Introduction to Differential Geometry

Robert Bartnik

January 1995

These notes are designed to give a heuristic guide to many of the basic constructions of differential geometry. They are by no means complete; nor are they at all exhaustive. Some of the elementary topics which would be covered by a more complete guide are: geodesics and conjugate points; Lie derivative and the flow of a vector field; coordinate construction techniques; Gauß-Bonnet Theorem; Bochner formulae; de Rham cohomology; Lie groups. Despite these and other omissions, I hope that the notes prove useful in motivating the basic geometric constructions on a manifold.

References

1. M do Carmo, *Differential Geometry of Curves and Surfaces*, Prentice Hall 1976

2. S Kobayashi and K Nomizu, *Foundations of Differential Geometry Volume 1*, Wiley 1963

3. J Milnor, *Morse Theory*, Princeton UP 1963

4. B O'Neill, *Elementary Differential Geometry*, Academic Press 1976

5. M Spivak, *A Comprehensive Introduction to Differential Geometry, Volumes I–V*, Publish or Perish 1972

1 Surfaces

Outline:

Parameterised surfaces in $\mathbf{R}^3$; tangent vectors; metric tensor; normal vector; directional derivative; covariant derivative; second fundamental form; principal curvatures; mean and Gauß identity; Codazzi-Mainardi identity; Gauß *theorem egregium.*

Suppose that $S \subset \mathbf{R}^3$ is a surface, with coordinate chart (or local parameterisation)

$$X : (u, v) \longmapsto X(u, v) = (x(u, v), y(u, v), z(u, v))^t \in S.$$

A fundamentally important observation is that most of the quantities we shall construct to describe the geometry of S are independent of the choice of coordinate chart. An index notation will be very useful: we introduce u^a, $a = 1, 2$ by

$$u^1 = u, \ u^2 = v.$$

One advantage of an index notation is that the generalisation of many of our calculations to the case of n–dimensional surfaces in $\mathbf{R}^{n+1}$, $n \geq 2$, is then very simple; another advantage is that we may use the Einstein summation convention:

An expression containing a repeated index (for example, $V^a \frac{\partial f}{\partial u^a}$), implies a summation over that index,

$$V^a \frac{\partial f}{\partial u^a} = \sum_{a=1}^{n} V^a \frac{\partial f}{\partial u^a}.$$

The summation convention allows us to express concisely many otherwise lengthy and repetitive formulae.

The *coordinate tangent vectors* to S are the vectors X_1, X_2 in $\mathbf{R}^3$ defined by

$$\begin{aligned} X_1 &= X_u = \frac{\partial X}{\partial u} = \frac{\partial X}{\partial u^1}, \\ X_2 &= X_v = \frac{\partial X}{\partial v} = \frac{\partial X}{\partial u^2}. \end{aligned} \tag{1}$$

Any tangent vector to S can be written uniquely as a linear combination of the basis vectors X_a, $a = 1, 2$,

$$V = V^a X_a = \sum_{1=1}^{2} V^a X_a$$

where V^a, $a = 1, 2$, are the coefficients of V in the basis X_a.

Using the inner product $\langle \cdot, \cdot \rangle$ for vectors in $\mathbf{R}^3$ we define the *metric* or *first fundamental form* of S, by

$$g(V, W) = \langle V, W \rangle \tag{2}$$

for any tangent vectors V, W on S. By linearity we may express the metric in terms of the components

$$g_{ab} = g(X_a, X_b) = \langle X_a, X_b \rangle$$

of the metric with respect to the coordinates u^a, for example

$$g(V, W) = \sum_{a,b=1}^{2} V^a W^b g_{ab} = V^a W^b g_{ab}. \tag{3}$$

The classical notation for the first fundamental form

$$E = g_{11}, \ F = g_{12}, \ G = g_{22}, \tag{4}$$

may still be found in many older books on surface theory.

One application of the metric is to describe the length of a curve given in terms of the coordinates u^a. Thus, suppose

$$\gamma : t \longmapsto (\gamma^1(t), \ \gamma^2(t)), \quad 0 \le t \le 1$$

defines a curve in S; more precisely, $\gamma : [0; 1] \longrightarrow \mathbf{R}^2$ describes a curve in a coordinate chart of S, and the curve in $S \subset \mathbf{R}^3$ is given by

$$\tilde{\gamma} = X \circ \gamma : t \longmapsto X(\gamma(t)) \in \mathbf{R}^3.$$

Then the length of γ is given by

$$\begin{aligned} \text{length } (\gamma) &= \int_0^1 \left| \frac{d}{dt} \tilde{\gamma}(t) \right| dt \\ &= \int_0^1 \sqrt{\dot{\gamma}^a(t) \dot{\gamma}^b(t) g_{ab}(\gamma(t))} \, dt \\ &= \int_0^1 \sqrt{\dot{\gamma}^a \dot{\gamma}^b g_{ab}} \, dt, \end{aligned} \tag{5}$$

since the tangent vector $d\tilde{\gamma}/dt \in \mathbf{R}^3$ satisfies

$$\frac{d\tilde{\gamma}}{dt} = \frac{d\gamma^a}{dt} X_a = \dot{\gamma}^a X_a$$

by the chain rule.

A unit normal vector N to S is determined up to $\pm N$, and may be described using the vector cross product in $\mathbf{R}^3$ by the formula

$$N = \frac{X_u \times X_v}{|X_u \times X_v|}.$$

This formula does not generalise so easily [1] to the case of hypersurfaces in $\mathbf{R}^{n+1}$, $n \geq 3$, but this is not a significant problem, since the existence of a normal vector is not in question.

If $f : \mathbf{R}^n \longrightarrow \mathbf{R}$, and if $\gamma : t \longmapsto \gamma(t) \in \mathbf{R}^n$ is a curve in $\mathbf{R}^n$, then $f \circ \gamma : \mathbf{R} \longrightarrow \mathbf{R}$ is a function of one variable and the chain rule gives

$$\begin{aligned} \frac{d}{dt} f \circ \gamma(t) &= \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} \frac{d\gamma^i}{dt} \\ &= \dot{\gamma}^i \frac{\partial f}{\partial x^i}, \end{aligned} \tag{6}$$

which depends on the tangent vector $\dot{\gamma} \in \mathbf{R}^n$ to the curve γ. We call this the *directional derivative* of f in the direction $\dot{\gamma}$,

$$D_{\dot{\gamma}} f = \dot{\gamma}^i \frac{\partial f}{\partial x^i}. \tag{7}$$

If V is a vector in $\mathbf{R}^n$, based at a point $x \in \mathbf{R}^n$, then we may always find a curve through x in the direction V (for example, $\gamma(t) = x + tV$), and then we define the directional derivative by

$$D_V f = \dot{\gamma}^i(0) \frac{\partial f}{\partial x^i}(x), \tag{8}$$

where $x = \gamma(0)$ and $V = \dot{\gamma}(0)$. Notice that (7) shows that this expression is independent of the choice of curve representing V, so the definition (8) is unambiguous. We may rewrite (8) in the elegant form

$$D_V f = V^i \frac{\partial f}{\partial x^i},$$

which makes the relation with the chain rule very explicit.

The definition of directional derivative of a function may be easily extended to vector fields in $\mathbf{R}^n$. Thus, for example, if Y, Z are two vector fields in $\mathbf{R}^3$, so $Y = (Y^1(x,y,z), Y^2(x,y,z), Y^3(x,y,z))$, then the directional derivative of Z in the direction Y is the vector field

$$\begin{aligned} D_Y Z &= Y^1 \frac{\partial}{\partial x} Z + Y^2 \frac{\partial}{\partial y} Z + Y^3 \frac{\partial}{\partial z} Z \\ &= Y^i \frac{\partial}{\partial x^i} Z. \end{aligned}$$

[1] The generalisation defines the normal vector using n (coordinate) tangent vectors $X_1, \ldots, X_n$ and the $(n+1)$ cofactors of the $(n+1) \times n$ matrix $[X_1 \cdots X_n]$

If Y, Z are vector fields tangent to the surface S, then we may decompose $D_Y Z$ into components tangential and normal to S,

$$D_Y Z = \nabla_Y Z + \mathrm{II}(Y, Z)N \tag{9}$$

where the tangential component

$$\nabla_Y Z = (D_Y Z)^{tangential} \tag{10}$$

is called the *covariant derivative* on S of the vector field Z in the direction Y, and the normal component

$$\mathrm{II}(Y, Z) = \langle D_Y Z, N \rangle \tag{11}$$

is the *second fundamental form* of S. Since

$$\begin{aligned} 0 = D_Y(\langle Z, N \rangle) &= \langle D_Y Z, N \rangle + \langle Z, D_Y N \rangle, \\ 0 = D_Y(\langle N, N \rangle) &= 2\langle N, D_Y N \rangle, \end{aligned} \tag{12}$$

it follows that

$$\mathrm{II}(Y, Z) = -\langle Z, D_Y N \rangle \tag{13}$$

and thus we may interpret II geometrically as describing the "bending" of the normal vector as we move around the surface. It is clear from (13) that $\mathrm{II}(Y, Z)$ depends linearly on the tangent vectors Y, Z;

$$\mathrm{II}(Y, Z) = Y^a Z^b \mathrm{II}(X_a, X_b) \tag{14}$$

where $Y = Y^a X_a$ is the expansion of the tangent vector to S in the basis X_a, $a = 1, 2$ of the tangent vectors to S. Noting that

$$\begin{aligned} D_{X_a} X_b &= \frac{\partial}{\partial u^a} X_b \\ &= \frac{\partial^2}{\partial u^a \partial u^b} X \\ &= X_{ab}, \end{aligned} \tag{15}$$

we obtain the useful formula

$$\mathrm{II}_{ab} = \mathrm{II}(X_a, X_b) = \langle X_{ab}, N \rangle. \tag{16}$$

Since $X_{ab} = \partial^2 X / \partial u^a \partial u^b = X_{ba}$, this implies $\mathrm{II}_{ab} = \mathrm{II}_{ba}$ or in terms of general tangent vectors Y, Z,

$$\mathrm{II}(Y, Z) = \mathrm{II}(Z, Y). \tag{17}$$

Thus the second fundamental form is a *symmetric bilinear form* on tangent vectors to S.

The *principal curvatures* of S are the eigenvalues of II with respect to the metric g; equivalently, they are the roots of the polynomial equation

$$p(\lambda) = \det(\mathrm{II} - \lambda g) = 0. \tag{18}$$

The *principal vectors* $e_1,\ e_2$ of II are tangent vectors which are *orthonormal* with respect to g,

$$g(e_1,\ e_2) = 0,\ g(e_1,\ e_1) = g(e_2,\ e_2) = 1,$$

and which diagonalise II,

$$\mathrm{II}(e_1,\ e_2) = 0,\ \mathrm{II}(e_1,\ e_1) = \lambda_1,\ \mathrm{II}(e_2,\ e_2) = \lambda_2$$

where $\lambda_1,\ \lambda_2$ are the principal curvatures. These relations may be written more succinctly as

$$\begin{aligned} g(e_a, e_b) &= \delta_{ab} & (19) \\ \mathrm{II}(e_a, e_b) &= \lambda_a \delta_{ab}, & (20) \end{aligned}$$

where $\delta_{ab} = 0$ if $a \neq b$, $\delta_{ab} = 1$ if $a = b$, is the Kronecker delta, and where there is no summation implied by the repeated index a in (20).

From (12) we see that $D_Y N$ is tangent to S and depends linearly on Y, so the *Weingarten map*

$$Y \longmapsto D_Y N \tag{21}$$

is a linear transformation of tangent vectors to S, which may be described using matrices with respect to the basis vectors X_a using (13) by

$$D_Y N = -Y^a \mathrm{II}_{ab} g^{bc} X_c, \tag{22}$$

where $(g^{ab}) = (g_{ab})^{-1}$ is the *inverse metric.* Another interpretation of the principal curvatures is as the negative eigenvalues of the Weingarten map (21), or using (22), as the eigenvalues of the matrix

$$\mathrm{II}_a^b = g^{bc} \mathrm{II}_{ac}. \tag{23}$$

The principal vectors are then the eigenvectors of II_a^b, normalised to unit length — because II_{ab} is symmetric, the Principal Axis Theorem of elementary linear algebra implies that the eigenvectors of II_a^b are orthogonal with respect to g_{ab}.

The mean curvature H and Gauß curvature K of S are defined using symmetric functions of the principal curvatures $\lambda_1,\ \lambda_2$ by

$$2H = \lambda_1 + \lambda_2 = g^{ab}\mathrm{II}_{ab} = tr_g\mathrm{II}, \tag{24}$$

$$K = \lambda_1\lambda_2 = \det(\mathrm{II}_{ab})/\det(g_{ab}). \tag{25}$$

In terms of the classical notation

$$e = \mathrm{II}_{11},\ f = \mathrm{II}_{12},\ g = \mathrm{II}_{22},$$

we have the formulae

$$2H = \frac{Eg + eG - 2fF}{EG - F^2}, \tag{26}$$

$$K = \frac{eg - f^2}{EG - F^2}. \tag{27}$$

Unlike the second fundamental form $\mathrm{II}(Y, Z)$, the covariant derivative $\nabla_Y Z$ cannot depend only on the value of the vectors Y, Z at a point (see (14)), but must involve the derivative of the coefficients of Z, since the total directional derivative $D_Y Z$ involves the derivative of Z. Explicitly, by expanding Y, Z in the basis X_a we obtain

$$\begin{aligned} \nabla_Y Z &= \left(Y^a \frac{\partial}{\partial u^a} Z^b\right) X_b + \left(Y^a Z^b \Gamma^c_{ab}\right) X_c \\ &= Y^a \left(\frac{\partial}{\partial u^a} Z^b + Z^c \Gamma^b_{ac}\right) X_b, \end{aligned} \tag{28}$$

where we have defined the *Christoffel symbol* Γ^c_{ab} by

$$\nabla_{X_a} X_b = \Gamma^c_{ab} X_c. \tag{29}$$

Defining $\Gamma_{abc} = g_{cd}\Gamma^d_{ab}$ we obtain using (15)

$$\begin{aligned} \Gamma_{abc} &= g_{cd}\Gamma^d_{ab} = \langle \nabla_{X_a} X_b, X_c \rangle \\ &= \langle D_{X_a} X_b, X_c \rangle \\ &= \langle X_{ab}, X_c \rangle \end{aligned} \tag{30}$$

which gives a simple formula for computing the Christoffel symbol $\Gamma^c_{ab} = g^{cd}\Gamma_{abd}$ and hence the covariant derivative (28). Remarkably, the Christoffel symbol may be expressed by formulae which use only the metric g_{ab},

$$\begin{aligned} \Gamma_{abc} &= \frac{1}{2}\left(\frac{\partial}{\partial u^a} g_{bc} + \frac{\partial}{\partial u^b} g_{ac} - \frac{\partial}{\partial u^c} g_{ab}\right), \\ \Gamma^c_{ab} &= \frac{1}{2} g^{cd}\left(\frac{\partial}{\partial u^a} g_{bd} + \frac{\partial}{\partial u^b} g_{ad} - \frac{\partial}{\partial u^d} g_{ab}\right). \end{aligned} \tag{31}$$

These formulae follow from the useful identity

$$D_V(g(Y,Z)) = g(\nabla_V Y, Z) + g(Y, \nabla_V Z) \tag{32}$$

for tangent vectors V, Y, Z to S, which implies in particular

$$\begin{aligned} \frac{\partial}{\partial u^a} g_{bc} &= D_{X_a}(g(X_b, X_c)) \\ &= g(\nabla_{X_a} X_b, X_c) + g(X_b, \nabla_{X_a} X_c). \end{aligned} \tag{33}$$

We say that the covariant derivative ∇ is *metric-compatible* if (32) holds.

Because (31) involves only the metric g_{ab} and makes no explicit use of the embedding X of S in $\mathbf{R}^3$, these formulae may be used to define the covariant derivative for an abstract manifold, as described in later lectures.

It is often very useful to consider a tangent vector V as equivalent to the differential operator D_V on functions. The *Lie bracket* $[V, W]$ of two vector fields V, W on $\mathbf{R}^3$ for example is defined via its differential operator $D_{[V,W]}$ on functions by

$$\begin{aligned} D_{[V,W]} f &= D_V(D_W f) - D_W(D_V f) \\ &= [D_V, D_W] f, \end{aligned} \tag{34}$$

where $[D_V, D_W]$ denotes the commutator of the differential operators D_V, D_W. By expanding $V = V^i \dfrac{\partial}{\partial x^i}$, $W = W^j \dfrac{\partial}{\partial x^j}$ we find that

$$D_{[V,W]} f = \left(V^j \frac{\partial}{\partial x^j} W^i - W^j \frac{\partial}{\partial x^j} V^i \right) \frac{\partial}{\partial x^i} f,$$

using the fact that partial derivatives commute, and thus we see that $[V, W]$ is a vector field with coefficients

$$[V, W]^i = V^j \frac{\partial}{\partial x^j} W^i - W^j \frac{\partial}{\partial x^j} V^i. \tag{35}$$

If Y, Z are tangent vectors to S, and Y^a, Z^b are their coefficients with respect to the basis X_a $a = 1, 2$ of tangent vectors to S (rather than the basis $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, 2, 3$ of tangent vectors to $\mathbf{R}^3$), then their Lie bracket is given by

$$[Y, Z]^a = Y^b \frac{\partial}{\partial u^b} Z^a - Z^b \frac{\partial}{\partial u^b} Y^a, \tag{36}$$

since the Lie bracket of coordinate tangent vectors vanishes

$$[X_a, X_b] = 0. \tag{37}$$

This can be seen most easily by noting that

$$D_{X_a} D_{X_b} f = \frac{\partial}{\partial u^a} \left(\frac{\partial}{\partial u^b} f(u^1, u^2) \right), \tag{38}$$

since f can be expressed as a function of the coordinates (u^1, u^2) on S. Now combining (28), (31), (36) and (37) yields the identity

$$\nabla_Y Z - \nabla_Z Y = [Y, Z], \tag{39}$$

which we paraphrase by saying that ∇ is *torsion-free*.

The *Riemann curvature tensor* $R(U, V, Y, Z)$ evaluated on tangent vectors U, V, Y, Z is defined by

$$R(U, V, Y, Z) = g((\nabla_U \nabla_V - \nabla_V \nabla_U - \nabla_{[U,V]})Y, Z); \tag{40}$$

remarkably, this expression which apparently involves second derivatives of the coefficients of Y for example, is in fact linear in just the coefficients (zero'th derivatives) alone of the vectors U, V, Y, Z. Thus we have

$$R(U, V, Y, Z) = U^a V^b Y^c Z^d R_{abcd} \tag{41}$$

where

$$\begin{aligned} R_{abcd} &= R(X_a, X_b, X_c, X_d) \\ &= g((\nabla_{X_a} \nabla_{X_b} - \nabla_{X_b} \nabla_{X_a}) X_c, X_d) \\ &= \frac{\partial}{\partial u^a} \Gamma_{bcd} - \frac{\partial}{\partial u^b} \Gamma_{acd} + \Gamma^e_{ac} \Gamma_{bde} - \Gamma^e_{bc} \Gamma_{ade} \end{aligned} \tag{42}$$

since $[X_a, X_b] = 0$. Notice in particular from (31) and (42) that the curvature R_{abcd} may be expressed by a formula which involves only the metric g_{ab} and its first and second derivatives. Further properties of the Riemann curvature will be developed later.

Alternatively we may use (42) to express R_{abcd} in terms of X and its derivatives $X_a = \partial X / \partial u^a$, $X_{ab} = \partial^2 X / \partial u^a \partial u^b$ and $X_{abc} = \partial^3 X / \partial u^a \partial u^b \partial u^c$. By systematically expanding into tangential and normal components we may obtain some fundamental identities of surface theory. From (9), (15), (16) and (29) we have

$$X_{ab} = \mathrm{II}_{ab} N + \Gamma^c_{ab} X_c. \tag{43}$$

Substituting (43) into the identity

$$0 = X_{cba} - X_{cab} = D_{X_a}(X_{cb}) - D_{X_b}(X_{ca}) \tag{44}$$

and taking the normal component gives the *Codazzi–Mainardi* identity

$$\nabla_a \mathrm{II}_{bc} - \nabla_b \mathrm{II}_{ac} = 0, \tag{45}$$

where $\nabla\mathrm{II}$ is the covariant derivative of II,

$$\nabla_a \mathrm{II}_{bc} = \frac{\partial}{\partial u^a}\mathrm{II}_{bc} - \Gamma^d_{ab}\mathrm{II}_{dc} - \Gamma^d_{ac}\mathrm{II}_{bd}, \tag{46}$$

which should be compared with the expression (34) for the covariant derivative of a vector Z.

The tangential component of (44) gives the *Gauß identity*

$$R_{abcd} = \mathrm{II}_{ad}\mathrm{II}_{bc} - \mathrm{II}_{ac}\mathrm{II}_{bd}. \tag{47}$$

In particular, this shows that

$$R_{abcd} = -R_{bacd} = -R_{abdc} \tag{48}$$

and thus, for surfaces S in $\mathbf{R}^3$, essentially the only non–vanishing curvature component is R_{1221}. From (47) we derive a relation between the Gauß curvature $K = \lambda_1\lambda_2$, defined using the bending of the embedding $S \subset \mathbf{R}^3$ and the Riemann curvature, defined using the intrinsic length measure g_{ab}, namely

$$\lambda_1\lambda_2 = K = \frac{R_{1221}}{\det(g_{ab})} = \frac{R_{1221}}{g_{11}g_{22} - (g_{12})^2}. \tag{49}$$

This is the *theorem egregium* of Gauß.

2 Manifolds

Outline:

> examples of manifolds; motivation; coordinate charts; transition functions; smooth functions; definitions of a manifold; Implicit Function Theorem; submanifolds of $\mathbf{R}^m$; embedded and immersed submanifolds; tangent vectors.

The aim of this chapter is to introduce the fundamental concept of a *manifold.* The systematic formulation of this definition was a major achievement of early 20th century geometry, and laid the foundations for a vast amount of work in topology, analysis and geometry.

Roughly speaking, a manifold is an n–dimensional surface, but without the representation into Euclidean space that proved so useful to us when we described surfaces S in $\mathbf{R}^3$. Some examples of useful and common manifolds are

- the n–sphere $S^n = \{x \in \mathbf{R}^{n+1};\ |x| = 1\}$;
- the n–torus $T^n = S^1 \times S^1 \times \cdots \times S^1$, which may also be considered as a quotient space

  $$T^n = \mathbf{R}^n/\mathbf{Z}^n,$$

  or space of equivalence classes of points $x, y \in \mathbf{R}^n$ under the equivalence

  $$x \sim y \quad \Longleftrightarrow \quad x - y = (m_1, \cdots, m_n) \in \mathbf{Z}^n;$$

- Real Projective Space $\mathbf{RP}^n = S^n/(x \sim -x)$;
- the space of lines in $\mathbf{R}^2$ (this turns out to be the same as $\mathbf{RP}^2$); [2]
- the quotient space $SU(2)/U(1)$, which turns out to be the same as S^2;
- the set $\{(u, v) \in S^2 \times S^2;\ u \perp v\}$, which turns out to be the same as $SO(3)$, the real orthogonal group.

[2] More precisely and more generally, the space of affine lines in $\mathbf{R}^n$ is an open dense subset of $G_{2,(n+1)}$, the Grassmanian of two planes (through 0) in $\mathbf{R}^{n+1}$. This may be seen by identifying a line $\ell \subset \mathbf{R}^n$ with the line $(\ell, 1) \subset \mathbf{R}^{n+1}$, which spans a unique 2-plane through 0 in $\mathbf{R}^{n+1}$. The missing lines correspond to the $G_{2,n}$ at infinity in $G_{2,(n+1)}$.

For some of these examples, it's not immediately obvious how they might be realised as subsets of some Euclidean space, and even less obvious that any such representation is an essential feature of the set. Instead, we aim to consider these sets independent of any particular representation as a subset of some $\mathbf{R}^m$.

A key observation on the path to constructing such an intrinsic definition of "surface" is that for a parameterised surface $X : (u^1, u^2) \longmapsto X(u^1, u^2) \in S \subset \mathbf{R}^3$, many (but not all!) of the computations may be written solely in terms of the coordinates (u^a) and quantities which may be defined as functions of (u^a), such as g_{ab}, R_{abcd}, etc. The definition of manifold below will start with coordinate charts, but (somehow) not make any reference to any ambient Euclidean space.

For another clue about the general definition of manifold, consider the definition of a C^∞ (or $C^k, k \geq 1$) function on $S \subset \mathbf{R}^3$. If $f : S \longrightarrow \mathbf{R}$, then one possible definition of C^∞ would require that f be the restriction of some C^∞ function $\tilde{f} : \mathbf{R}^3 \longrightarrow \mathbf{R}$. This obvious definition has the aesthetic drawback that it involves values of $\tilde{f}$ at points which do not lie on S. Note that this idea of extension away from S is implicitly used in many of the surface theory calculations of the previous chapter, for example, in constructing the covariant derivative $\nabla_Y Z$ of two tangential vector fields from the $\mathbf{R}^3$ directional ($\mathbf{R}^3$–covariant) derivative $D_Y Z = D_{\tilde{Y}} \tilde{Z}$.

An alternative definition of C^∞ uses the coordinates (u^a): we can require that the composition

$$\begin{aligned} (u^a) \longmapsto X(u^a) \quad &\longmapsto \quad f(X(u^a)), \\ f \circ X : U \subset \mathbf{R}^2 \quad &\longrightarrow \quad \mathbf{R} \end{aligned}$$

is C^∞ as a map of Euclidean spaces, where it is classical what is meant by "$f \circ X$ is C^∞". Similarly we may define spaces of Hölder–continuous functions $C^{k,\alpha}$, or Sobolev spaces $W^{k,p}$ — but in these cases, setting up the Banach or Hilbert space norms requires further work.

A major problem with working with a coordinate definition is that of ensuring that the definition is independent of the choice of coordinates; for a surface such as the 2–sphere $S \subset \mathbf{R}^3$, not only are many different choices of coordinates available, but also the whole surface cannot be covered by a single coordinate system. Thus we must consider the effect of a change in coordinates. Suppose $X : (u^a) \longmapsto X(u) \in S$, and $Y : (v^b) \longmapsto Y(v^b) \in S$ are parameterisations of $S \subset \mathbf{R}^3$, then if the regions of S covered by the two parameterisations overlap, we may consider the *transition function*

$$u \overset{X}{\longmapsto} X(u) = Y(v) \overset{Y^{-1}}{\longmapsto} v,$$

or heuristically, $v = v(u) = (Y^{-1} \circ X)(u)$. Since

$$f \circ X = (f \circ Y) \circ (Y^{-1} \circ X),$$

the two coordinate representations of $f : S \longrightarrow \mathbf{R}$ are related by the transition function $u \longmapsto v(u) = (Y^{-1} \circ X)(u)$, and we may relate differentiability of f in u to differentiability of f in v by using the chain rule of multivariable calculus and the (assumed) smoothness of the transition function $u \longmapsto v(u)$. This imposes a compatibility condition on the coordinate choices, namely that the resulting transition functions have sufficient regularity (eg C^∞, C^k etc).

We can now construct the definition of a *manifold.* An n–dimensional (topological) manifold M is a set with a topology which is

(a) *Hausdorff* (i.e. if $x, y \in M$, $x \neq y$, then there are open sets $U \subset M$, $V \subset M$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$);

(b) *Separable* (i.e. the topology on M has a basis of open sets which is countable); and

(c) *Locally Euclidean* (i.e. for any $x \in M$, there is an open set $x \in U \subset M$ and a homeomorphism $\phi : U \longrightarrow \phi(U) \in \mathbf{R}^n$).

(Recall that a homeomorphism is a continuous bijection with a continuous inverse). The condition of Hausdorff can be relaxed, at the cost of allowing some relatively bizarre spaces to qualify as manifolds. The separability condition is needed to ensure paracompactness, which in turn is used to ensure the existence of partitions of unity; these are essential in many constructions, such as deriving the existence of a Riemannian metric on M, and in defining integration on M.

The map

$$\phi : U \subset M \longrightarrow \mathbf{R}^n$$

is called a *coordinate chart* (about x). The coordinate charts $\phi : U \longrightarrow \mathbf{R}^n$, $\psi : V \longrightarrow \mathbf{R}^n$ are C^k*–compatible*, $k \leq \infty$, if the *transition function*

$$\phi \circ \psi^{-1} : \psi(U \cap V) \longrightarrow \mathbf{R}^n$$

is C^k as a map between Euclidean spaces. Notice that the coordinate charts we are using ($\phi : U \subset M \longrightarrow \mathbf{R}^n$) go the opposite direction to the parameterisations $X : U \subset \mathbf{R}^2 \longrightarrow S \subset \mathbf{R}^3$ we used when considering surfaces in $\mathbf{R}^3$; this turns out to be more convenient.

A C^k*–atlas* of the manifold M is a family of coordinate charts

$$\Phi = \{(\phi_\alpha, U_\alpha) : \alpha \in A\}$$

(where A is the indexing set of the family), such that the sets U_α, $\alpha \in A$, cover M

$$M = \bigcup_{\alpha \in A} U_\alpha,$$

and such that the transition functions $\phi_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$

$$\phi_{\alpha\beta} : \phi_\beta(U_\alpha \cap U_\beta) \longrightarrow \mathbf{R}^n, \quad \alpha, \beta \in A$$

are all C^k. A *maximal* C^k atlas is an atlas which contains all C^k compatible coordinate charts; because

$$\phi_{\alpha\gamma} = \phi_{\alpha\beta} \circ \phi_{\beta\gamma},$$

at least where both functions are defined, and because the composition of C^k functions is again C^k, it follows that in order to construct a maximal C^k atlas, it suffices to find just one family of C^k–compatible coordinate charts which covers M. Finally, a C^k *manifold* is a topological manifold with a C^k maximal atlas.

Henceforth, for simplicity we will consider all manifolds to be C^∞, unless explicitly indicated otherwise; the changes needed to consider C^k manifolds are usually very minor. We also use *smooth* as a synonym for C^∞.

It's easy to see that $\mathbf{R}^n$ is a smooth manifold, since the identity map $Id : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ defines a chart which covers all of $\mathbf{R}^n$ and hence gives a C^∞ atlas.

The definition of a smooth function $f : M \longrightarrow \mathbf{R}$ is now clear: we must have the composition

$$f_\alpha = f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \longrightarrow \mathbf{R}$$

a smooth function, for every coordinate chart (ϕ_α, U_α) on M. Again, because

$$f_\beta = f_\alpha \circ \phi_{\alpha\beta}$$

where both functions are defined, and because $\phi_{\alpha\beta} \in C^\infty$ always, it follows that this definition is consistent, and smoothness need only be verified on single atlas of charts. Similarly, a map $f : M^m \longrightarrow N^n$ between manifolds is C^∞ if the maps

$$\psi_i \circ f \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha) \xrightarrow{\phi_\alpha^{-1}} M \xrightarrow{f} N \xrightarrow{\psi_i} \mathbf{R}^n$$

are C^∞, for all charts (ϕ_α, U_α) on M and (ψ_i, V_i) on N. Now, an important example of maps between two manifolds is provided by curves; a C^∞ *curve* in M is a C^∞ map

$$\gamma : \mathbf{R} \longrightarrow M.$$

The graph of a smooth function $f : \mathbf{R}^n \longrightarrow \mathbf{R}^p$,

$$\text{graph}(f) = \{(x, f(x)) \in \mathbf{R}^{n+p},\ x \in \mathbf{R}^n\}$$

is a smooth manifold if we give graph(f) the topology induced by the *inclusion*

$$\text{graph}(f) \hookrightarrow \mathbf{R}^{n+p},$$

since we have a C^∞ atlas given by the single coordinate chart

$$\phi : \text{graph}(f) \longrightarrow \mathbf{R}^n, \quad (x, f(x)) \stackrel{\phi}{\longmapsto} x.$$

(Subtle point: if $f \in C^k$, $k < \infty$ is not smooth, then this construction still shows graph(f) is a C^∞ manifold, but the inclusion graph$(f) \hookrightarrow \mathbf{R}^{n+p}$ will no longer be a C^∞ map between manifolds).

The space $S^n \subset \mathbf{R}^{n+1}$ is not a graph, but it can be locally represented as a graph (over some coordinate n–plane, for example). This provides a covering of S^n by coordinate charts, and it is easy to check that the resulting transition functions are C^∞ and hence S^n with the topology induced from $\mathbf{R}^{n+1}$ is a smooth manifold. More generally, any connected subset of $\mathbf{R}^k$ which can locally be represented as a graph of a C^∞ function is a smooth manifold, by the same argument.

This leads to the general question: when can the level set $F^{-1}(0) = \{x \in \mathbf{R}^n,\ F(x) = 0\}$, where $F : \mathbf{R}^n \longrightarrow \mathbf{R}^m$, $m < n$, be given a manifold structure? By the above discussion, it suffices to show that $F^{-1}(0)$ is locally described as a graph, over some coordinate plane for example. Conditions which ensure this is possible are provided by

Theorem 1 (Implicit Function Theorem) .

Suppose $f : \mathbf{R}^n \longrightarrow \mathbf{R}^m$, $m \leq n$, is C^∞, let $\mathbf{R}^n = \mathbf{R}^{n-m} \times \mathbf{R}^m = \{(u, v) : u \in \mathbf{R}^{n-m},\ v \in \mathbf{R}^m\}$ and suppose $D_v f$ is an invertible $m \times m$ matrix at $p_0 = (u_0, v_0)$ with $f(u_0, v_0) = 0$. Then there is an open neighbourhood $U \subset \mathbf{R}^{m-n}$ of u_0 and a smooth function $g : U \longrightarrow \mathbf{R}^m$ such that $g(u_0) = v_0$ and

$$f(u, g(u)) = 0, \quad \forall u \in U.$$

Proof : Consider $F : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, $F(u, v) = (u, f(u, v))$. Clearly F is C^∞ and

$$DF = \begin{bmatrix} I_{n-m} & D_u f \\ 0 & D_v f \end{bmatrix}$$

is invertible where $D_v f$ is invertible. In particular, $DF(u_0, v_0)$ is invertible, and the Inverse Function Theorem gives a C^∞ function $G : \mathbf{R}^n \longrightarrow \mathbf{R}^n$ which is an inverse of F in some neighbourhood of (u_0, v_0),

$$F(G(x,y)) = (x,y) \in \mathbf{R}^{n-m} \times \mathbf{R}^m,$$

for (x, y) near $(u_0, f(u_0, v_0)) = (u_0, 0)$. Write $G(x,y) = (g_1(x,y), g_2(x,y))$, so

$$\begin{aligned} F(G(x,y)) &= F(g_1, g_2) = (g_1(x,y), f(g_1(x,y), g_2(x,y))) \\ &= (x,y). \end{aligned}$$

Hence $g_1(x,y) = x$, and

$$y = f(x, g_2(x,y))$$

for (x, y) near $(u_0, 0)$. Defining $g(u) = g_2(u, 0)$ gives

$$0 = f(u, g(u)) \quad \text{for } u \text{ near } u_0,$$

and thus g is the required C^∞ inverse function. ■

The practically useful form of this is

Corollary 2 *Suppose $f : \mathbf{R}^n \longrightarrow \mathbf{R}^m$, $m \leq n$ and $y_0 \in \mathbf{R}^m$ are such that*

$$\textit{rank } Df(x) = m$$

at all points x for which $f(x) = y_0$. Then $f^{-1}(y_0) = \{x \in \mathbf{R}^n;\ f(x) = y_0\}$ is a smooth manifold under the topology induced from $\mathbf{R}^n$.

Proof : The rank condition implies that $D_v f$ is invertible, for v in some coordinate plane, and hence by the Implicit Function Theorem, $f^{-1}(y_0)$ is always locally representable as a graph over some coordinate plane. ■

Another way of stating this is that $f^{-1}(y_0)$ is an *embedded submanifold.* More generally, $f : M^m \longrightarrow N^n$ is an *embedding* if the induced topology on M from its inclusion in N coincides with the topology on M, and if $f : M \longrightarrow f(M)$ is a *diffeomorphism*, i.e. a smooth map with a smooth inverse. If $f : M \longrightarrow N$ is locally an embedding (i.e. for every $x \in M$, there is a neighbourhood $x \in U \subset M$ such that $f|_U$ is an embedding), then f is an *immersion.* Note that an immersed submanifold can still have self-intersections.

Example: It was claimed earlier that the set $M = \{(u,v) \in S^2 \times S^2, u \perp v\}$ is a manifold. To show this, note that $M = f^{-1}(0)$ where

$$f : \mathbf{R}^6 \longrightarrow \mathbf{R}^3, \qquad f(u,v) = (|u|^2 - 1, |v|^2 - 1, \langle u, v \rangle)^t, \quad u, v \in \mathbf{R}^3,$$

and thus

$$Df = \begin{bmatrix} 2u^t & 0 \\ 0 & 2v^t \\ v^t & u^t \end{bmatrix}.$$

Now $\operatorname{rank}(Df) = 3$ at points (u, v) where $f(u, v) = 0$, so the Corollary applies to show M is a manifold. The identification with $SO(3)$ is obtained by noting that points $(u, v) \in M$ are in one-to-one correspondence with orthonormal frames, by $(u, v) \longleftrightarrow (u, v, u \times v)$.

3 Vectors and Tensors

Outline:

tangent vectors and covectors; coordinate vectors; differentials; tangent and cotangent bundles; vector fields; tensor and exterior algebras; push forward and pull back maps; Riemannian manifold.

Now that we have a general definition of manifold, we turn to the problem of constructing appropriate generalisations of geometric entities such as tangent vectors and metrics. We start by defining tangent vectors.

Definition 3 *A* tangent vector *at $p_0 \in M$ is an equivalence class of curves, under the equivalence relation $\gamma \sim \delta$ defined by*

$$\gamma(0) = \delta(0) = p_0$$

and for some coordinate chart $\phi : U \longrightarrow \mathbf{R}^n$, $p_0 \in U$,

$$(\phi \circ \gamma)'(0) = (\phi \circ \delta)'(0). \tag{50}$$

Here $\phi \circ \gamma : \mathbf{R} \longrightarrow \mathbf{R}^n$ is a curve in $\mathbf{R}^n$, and $(\phi \circ \gamma)'(0)$ is its tangent vector in $\mathbf{R}^n$ at $t = 0$. Note that the chain rule ensures that the condition (50) is in fact independent of the choice of chart (ϕ, U) and hence the equivalence relation $\sim$ is well defined.

The directional derivative is defined using representing curves. If v is a tangent vector at $x \in M$, and $f : M \longrightarrow \mathbf{R}$, then

$$D_v(f)(x) = \frac{d}{dt} f \circ \gamma(t)\bigg|_{t=0}$$

where $\gamma : \mathbf{R} \longrightarrow M$ is any curve representing v. Again the chain rule and (50) combine to show this definition is independent of the choice of representing curve γ for v. Explicitly, if $\phi : U \longrightarrow \mathbf{R}^n$ is a chart about p_0, then we set

$$f \circ \gamma = f \circ \phi^{-1} \circ \phi \circ \gamma = \tilde{f} \circ \tilde{\gamma}$$

where $\tilde{f} = f \circ \phi^{-1} : \phi(U) \longrightarrow \mathbf{R}$ and $\tilde{\gamma} = \phi \circ \gamma : \mathbf{R} \longrightarrow \mathbf{R}^n$ are both maps between Euclidean spaces. Then

$$\begin{aligned}
\frac{d}{dt}(f \circ \gamma)(0) &= D\tilde{f} \cdot \frac{d\tilde{\gamma}}{dt}(0) \\
&= D\tilde{f}(\phi^{-1}(x)) \cdot \frac{d\tilde{\delta}}{dt}\sigma(0) \qquad \text{by (50)} \\
&= \frac{d}{dt}(f \circ \delta)(0),
\end{aligned}$$

where $\delta : \mathbf{R} \longrightarrow M$ is another representative of v.

If $\phi : U \longrightarrow \mathbf{R}^n$, $U \subset M$ is a coordinate chart, then we define the coordinate tangent vectors (of the chart ϕ) as the tangent vectors of the coordinate lines $\gamma_i : \mathbf{R} \longrightarrow M$, $i = 1, \cdots, n$

$$\gamma_i(t) = \phi^{-1}(\phi(p_0) + te_i),$$

where $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$, $i = 1, \cdots, n$ are the standard basis vectors in $\mathbf{R}^n$. If we (temporarily) continue with the notation $\tilde{f} = f \circ \phi^{-1} : \phi(U) \longrightarrow \mathbf{R}$ and let $x = (x^1, \cdots x^n)$ be the usual Cartesian coordinates on $\phi(U) \subset \mathbf{R}^n$ so that ϕ has the coordinate representation $\phi = (\phi^1, \cdots, \phi^n)$, then

$$\begin{aligned} \frac{d}{dt}(f \circ \gamma_i)(0) &= \frac{d}{dt}\tilde{f}(\phi(p_0) + te_i)\Big|_{t=0} \\ &= \frac{\partial \tilde{f}}{\partial x^i}(\phi(p_0)); \end{aligned} \tag{51}$$

that is, the directional derivative in the coordinate tangent vector direction is just the usual partial derivative, when all calculations are translated to $\mathbf{R}^n$ using the chart ϕ. This will be an extremely useful observation. For example, we adopt the notation ∂_i for the coordinate tangent vector, and then (51) may be rewritten as

$$D_{\partial_i} f(p_0) = \frac{\partial \tilde{f}}{\partial x^i}(\phi(p_0)).$$

If we now agree to forego the $\tilde{f}$ notation, then this can be written more simply as

$$D_{\partial_i} f = \frac{\partial}{\partial x^i} f = \partial_i f. \tag{52}$$

We now show that the coordinate tangent vectors form a basis for the vector space of tangent vectors (at the point p_0). If v is a tangent vector at p_0 with representing curve $\gamma : \mathbf{R} \longrightarrow M$ and $f : M \longrightarrow \mathbf{R}$ is any function, then

$$D_v f = \frac{d}{dt}\tilde{f} \circ \tilde{\gamma}(0)$$

and in the x^i coordinates, $\tilde{\gamma}(t) = (\phi \circ \gamma)(t) = (\tilde{\gamma}^1(t), \cdots, \tilde{\gamma}^n(t))$. By the chain rule and (52),

$$\begin{aligned} D_v f &= \frac{\partial \tilde{f}}{\partial x^i} \frac{d\tilde{\gamma}^i}{dt}(0) \\ &= \dot{\tilde{\gamma}}^i(0)\, D_{\partial_i} f. \end{aligned} \tag{53}$$

If we let $v^i = \dot{\gamma}^i(0)$, then the curve

$$\begin{aligned}\delta(t) &= \phi^{-1}(\phi(p_0) + t\sum_{i=1}^{n} v^i e_i) \\ &= \phi^{-1}(\phi(p_0) + t(v^1, \cdots, v^n))\end{aligned}$$

satisfies from (53),

$$D_v f = \frac{d}{dt}(f \circ \delta)(0)$$

and hence δ is a representing curve for v, since two vectors are the same exactly when their directional derivative operators agree on all smooth functions. The identification of v with $(v^1, \cdots, v^n) \in \mathbf{R}^n$ via the "standard representing curve" δ then gives a vector space structure to the set of tangent vectors at p_0 which satisfies, by (53),

$$v = v^i \partial_i. \tag{54}$$

That is, $\{\partial_1, \cdots, \partial_n\}$ is a basis for the tangent vectors. It is not hard to verify that this vector space structure is independent of the choice of coordinate chart.

If we let $r_i : \mathbf{R}^n \longrightarrow \mathbf{R}$, $i = 1, \cdots, n$ denote the standard Cartesian coordinate functions and if (ϕ, U) is a chart on M, then the n functions

$$x^i = r_i \circ \phi : U \longrightarrow \mathbf{R}, \quad i = 1, \cdots, n \tag{55}$$

are called the *coordinate functions* of (ϕ, U) and we may write $\phi = (x^1, \cdots, x^n)$. This notation can be very usefully abused. For example, if (ψ, V) is another chart with $p_0 \in U \cap V$ and if we let $y^i = r_i \circ \psi$ denote the coordinate functions of ψ, then the respective coordinate tangent vectors ∂_{x_i}, ∂_{y^i} are related by

$$\partial_{x^i} = \frac{\partial y^j}{\partial x^i} \partial_{y^j} \tag{56}$$

where the function $y(x) = (y^1(x^1, \cdots, x^n), \cdots, y^n(x^1, \cdots, x^n))$, $y : \phi(U \cap V) \longrightarrow \mathbf{R}^n$, is just the transition function,

$$y = \psi \circ \phi^{-1} = (r_i \circ \psi \circ \phi^{-1}). \tag{57}$$

Appropriately, the proof of (56) follows from the chain rule. Let $x_0 = \phi(p_0)$, then

$$\begin{aligned}D_{\partial_{x^i}} f &= \frac{\partial}{\partial x^i}(f \circ \phi^{-1})(x_0) \\ &= \frac{\partial}{\partial x^i}((f \circ \psi^{-1}) \circ (\psi \circ \phi^{-1}))\ (x_0) \\ &= \frac{\partial}{\partial y^j}(f \circ \psi^{-1})(y(x_0)) \cdot \frac{\partial y^j}{\partial x^i}\ (x_0) \\ &= \frac{\partial y^j}{\partial x^i} D_{\partial_{y^j}} f.\end{aligned}$$

The vector space of tangent vectors at $p \in M$ is denoted T_pM, and the space of all tangent vectors on M is $TM = \bigcup_{p\in M} T_pM$; TM has a natural manifold structure based on charts derived from coordinate charts on M. There is a natural projection $TM \longrightarrow M$ defined by $(p, v) \mapsto p$ where $v \in T_pM$, so that the *fibre* over $p \in M$ is the tangent space at p. A *section* of TM is a lifting $p \mapsto v_p$, or in other words, a *vector field* on M. The space of all smooth vector fields is denoted $\mathcal{X}(M)$.

Constructions with a vector space V (such as forming the dual space V^*, the tensor products $V \otimes \cdots \otimes V \otimes V^* \otimes \cdots \otimes V^*$, and the exterior products $\bigwedge^k V$) can be easily carried over to the tangent bundle TM, by simply applying them to each vector space $T_pM, p \in M$. We now briefly review these vector space constructions.

If V is a vector space then the dual space V^* is the vector space of linear functionals $\eta : V \to \mathbf{R}$. The dual basis of V^*, dual to a basis $v_1, \cdots, v_n$ of V, is the set of linear functionals $\{\eta^j, j = 1, \ldots, n\}$ satisfying

$$\eta^j(v_i) = \delta_i^j, \qquad i = 1, \ldots, n.$$

In particular, if $y = a^i v_i \in V$, then $\eta^j(y) = a^i$.

The *dual* vector space of T_pM is called the space of cotangent vectors or simply *covectors* and is denoted by T_p^*M. The cotangent bundle T^*M is the space of all covectors, $T^*M = \bigcup_{p\in M} T_p^*M$, and also is naturally a manifold, of dimension $2n$. Just as tangent vectors are naturally associated with curves in the manifold, there is a geometric interpretation of cotangent vectors, based on functions:

Given any function $f : M \longrightarrow \mathbf{R}$, there is a linear functional *df* on tangent vectors defined for $v \in T_pM$ by

$$df(v) = D_v f. \tag{58}$$

The linearity of *df* follows from the coordinate representation (53). Denoting the restriction of *df* to vectors at the point p by df_p (similarly we may indicate that a vector v is based at p by writing v_p), we have

$$df_p \in T_p^*M.$$

A natural basis for T_p^*M can be derived from a coordinate chart. The differentials dx^i of the coordinate functions $x^i : U \longrightarrow \mathbf{R}$ satisfy

$$\begin{aligned} dx^i(\partial_j) &= D_{\partial_j}(x^i) \\ &= \frac{\partial}{\partial x^j}\tilde{x}^i \end{aligned}$$

where $\tilde{x}^i = x^i \circ \phi^{-1} : \mathbf{R}^n \longrightarrow \mathbf{R}$, $\tilde{x}^i = r_i$, and hence

$$dx^i(\partial_j) = \delta^i_j, \quad 1 \leq i, j \leq n.$$

In other words, $\{dx^1, \cdots, dx^n\}$ is the basis of $T^*_p M$ dual to the coordinate vector basis $\{\partial_1, \cdots, \partial_n\}$ of $T_p M$.

The expansion of df in the basis $dx^i, i = 1, \ldots, n$ is obtained by noting that

$$df(\partial_i) = D_{\partial_i} f = \frac{\partial f}{\partial x^i},$$

and hence

$$df = \frac{\partial f}{\partial x^i}\, dx^i, \tag{59}$$

which should be compared with the chain rule.

The *tensor product* of two vector spaces V, W, with bases $\{v_1, \cdots, v_n\}$, $\{w_1, \cdots, w_m\}$ respectively, is the vector space $V \otimes W$ with basis

$$\{v_i \otimes w_a, 1 \leq i \leq n,\ 1 \leq a \leq m\};$$

hence $V \otimes W$ has dimension nm, and there is a natural product operation

$$\otimes : V \times W \longrightarrow V \otimes W$$

which is linear in each factor. If $y = y^i v_i \in V$, $z = z^a w_a \in W$, then

$$y \otimes z = \sum_{i=1}^{n} \sum_{a=1}^{m} (y^i z^a)\, v_i \otimes w_a;$$

it is straightforward to verify that this definition is independent of the choices of bases of V, W. Iterating this construction gives the space of (r, s)–tensors,

$$\bigotimes{}^{r,s}(V) = \underbrace{V \otimes \cdots \otimes V}_{r} \otimes \underbrace{V^* \otimes \cdots \otimes V^*}_{s}, \tag{60}$$

which are well–defined since the tensor product $\otimes$ is associative, $(U \otimes V) \otimes W = U \otimes (V \otimes W)$. However, the tensor product is not commutative, since there is no unique identification of $V \otimes V \otimes W$ with $W \otimes V \otimes V$ for example. Alternatively, $v_i \otimes v_j \neq v_j \otimes v_i$, so

$$\begin{aligned} (y^i v_i) \otimes (z^j v_j) &= y^i z^j\ v_i \otimes v_j \\ &\neq y^j z^i\ v_i \otimes v_j = (z^i v_i) \otimes (y^j v_j). \end{aligned}$$

Tensors in $\bigotimes^s V^*$ are identified with multilinear functionals on V. For example if $\alpha, \beta \in V^*$ then $\alpha \otimes \beta$ is the bilinear functional defined by

$$(\alpha \otimes \beta)(y, z) = \alpha(y)\, \beta(z), \qquad \forall\, y, z \in V.$$

A section of the bundle $T^{(0,2)}M$ is a choice of $(0,2)$–tensor or bilinear form on tangent vectors, at each point $p \in M$. We have encountered two examples of such animals in the lecture on surfaces, namely the *metric* tensor,

$$g = g_{ij}\, dx^i \otimes dx^j \tag{61}$$

which satisfies $g_{ij} = g_{ji}$ (symmetric) and $g_{ij} > 0$ (positive definite). Here g_{ij} are the coefficients of the metric tensor with respect to the local basis $dx^i \otimes dx^j$, $1 \leq i, j \leq n$, of $T_p^{(0,2)}M$, and thus by duality of ∂_i and dx^j we have $g_{ij} = g(\partial_i, \partial_j)$.

A *Riemannian* manifold is an n-dimensional manifold with a metric g — a symmetric positive definite bilinear form on vectors. Note that this definition of metric does not make any reference to any relation with the metric induced from an immersion in any Euclidean space; in fact, *a priori* it is far from clear whether or not a given metric can be realised by an immersion into some $\mathbf{R}^k$. That this is indeed true is the content of a deep theorem of John Nash.

The second example $(0,2)$–tensor is the second fundamental form II. Again this is also symmetric, but not in general positive definite. In local coordinates (u^a) on the surface S we have

$$\mathrm{II} = \mathrm{II}_{ab}\, du^a \otimes du^b,$$

where $\mathrm{II}_{ab} = \langle N, X_{ab} \rangle$.

Defining $\bigotimes^0 V = \mathbf{R}$, the tensor algebra space

$$\bigotimes V = \bigoplus_{r=0}^{\infty} \otimes^r V = \mathbf{R} \oplus V \oplus (V \otimes V) \oplus \cdots$$

has the structure of an associative algebra over $\mathbf{R}$ with identity $1 \in \bigotimes^0 V$, and product operation $\otimes$. Taking the quotient of $\bigotimes V$ by the two-sided ideal $\langle v \otimes v \rangle$ generated by elements of the form $v \otimes v$, $v \in V$, gives the *exterior algebra* of V,

$$\bigwedge V = \bigotimes V \big/ \langle v \otimes v \rangle. \tag{62}$$

The tensor product $\otimes$ in $\bigotimes V$ descends to $\bigwedge V$ to give the wedge product $\wedge$ on $\bigwedge V$; since

$$v \wedge v = 0 \qquad \text{for all } v \in V \tag{63}$$

in $\bigwedge V$ by the definition (62), it follows from linearity that

$$v \wedge w = -w \wedge v \quad \text{for all } v, w \in V. \tag{64}$$

Consequently a basis of $\bigwedge^k V = \bigotimes^k(V) \big/ \langle v \otimes v \rangle$ is given by

$$v_{i_1} \wedge \cdots \wedge v_{i_k}, \quad 1 \leq i_1 < i_2 < \cdots < i_k \leq n$$

and hence

$$\dim \bigwedge^k V = \binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad k \leq n$$

and $\bigwedge^k V = 0$ for $k > n$. It follows by repeated application of (64) that

$$\beta \wedge \alpha = (-1)^{k\ell} \alpha \wedge \beta, \qquad \text{for } \alpha \in \bigwedge^k V, \ \beta \in \bigwedge^\ell V. \tag{65}$$

We may identify $\bigwedge^k V^*$ as the space of *alternating*, or totally antisymmetric, multilinear forms on V in one of two ways. We shall use the dual pairing

$$(\alpha \wedge \beta)(v, w) = \alpha(v)\beta(w) - \alpha(w)\beta(v) \quad \text{for } \alpha, \beta \in \bigwedge^1 V, \ v, w \in V,$$

and more generally,

$$(\alpha_1 \wedge \cdots \wedge \alpha_k)(w_1, \ldots, w_k) = \sum_{\sigma \in perm(1,\ldots,k)} (-1)^{sgn(\sigma)} \alpha_1(w_{\sigma(1)}) \cdots \alpha_k(w_{\sigma(k)}). \tag{66}$$

The other identification would insert a factor of $1/2$ in the first relation, and $1/k!$ in the general relation (66). The convention adopted here has the advantage that the bases v_I, $I = (i_1, \ldots, i_k)$, $1 \leq i_1 < \cdots < i_k \leq n$ and η^I are dual, $\eta^I(v_J) = \delta^I_J$. The *interior* or cut product $\iota_v : \bigwedge^{k+1} V^* \longrightarrow \bigwedge^k V^*$ is defined using the identification with alternating forms by

$$(\iota_v \alpha)(w_1, \ldots, w_k) = \alpha(v, w_1, \ldots, w_k), \tag{67}$$

where $v \in V$ is any vector. Note that $\iota_v \circ \iota_v = 0$ by the alternating property. The *exterior* or wedge product $\epsilon_\lambda : \wedge^k V^* \longrightarrow \wedge^{k+1} V^*$ is defined similarly,

$$\epsilon_\lambda \alpha = \lambda \wedge \alpha, \quad \lambda \in V^*, \tag{68}$$

and we note the interesting identity

$$\epsilon_\lambda \iota_v + \iota_v \epsilon_\lambda = \lambda(v) \quad \lambda \in V^*, v \in V. \tag{69}$$

As before, we may apply the exterior product construction to the tangent or cotangent bundles, yielding the spaces $\bigwedge TM$, and $\bigwedge T^*M$. The space $\mathcal{A}^k(M)$ of sections of the bundle $\bigwedge^k T^*M$ is particularly important — sections of $\mathcal{A}^k(M)$ are called *k–forms.* Note that 1–form is thus a synonym for cotangent vector field, since $\bigwedge^1 V = V$.

A map $\phi : M^m \longrightarrow N^n$ of manifolds may be used to transport objects from one manifold to the other. For example, if $\gamma : \mathbf{R} \to M$ is a curve through $p \in M$ then $\phi \circ \gamma : \mathbf{R} \to N$ is a curve through $\phi(p) \in N$. The *push forward* $\phi_* v_p$ of the tangent vector $v_p = \gamma'(0)$ to γ at p is then the tangent vector at $\phi(p)$ to $\phi \circ \gamma$. Other notations for push forward are

$$\phi_* v = (\phi \circ \gamma)'(0) = d\phi(v) = T\phi(v).$$

Note that although individual vectors may be pushed forward, the push forward of a vector field is not a vector field — a point $q \in N$ may have more than one preimage under ϕ, or none at all. One example of push forward of a vector has already been given — the tangent vectors X_a to a surface may be viewed as the push forward of the coordinate tangent vectors in $U \subset \mathbf{R}^2$,

$$X_a = X_*(\partial_{u^a}) = X_*(\partial_a).$$

If $\alpha \in T^*_{\phi(p)}N$ is a covector on N, then $\phi^*\alpha$ is the covector at $p \in M$ defined by

$$(\phi^*\alpha)(v) = \alpha(\phi_* v), \qquad \forall\ v \in T_pM, \tag{70}$$

and is called the *pull back* of α. Unlike the push forward of a vector field, the pull back of a covector field is always a covector field. Note that if $f : N \to \mathbf{R}$ then we may also define the pull back of f by $\phi^* f = f \circ \phi : M \to \mathbf{R}$, and then we have an interesting relationship with the differential,

$$d(\phi^* f) = \phi^*(df). \tag{71}$$

We also note that ϕ^* respects the algebra structure of $\bigwedge M$ ie. $\phi^*(\alpha \wedge \beta) = \phi^*(\alpha) \wedge \phi^*(\beta)$, which follows from the linearity of ϕ^*, ϕ_* and the definition of ϕ^* applied to $\bigwedge^k T^*N$. In particular, $\phi^* : \mathcal{A}^k(N) \longrightarrow \mathcal{A}^k(M)$.

4 Curvature

Outline:

Covariant derivative; curvature tensor; gradient and Laplace operators; exterior derivative; Cartan calculus; connection matrix; structure equations; Bianchi identities.

Using the directional derivative and a metric on a manifold allows us to generalise several geometric constructions of surface theory, in particular, the covariant derivative and curvature.

The covariant derivative on a surface is a map $\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ satisfying the basic derivation rules,

$$\begin{aligned} \nabla_X fY &= D_X f\, Y + f\nabla_X Y, && (72)\\ \nabla_{fX} Y &= f\nabla_X Y, && (73) \end{aligned}$$

for any vector fields $X, Y \in \mathcal{X}(M)$ and function $f \in C^\infty(M)$, and the metric compatibility and torsion-free conditions

$$\begin{aligned} D_X(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z), && (74)\\ [X,Y] &= \nabla_X Y - \nabla_Y X. && (75) \end{aligned}$$

The *Levi–Civita identity*

$$\begin{aligned} g(\nabla_X Y, Z) &= \tfrac{1}{2}\big(D_X(g(Y,Z)) + D_Y(g(X,Z)) - D_Z(g(X,Y))\\ &\qquad -g(Y,[X,Z]) - g(X,[Y,Z]) + g(Z,[X,Y])\big) \qquad (76) \end{aligned}$$

defines the covariant derivative on any Riemannian manifold, and is easily seen to have the required properties (72), (73), (74), (75). It also follows that ∇ is uniquely determined by these properties.

There are two interesting special cases of (76). If we restrict attention to coordinate tangent vector fields ∂_i, then $[\partial_i, \partial_j] = 0$ and (76) gives the Christoffel symbol formula

$$\begin{aligned} \Gamma_{ijk} &= g(\nabla_{\partial_i}\partial_j, \partial_k)\\ &= \tfrac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}). \qquad (77) \end{aligned}$$

If on the other hand we restrict attention to vector fields $e_1, \ldots, e_n$ forming an orthonormal frame

$$g(e_i, e_j) = \delta_{ij}$$

(or more generally, $g(e_i, e_j) = const$), then the first three terms in (76) drop out and we obtain

$$g(\nabla_{e_i} e_j, e_k) = \tfrac{1}{2}\left(g(e_k, [e_i, e_j]) - g(e_i, [e_j, e_k]) - g(e_j, [e_i, e_k])\right). \tag{78}$$

In particular, the *connection 1-forms* ω_{ij} defined by

$$\omega_{ij}(X) = g(e_i, \nabla_X e_j), \tag{79}$$

are antisymmetric $\omega_{ji} = -\omega_{ij}$ and may be computed explicitly by

$$\omega_{ij}(e_k) = \tfrac{1}{2}\left(g([e_i, e_j], e_k) - g(e_i, [e_j, e_k]) + g(e_j, [e_i, e_k])\right). \tag{80}$$

The covariant derivative may be extended to act on tensors more general than just vector fields, by requiring the obvious linearity and derivation rules (cf. (72)), together with a *Leibnitz* or product rule property. Thus, for example, ∇ acting on $(2,0)$-tensors satisfies

$$\nabla(Y \otimes Z) = (\nabla Y) \otimes Z + Y \otimes (\nabla Z),$$

and ∇ acting on a cotangent field α satisfies

$$D_Y(\alpha(Z)) = (\nabla_Y \alpha)(Z) + \alpha(\nabla_Y Z).$$

Obvious extensions of these properties define ∇ on all (r,s)-tensors. Note that if we regard the direction of the covariant derivative as unspecified, then the covariant derivative of an (r,s)-tensor is an $(r, s+1)$-tensor.

An index notation is widely used for the covariant derivative. For example, if $Y = Y^i \partial_i$ is a vector field, then the covariant derivative is the $(1,1)$-tensor

$$\begin{aligned} \nabla Y &= Y^i_{;j}\, \partial_i \otimes dx^j \\ &= \left(\partial_j(Y^i) + Y^k \Gamma^i_{jk}\right) \partial_i \otimes dx^j, \end{aligned} \tag{81}$$

and the covariant derivative of a covector field is

$$\begin{aligned} \nabla \alpha &= \alpha_{i;j}\, dx^i \otimes dx^j \\ &= \left(\partial_j(\alpha_i) - \alpha_k \Gamma^k_{ji}\right) dx^i \otimes dx^j. \end{aligned} \tag{82}$$

The formula (46) for the covariant derivative of the $(0,2)$-tensor $\mathrm{I\!I}$ is a direct generalisation of (82). More directly exploiting the Leibnitz rule, we have a coordinate-free description of the covariant derivative of $\mathrm{I\!I}$:

$$(\nabla_X \mathrm{I\!I})(Y, Z) = D_X(\mathrm{I\!I}(Y, Z)) - \mathrm{I\!I}(\nabla_X Y, Z) - \mathrm{I\!I}(Y, \nabla_X Z). \tag{83}$$

Comparing (74) with (83), we see that the metric compatibility condition is equivalent to $\nabla g = 0$.

The Riemann curvature tensor is defined as before on vector fields $X, Y, Z, W \in \mathcal{X}(M)$ by

$$R(X,Y,Z,W) = g((\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z, W). \tag{84}$$

That this is in fact a tensor, ie. a multilinear function in each of the four slots, follows from the definition, the derivation property (72) and the definition of the Lie bracket. First, it is immediately clear that $R(X,Y,Z,fW) = fR(X,Y,Z,W)$ for any function $f \in C^\infty(M)$. Since

$$[fX, Y] = f[X,Y] - D_Y f\, X \tag{85}$$

we have

$$R(fX,Y,Z,W) = fR(X,Y,Z,W) + g(-D_Y f\, \nabla_X Z - \nabla_{(-D_Y f\, X)} Z, W) = fR(X,Y,Z,W).$$

This shows also that $R(X, fY, Z, W) = fR(X,Y,Z,W)$, by the antisymmetry of R in the first two slots. The final most remarkable cancellation is verified as follows:

$$\begin{aligned} R(X,Y,fZ,W) &= g(\nabla_X(f\nabla_Y Z + D_Y f\, Z) - \nabla_Y(f\nabla_X Z + D_X f\, Z),\, W) \\ &\quad - g(f\nabla_{[X,Y]}Z, W) - g(D_{[X,Y]} f\, Z, W) \\ &= g(D_X f\, \nabla_Y Z + D_X D_Y f\, Z + D_Y f\, \nabla_X Z, W) \\ &\quad - g(D_Y f\, \nabla_X Z + D_Y D_X f\, Z + D_X f\, \nabla_Y Z, W) \\ &\quad - g(D_{[X,Y]} f\, Z, W) + fR(X,Y,Z,W) \\ &= fR(X,Y,Z,W). \end{aligned}$$

Thus the Riemann curvature depends only on the values of the vectors and not on the first or second derivatives; in other words, $Riem = R(\cdot,\cdot,\cdot,\cdot)$ is a $(0,4)$-tensor. By the linearity property and expanding all vectors in a basis frame of coordinate vectors, we may write $R(X,Y,Z,W)$ in index form

$$\begin{aligned} R(X,Y,Z,W) &= X^i Y^j Z^k W^\ell R_{ijk\ell}, \\ R_{ijk\ell} &= R(\partial_i, \partial_j, \partial_k, \partial_\ell) \\ &= g((\nabla_{\partial_i}\nabla_{\partial_j} - \nabla_{\partial_j}\nabla_{\partial_i})\partial_k,\, \partial_\ell) \\ &= \partial_i(\Gamma_{jk\ell}) - \partial_j(\Gamma_{ik\ell}) - \Gamma^p_{jk}\Gamma_{i\ell p} + \Gamma^p_{ik}\Gamma_{j\ell p}. \end{aligned} \tag{86}$$

The expressions (75), (86) show that $R_{ijk\ell}$ depends polynomially on the first and second derivatives of the metric g_{ij}. Clearly, the curvature tensor of $\mathbf{R}^n$ with the standard metric $g_{ij} = \delta_{ij}$ vanishes, as might have been expected.

Note that it is not essential that indices refer to a basis frame of coordinate vectors $\partial_1, \ldots, \partial_n$ — already with the connection 1-form ω_{ij} we have seen an example where the indices refer to an orthonormal frame of vectors $e_1, \ldots, e_n$. In general, the frame used to define the indices in any formula will be either understood, or explicitly mentioned. Particularly in Riemannian geometry, it is often much more useful to perform computations in an orthonormal frame rather than a coordinate frame.

The curvature tensor has a number of symmetries, the first being the obvious $R(X,Y,Z,W) = -R(Y,X,Z,W)$. Expanding out the identity

$$\left(D_X D_Y - D_Y D_X - D_{[X,Y]}\right) g(Z,W) = 0$$

shows that

$$R(X,Y,Z,W) = -R(X,Y,W,Z), \tag{87}$$

whilst the first Bianchi identity

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0 \tag{88}$$

follows from the definition by a straightforward computation,

$$\begin{aligned}
&R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) \\
&\quad= g(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\
&\qquad + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X - \nabla_{[Y,Z]} X \\
&\qquad + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y - \nabla_{[Z,X]} Y, W) \\
&\quad= g(\nabla_X [Y,Z] - \nabla_Y [X,Z] + \nabla_Z [X,Y] \\
&\qquad - \nabla_{[X,Y]} Z - \nabla_{[Y,Z]} X - \nabla_{[Z,X]} Y, W) \\
&\quad= g([X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]], W) \\
&\quad= 0,
\end{aligned}$$

since the Lie bracket satisfies the easily verified Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0. \tag{89}$$

Finally, combining (87), (88) gives a symmetry between the (12) and (34) positions,

$$R(X,Y,Z,W) = R(Z,W,X,Y) \tag{90}$$

as follows:

$$2R(X,Y,Z,W) \;=\; -R(Y,Z,X,W) - R(Z,X,Y,W)$$

$$
\begin{aligned}
& + R(Y,W,X,Z) + R(W,X,Y,Z) \\
= \quad & R(X,Y,Z,W) + R(X,Y,Z,W) \\
& + R(Y,Z,W,X) + R(Z,X,W,Y) \\
= \quad & - R(Z,W,Y,X) - R(W,Z,X,Y) \\
= \quad & 2R(Z,W,X,Y).
\end{aligned}
$$

The definition of *Riem* can be written in terms of a commutator of covariant derivatives $R(X,Y,Z,W) = g(R(X,Y)Z,W)$, where

$$R(X,Y)Z = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]})Z. \tag{91}$$

In index notation and with a coordinate frame we have

$$R(X,Y)Z = X^i Y^j Z^k \, R_{ijk}{}^{\ell} \, \partial_\ell$$

where $R_{ijk}{}^{\ell} = R_{ijkp} g^{p\ell}$ and $(g_{ij}) = (g_{ij}^{-1})$ is the inverse metric. This process of raising and lowering of indices using the metric and its inverse, corresponds to the canonical isomorphisms between T_pM and T_p^*M defined by the inner product g. Using the $\cdot_{;i}$ notation for covariant derivative yields the so-called *Ricci identity*

$$Z^k_{;ji} - Z^k_{;ij} = Z^p R_{ijp}{}^{k}. \tag{92}$$

Applying the metric isomorphism to the differential df of a function $f \in C^\infty(M)$ gives the gradient operator,

$$\mathrm{grad} f = f_{;i} g^{ij} \, \partial_j = \nabla^j f \, \partial_j = f^{;j} \partial_j, \tag{93}$$

where $f_{;i} = \partial_i f$. Another way of stating the definition of $\mathrm{grad} f$ is

$$g(\mathrm{grad} f, X) = D_X f = df(X), \quad \text{for all vectors } X. \tag{94}$$

Taking the covariant derivative of $\mathrm{grad} f$ gives the *Hessian*, or second covariant derivative matrix of f,

$$\nabla^2_{ij} f = f_{;ij} = \partial_i \partial_j f - \Gamma^k_{ij} \partial_k f, \tag{95}$$

and the trace of $\nabla^2 f$ is the *Laplace-Beltrami operator* of the metric g

$$
\begin{aligned}
\Delta_g f &= g^{ij} f_{;ij} \\
&= g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial f}{\partial x^k} \right) \\
&= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j} \right),
\end{aligned} \tag{96}
$$

where $\sqrt{g} = \sqrt{\det g}$.

Another method for describing the curvature and covariant derivatives was developed by E. Cartan, and uses orthonormal frames and forms, rather than vectors. Because forms behave better than vectors under maps of manifolds, the Cartan calculus is frequently better suited to actual computations.

For this we need the *exterior derivative* operator

$$d : \mathcal{A}^k(M) \longrightarrow \mathcal{A}^{k+1}(M), \quad k \geq 0, \tag{97}$$

which is a linear first order differential operator satisfying

(a) $d(f) = df$, for $f \in C^\infty(M)$;

(b) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$, for $\alpha \in \mathcal{A}^k(M)$;

(c) $d^2 = 0$.

These properties uniquely determine d, since they lead to an explicit expression for d in any local coordinate system. Firstly, (a) determines d acting on $\mathcal{A}(M) = C^\infty(M)$. By (b) and (c), for any two functions $f, g \in C^\infty(M)$ we have

$$\begin{aligned} d(f\,dg) &= df \wedge dg + f\,d(dg) \\ &= df \wedge dg, \end{aligned}$$

which determines d on $\mathcal{A}^1(M)$ since all 1-forms may be written as a linear combination of terms of the form $f\,dg$. Now

$$dg_1 \wedge \cdots \wedge dg_k = d(g_1\,dg_2 \wedge \cdots \wedge dg_k) - g_1\,d(dg_2 \wedge \cdots \wedge dg_k), \quad k \geq 2,$$

and thus by induction it follows that

$$d(dg_1 \wedge \cdots \wedge dg_k) = 0$$

for any functions $g_1, \ldots, g_k$. Hence

$$d(f\,dg_1 \wedge \cdots \wedge dg_k) = df \wedge dg_1 \wedge \cdots \wedge dg_k \tag{98}$$

which shows that the conditions (a), (b), (c) determine d on $\mathcal{A}^k(M)$. To show that this definition is consistent, we note that any k-form may be written as $\alpha = \alpha\,dx^{i_1} \wedge \ldots \wedge dx^{i_k} = \alpha_I\,dx^I$ where the coefficients $\alpha_{i_1 \cdots i_k} = \alpha_I$ are uniquely determined and $I = (i_1 < \cdots < i_k)$ is a multi-index. Then the local definition

$$d\alpha = \frac{\partial \alpha_I}{\partial x^i} dx^i \wedge dx^I, \tag{99}$$

satisfies the required properties, and (98) shows that any definition of d is unique (since the differentials dg_j are defined already), hence there exists a unique operator with the properties (a), (b), (c). By iterating (99) we see that the property $d^2 = 0$ reflects the commutativity of second partial derivatives.

By combining (71) and (98) we see that d behaves naturally under pullback by a map $\phi : M \to N$,

$$\phi^*(d\alpha) = d(\phi^*\alpha), \quad \forall \alpha \in \mathcal{A}(N). \tag{100}$$

By computing in local coordinates, we can easily show that for any $\alpha \in \mathcal{A}^1(M)$ and vector fields $X, Y \in \mathcal{X}(M)$,

$$d\alpha(X,Y) = D_X(\alpha(Y)) - D_Y(\alpha(X)) - \alpha([X,Y]). \tag{101}$$

A standard exercise relates d acting on $\mathcal{A}(\mathbf{R}^3)$ to the classical differential operators div, grad and curl.

Now suppose $e_1, \ldots, e_n$ is an orthonormal frame, and let $\theta_1, \ldots, \theta_n$ be the dual covector frame. From (101) we have

$$\begin{aligned} d\theta_i(e_j, e_k) &= -\,\theta_i([e_j, e_k]) \\ &= -\,g(e_i, \nabla_{e_j} e_k - \nabla_{e_k} e_j) \\ &= \omega_{ij}(e_k) - \omega_{ik}(e_j) \\ &= -\,(\omega_{i\ell} \wedge \theta_\ell)\,(e_j, e_k), \end{aligned}$$

which gives Cartan's first structure equation

$$d\theta_i + \omega_{ij} \wedge \theta_j = 0, \tag{102}$$

where $\omega_{ij} = g(e_i, \nabla e_j)$ is the connection 1-form. A similar computation yields the second structure equation,

$$d\omega_{ij} + \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \tag{103}$$

where Ω_{ij} is the curvature 2-form and is related to the Riemann curvature tensor by

$$\Omega_{ij} = -\tfrac{1}{2} R_{ijk\ell}\theta_k \wedge \theta_\ell, \tag{104}$$

where the indices in $R_{ijk\ell}$ refer to components in the basis $e_1, \ldots, e_n$.

The first Bianchi identity (88) is equivalent to

$$\Omega_{ij} \wedge \theta_j = 0 \tag{105}$$

and is derived (much more easily this time!) by taking the exterior derivative of (102). Taking the exterior derivative of Ω_{ij} using (103) gives the Cartan form of the *second Bianchi identity*

$$d\Omega_{ij} + \omega_{ik} \wedge \Omega_{kj} - \Omega_{ik} \wedge \omega_{kj} = 0, \tag{106}$$

which is equivalent to

$$(\nabla_X R)(Y, Z)W + (\nabla_Y R)(Z, X)W + (\nabla_Z R)(X, Y)W = 0, \tag{107}$$

a form which can be verified directly, but not so simply, from the definition of *Riem* and ∇*Riem*. The index notation versions of (105) and (106) are

$$R_{[ijk]\ell} = 0, \qquad R_{ij[k\ell;m]} = 0, \tag{108}$$

where enclosing indices in square brackets indicates a total antisymmetrisation over those indices, ie. $[ijk] = \frac{1}{6}(ijk - ikj + jki - jik + kij - kji)$.

5 Integration and Stokes' Theorem

Outline:

> Orientation of a manifold; volume form; integration of an n-form; manifold with boundary; Stokes' Theorem; divergence, Gauß and Stokes' formulae; integration by parts; Green's formulae.

As the first step in understanding the process of integration on a manifold M^n, assume $\omega \in \mathcal{A}^n(M)$ is supported in a single coordinate neighbourhood $U \subset M$. Let (x^i) be the local coordinates and $\partial_i = \partial_{x^i}$ be the coordinate tangent vectors. We define the integral of ω over M by

$$\int_M \omega = \int_{\mathbf{R}^n} \omega(\partial_1 \wedge \ldots \wedge \partial_n)\, d\mathcal{L}^n_x, \tag{109}$$

where $d\mathcal{L}^n_x$ is the n-dimensional Lebesgue measure and $\omega(\partial_1 \wedge \ldots \wedge \partial_n) = \omega(\partial_1, \ldots, \partial_n)$ is the multilinear pairing between ω and the infinitesimal coordinate parallelepiped $\partial_1 \wedge \ldots \wedge \partial_n$, normalised by $(dx^1 \wedge \ldots \wedge dx^n)(\partial_1 \wedge \ldots \wedge \partial_n) = 1$. To see that this definition is independent of the choice of coordinate chart, suppose $y = (y^i) = y(x)$ is some other coordinate chart covering the support of ω. The alternating property and (56) imply that

$$\begin{aligned}
&\int_{\mathbf{R}^n} \omega(\partial_{y^1} \wedge \ldots \wedge \partial_{y^n}) d\mathcal{L}^n_y \\
&= \int_{\mathbf{R}^n} \omega\left(\frac{\partial x^{i_1}}{\partial y^1}\partial_{x^{i_1}} \wedge \ldots \wedge \frac{\partial x^{i_n}}{\partial y^n}\partial_{x^{i_n}}\right) d\mathcal{L}^n_y \\
&= \int_{\mathbf{R}^n} \omega\left(\det\left(\frac{\partial x^i}{\partial y^j}\right) \partial_{x^1} \wedge \ldots \wedge \partial_{x^n}\right) d\mathcal{L}^n_y \\
&= \int_{\mathbf{R}^n} \omega\left(\partial_{x^1} \wedge \ldots \wedge \partial_{x^n}\right) \det\left(\frac{\partial x}{\partial y}\right) d\mathcal{L}^n_y \\
&= \int_{\mathbf{R}^n} \omega\left(\partial_{x^1} \wedge \ldots \wedge \partial_{x^n}\right) d\mathcal{L}^n_x,
\end{aligned} \tag{110}$$

by the change of variables formula for Lebesgue measure

$$d\mathcal{L}^n_y = \left|\det \frac{\partial y}{\partial x}\right| d\mathcal{L}^n_x, \tag{111}$$

<u>assuming</u> we also have the orientation condition

$$\det \frac{\partial y}{\partial x} > 0, \tag{112}$$

restricting the admissible coordinate charts.

We say that M is *orientable* if there is $\mu \in \mathcal{A}^n(M)$ such that $\mu \neq 0$ everywhere in M; an oriented coordinate chart then is required to satisfy $\mu(\partial_1 \wedge \ldots \wedge \partial_n) > 0$. Equivalently, an orientable manifold is one with a covering by coordinate charts satisfying (112).

The problem of integrating a general $\omega \in \mathcal{A}^n(M)$ is reduced to the case of support within a coordinate neighbourhood by the use of a *partition of unity*: a family of functions $\lambda_\alpha \in C^\infty(M)$, $\alpha \in A$, satisfying $0 \leq \lambda_\alpha(p) \leq 1$ and $\sum_\alpha \lambda_\alpha(p) = 1$ for all $p \in M$, where only a finite number of terms in the sum are non-zero. If M is separable, then a partition of unity exists subordinate to any locally finite covering of M.

Writing $\omega = \sum_\alpha \lambda_\alpha \omega$ and using the linearity of $\int_M$ then determines $\int_M \omega$. Note that this definition of $\int_M \omega$ depends strongly on the choice of orientation $\mu \in \mathcal{A}^n(M)$ of M: denoting by $-M$ the manifold with the reverse orientation $-\mu$, we have $\int_{(-M)} \omega = -\int_M \omega$.

If $\phi : M^m \longrightarrow N^n$ and if $\alpha \in \mathcal{A}^m(N)$ then the integral of α over M is defined naturally by pull-back:

$$\int_{\phi(M)} \alpha = \int_M \phi^*(\alpha). \tag{113}$$

This applies in particular to the situation where M is a submanifold of N and ϕ is the inclusion map, which shows that there is a natural dual pairing between $\mathcal{A}^k(N)$ and k-dimensional submanifolds of N, given by integration along the submanifold.

On an oriented Riemannian manifold we may normalise the orientation form to satisfy $||\mu|| = 1$, where $||\mu||$ denotes the metric length. This normalised n-form is called the *volume form* of M. In local (oriented) coordinates we have

$$\mu = \sqrt{\det(g)}\, dx^1 \wedge \ldots \wedge dx^n, \tag{114}$$

where $\det(g) = \det(g_{ij})$, $g_{ij} = g(\partial_i, \partial_j)$. Under a change of coordinates $y = y(x)$ we have

$$\begin{aligned} \det(g(\partial_{x^i}, \partial_{x^j})) &= \det\left(\frac{\partial y^k}{\partial x^i} g(\partial_{y^k}, \partial_{y^\ell}) \frac{\partial y^\ell}{\partial x^j}\right) \\ &= \left|\det \frac{\partial y^k}{\partial x^i}\right|^2 \det(g(\partial_{y^k}, \partial_{y^\ell})), \end{aligned}$$

and thus the volume form μ defined by (114) is independent of the choice of oriented coordinate system.

If M is Riemannian but not oriented then it is still possible to define an integral for functions (but not n-forms), by using the measure $\sqrt{\det(g)}\, d\mathcal{L}^n_x$ in coordinate patches.

Although viable and sometimes useful, this non-oriented integral does not satisfy the very elegant Stokes' Theorem, a.k.a. the generalised fundamental theorem of calculus.

In order to state Stokes' theorem, we need a definition of *manifold with boundary.* This is a pair $(M, \partial M)$ where M is a smooth manifold under the induced topology from $M \subset M \cup \partial M$, and where points in ∂M admit boundary coordinate charts. A boundary coordinate chart is (ϕ, U), $U \subset M \cup \partial M$, where $\phi : U \to \bar{\mathbf{R}}^n_+ = \{(x^1, \dots, x^n), x^n \geq 0\}$ and as usual we require the boundary charts to be C^∞-compatible in the interior set M, and such that the charts $(\phi|_{\partial M}, U \cap \partial M)$ define a C^∞ structure on ∂M. The definitions of tangent vector, smooth function etc, extend naturally to the case of a manifold with boundary. Note, however, that $C^\infty(M \cup \partial M) \neq C^\infty(M)$.

If M is oriented by $\mu \in \mathcal{A}^n(M)$, then there is a natural induced orientation on ∂M defined by $(-1)^n \partial_1 \wedge \dots \partial_{n-1}$, where $(x^1, \dots, x^n)$ is any oriented boundary coordinate chart with the boundary defined by $x^n \geq 0$ (ie. ∂_n is inward pointing). The induced orientation form on ∂M is then $\tilde{\mu} = -\iota_{\partial_n} \mu$.

Theorem 4 (Stokes) *Suppose* $(M, \partial M)$ *is an oriented manifold with boundary, and* $\omega \in \mathcal{A}^{n-1}(M)$. *Then*

$$\int_M d\omega = \oint_{\partial M} \omega. \tag{115}$$

Proof : By invoking a partition of unity we may distinguish two cases, with ω supported in either an interior coordinate chart or a boundary chart. In the interior case, by linearity we may assume $\omega = f(x) dx^2 \wedge \dots \wedge dx^n$ with $f \in C_c^\infty(\mathbf{R}^n)$, and then $d\omega = \frac{\partial}{\partial x^1} f \, dx^1 \wedge \dots \wedge dx^n$. By Fubini's theorem we have

$$\begin{aligned}
\int_M d\omega &= \int_{\mathbf{R}^n} d\omega(\partial_1 \wedge \dots \wedge \partial_n) \, d\mathcal{L}^n_x \\
&= \int_{\mathbf{R}^n} \frac{\partial f}{\partial x^1} \, d\mathcal{L}^n_x \\
&= \int_{\mathbf{R}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x^1} \, dx^1 \right) dx^2 \cdots dx^n,
\end{aligned}$$

which vanishes by the fundamental theorem of calculus, since f has compact support.

If ω is supported in a boundary chart (ϕ, U) with boundary defined by $x^n \geq 0$, then we distinguish two subcases,

(a) $\omega = f(x) \, dx^1 \wedge \dots \wedge dx^{n-1}$; and

(b) $\omega = f(x) \, dx^2 \wedge \dots \wedge dx^n$.

In case (a) we have

$$d\omega = (-1)^{n-1}\frac{\partial f}{\partial x^n}dx^1 \wedge \ldots \wedge dx^n,$$

and thus

$$\begin{aligned}\int_M d\omega &= \int_{\mathbf{R}^n_+} d\omega(\partial_1 \wedge \ldots \wedge \partial_n) d\mathcal{L}^n_x \\ &= (-1)^{n-1}\int_{\mathbf{R}^{n-1}}\left(\int_0^\infty \frac{\partial f}{\partial x^n}\,dx^n\right) dx^1 \cdots dx^{n-1} \\ &= (-1)^{n-1}\int_{\mathbf{R}^{n-1}}[f(x',x^n)]|_0^\infty \,dx^1 \cdots dx^{n-1} \\ &= (-1)^n \int_{\mathbf{R}^{n-1}} f(x',0)\, d\mathcal{L}^{n-1}_{x'}.\end{aligned}$$

Since

$$\omega((-1)^n\partial_{x^1} \wedge \ldots \wedge \partial_{x^{n-1}}) = (-1)^n f(x',0)$$

on ∂M, we see that the final expression is just $\oint_{\partial M}\omega$ as required.

The final subcase (b) with ω not involving dx^n is computed similarly, but the interior integral is now $\int_{-\infty}^{\infty} \frac{\partial}{\partial x^1} f(x',0)\,dx^1$ which again vanishes by the fundamental theorem of calculus and the compact support of f. ■

If M is an oriented Riemannian manifold and μ is the metric volume form, then we can define the *divergence* of a vector field $X \in \mathcal{X}(M)$ by

$$\mathrm{div}X\,\mu = d(\iota_X\mu). \tag{116}$$

In coordinates,

$$\begin{aligned}\mathrm{div}X &= \frac{1}{\sqrt{\det g}}\frac{\partial}{\partial x^i}\left(\sqrt{\det g}\,X^i\right) \\ &= X^i_{;i}.\end{aligned} \tag{117}$$

The divergence form of Stokes's theorem is thus

$$\int_M \mathrm{div}X\,\mu = \oint_{\partial M}\iota_X\mu, \tag{118}$$

which we may write in a more familiar form by noting that the induced oriented volume form on ∂M is defined by $\tilde{\mu} = \iota_n\mu$, where $n \in T_p(M \cup \partial M)$, $p \in \partial M$, is the *outward* pointing unit normal to ∂M. This implies that

$$\iota_X\mu = g(X,n)\tilde{\mu} + \text{ a term involving } dx^n,$$

so that

$$\int_M \mathrm{div}X\,\mu = \oint_{\partial M} g(X,n)\tilde{\mu}. \tag{119}$$

This reduces to the well-known Gauß and Green's formulae in dimensions 3 and 2 respectively.

A more general form of (119) is sometimes useful, for example when integrating along a null hypersurface in a Lorentzian manifold, where the metric volume form vanishes and it is not possible to define a unit length normal vector. If μ and $\tilde{\mu}$ are orientation forms on M and ∂M respectively, then there is a 1-form ν such that $\mu = \nu \wedge \tilde{\mu}$; ν is called a conormal to ∂M. Then (118) gives

$$\int_M d(\iota_X \mu) = \oint_{\partial M} \nu(X)\, \tilde{\mu}. \tag{120}$$

To derive the classical Stokes' theorem for the integral of $\mathrm{curl}\, V$ over a surface S in $\mathbf{R}^3$ we need to pull back from the surface to $\mathbf{R}^2$ with the parameterisation map $X : U \subset \mathbf{R}^2 \longrightarrow S \subset \mathbf{R}^3$. Details are left as an exercise.

Several useful identities may be obtained from the divergence form of Stokes' theorem. Choosing a vector field of the form fX, where $f \in C^\infty(M)$ and $X \in \mathcal{X}(M)$, we find

$$\int_M (f \mathrm{div} X + D_X(f))\, dv_M = \oint_{\partial M} f\, g(X, n)\, dS, \tag{121}$$

where we use the common notation $dv_M = \mu$ and $dS = \tilde{\mu}$ for the volume forms of M and ∂M respectively.

Comparing (93), (96) and (117) we see that

$$\Delta_g f = \mathrm{div}(\mathrm{grad} f); \tag{122}$$

the divergence identity applied with $X = \mathrm{grad} f$ gives

$$\int_M \Delta_g f\, dv = \oint_{\partial M} D_n f\, dS, \tag{123}$$

where $D_n f$ is the outer normal derivative of f. Combining (121) and (123) gives the Green's identities for two functions $\phi, \psi \in C^\infty(M)$,

$$\begin{aligned} \int_M \psi \Delta_g \phi\, dv &= -\int_M g(\mathrm{grad}\phi, \mathrm{grad}\psi)\, dv + \oint_{\partial M} \psi\, D_n \phi\, dS, & (124) \\ \int_M (\psi \Delta_g \phi - \phi \Delta_g \psi)\, dv &= \oint_{\partial M} (\psi D_n \phi - \phi D_n \psi)\, dS. & (125) \end{aligned}$$

These identities enable us to integrate by parts over a manifold, and turn out to be very useful in studying the functions and operators associated with M. For example, we have

Corollary 5 *Suppose M is a connected compact oriented manifold without boundary. The only harmonic functions (ie. satisfying $\Delta_g \phi = 0$) on M are constants.*

Proof : Applying (124) with $\phi = \psi$ gives

$$\int_M \phi \Delta_g \phi \, dv + \int_M g(\text{grad}\phi, \text{grad}\phi) \, dv = \oint_{\partial M} \phi \, D_n \phi \, dS.$$

Since $\partial M = \emptyset$ and $\Delta_g \phi = 0$ this gives

$$\int_M g(\text{grad}\phi, \text{grad}\phi) \, dv = 0.$$

Since g is positive definite, it follows that $\text{grad}\phi = 0$ identically, hence ϕ is constant.
■

In a similar fashion we may prove uniqueness for solutions $u \in C^2(M)$ of the Dirichlet problem

$$\left.\begin{array}{rcll} \Delta_g u & = & f & \text{in } M \\ u & = & \phi & \text{on } \partial M \end{array}\right\} \tag{126}$$

where $f \in C^0(M)$ and $\phi \in C^0(\partial M)$ are given. For, suppose u_1, u_2 are two solutions of (126), then the difference $w = u_1 - u_2$ satisfies the Dirichlet problem with $f = 0$ and zero boundary conditions. The previous argument carries over and shows that w is constant, hence $w = 0$ identically by the zero boundary conditions.