12 Complete Minimal Surfaces of Finite Total Curvature

To have a better understanding of a complete immersed minimal surface of finite total curvature, we will prove a theorem due to Jorge and Meeks which says that if one looks at the surface from infinity, then the surface looks like a finite number of planes passing through the origin.

Let $X : M \cong S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ be an immersed complete surface. Let $S^2(r)$ be the sphere centred at $(0,0,0)$ with radius r. Let $Y_r = X(M) \cap S^2(r)$ and

$$W_r = \frac{1}{r} Y_r \subset S^2.$$

Theorem 12.1 ([38]) *Suppose that the Gauss map on M extends continuously to S_k. Then*

1. *$X : M \cong S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ is proper.*
2. *For large r, $W_r = \{\gamma_1^r, \cdots, \gamma_n^r\}$ consists of n immersed closed curves on S^2.*
3. *γ_i^r converges in the C^1 sense to a geodesic of S^2 with multiplicity $I_i \geq 1$ as r goes to infinity.*
4. *If X is a minimal surface then the convergence in 3 is C^∞.*
5. *X is embedded at an end corresponding to p_i if and only if $I_i = 1$.*

Proof. We need only consider a neighbourhood of a puncture p. Let $D^* = D - \{p\}$ be a punctured disk and ∂D be compact. Suppose that

$$N = \lim_{|z| \to 0} N(z),$$

and that

$$N \bullet N(z) = \cos\theta \geq \frac{\sqrt{3}}{2} \text{ for } 0 \leq \theta \leq \frac{\pi}{6} \tag{12.52}$$

for all $z \in D^*$. Let π be a plane containing the line generated by N and let $\Gamma = X^{-1}(\pi)$. Since $N \bullet N(z) \geq \sqrt{3}/2$, X is transversal to π. It follows that Γ consists of points in ∂D and connected curves (in fact, the interior of $X^{-1}(\pi)$ is a one-dimensional manifold). Let γ be a connected component of Γ that is a curve.

We will consider coordinates (t, y) in π such that the y-axis is the line generated by N. It follows from (12.52) that the tangent vector of $X(\gamma)$ is never collinear with N. Thus $X(\gamma)$ is the graph of a function $y(t)$. The angle between the normal vector $(-y', 1)$ of $X(\gamma)$ and N is less than or equal to θ. Therefore

$$\frac{1}{\sqrt{1 + y'(t)^2}} \geq \cos\theta,$$

which implies that

$$|y'(t)| \leq \tan\theta, \quad \text{for all } t. \tag{12.53}$$

If γ is compact it follows that the extremal points of $X(\gamma)$ are contained in $X(\partial D)$. Let $x_1 = X(\gamma(t_1)) \in X(\partial D)$ and $x_2 = X(\gamma(t_2)) \in X(\partial D)$, and $x = X(\gamma(t))$, $t \in (t_1, t_2)$. Then

$$|x| \leq |x_1| + |x - x_1| \leq |x_1| + \int_{t_1}^{t_2} \sqrt{1 + y'(s)^2}\, ds \leq |x_1| + \frac{2}{\sqrt{3}}|t_2 - t_1| \leq |x_1| + 2|x_2 - x_1|.$$

Thus

$$\sup_{x \in X(\gamma)} |x| \leq d_0 + 2d_1 \tag{12.54}$$

where

$$d_0 = \sup_{x \in X(\partial D)} |x|, \quad d_1 = \text{diameter of } X(\partial D).$$

If γ is non-compact in D^*, then it must be a divergent curve; hence $X(\gamma)$ has infinite arc length (X is complete). It follows from (12.53) that $y(t)$ is defined in the interval $(-\infty, a]$, $[a, \infty)$ or $(-\infty, \infty)$.

Let C_r be the solid cylinder of radius r whose axis is the line generated by N. Let $\tilde{A}$ be the annulus D^* with the metric induced by X so that $X : \tilde{A} \to \mathbf{R}^3$ is an isometric immersion. Note that $\partial\tilde{A} = \partial D$.

Claim : $X^{-1}(C_r)$ is a compact set of $\tilde{A}$. In particular, the immersion $X : D^* \hookrightarrow \mathbf{R}^3$ is proper.

Proof of the Claim : We will denote by $\tilde{\rho}$ the distance on $\tilde{A}$. Choose $r > 0$ such that $X(\partial D)$ is contanined in C_r. Let $\tilde{x} \in \tilde{A}$ be such that $X(\tilde{x}) = x$. Let π' be the plane passing through x and the line generated by N. Consider a connected curve γ in $X^{-1}(\pi')$ containing $\tilde{x}$. We know that $X(\gamma)$ is the graph of a function $y(t)$ in π' with $x = (t_0, y(t_0))$. Observe that $|t_0| \leq r$. If the domain of $y(t)$ is the interval $(-\infty, a]$ or $[a, \infty)$, then $(a, y(a)) \in X(\partial D)$ and

$$\tilde{\rho}(\tilde{x}, \partial\tilde{A}) \leq \left| \int_a^{t_0} \sqrt{1 + y'(t)^2}\, dt \right| \leq 2r\sec\theta \leq 4r. \tag{12.55}$$

Assume now that t varies from $-\infty$ to ∞. Let π_t be the plane passing through the point $(t, y(t))$ of $X(\gamma)$, orthogonal to π' and parallel to the line generated by N. Let γ_{t_0} be the connected curve in $X^{-1}(\pi_{t_0})$ that contains the point $\tilde{x}$. If γ_{t_0} intersects $\partial\tilde{A}$, then (12.55) holds. We assert that there exists $t \in (-r, r)$ such that $X^{-1}(\pi_t)$ contains some curve γ_t intersecting both γ and $\partial\tilde{A}$. If not, then $X(\gamma_{t_0})$ is a graph in π_{t_0} over the t-axis of π_{t_0}. As t_0 varies along the t-axis of π', $X(\gamma_{t_0})$ describes some surface M_0 that is a graph over the plane orthogonal to the vector N. Then $X^{-1}(M_0)$ contains some connected component of $\tilde{A}$ without boundary which contradicts the fact that $\tilde{A}$ is connected and has boundary. Thus for some t, γ_t intersects $\partial\tilde{A}$; if $|t| \geq r$ then $\pi_t \cap X(\partial\tilde{A}) = \emptyset$, hence $|t| < r$. This proves the assertion.

If γ_t is given by the assertion above, then in the same way as in (12.55), letting $x' \in \gamma_t$ be such that $X(x') = (t, y(t))$, we have

$$\tilde{\rho}(x', \partial\tilde{A}) \leq 4r.$$

Let t_1 be a point on the t-axis of π' such that $X(\gamma_t) \cap \pi' = (t, y(t))$. It follows easily from the triangle inequality that

$$\tilde{\rho}(\tilde{x}, \partial\tilde{A}) \leq 4r + \left| \int_{t_0}^{t_1} \sqrt{1 + y'(t)^2}\, dt \right| \leq 8r,$$

which proves the claim.

Now let $r_0 = d_0 + 2d_1$ where d_0 and d_1 are defined after (12.54). Then $X(\partial\tilde{A})$ is contained inside the solid cylinder C_{r_0}. By the above claim, the set $X^{-1}(C_{r_0})$ is compact. Set

$$r_1 = \sup_{z \in X^{-1}(C_{r_0})} \{|X(z)|\}$$

and $r_2 > \max\{r_0, r_1\}$ such that

$$\frac{r_0 + r_1}{r_2} + \tan\frac{\pi}{6} < \frac{\sqrt{3}}{2}.$$

Then $X(\partial D)$ is contained inside the sphere $S^2(r_2)$ of radius r_2 and centred at the origin. By the claim and by the fact that $\lim_{|z| \to 0} N(z) = N$, there exists a subannulus $A' \subset D^*$ such that

1. (12.52) holds for $z \in A'$,

2. $X(z)$ is outside C_{r_2} for $z \in A'$.

Let π be the plane containing $X(z)$ and the axis of C_{r_0} for $z \in A'$. Let γ be a connected component of $X^{-1}(\pi)$ containing z. The $X(\gamma)$ is a graph generated by $y(t)$ in π. By the transversality of π and $X(D^*)$ and the fact $X(\partial D) \subset C_{r_0}$, $X(\gamma)$ intersets C_{r_0}. Then y is defined at r_0 or $-r_0$. We may assume that y is defined at r_0. Then

$$|y(r_0)| \leq |(r_0, y(r_0))| \leq r_1.$$

Let $z \in A'$ and $X(z) = (r, y(r))$, $r > r_0$. By (12.53) it follows that

$$|y(r)| \leq |y(r_0)| + \left| \int_{r_0}^{r} y'(t) dt \right| \leq r_0 + r_1 + r\tan\theta.$$

Then, if $X(z) = (r, y(r))$, we have

$$\left| \frac{X(z)}{|X(z)|} \bullet N \right| = \frac{|y(r)|}{\sqrt{r^2 + y^2(r)}} \leq \frac{r_0 + r_1}{r} + \tan\theta < \frac{\sqrt{3}}{2}, \quad z \in A'. \tag{12.56}$$

Set $r_3 > \sup_{z\in(D^*-A')}\{|X(z)|\}$.

We now prove that X and $S^2(r)$ are transverse for $r \geq r_3$. If X and $S^2(r)$ are not transverse, then there exists $z \in X^{-1}(S^2(r))$ such that

$$N(z) = \frac{X(z)}{|X(z)|}.$$

Since $X(D^* - A')$ lies inside $S^2(r)$, we have that $z \in A'$ and (12.52) and (12.56) give a contradiction. Thus X is transverse to $S^2(r)$ for all $r \geq r_3$. We restrict X to A'.

Then by the *claim*, the function $h : A' \to \mathbf{R}$ defined by

$$h(z) = |X(z)|^2$$

is proper. If $z \in A'$ is a critical point of h, then $Dh(z) = (0, 0)$, which means that $X_x(z)$ and $X_y(z)$ are perpendicular to $X(z)$, and so $N(z) = X(z)/|X(z)|$, contradicting to X and $S^2(r)$ are transverse. This contradiction proves that h does not have critical points. If $r > r_3$, then $h^{-1}(r^2)$ is a compact curve that does not intersect $\partial A'$. Hence $h^{-1}(r^2)$ is a finite collection of Jordan curves. If $h^{-1}(r^2)$ has more than one Jordan curve, then there is a compact domain $\Omega \subset A'$ such that $\partial\Omega$ is the union of Jordan curves of $h^{-1}(r^2)$. Then h has a maximum or minimum, hence a critical point, in the interior of Ω, which has already been proved impossible. This shows that $h^{-1}(r^2)$ is a single Jordan curve. Hence

$$\Gamma^r := X(D^*) \cap S^2(r)$$

is an immersion of S^1 and this proves item 2 in this theorem.

We observe that θ of (12.56) goes to zero as r goes to infinity. In fact θ depends on r_0, but we can let $r_0 \to \infty$ and set $r > r_0^2$. By (12.56) the curve $\gamma^r = 1/r\Gamma^r$ is contained in a strip of S^2 that converges to a great circle S as r goes to infinity. Also, by (12.52), the angle between the tangent vector of Γ^r and N goes to $\pi/2$ as r goes to infinity. Hence, Γ^r makes at least one loop around the direction N and γ^r converges C^0 to S as r goes to infinity.

Let $\alpha(\phi)$, $\phi \in \mathbf{R}$, be a parametrisation by arc length of the great circle S. Let β_r be a parametrisation of γ^r such that $\beta_r(\phi)$ lies in the great circle of S^2 that contains N and $\alpha(\phi)$. We have that

$$\left(\frac{\beta_r'}{|\beta_r'|}\bullet\alpha'\right)^2 = 1 - \left(\frac{\beta_r'}{|\beta_r'|}\bullet\alpha\right)^2 - \left(\frac{\beta_r'}{|\beta_r'|}\bullet N\right)^2.$$

As β_r' is orthogonal to $N(\beta_r)$, we have that $\beta_r'/|\beta_r'|\bullet N$ goes to zero as r goes to infinity. Since γ^r converges in the C^0 sense to α, it follows that

$$\lim_{r\to\infty} \beta_r'/|\beta_r'|\bullet\alpha = \lim_{r\to\infty} \beta_r'/|\beta_r'|\bullet \lim_{r\to\infty} \beta_r/|\beta_r| = 0.$$

Therefore γ^r converges in the C^1 sense to the great circle S, with multiplicity, and item 3 is proved.

We now prove that if X is minimal, then γ^r converges in the C^∞ sense to S. Let π be the plane orthogonal to N and containing the origin. Let Ω be the annulus $\{p \in \pi \mid 1/2 \leq |p| \leq 2\}$. Set

$$M_r := (1/rX(D^*)) \cap (\Omega \times \mathbf{R}).$$

The orthogonal projection of M_r onto Ω is a covering of Ω and locally we may write M_r as a graph of a function f_r defined over an angular sector of Ω. It follows from the C^0 convergence of M_r and convergence properties for minimal surfaces (see, e.g. Corollary (16.7) in [21]) that all derivatives of f_r of order less then $j+1$, j an integer, are uniformly bounded by a constant K_{j+1}. Since f_r converges in the C^0 sense to $f = 0$ and the inclusion map of the space of C^{j+1} functions into the space of C^j functions is absolutely continuous, it follows that f_r converges in the C^j sense to $f = 0$. In particular, the intersection of M_r with S^2 converges in the C^j sense to S with multiplicity for all j. This completes the proof of the theorem. $\square$

Now let $X : M = S_k - \{p_1, \cdots, p_n\} \hookrightarrow \mathbf{R}^3$ be a complete minimal surface of finite total curvature. Let $E_i = X : D_i - \{p_i\}$ be the end corresponding to p_i. Let $W_i^r = 1/rX(D_i - \{p_i\}) \cap S^2(r)$ and $\Gamma_i^r = X^{-1}(rW_i^r)$. Jorge and Meeks' theorem says that Γ_i^r is a Jordan curve in $D_i - \{p_i\}$ for r large enough and W_i^r converges to a great circle of S^2 with multiplicity I_i. We will define the multiplicity of E_i to be I_i. Clearly $I_i = 1$ if and only if E_i is embedded. An application of Jorge-Meeks' theorem is that we can get a total curvature formula via the genus k, the number of punctures n, and the multiplicities I_i.