A LIMIT THEOREM FOR TESTING WITH RANDOMLY CENSORED DATA

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1. Introduction

Let $X_1,\ldots,X_n$ be independent identically distributed random variables (r.v.'s) and $Y_1,\ldots,Y_n$ be independent r.v.'s., independent of $X_1,\ldots,X_n$. Let

$$F(x) = P(X_i > x), \; x \geq 0$$

$$G_i(y) = P(Y_i > y), \; y \geq 0, \; 1 \leq i \leq n \; .$$

Let

$$\delta_i = [X_i < Y_i], \; Z_i = \min(X_i, Y_i), \; 1 \leq i \leq n \; .$$

Here, [A] denotes indicator of event A. The X_i's are true survival times, the Y_i's are censoring times and one observes $\{(\delta_i, Z_i), 1 \leq i \leq n\}$. This is the so-called random censoring model where often one is interested in making inferences about F or about some function of F based on $\{(\delta_i, Z_i), 1 \leq i \leq n\}$. In order to describe the specific problems to be considered here we need the following definitions. In all of these definitions $F(0) = 1$.

DEFINITION 1:

A life distribution $1-F$ is said to be New Better Than Used (NBU) if and only if

(1) $$F(x+y) \leq F(x)\, F(y),\quad x,y \geq 0 \quad .$$

DEFINITION 2:

A life distribution $1-F$ with $0<\mu<\infty$ is said to be Decreasing Mean Residual Life (DMRL) if and only if

(2) $$J(s)\, F(t) \geq J(t)\, F(s),\quad 0 \leq s \leq t < \infty ,$$

where

$$J(s) = \int_s^\infty F(t)dt \quad .$$

DEFINITION 3:

A life distribution $1-F$ with mean $0<\mu<\infty$ is said to be New Better Than Used in Expectation (NBUE) if, and only if

(3) $$J(t) \leq \mu\, F(t),\quad t \geq 0 \quad .$$

It is clear that DMRL $\subset$ NBUE. An NBU F with $0 < \mu < \infty$ is also NBUE. An equality obtains in (1) if and only if F is an exponential whereas equality is obtained in (2) and (3) only by an exponential distribution among all continuous F's with $0 < \mu < \infty$.

The probabilistic aspects of the above classes of life distributions have been extensively studied by Bryson and Siddiqui (1969), Bryson (1974), Marshall and Proschan (1972), Barlow and Proschan (1975), among others.

It is of interest, as discussed in Hollander and Proschan (H-P) (1972,1975), Koul (1977,1978a,b) and Koul and Susarla (1980), to test

$$H_0: F(x) = e^{-\lambda x}, \ x \geq 0, \ \lambda > 0 \text{ unknown}$$

against the alternatives

$$H_1: F \text{ is NBU, not an exponential}$$

or

$$H_2: F \text{ is DMRL, not an exponential}$$

or

$$H_3: F \text{ is NBUE, not an exponential} \quad .$$

The papers of Hollander and Proschan and Koul discuss some tests of H_0 vs H_1,H_2 and H_3 when the observations are not censored. The paper of Koul and Susarla discusses a test of H_0 vs H_3 and that of Chen, Hollander and Langberg (1980) discusses a test of H_0 vs H_2 for randomly censored data.

In this paper we present two tests of H_0 vs H_1 and a new test of H_0 vs H_2 for randomly censored data. Besides these the paper contains a limit theorem which is useful in deriving the asymptotic distribution, under H_0 and under alternatives, of the test statistics for the above problems. The theorem partly unifies the proofs of the asymptotic normality of these statistics under random censoring and it also has applications to other problems, such as the estimation of moments of F.

Section 2 contains the main theorem, the tests of H_0 vs H_1 and H_2 based on $\{(\delta_i, Z_i),\ 1 \leq i \leq n\}$, and theorems stating their asymptotic normality along with some proofs. Section 3 has a discussion about the asymptotic Pitman efficiency of some of the tests. Section 4 contains the proof of the main theorem.

NOTATION. The symbols $\sum$ and Π stand, respectively, for the summation and product over the indices $1 \leq i \leq n$. For any function g and set A, $\int_A g$ denotes $\int_A g(x)dx$. All limits are taken as $n \to \infty$. $\bar{G} = n^{-1} \sum G_i$. By $o(1)(o_p(1))$ is meant a sequence of numbers (r.v.'s) that converges to 0 (in probability). z_t denotes the t^{th} percentile of N(0,1) distribution. For any function H, H^{-1} will stand for 1/H. The symbol := stands for "by definition".

2. The Main Theorem and Test Statistics

Let

$$\hat{F}(t) = \frac{1}{1+n} N(t) \cdot \Pi \left\{ \frac{1+N(Z_i)}{2+N(Z_i)} \right\}^{[\delta_i = 0,\ Z_i \leq t]}, \quad t \geq 0 \tag{4}$$

denote a modified product limit estimator of F, where

$$N(t) = \sum [Z_i > t] , \quad t \geq 0 \quad .$$

Let $\{h_n\}$ be a sequence of non-random functions on $(0,\infty)$ and $\{t_n\}$ be a sequence of positive real numbers, $t_n \uparrow \infty$.

THE MAIN THEOREM:

Let $\{F_n\}$ be a sequence of survival functions and $G_1,\ldots,G_n$ the censoring survival functions. Assume that the following conditions hold:

(C1) $\quad n^{-1/2}(\ln n)^2 \int_0^{t_n} |h_n(x)| \{\bar{G}(x)\}^{-1} \left(-\int_0^x F_n(H)^{-4}\, d\bar{G}\right)dx = o(1)$,

(C2) $\quad \limsup \sigma_n^2 < \infty$, where $H = F_n \bar{G}$ and

$$\sigma_n^2 = -\int_0^{t_n} F_n^{-2}(x) \, \{\bar{G}(x)\}^{-1} \left(\int_x^{t_n} F_n h_n \right)^2 dF_n(x) \quad . \tag{5}$$

Then

$$n^{1/2} \sigma_n^{-1} \int_0^{t_n} (\hat{F} - F_n) \, h_n \to_d N(0,1) \quad . \tag{6}$$

Typically $t_n = c(\ell n\, n)^a$, $c > 0$, $0 \leq a < 1$ satisfies (C1) and (C2) for a large class of $\{F_n\}$ and $\{G_i\}$. The proof of this theorem is sketched in Section 4. In this section we now present some important applications of this theorem to the testing problems mentioned in Section 1. First consider the problem of testing:

(a) H_0 versus H_1. Two measures of departure of H_1 from H_0, for a given F, are

$$\Delta_1(F) := \int_0^\infty \int_0^\infty D(s,t) \, dsdt = \int_0^\infty s \, F(s) - \left(\int_0^\infty F\right)^2$$

and

$$\Delta_2(F) := \int_0^\infty \int_0^\infty D(s,t) \, dF(s) \, dF(t) = \int_0^\infty \int_0^\infty F(s+t) \, dF(s) \, dF(t) - 1/4$$

where

$$D(s,t) := F(s+t) - F(s)\,F(t), \quad s,t \geq 0 \quad .$$

The measure Δ_2 was considered by H-P (1972) in the case of no censoring. For some other measures see Koul (1978a,b). Observe that $\Delta_j(F) = 0$, $j = 1,2$ if F is in H_0 and $\Delta_j(F) < 0$, $j = 1,2$ if F is continuous in H_1. The smaller $\Delta_j(F)$ is for a given F, the more there is evidence in favor of F in H_1, $j = 1,2$. Therefore, it is natural to base tests of H_0 vs H_1 on $\Delta_j(\hat{F})$, $j = 1,2$, where $\hat{F}$

is given by (4). Because of the bad tail behavior of $\hat{F}$, we instead consider $\Delta_j(\hat{F},M_n)$, where

$$\Delta_1(F,M_n) = \int_0^{M_n}\int_0^{M_n} D(s,t)\, dsdt ,$$

and

$$\Delta_2(F,M_n) = \int_0^{M_n}\int_0^{M_n} D(s,t)\, dF(s)\, dF(t) ,$$

and where $M_n \uparrow \infty$.

The test j rejects for small values of $\Delta_j(\hat{F},M_n)$, $j = 1,2$. The following theorem gives the asymptotic distribution of $\Delta_j(\hat{F},M_n)$, $j = 1,2$ for a sequence $\{F_n\}$ of survival distributions in $H_0 \cup H_1$ and for non-identically distributed censoring r.v.'s. Let $\mu_n = \int_0^\infty F_n$, $\gamma_n = \int_0^\infty sF_n(s)\, ds$. Note that now $X_1,X_2,\ldots,X_n$ are i.i.d. F_n, $n \geq 1$.

THEOREM 2:

(a) Let

$$h_{n1}(x) = (x - 2\mu_n)\,[0 < x < M_n] + (2M_n - x)\,[M_n \leq x < 2M_n] .$$

Assume that $\{F_n\}$ in $H_0 \cup H_1$, and $G_1,\ldots,G_n$ satisfy (C1) and (C2) with $h_n = h_{n1}$ and $t_n = 2M_n$. Also assume that

$$\limsup \mu_n < \infty \quad , \tag{7}$$

$$\limsup \gamma_n < \infty \quad . \tag{8}$$

Then

$$n^{1/2}\sigma_{n1}^{-1}\{\Delta_1(\hat{F},M_n) - \Delta_1(F_n,M_n)\} = n^{1/2}\sigma_{n1}^{-1}\int_0^{M_n}(\hat{F} - F_n)\, h_{n1} + o_p(1) \tag{9}$$
$$\to_d N(0,1) ,$$

where σ_{n1} is the σ_n of (5) with h_n replaced by h_{n1}.

(b) Assume that $\{F_n\}$ in $H_0 \cup H_1$ have densities $\{f_n\}$ and set

$$h_{n2}(x) = [0 < x < M_n] \int_0^x f_n(x-t)\, f_n(t)\, dt + [M_n \le x < 2M_n] \int_{M_n - x}^{M_n} f_n(x-t)\, f_n(t) dt - 2 \int_0^{M_n} f_n(x+t)\, f_n(t) dt \; .$$

Assume that (C1) and (C2) are satisfied by $\{F_n\}$, $G_1,\ldots,G_n$, and h_n replaced by h_{n2}. Then

$$\text{(10)} \qquad n^{1/2}\, \sigma_{n2}^{-1}\, \{\Delta_2(\hat{F},M_n) - \Delta_2(F_n,M_n)\} = n^{1/2}\, \sigma_{n2}^{-1} \int_0^{M_n} (\hat{F} - F_n)\, h_{n2} + o_p(1)$$

$$\to_d N(0,1) \quad ,$$

where σ_{n2} is the σ_n of (5) with $h_n = h_{n2}$.

Outline of Proof: Due to the limited space, we only sketch a proof of (9). The details for (10) are similar in nature. Write M for M_n and observe that

$$n^{1/2}(\Delta_1(\hat{F},M) - \Delta_1(F_n,M)) = n^{1/2} \int_0^M \int_0^M \{\hat{F}(s+t) - F_n(s+t)\}\, ds\, dt$$

$$- n^{1/2} \{ (\int_0^M \hat{F})^2 - (\int_0^M F_n)^2\} = A_n - B_n, \text{ say} \quad .$$

One can check, using (7), that if (C1) and (C2) hold with $h_n = h_{n1}$, then they also hold with $h_n = 1$. Therefore, by (6)

$$\int_0^{M_n} (\hat{F} - F_n) = o_p(1) \quad .$$

This, in turn, implies that

$$(11)\qquad B_n = n^{1/2}\int_0^M (\hat{F}-F_n)\cdot\{\int_0^M (\hat{F}+F_n)\} = 2\int_0^M F_n \cdot n^{1/2}\int_0^M (\hat{F}-F_n) + o_p(1)\cdot$$

$$= 2\,\mu_n\, n^{1/2}\int_0^M (\hat{F}-F_n) + o_p(1) \quad .$$

Direct integration yields that

$$(12)\qquad A_n = n^{1/2}\int_0^{2M}\{u[0<u<M] + (2M-u)\;[M\leq u<2M]\}\;(\hat{F}(u) - F_n(u))\,du \quad .$$

Therefore, (12) and (11) yield the equality of (9), whereas the convergence in distribution to N(0,1) follows from the Main Theorem.

REMARK 1. Under H_0, $\mu_n = \lambda^{-1}$ and $h_{n1}(x) \to (x-2/\lambda)$.

Actually, if

$$(13)\qquad \limsup \int_0^\infty x^2\, e^{-x}\,\{\bar{G}(x/\lambda)\}^{-1}\,dx < \infty \,,$$

then one can show that

$$(14)\qquad \sigma_{n1}^{\;2} = \lambda^{-4}\int_0^\infty (x-1)^2\, e^{-x}\,\{\bar{G}(x/\lambda)\}^{-1}\,dx + o(1) \quad .$$

Also, if

$$(15)\qquad \limsup \int_0^\infty y^2\, e^{-3y}\,\{\bar{G}(y/\lambda)\}^{-1}\,dy < \infty \,,$$

then

$$(16) \qquad \sigma_{n2}^2 = 4^{-1} \int_0^\infty (y - 1/2)^2 \, e^{-3y} \, \{\bar{G}(y/\lambda)\}^{-1} \, dy + o(1) \ .$$

Thus (13) and (15) imply (C2), respectively, for h_{n1} and h_{n2} under H_0. A sufficient condition for (C1) to hold for both, h_{n1} and h_{n2}, under H_0 is that

$$(17) \qquad n^{-1/2} \, (\ell n \, n)^2 \, M_n^2 \, e^{3M_n\lambda} \, [\{\bar{G}(M_n)\}^{-3} - 1] = o(1) \ .$$

From (14) and (16) it is clear that the asymptotic null distribution of the proposed tests depend on λ and $\bar{G}$. To implement the tests we estimate λ by

$$\hat{\lambda} = \sum \delta_i / \sum Z_i$$

and $\bar{G}$ by

$$\hat{\bar{G}}(t) = \prod_{j=1}^{n} \left\{ \frac{1+N(Z_j)}{2+N(Z_j)} \right\}^{[\delta_j = 0, \ Z_j \le t]} , \ t \ge 0 \ .$$

It is easy to check that $\hat{\lambda}$ is a consistent estimator of λ under H_0 as long as

$$0 < \liminf \lambda \int_0^\infty \bar{G}(t) \, e^{-\lambda t} \, dt \ \le \limsup \lambda \int \bar{G}(t) \, e^{-\lambda t} \, dt < 1 \ .$$

That $\hat{\bar{G}}$ is a consistent estimator of $\bar{G}$, under H_0, can be deduced from Koul, Susarla and Van Ryzin (1981).

Let

$$\hat{\sigma}_{n1}^2 = \hat{\lambda}^{-4} \int_0^{N_n} (x - 1)^2 \, e^{-x} \, \{\hat{\bar{G}}(x/\hat{\lambda})\}^{-1} \, dx \ ,$$

$$\hat{\sigma}_{n2}^2 = 4^{-1} \int_0^{N_n} (y - 1/2)^2 \, e^{-3y} \, \{\hat{\bar{G}}(y/\hat{\lambda})\}^{-1} \, dy \ ,$$

where $N_n \uparrow \infty$.

Under (13), (15) and (17), with M_n replaced by N_n, one can show that $\hat{\sigma}_{nj}^2 = \sigma_{nj}^2 + o_p(1)$, $j=1,2$. Consequently the test that rejects H_0 when $\Delta_j(\hat{F},M_n) \leq z_\delta \hat{\sigma}_{nj}/n^{1/2}$ has the asymptotic size δ, $j=1,2$. Next, consider

(b) H_0 vs H_2. Two reasonable measures of the deviation of H_2 from H_0, for a given F, are

$$\Delta_3(F) = \int\int [0 < s \leq t < \infty] \; E(s,t) \; ds\, dt$$

and

$$\Delta_4(F) = \int\int [0 < s \leq t < \infty] \; E(s,t) \; dF(s) \; dF(t) = \int (3F^2 - F - 2F^4)/6 \; ,$$

where

$$E(s,t) = F(t)\; J(s) - F(s)\; J(t), \; 0 < s \leq t < \infty \; .$$

Let $\Delta_3(F,M) = \int\int [0 < s \leq t \leq M] \; E(s,t) ds\, dt$ and define $\Delta_4(F,M)$, similarly. The test j rejects H_0 in favor of H_2 if $\Delta_j(\hat{F},M_n)$ is large, $j=3,4$. The following theorem gives the asymptotic normality of these test statistics for a sequence $\{F_n\}$ in $H_0 \cup H_2$ and for non-identically distributed censoring variables. Note that a variant of the Δ_4-test was suggested by Chen, Hollander and Langberg (1980) but they do not discuss the asymptotic distribution under sequences of alternatives.

THEOREM 3:

(a) Let $\{F_n\}$ be in $H_0 \cup H_2$. For $s \leq M_n$,

$$h_{n3}(s) := 2\int_0^{M_n} \min(s,x) \; F_n(x) dx - \int_0^{M_n} (x-s) \; F_n(x) dx - \int_0^s (s-x) F_n(x) dx.$$

Assume $\{F_n\}$, $G_1,,,.G_n$ and h_{n3} satisfy (C1) and (C2). Also, assume that (8) holds.

Then

$$n^{1/2}\,\sigma_{n3}^{-1}\,(\Delta_3(\hat{F},M_n) - \Delta_3(F_n,M_n)) = n^{1/2}\,\sigma_{n3}^{-1}\int_0^{M_n} (\hat{F}-F_n)\,h_{n3} + o_p(1)$$

$$\to_d N(0,1) \quad ,$$

where σ_{n3} is the σ_n of (5) with $h_n = h_{n3}$.

(b) Let

$$h_{n4} := (6F_n - 1 - 8F_n^{\,3})/6 \text{ on } [0,M_n] \ .$$

Assume $\{F_n\}$, h_{n4} and $\{G_i\}$ satisfy (C1) and (C2). Also, assume that (7) holds. Then

$$n^{1/2}\,\sigma_{n4}^{-1}\,(\Delta_4(\hat{F},M_n) - \Delta_4(F_n,M_n)) = n^{1/2}\,\sigma_{n4}^{-1}\int_0^{M_n} (\hat{F}-F_n)\,h_{n4} + o_p(1)$$

$$\to_d N(0,1) \quad ,$$

where σ_{n4} is the σ_n of (5) with $h_n = h_{n4}$.

REMARK 2. As in Remark 1, it can be shown that under (13) and under H_0 with a fixed λ,

$$\sigma_{n3}^2 = \lambda^{-6}\int_0^{\infty} e^{-t}\,(2 - 2e^{-t} - t)^2\,\{\bar{G}(t/\lambda)\}^{-1}\,dt + o(1)$$

and that if

$$\limsup \int_0^{\infty} e^{-\lambda}\,\{\bar{G}(s/\lambda)\}^{-1}\,ds < \infty \ ,$$

then

$$\sigma_{n4}^2 = (36\lambda^2)^{-1}\int_0^{\infty} e^{-s}\,(3e^{-s} - 1 - 2e^{-3s})^2\,\{\bar{G}(s)/\lambda)\}^{-1}\,ds + o(1) \ .$$

Moreover, a sufficient condition for (C1) to hold under H_0, for both h_{n3} and h_{n4}, is (17).

Let

$$\hat{\sigma}_{n3}^{2} = (\hat{\lambda})^{-6} \int_0^{N_n} e^{-t} (2-2e^{-t}-t)^2 \{\hat{\bar{G}}(t/\hat{\lambda})\}^{-1} dt \quad ,$$

$$\hat{\sigma}_{n4}^{2} = (36\hat{\lambda}^2)^{-1} \int_0^{N_n} e^{-t} (3e^{-t}-1-ze^{-3t})^2 \{\hat{\bar{G}}(t/\hat{\lambda})\}^{-1} dt \ .$$

Then the test that rejects H_0 when $\Delta_j(\hat{F},M_n) \geq z_{1-\alpha}\, \hat{\sigma}_{nj}/n^{1/2}$ has the asymptotic size α, $j=3,4$. Both of these tests are consistent against a fixed F in H_2 and for all those censoring distributions for which (17) holds and an analogue of (C2) holds for h_{n3} and h_{n4} at the given F.

3. Asymptotic Efficiency

Consider the problem of testing H_0 vs a sequence of alternatives $\{F_{\theta_n}\} \varepsilon H_1$ when there is no censoring. In this case one can base tests on $\tilde{\Delta}_j = \Delta_j(\tilde{F})$, $j=1,2$, where $\tilde{F}(x) = n^{-1} \sum[X_i > x]$, $x \geq 0$. Observe that $\tilde{\Delta}_1 = (2n)^{-1} \sum X_i^2 - \bar{X}^2$. The $\tilde{\Delta}_2$ is a priori scale invariant while a scale invariant analogue of $\tilde{\Delta}_1$ is

$$\Delta_1^* = \tilde{\Delta}_1 / \bar{X}^2 \quad .$$

Note that an analogue of Δ_1^* under random censoring is $\hat{\lambda}^2\Delta_1(\hat{F},M_n)$. We did not consider this statistic in the previous section because its asymptotic null distribution still depends on λ as does that of $\Delta_1(\hat{F},M_n)/(\int_0^{M_n} \hat{F})^2$.

Using the standard central limit theorem one has

$$\text{(18)} \qquad n^{1/2} (\Delta_1^* - \mu_n^{-2} \Delta_1(F_{\theta_n})) \to N(0,1) \ , \ (\mu_n = \int_0^\infty F_{\theta_n}) \quad ,$$

for all $\{F_{\theta_n}\} \varepsilon H_1$ which are contiguous to F_{θ_0}. Note that implicit in (18) is the assumption that $\Delta_1(F_{\theta_n}) < \infty$ for all n which amounts to assuming the finiteness of the second moments (see (8)) whereas no such assumption is needed for $\tilde{\Delta}_2$-test.

Now if $\dot{\Delta}_j(\theta) = \partial\Delta_j(F_\theta)/\partial\theta$, $j = 1,2$, then it follows that the asymptotic relative Pitman efficiency of Δ_1^*-test relative to the $\tilde{\Delta}_2$-test is

$$e(1,2) = \frac{5}{432}\{\dot{\Delta}_1(\theta_0)/\dot{\Delta}_2(\theta_0)\}^2 .$$

Consider the alternatives: (a1). $F_{\theta_n}(x) = e^{-x-x^2\theta_n/2}$, $\theta_n = \delta_n^{-1/2}$, $\delta > 0$, $x \geq 0$. Then $\theta_0 = 0$ and $\dot{\Delta}_1(0) = 1$, $\dot{\Delta}_2(0) = 1/16$ and $e(1,2) = (5x256)/432 = 2.96$.

(a2). If $F_{\theta_n}(x) = \exp(-x^{\theta_n})$, $\theta_n = 1 + \delta n^{-1/2}$, $\delta \geq 0$, then $\theta_0 = 1$ and $\dot{\Delta}_1(0) = 1$, $\dot{\Delta}_2(1) = 1/8$ and $e(1,2) = .74$.

Now suppose there is random censoring with $\bar{G}(x) \equiv e^{-\theta x}$, $\theta < \lambda$. Then from (14)

$$\sigma_{n1}^2 \to \lambda^{-4}\int (x-1)^2 e^{-\alpha x}dx \qquad (\alpha = 1-(\theta/\lambda))$$

$$= \lambda^{-4}[2\alpha^{-3} - 2\alpha^{-2} + \alpha^{-1}]$$

$$= (1+r^2)/\lambda^4\alpha^3 = \sigma_1^2, \text{ (say)} . \qquad (r = \theta/\lambda)$$

Also, from (16)

$$\sigma_{n2}^2 \to 4^{-1}\int (x-1/2)^2 e^{-(3-r)x} dx \qquad (\beta = 3-r)$$

$$= 4^{-1}[2\beta^{-3} - \beta^{-2} + 4^{-1}\beta^{-1}]$$

$$= (5-2r+r^2)/16\beta^3 = \sigma_2^2, \text{ (say)} .$$

Note that $\dot{\Delta}_j$'s do not change. Then at the alternative (a1),

$$e(1,2) = 256.\ (\sigma_2^2/\sigma_1^2) = 256.\ \frac{(5-2r+r^2)\ \lambda^4(1-r)^3}{16(3-r)^3\ (1+r^2)}$$

$$\to 2.96 \qquad \text{as } r \to 0 \quad (\text{i.e., } \theta \to 0)$$

$$\to 0 \qquad \text{as } r \to 1 \quad (\text{i.e., } \theta \to \lambda) \quad .$$

Thus, for example, if censoring distributions are almost like the exponential (λ) distributions, then Δ_2-test would be preferred.

In general, if $\bar{G}$, the average of censoring distributions, has lighter right tail compared to the exponential tails, we suggest using the test based on $\Delta_1(\hat{F},M_n)$ with $M_n = c(\ell n\, n)^a$, $c > 0$, $0 < a < 1$.

4. Proof of the Main Theorem

The technical details of the proof are similar to those in Section 7 of Koul, Susarla and Van Ryzin (1981). We provide only a sketch of the proof here. Write $\hat{F} = \hat{H}\hat{W}$ where $(n+1)\ \hat{H} = 1+N$ and $\hat{W} = \hat{\bar{G}}^{-1}$, the second factor in (4). Write M for M_n. Observe that

$$\hat{F} - F_n = \bar{G}^{-1}\ (\hat{H} - H) + \hat{H}\ (\hat{W} - \bar{G}^{-1}),\quad H = \bar{G}F_n \ .$$

Hence,

$$n^{1/2} \int_0^M (\hat{F} - F_n)h_n = n^{1/2}\ [\int_0^M \hat{H}(\hat{W} - \bar{G}^{-1})h_n + \int_0^M \bar{G}^{-1}\ (\hat{H} - H)h_n]$$

$$= I + II,\ (\text{say}) \quad .$$

The term II is a sum of centered independent r.v's. We only need to approximate I by a sum of independent r.v's. To this end we write $W = \exp(\ell n\ W)$, $\bar{G}^{-1} = \exp(-\ell n\ \bar{G})$, and use a Taylor expansion to obtain

$$(19)\qquad |(\hat{W}-\bar{G}^{-1}) - \bar{G}^{-1}(\ell n\ \hat{W} + \ell n\ \bar{G})\ | \leq 2\bar{G}^{-1}(\ell n\ \hat{W} + \ell n\ \bar{G})^{2} .$$

From the details similar to those in Section 7 of Koul, Susarla and Van Ryzin (use Lemma 7.1 with $p_i = F_n G_i$ and the details similar to those in the proof of Lemma 7.2), one obtains

$$E(\text{RHS (19)}) \leq -k_1\ n^{-1}\ \bar{G}^{-1} \int_0^{\cdot} F_n\ H^{-4}\ d\bar{G}, \quad (\text{for some constant } k_1) .$$

Therefore,

$$I = n^{1/2} \int_0^M h_n\ \hat{H}\ \bar{G}^{-1}\ (\ell n\ \hat{W} + \ell n\ \bar{G}) + o_p(1).$$

provided $n^{1/2} \int_0^M |h_n(x)|\bar{G}^{-1}(x)(\int_0^x F_n\ H^{-4}\ d\bar{G})dx = o(1)$, which in turn is implied by (C1). The next step is to approximate $\ell n\ \hat{W}$. Again, carrying out details similar to those in Koul, Susarla and Van Ryzin, one obtains

$$(20)\qquad I = \int_0^M F_n h_n\ n^{1/2}\ \{ \int_0^x (2H - H_n)H^{-2}\ dH_n^* + \ell n\ \bar{G}(x)\} + o_p(1) ,$$

where

$$H_n = n^{-1}N,\ nH_n^*(\cdot) = \sum(1-\delta_i)\ [Z_i \leq \ \cdot] .$$

The first r.v. on the right-hand side of (20) can be expressed as a U-statistic and, hence, by the projection technique, one can show that, under (C1),

$$(21)\qquad I = \int_0^M h_n(x)\ F_n(x)\ n^{-1/2} \sum_i \{[(1-\delta_i)][Z_i \leq x]\ H^{-1}(Z_i) - \int^{x\wedge Z_i} H^{-2}d\bar{G}$$

$$+ \ell n\ \bar{G}(x)\}dx + o_p(1) .$$

Combining (21) with (19), the final approximating r.v. is the sum of II and the first r.v. on the right hand side of (21). Its variance is σ_n^2 and (C2) implies the asymptotic normality (6) by the Lindeberg-Feller CLT.

ACKNOWLEDGEMENT

The research of author (VS) is supported by the National Institutes of Health Grant No. 1-RO-1-GM-28405. This author is on leave from Suny, Binghamton, New York.

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