\omega = \nu$. To end, it is sufficient to prove the following implication: If P_ξ belongs to the ideal J_μ, then for every ω such that $\mathfrak{p}_k(\xi) > \mathfrak{p}_k(\omega)$, the polynomial P_ω belongs to the ideal J_μ. By Lemma 2.2 we get $P_\omega = fP_{\mathfrak{p}_k(\omega)} + \sum m_\xi P_\xi$, where the summation is over ξ such that $\mathfrak{p}_k(\xi) > \mathfrak{p}_k(\omega)$, $m_\xi \in Z$, and $f \in R/IR$. We know that the terms in the sum on the right-hand side are in J_μ. Moreover it is not hard to check that $\omega \in \mathcal{I}(\mu)-\mathcal{I}(\nu)$ if and only if $\mathfrak{p}_k(\omega) \in \mathcal{I}(\mu)-\mathcal{I}(\nu)$. Therefore $fP_{\mathfrak{p}_k(\omega)} \in J_\mu$, and the proof is complete.

3. The nilpotent description of the cohomology ring of a Schubert subvariety of GL_n/P

Let G be a complex reductive linear algebraic group, B a Borel subgroup of G, and P a parabolic subgroup of G which contains B. Let $\mathfrak{n}$ be a regular nilpotent element of the Lie algebra $\mathfrak{g}$ of G which is taken from the Lie algebra $\mathfrak{b}$ of B, and let $\widetilde{V}$ (respectively V) be the vector field induced from the $\mathbf{C}$-action $\exp(t\mathfrak{n})$ on G/B (respectively G/P). By the Jacobson-Morosov Lemma (see [8]) $\widetilde{V}$ (respectively V) satisfies property (B), and in fact the above $\mathbf{C}^*$-action is induced from a one-parameter subgroup of B via the left multiplication. We also note that $\widetilde{V}$ (respectively V) has only one zero $x_0 = B$ (respectively P). Thus we can talk about the nilpotent description of any B-invariant subvariety of G/B (respectively G/P). In the following proposition we shall use the fact that the fixed point scheme $X^{\mathbf{C}}$ of a holomorphic $\mathbf{C}$-action $\sigma\colon \mathbf{C}\times X \to X$ on a complex manifold X is equal to the zero scheme of the vector field V associated to σ. This result appears to be not commonly known; a proof can be found in [3].

Proposition 3.1. *If the cohomology ring of any Schubert subvariety of G/B has a nilpotent description, then the cohomology ring of any Schubert subvariety of G/P has also a nilpotent description.*

Proof. Let $\widetilde{Z}$ (respectively Z) denote the zero scheme of $\widetilde{V}$ (respectively V), and let $Y_\sigma = \overline{B\sigma P}$ be the Schubert subvariety of G/P. Let $\pi\colon B/G \to B/P$ denote the natural projection map. It is well known that the inverse image scheme $\tau^{-1}(Y_\sigma)$ of Y_σ is a Schubert subvariety $X_{\sigma\tau} = \overline{B\sigma\tau B}$ of G/B, and the restriction map $\rho := \pi|\colon X_{\sigma\tau} \to Y_\sigma$ is a P/B fibration (see [9], for example). Thus the fiber product map $(Y_\sigma \cap Z)\times_{Y_\sigma} X_{\sigma\tau} \to Y_\sigma \cap Z$ induced by ρ is also a P/B fibration. This implies that $(Y_\sigma \cap Z)\times_{Y_\sigma} X_{\sigma\tau}$ is B-equivariantly isomorphic to $(Y_\sigma\cap Z)\times P/B$, because $\dim Y_\sigma \cap Z = 0$. Since ρ is a surjective B-equivariant map, the fixed point scheme $((Y_\sigma\cap Z)\times_{Y_\sigma} X_{\sigma\tau})^{\mathbf{C}}$ of the $\mathbf{C}$-action induced by $\exp(t\mathfrak{n})$ on $Y_\sigma \cap Z)\times_{Y_\sigma} X_{\sigma\tau}$ is isomorphic to $X_{\sigma\tau}\cap\widetilde{Z}$. This gives us $(Y_\sigma\cap Z)\times (P/B)^{\mathbf{C}} \cong X_{\sigma\tau}\cap Z$. Let ρ_1 denote the map $X_{\sigma\tau}\cap\widetilde{Z} \to Y_\sigma \cap Z$, induced by the projection $\rho\colon X_{\sigma\tau}\to Y_\sigma$. It follows from above that the comorphism $(\rho_1)^*: A(Y_\sigma \cap Z) \to A(X_{\sigma\tau} \cap \tilde{Z})$ is an inclusion. On the other hand, we have the following commutative diagram of graded algebra homomorphisms:

$$\begin{array}{ccc} A(X_{\sigma\tau} \cap \tilde{Z}) & \xrightarrow{\simeq} & H^*(X_{\sigma\tau}; \mathbf{C}) \\ {\scriptstyle (\rho_1)^*}\uparrow & & \uparrow \\ \overline{\psi}: A(Y_\sigma \cap Z) & \longrightarrow & H^*(Y_\sigma; \mathbf{C}) \end{array}$$

(see [1], for example). It follows from the diagram that $\overline{\psi}$ is injective, and therefore it is an isomorphism.

Theorem 3.1. *The cohomology ring of any Schubert subvariety of* GL_n/B *has a nilpotent description.*

Proof. Let $X_\omega = \overline{B\omega B}$ be the Schubert subvariety of GL_n/B associated to ω in W, and let J_ω be the ideal of $A(U) = \mathbf{C}[z_{i,j}: 1 \le j < i \le n]$ generated by those Plücker coordinates $\det[z_{i_m,j}]$, $1 \le m$, $j \le k$, where $(i_1, \cdots, i_k) \in I_k$ and $\sigma(i_1, \cdots, i_k) \nleq \omega$ in W. It is well known that J_ω is the ideal of the Schubert variety X_ω in the affine neighborhood U of $x_0 = B$ (see [9, Theorem 9.1], for example). This implies that if f is in $\overline{J}_\omega$, then $f = 0$ in $A(X_\omega \cap Z)$, where $\overline{J}_\omega$ is the ideal of $A(Z)$ generated by $[i_1, \cdots, i_k]$ such that $\sigma(i_1, \cdots, i_k) \nleq \omega$ in W. By using Theorems 1.2 and 2.2, we obtain $j^*(\overline{\varphi}(P_\tau)) = 0$ whenever $\tau \nleq \omega$ in W. Here j stands for the natural inclusion $X_\omega \cap Z \to Z$, and $\overline{\varphi}$ is the isomorphism $R/IR \cong A(Z)$ given in Theorem 1.2. It follows from this fact that C-vector space $A(X_\omega \cap Z)$ is spanned by the set $\{j^*(\overline{\varphi}(P_\xi)): \xi \le \omega\}$. Since $\{P_\sigma: \sigma \in W\}$ is a basis of R/IR, we get $\dim_{\mathbf{C}} A(X_\omega \cap Z) \le \text{cardinality}\{\xi \in W: \xi \le \omega\} = \dim_{\mathbf{C}} H^*(X_\omega; \mathbf{C})$. Thus the surjective map $\overline{\psi}: A(X_\omega \cap Z) \to H^*(X_\omega; \mathbf{C})$ is an isomorphism.

Corollary. *The cohomology ring of any Schubert subvariety of the partial flag manifold* GL_n/P *has a nilpotent description.*

Proof. The corollary follows from Proposition 3.1 and Theorem 3.1.

References

[1] E. Akyildiz, SL_2 *actions and cohomology of Schubert varieties*, Topics in Algebra, Banach Center Publications, Vol. 26, Part 2, Warsaw, 1990, 13–26.

[2] E. Akyildiz & Y. Akyildiz, *The relations of Plücker coordinates to Schubert calculus*, J. Differential Geometry **29** (1989) 135–142.

[3] E. Akyildiz & B. Aubertin, *Zero scheme of a vector field is equal to fixed point scheme*, IV. Ulusal Matematik Sempozyumu Antakya (1991), to appear.

[4] E. Akyildiz, J. B. Carrell & D. I. Lieberman, *Zeros of holomorphic vector fields on singular spaces and intersection rings of Schubert varieties*, Compositio Math. **57** (1986) 237–248.

[5] E. Akyildiz & J. B. Carrell, *Cohomology of projective varieties with regular* SL_2 *actions*, Manuscripta Math. **58** (1987) 473–486.

[6] I. N. Bernstein, I. M. Gelfand & S. I. Gelfand, *Schubert cells and cohomology of the space* G/P, Russian Math. Surveys **28** (1973) 1–26.

[7] M. Demazure, *Desingularisation des varietes de Schubert generalisees*, Ann. Sci. École Norm. Sup (4) **7** (1974) 53–88.

[8] B. Kostant, *The principal three-dimensional subgroup and the Betti numbers of complex semisimple Lie group*, Amer. J. Math. **81** (1959) 973–1032.

[9] V. Lakshmibai & C. S. Seshadri, *Geometry of* $G/P - V$, J. Algebra **100** (1986) 462–557.

[10] A. Lascoux & M. P. Schützenberger, *Symmetry and flag manifolds*, Lecture Notes in Math., Vol. 996, Springer, 1983, 118–144.

[11] D. Monk, *The geometry of flag manifolds*, Proc. London Math. Soc. **9** (1959) 253–286.

Middle East Technical University, Turkey
Universite Paris-VII
Polish Academy of Sciences