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RIEMANNIAN CURVATURES OF THE FOUR BASIC CLASSES OF REAL HYPERSURFACES OF A COMPLEX SPACE FORM

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Abstract. Any real hypersurface of a Kähler manifold carries a natural almost contact metric structure. There are four basic classes of real hypersurfaces of a Kähler manifold with respect to the induced almost contact metric structure. In this paper we study the basic classes of real hypersurfaces of a complex space form in terms of their Riemannian curvatures.

1. Introduction

Let $\bar{M}^{2n+2}(J,G)$ be an almost Hermitian manifold with almost complex structure J and Riemmannian metric G: $J^2 = -\operatorname{Id}$, $G(J\bar{X}, J\bar{Y}) = G(\bar{X},\bar{Y})$, $\bar{X},\bar{Y} \in \mathfrak{X}\bar{M}^{2n+2}$.

If M^{2n+1} is a hypersurface in $\bar{M}^{2n+2}$ with a unit normal vector field N, then there arises naturally an almost contact metric structure (φ,ξ,η,g) on M^{2n+1} in the following way [3, 10, 12]:

$$\xi = -JN, \quad g = G_{|M}, \quad \varphi = J - \eta\otimes N\,,$$
$$\eta(X) = g(\xi, X), \quad X \in \mathfrak{X}M^{2n+1}\,.$$

Let ∇ and ∇' be the Levi-Civita connections on M^{2n+1} and $\bar{M}^{2n+2}$, respectively. We denote by Φ the fundamental 2-form of the structure (φ,ξ,η,g)

$$\Phi(X,Y) = g(X,\varphi Y)\,, \qquad X,Y \in \mathfrak{X}M^{2n+1}$$

and by $F' = -\nabla'\Phi, F = \nabla\Phi$. If A is the shape operator and $h(X,Y) = g(AX,Y)$, $\quad X,Y \in \mathfrak{X}M^{2n+1}$ is the second fundamental tensor of M^{2n+1}, then the Gauss and Weingarten formulas

$$\nabla'_X Y = \nabla_X Y + h(X,Y)N\,, \qquad \nabla'_X N = -AX\,, \quad X,Y \in \mathfrak{X}M^{2n+1}$$

imply immediately

$$\begin{gathered}(\nabla'_X \eta)Y = (\nabla_X \eta)Y\,,\\ (\nabla'_X J)Y = (\nabla_X \varphi)Y + \{h(X,\varphi Y) + (\nabla_X \eta)Y\}N - \eta(Y)AX + h(X,Y)\xi\,,\\ F'(X,Y,Z) = F(X,Y,Z) + \eta(Z)h(X,Y) - \eta(Y)h(X,Z)\,.\end{gathered}$$

For any point $p \in M$ we have

$$\begin{aligned} T_p\bar{M} &= T_pM \oplus N_p\,, \qquad & N_p &\perp T_pM\,,\\ T_pM &= D_p \oplus \xi_p\,, \qquad & \xi_p &\perp D_p\,. \end{aligned}$$

$D = \{D_p, p \in M\}$ is the *contact distribution*, and $\{\xi_p;\, p \in M\}$ is the *vertical distribution* of M. There exists a second shape operator $\mathcal{A}$, acting in M. The actions of the shape operators A and $\mathcal{A}$ are as follows:

$$A: \begin{array}{l} T_p\bar{M} \to T_pM \\ X \mapsto AX = -\nabla'_X N\,, \end{array} \qquad \mathcal{A}: \begin{array}{l} T_pM \to D_p \\ X \mapsto \mathcal{A}X = \nabla_X \xi\,. \end{array}$$

If $\bar{M}^{2n+2}(J,G)$ is Kählerian ($\nabla' J = 0$), then the following formulas are an immediate consequence from the above formulas [12]:

$$\begin{gathered}(\nabla_X \varphi)Y = \eta(Y)AX - h(X,Y)\xi\,,\\ (\nabla_X \eta)Y = F(X,\varphi Y,\xi) = -h(X,\varphi Y)\,,\\ AX = -\varphi\mathcal{A}X + h(X,\xi)\xi, \qquad \varphi AX = \mathcal{A}X\,,\end{gathered}$$

$$\begin{aligned} 2\,\mathrm{d}\eta(X,Y) &= h(Y,\varphi X) - h(X,\varphi Y)\\ &= 2\,\mathrm{d}\eta(\varphi X,\varphi Y) + \eta(X)h(\varphi Y,\xi) - \eta(Y)h(\varphi X,\xi)\,, \end{aligned} \tag{1}$$

$$\begin{aligned} (\mathcal{L}_\xi g)(X,Y) &= -\{h(Y,\varphi X) + h(X,\varphi Y)\}\\ &= -(\mathcal{L}_\xi g)(\varphi X,\varphi Y) - \eta(X)h(\varphi Y,\xi) - \eta(Y)h(\varphi X,\xi), \end{aligned} \tag{2}$$

$$\mathrm{d}\Phi = 0\,,$$

$$\begin{aligned} N(X,Y) &= [\varphi,\varphi](X,Y) + 2\,\mathrm{d}\eta(X,Y)\xi\\ &= \eta(X)(\varphi AY - A\varphi Y) - \eta(Y)(\varphi AX - A\varphi X)\,,\\ N(X,Y,Z) &= -\eta(X)\{h(Y,\varphi Z) + h(Z,\varphi Y)\}\\ &\quad + \eta(Y)\{h(X,\varphi Z) + h(Z,\varphi X)\}\,, \end{aligned}$$

$$F(X,Y,Z) = \eta(Y)h(X,Z) - \eta(Z)h(X,Y)\,, \tag{3}$$

$$h(X,Y) = -F(X,Y,\xi) = -p(X,Y)\,, \tag{4}$$

$$h(\xi,Z) = F(\xi,\xi,Z) = \omega(Z) = (\mathcal{L}_\xi g)(\varphi Z,\xi) = -2\,\mathrm{d}\eta(\varphi Z,\xi)\,.$$

The function $\nu = h(\xi, \xi)$ cannot be expressed by F.

If, moreover, $\bar{M}^{2n+2}(c)$ is a complex space form, i. e. Kähler manifold with constant holomorphic sectional curvature c, then the equations of Gauss and Codazzi are [2]

$$
\begin{aligned}
R(X,Y,Z,U) = \Big\{ \frac{c}{4}[g(Y,*) \wedge g(X,*) + \Phi(Y,*) \wedge \Phi(X,*) \\
- 2\Phi(X,Y).\Phi] + h(Y,*) \wedge h(X,*) \Big\}(Z,U)\,, \\
g((\nabla_X A)Y - (\nabla_Y A)X, Z) = (\nabla_X h)(Y,Z) - (\nabla_Y h)(X,Z) \\
= \frac{c}{4}[\eta \wedge \Phi(Z,*)](X,Y) + 2\eta(Z)\Phi(X,Y)\,.
\end{aligned}
$$

Alexiev and Ganchev introduced in [1] twelve basic classes W_i, $i = 1, \dots, 12$, of almost contact metric manifolds with respect to the symmetries of the tensor field F. By using this classification Ganchev and Hristov obtained in [6] the sixteen classes of real hypersurfaces of a Kähler manifold and described them in terms of their second fundamental tensor.

Theorem 1.1. ([6]) *Let $\bar{M}^{2n+2}(J, G), (n \geq 2)$ be a Kähler manifold. Any real hypersurface $M^{2n+1}(\varphi, \xi, \eta, g)$ of $\bar{M}^{2n+2}$ is in the class*

$$W_1 \oplus W_2 \oplus W_4 \oplus W_6\,.$$

The four basic classes W_1, W_2, W_4 and W_6 generate sixteen classes of real hypersurfaces, which are characterized in terms of their second fundamental tensor h as follows:

i) W_0: $(W_0 \subset W_i,\ i = 1,2,4,6)$ $\quad h \equiv h_0$,
$h_0(X,Y) = \nu(\eta \otimes \eta)(X,Y)$;

ii) W_1: $h \equiv h_1 + h_0$, $\quad h_1(X,Y) = (\eta \otimes \omega + \omega \otimes \eta)(X,Y)$;

iii) W_2: $h \equiv h_2 + h_0$, $\quad h_2(X,Y) = -\frac{f(\xi)}{2n} g(hX, hY)$, $\quad \operatorname{Tr} h \neq 0$;

iv) W_4: $h \equiv h_4 + h_0$, $\quad h_4(X,Y) = \frac{f(\xi)}{2n} g(hX, hY) - \mathrm{d}\eta(\varphi X, hY)$,
$\operatorname{Tr} h = 0$;

v) W_6: $h \equiv h_6 + h_0$, $\quad h_6(X,Y) = \frac{1}{2}(\mathcal{L}_\xi g)(\varphi X, hY)$;

vi)-xi) $W_i \oplus W_j$: $h \equiv h_i + h_j + h_0$, $\quad i,j \in \{1,2,4,6\}, i \neq j$;

xii)-xv) $W_i \oplus W_j \oplus W_k$: $h \equiv h_i + h_j + h_k + h_0$, $\quad i,j,k \in \{1,2,4,6\}$,
$i \neq j \neq k$;

xvi) $W_1 \oplus W_2 \oplus W_4 \oplus W_6$: $h \equiv h_1 + h_2 + h_4 + h_6 + h_0$.

Remark 1.1. *If $n = 1$, then it follows that the class W_4 is empty [4]. In this case there arise 8 classes of real hypersurfaces of a 4-dimentional Kähler manifold.*

Remark 1.2. *With respect to the theory of almost contact metric manifolds, the class:*

- *W_0 is exactly the class of cosymplectic manifolds, characterized by*

$$F = 0\,, \qquad W_0 \subset W_i\,, \qquad i = 1, 2, \ldots, 12\,,$$

- *W_2 is exactly the class of α-Sasakian manifolds, characterized by*

$$N = 0\,, \qquad \mathrm{d}\eta = \alpha\Phi\,, \qquad \alpha \neq 0\,,$$

- *$W_2 \oplus W_4$ is exactly the class of quasi-Sasakian manifolds, characterized by*

$$N = \mathrm{d}\Phi = 0\,,$$

- *$W_6 \oplus W_{11}$ is exactly the class of almost cosymplectic manifolds, characterized by*

$$\mathrm{d}\eta = \mathrm{d}\Phi = 0\,,$$

- *$W_2 \oplus W_3 \oplus W_4 \oplus W_5 \oplus W_9 \oplus W_{10}$ is exactly the class of normal manifolds, characterized by $N = 0$.*

Similar questions were considered in [4, 5].

2. Preliminaries

In this section we shall use the notion of a complexification of an almost contact metric vector space and of the tangent bundle of an almost contact metric manifold. We shall consider the complex linear extension of real tensors and their essential components (which may not be zero) with respect to the standard complex coordinates (e. g. [8]).

The following results are the odd-dimensional analogues of the well known decomposition of the curvature tensor in an almost Hermitian manifold [11].

Let $V^{2n+1}(\varphi, \xi, \eta, g)$ be an almost contact metric vector space and $V^{\mathrm{c}} = D^{\mathrm{c}} \oplus \{\xi\} = D^{1,0} \oplus D^{0,1} \oplus \{\xi\}$ be its complexification. $D^{1,0}$ and $D^{0,1}$ are $(+\mathrm{i})$- and $(-\mathrm{i})$-eigenspaces of φ. $D^{1,0} = \mathrm{span}\{Z_\alpha = e_\alpha - \mathrm{i}\varphi e_\alpha\}$, $D^{0,1} = \mathrm{span}\{Z_{\overline{\alpha}} = e_\alpha + \mathrm{i}\varphi e_\alpha\}$, $Z_0 \equiv \xi$, where $\{e_\alpha, \varphi e_\alpha, \xi, \alpha = 1, \ldots n\}$ is an orthonormal basis of V. $\mathcal{R}$ denotes the set of all Riemannian-like curvature tensors:

$$\mathcal{R} = \{R \in \otimes^4 V^*\,;\, R(X,Y,Z,U) = -R(Y,X,Z,U) = -R(X,Y,U,Z)\,,$$
$$\underset{(X,Y,Z)}{\sigma} R(X,Y,Z,U) = 0\}\,.$$

Further $\rho_R, \tilde{\rho}_R, \rho^*_R$ are the Ricci-type contractions associated with $R \in \mathcal{R}$ and $\tau_R, \tilde{\tau}_R, \tau^*_R$ are the corresponding scalar curvatures:

$$\rho_R(X,Y) = \sum_{i=1}^{2n+1} R(e_i, X, Y, e_i) = g(Q_R(X), Y) \qquad \tau_R = \sum_{i,j=1}^{2n+1} R(e_i, e_j, e_j, e_i)$$

$$\rho^*_R(X,Y) = \sum_{i=1}^{2n+1} R(e_i, X, Y, \varphi e_i) = g\left(Q^*_R(X), Y\right) \qquad \tau^*_R = \sum_{i,j=1}^{2n+1} R(e_i, e_j, \varphi e_j, \varphi e_i)$$

$$\tilde{\rho}_R\,(X,Y) = R(\xi, X, Y, \xi) = g\left(\tilde{Q}_R(X), Y\right) \qquad \tilde{\tau}_R = \sum_{i=1}^{2n+1} R(\xi, e_i, e_i, \xi)$$

where $\{e_i, e_{2n+1} \equiv \xi\}$ is an orthonormal basis of $T_pM, p \in M$.

Let $\mathcal{L}$ be the vector space of all (0, 2)-tensors over T_pM. In [9] it is proved that $\mathcal{L} = \bigoplus_{i=1}^{9} \mathcal{L}_i$, where $\mathcal{L}_i$ are mutually orthogonal, $U(n)\times 1$-invariant and irreducible spaces. $\mathcal{L}_i$ are described by the following symmetries for the essential components for $L \in \mathcal{L}$ with complex contraction $\mu = g^{\bar{\alpha}\beta} L_{\bar{\alpha}\beta}$:

$$\begin{aligned}
&\mathcal{L}_1 \colon L_{\alpha\bar{\beta}} = L_{\bar{\beta}\alpha} = \frac{\mu}{n} g_{\alpha\bar{\beta}}\,, \quad \mu = \bar{\mu}\,,\\
&\mathcal{L}_2 \colon L_{\alpha\bar{\beta}} = L_{\bar{\beta}\alpha}\,, \quad \mu = \bar{\mu}\,,\\
&\mathcal{L}_3 \colon L_{\alpha\beta} = L_{\beta\alpha}\\
&\mathcal{L}_4 \colon L_{\alpha\bar{\beta}} = -L_{\bar{\beta}\alpha} = \mathrm{i}\frac{\mu}{n} \Phi_{\alpha\bar{\beta}}\,, \quad \mu = -\bar{\mu}\,,\\
&\mathcal{L}_5 \colon L_{\alpha\bar{\beta}} = -L_{\bar{\beta}\alpha}\,, \quad \mu = -\bar{\mu}\,,\\
&\mathcal{L}_6 \colon L_{\alpha\beta} = -L_{\beta\alpha}\,,\\
&\mathcal{L}_7 \colon L_{\alpha 0} = L_{0\alpha}\,,\\
&\mathcal{L}_8 \colon L_{\alpha 0} = -L_{0\alpha}\,,\\
&\mathcal{L}_9 \colon L_{00}\,.
\end{aligned}$$

By using appropriate involutive isometries, commuting with the Lie group $\mathcal{U}(n)\times 1$, analogously to [11], we obtain the following

Theorem 2.1. (partial decomposition) *The space $\mathcal{R}$ is decomposable into mutually orthogonal and $\mathcal{U}(n)\times 1$-invariant factors as follows*

$$\mathcal{R} = h\mathcal{R}_1 \oplus h\mathcal{R}_1^\perp \oplus h\mathcal{R}_2^\perp \oplus h\mathcal{R}_3^\perp \oplus v\mathcal{R}_1 \oplus v\mathcal{R}_1^\perp \oplus v\mathcal{R}_2^\perp \oplus w\mathcal{R}_1 \oplus w\mathcal{R}_1^\perp\,.$$

The characteristic conditions for the factors in terms of the essential components, are

$$\begin{aligned}
h\mathcal{R} &= h\mathcal{R}_3 \oplus h\mathcal{R}_3^\perp = \{R\in\mathcal{R};\ R_{\alpha\beta\gamma\delta}, R_{\bar\alpha\beta\gamma\delta}, R_{\alpha\bar\beta\gamma\bar\delta}, R_{\alpha\beta\bar\gamma\bar\delta} - \text{essential}\},\\
h\mathcal{R}_3^\perp &= \{R\in h\mathcal{R};\ R_{\bar\alpha\beta\gamma\delta} - \text{essential}\},\\
h\mathcal{R}_3 &= h\mathcal{R}_2 \oplus h\mathcal{R}_2^\perp = \{R\in h\mathcal{R};\ R_{\bar\alpha\beta\gamma\delta}=0\},\\
h\mathcal{R}_2^\perp &= \{R\in h\mathcal{R}_3\,;\ R_{\alpha\beta\gamma\delta} - \text{essential}\},\\
h\mathcal{R}_2 &= h\mathcal{R}_1 \oplus h\mathcal{R}_1^\perp = \{R\in h\mathcal{R}_3\,;\ R_{\alpha\beta\gamma\delta}=0\},\\
h\mathcal{R}_1^\perp &= \{R\in h\mathcal{R}_2\,;\ R_{\alpha\beta\bar\gamma\bar\delta} = 2R_{\alpha\bar\gamma\beta\bar\delta} \iff R_{\alpha\bar\beta\gamma\bar\delta} = -R_{\alpha\bar\delta\gamma\bar\beta}\},\\
h\mathcal{R}_1 &= \{R\in h\mathcal{R}_2\,;\ R_{\alpha\beta\bar\gamma\bar\delta} = 0 \iff R_{\alpha\bar\beta\gamma\bar\delta} = R_{\alpha\bar\delta\gamma\bar\beta}\};
\end{aligned}$$

$$\begin{aligned}
v\mathcal{R} &= v\mathcal{R}_2 \oplus v\mathcal{R}_2^\perp = \{R\in\mathcal{R};\ R_{0\alpha\beta\gamma}, R_{0\bar\alpha\beta\gamma}, R_{0\alpha\beta\bar\gamma} - \text{essential}\},\\
v\mathcal{R}_2^\perp &= \{R\in v\mathcal{R};\ R_{0\alpha\beta\gamma} - \text{essential}\},\\
v\mathcal{R}_2 &= v\mathcal{R}_1 \oplus v\mathcal{R}_1^\perp = \{R\in v\mathcal{R};\ R_{0\alpha\beta\gamma}=0\},\\
v\mathcal{R}_1^\perp &= \{R\in v\mathcal{R}_2\,;\ R_{0\bar\alpha\beta\gamma} = 2R_{0\gamma\beta\bar\alpha} \iff R_{0\alpha\beta\bar\gamma} = -R_{0\beta\alpha\bar\gamma}\},\\
v\mathcal{R}_1 &= \{R\in v\mathcal{R}_2 z;\ R_{0\bar\alpha\beta\gamma} = 0 \iff R_{0\alpha\beta\bar\gamma} = R_{0\beta\alpha\bar\gamma}\};
\end{aligned}$$

$$\begin{aligned}
w\mathcal{R} &= w\mathcal{R}_1 \oplus w\mathcal{R}_1^\perp = \{R\in\mathcal{R};\ R_{0\alpha\beta0}, R_{0\bar\alpha\beta0} - \text{essential}\},\\
w\mathcal{R}_1^\perp &= \{R\in w\mathcal{R};\ R_{0\bar\alpha\beta0}=0, R_{0\alpha\beta0} - \text{essential}\},\\
w\mathcal{R}_1 &= \{R\in w\mathcal{R};\ R_{0\alpha\beta0}=0, R_{0\alpha\bar\beta0} - \text{essential}\}.
\end{aligned}$$

By using the self-conjugate projections $p_i, q_i, r_i,\ i=1,2,\ s,t,u,$ commuting with $\mathcal{U}(n)\times 1$, we describe their kernels and images, and following the scheme in [7, 11] we get

Theorem 2.2. (complete decomposition) *The space $\mathcal{R}$ is decomposable into 18 mutually orthogonal, $\mathcal{U}(n)\times 1$-invariant and irreducible factors:*

$$\begin{aligned}
\mathcal{R} = h\mathcal{R}_{11} &\oplus h\mathcal{R}_{12} \oplus h\mathcal{R}_{13} \oplus h\mathcal{R}_{11}^\perp \oplus h\mathcal{R}_{12}^\perp \oplus h\mathcal{R}_{13}^\perp \oplus h\mathcal{R}_2^\perp\\
&\oplus h\mathcal{R}_{31}^\perp \oplus h\mathcal{R}_{32}^\perp \oplus h\mathcal{R}_{33}^\perp\\
&\oplus v\mathcal{R}_{11} \oplus v\mathcal{R}_{12} \oplus v\mathcal{R}_{11}^\perp \oplus v\mathcal{R}_{12}^\perp \oplus v\mathcal{R}_2^\perp\\
&\oplus w\mathcal{R}_{11} \oplus w\mathcal{R}_{12} \oplus w\mathcal{R}_1^\perp,
\end{aligned}$$

describing by the essential components and equalities in the following table:

Factor	Essential components and equalities for R
$h\mathcal{R}_{11} = \ker(p_1)$	$R_{\alpha\bar\beta\gamma\bar\delta} = \frac{4\tau_R}{n(n+1)} 2(g_{\alpha\bar\beta}g_{\gamma\bar\delta} + g_{\gamma\bar\beta}g_{\alpha\bar\delta})$, $\rho_R \in \mathcal{L}_1, \rho^*_R \in \mathcal{L}_4, \tilde\rho_R = 0$
$h\mathcal{R}_{12} = \ker(p_2)$ $\subset \mathrm{Im}(p_1)$	$R_{\alpha\bar\beta\gamma\bar\delta} = \frac{2}{n+2} 2[g_{\alpha\bar\delta}(\rho_R)_{\bar\beta\gamma} + g_{\bar\beta\gamma}(\rho_R)_{\alpha\bar\delta} + g_{\gamma\bar\delta}(\rho_R)_{\alpha\bar\beta} + g_{\alpha\bar\beta}(\rho_R)_{\gamma\bar\delta}]$, $\rho_R \in \mathcal{L}_2, \rho^*_R \in \mathcal{L}_5, \tilde\rho_R = 0, \tau_R = \tau^*_R = 0$
$h\mathcal{R}_{13} = p_2(\mathrm{Im}(p_1))$	$W'_{\alpha\bar\beta\gamma\bar\delta} = [(p_2 \circ p_1)R]_{\alpha\bar\beta\gamma\bar\delta}, \quad \rho_{W'} = \rho^*_{W'} = \tilde\rho_{W'} = 0$
$h\mathcal{R}^\perp_{11} = \ker(q_1)$	$R_{\alpha\bar\beta\gamma\bar\delta} = \frac{4\tau^*_R}{n(n-1)} 2(g_{\alpha\bar\beta}g_{\gamma\bar\delta} - g_{\gamma\bar\beta}g_{\alpha\bar\delta})$, $\rho_R \in \mathcal{L}_1, \rho^*_R \in \mathcal{L}_4, \tilde\rho_R = 0$
$h\mathcal{R}^\perp_{12} = \ker(q_2)$ $\subset \mathrm{Im}(q_1)$	$R_{\alpha\bar\beta\gamma\bar\delta} = \frac{4}{3(n-2)} [g_{\alpha\bar\delta}(\rho_R)_{\gamma\bar\beta} + g_{\bar\beta\gamma}(\rho_R)_{\alpha\bar\delta} - g_{\gamma\bar\delta}(\rho_R)_{\alpha\bar\beta} - g_{\alpha\bar\beta}(\rho_R)_{\gamma\bar\delta}]$, $\rho_R \in \mathcal{L}_2, \rho^*_R \in \mathcal{L}_5, \tilde\rho_R = 0, \tau_R = \tau^*_R = 0$
$h\mathcal{R}^\perp_{13} = q_2(\mathrm{Im}(q_1))$	$W''_{\alpha\bar\beta\gamma\bar\delta} = [(q_2 \circ q_1)R]_{\alpha\bar\beta\gamma\bar\delta}, \quad \rho_{W''} = \rho^*_{W''} = \tilde\rho_{W''} = 0$
$h\mathcal{R}^\perp_2$	$R_{\alpha\beta\gamma\delta}$ — essential $\quad \rho_R = \rho^*_R = \tilde\rho_R = 0$
$h\mathcal{R}^\perp_{31} = \ker(r_1)$	$R_{\bar\alpha\beta\gamma\delta} = \frac{2}{n-1} [g_{\bar\alpha\delta}(\rho_R)_{\beta\gamma} - g_{\bar\alpha\gamma}(\rho_R)_{\beta\delta}]$, $\rho^*_R = \tilde\rho_R = 0, \rho_R \in \mathcal{L}_3$
$h\mathcal{R}^\perp_{32} = \ker(r_2)$ $\subset \mathrm{Im}(r_1)$	$R_{\bar\alpha\beta\gamma\delta} = -\frac{2}{n+1} [\Phi_{\bar\alpha\delta}(\rho^*_R)_{\beta\gamma} - \Phi_{\bar\alpha\gamma}(\rho^*_R)_{\beta\delta} - 2\Phi_{\bar\alpha\beta}(\rho^*_R)_{\gamma\delta}]$, $\rho_R = \tilde\rho_R = 0, \rho^*_R \in \mathcal{L}_6$
$h\mathcal{R}^\perp_{33} = r_2(\mathrm{Im}(r_1))$	$W'''_{\bar\alpha\beta\gamma\delta} = [(r_2 \circ r_1)R]_{\bar\alpha\beta\gamma\delta}, \quad \rho_{W'''} = \rho^*_{W'''} = \tilde\rho_{W'''} = 0$
$v\mathcal{R}_{11} = \ker(s)$	$R_{0\alpha\beta\bar\gamma} = -\frac{4}{n+1} [g_{\alpha\bar\gamma}(\rho_R)_{0\beta} + g_{\beta\bar\gamma}(\rho_R)_{0\alpha}]$ $\rho_R \in \mathcal{L}_7, \rho^*_R \in \mathcal{L}_8, \tilde\rho_R = 0$
$v\mathcal{R}_{12} = \mathrm{Im}(s)$	$V'_{0\alpha\beta\bar\gamma} = R_{0\alpha\beta\bar\gamma} + \frac{4}{n+1} [g_{\alpha\bar\gamma}(\rho_R)_{0\beta} + g_{\beta\bar\gamma}(\rho_R)_{0\alpha}]$ $\rho_{V'} = \rho^*_{V'} = \tilde\rho_{V'} = 0$
$v\mathcal{R}^\perp_{11} = \ker(t)$	$R_{0\alpha\beta\bar\gamma} = -\frac{4}{3(n-1)} [g_{\alpha\bar\gamma}(\rho_R)_{0\beta} - g_{\beta\bar\gamma}(\rho_R)_{0\alpha}]$ $\rho_R \in \mathcal{L}_7, \rho^*_R \in \mathcal{L}_8, \tilde\rho_R = 0$
$v\mathcal{R}^\perp_{12} = \mathrm{Im}(t)$	$V''_{0\alpha\beta\bar\gamma} = R_{0\alpha\beta\bar\gamma} + \frac{4}{3(n-1)} [g_{\alpha\bar\gamma}(\rho_R)_{0\beta} - g_{\beta\bar\gamma}(\rho_R)_{0\alpha}]$ $\rho_{V''} = \rho^*_{V''} = \tilde\rho_{V''} = 0$
$v\mathcal{R}^\perp_2$	$R_{0\alpha\beta\gamma}$ — essential $\quad \rho_R = \rho^*_R = \tilde\rho_R = 0$
$w\mathcal{R}_{11} = \ker(u)$	$R_{0\bar\alpha\beta 0} = \frac{2\tilde\tau_R}{n} g_{\bar\alpha\beta} \quad \rho_R \in \mathcal{L}_9, \tilde\rho_R \in \mathcal{L}_1, \rho^*_R = 0$
$w\mathcal{R}_{12} = \mathrm{Im}(u)$	$W_{0\bar\alpha\beta 0} = R_{0\bar\alpha\beta 0} - \frac{2\tilde\tau_R}{n} g_{\bar\alpha\beta} \quad \rho_W = \rho^*_W = 0, \tilde\rho_W \in \mathcal{L}_2, \tilde\tau_W = 0$
$w\mathcal{R}^\perp_1$	$R_{0\alpha\beta 0}$ — essential, $\rho_R = \rho^*_R = 0, \tilde\rho_R \in \mathcal{L}_3$

3. Riemannian Curvature Identities in the Basic Classes of Real Hypersurfaces of a Complex Space Form

In the sense of [8], we give the following

Lemma 3.1. *The characterization conditions for the basic classes of real hypersurfaces $M^{2n+1}(\varphi, \xi, \eta, g)$ of a Kähler manifold in terms of the essential components and essential complex equalities for the fundamental tensors, are as follows:*

- $W_1: \quad h_{\alpha 0} = L_{\alpha 0} = F_{00\alpha} = \omega_\alpha = -2\mathrm{i}\eta_{\alpha 0} = \mathrm{i}(\mathcal{L}_\xi g)_{\alpha 0} = -\mathrm{i}N_{\alpha 00}$
- $W_2: \quad h_{\alpha\bar\beta} = \dfrac{\mathrm{i}}{2}L_{\alpha\bar\beta} = -\mathrm{i}\eta_{\alpha\bar\beta} = -F_{\alpha\bar\beta 0} = -F_{\bar\beta\alpha 0} = -\dfrac{f(\xi)}{2n}g_{\alpha\bar\beta}$
- $W_4: \quad h_{\alpha\bar\beta} = \dfrac{\mathrm{i}}{2}L_{\alpha\bar\beta} = -\mathrm{i}\eta_{\alpha\bar\beta} = -F_{\alpha\bar\beta 0} = -F_{\bar\beta\alpha 0}, \quad f(\xi) = \operatorname{Tr} h = 0$
- $W_6: \quad h_{\alpha\beta} = \dfrac{\mathrm{i}}{2}(\mathcal{L}_\xi g)_{\alpha\beta} = -F_{\alpha\beta 0} = -F_{\beta\alpha 0} = \dfrac{\mathrm{i}}{2}N_{0\alpha\beta}, \quad L \equiv 0$

where L is the Levi form.

Using the relations between the shape operators, we get

Lemma 3.2. *The characterization conditions for the basic classes W_i, $i = 1, 2, 4, 6$ with respect to the shape operators $\mathcal{A}$ and A are as follows:*

W_1: $\mathcal{A} = \varphi A$, $g(\mathcal{A}X, Y) = -(\eta \otimes \omega \circ \varphi)(X, Y)$, $\quad \mathcal{A}^* = -(\omega \circ \varphi) \otimes \xi$, $\operatorname{Tr}\mathcal{A} = 0$, $\quad \operatorname{Tr}(\mathcal{A} \circ \varphi) = 0$

W_2: $\mathcal{A} = -\mathcal{A}^* = \varphi \circ \mathcal{A} \circ \varphi^{-1} = -\dfrac{\delta\Phi(\xi)}{2n}\varphi$, $\operatorname{Tr}\mathcal{A} = 0$, $\operatorname{Tr}(\mathcal{A} \circ \varphi) \neq 0$

or equivalently: $\varphi \circ A = A \circ \varphi = -\dfrac{\delta\Phi(\xi)}{2n}\varphi$, $\operatorname{Tr} A = \operatorname{Tr} h = -\operatorname{Tr}\mathcal{A} + \nu$

W_4: $\mathcal{A} = -\mathcal{A}^* = \varphi \circ \mathcal{A} \circ \varphi^{-1}$, $\quad \operatorname{Tr}\mathcal{A} = 0$, $\quad \operatorname{Tr}(\mathcal{A} \circ \varphi) = 0$
or equivalently: $\varphi \circ A = A \circ \varphi \quad \operatorname{Tr} A = \operatorname{Tr} h = \nu$

W_6: $\mathcal{A} = \mathcal{A}^* = -\varphi \circ \mathcal{A} \circ \varphi^{-1}$, $\quad \operatorname{Tr}\mathcal{A} = 0$, $\quad \operatorname{Tr}(\mathcal{A} \circ \varphi) = 0$
or equivalently: $\varphi \circ A = -A \circ \varphi \quad \operatorname{Tr} A = \operatorname{Tr} h = \nu$.

Here $\mathcal{A}^*$ is the operator, g-conjugate to A and $\delta\Phi$ is the codifferential of Φ.

The equation of Codazzi implies:

Lemma 3.3. *The essential components of the covariant derivatives of the shape operator A and of the second fundamental tensor h of a real hypersurface of a complex space form $\bar M^{2n+2}(c)$ satisfy the following equalities:*

i) $(\nabla_\alpha A)_\beta - (\nabla_\beta A)_\alpha = 0 \iff (\nabla_\alpha h)_{\beta C} - (\nabla_\beta h)_{\alpha C} = 0, C \in \{\gamma, \bar\gamma, 0\}$;

ii) $(\nabla_\alpha A)_{\bar\beta} - (\nabla_{\bar\beta} A)_\alpha = \frac{c}{2}\Phi_{\alpha\bar\beta}\xi \iff (\nabla_\alpha h)_{\bar\beta 0} - (\nabla_{\bar\beta} h)_{\alpha 0} = \frac{c}{2}\Phi_{\alpha\bar\beta} = -\frac{c}{2}\mathrm{i} g_{\alpha\bar\beta}$;

iii) $(\nabla_\alpha A)_0 - (\nabla_0 A)_\alpha = -\frac{c}{4}\varphi Z_\alpha = -\mathrm{i}\frac{c}{4}Z_\alpha \iff (\nabla_\alpha h)_{0\bar\beta} - (\nabla_0 h)_{\alpha\bar\beta} = \frac{c}{2}\Phi_{\alpha\bar\beta} = -\frac{c}{2}\mathrm{i} g_{\alpha\bar\beta}$.

The Gauss equation and Lemma 3.1 imply

Lemma 3.4. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be real hypersurface of a complex space form $\bar M^{2n+2}(c)$. Then the Riemannian curvature tensor R satisfies in terms of the essential components the following identities:*

1. $$R_{\alpha\bar\beta\gamma\bar\delta} = \frac{c}{2}(g_{\alpha\bar\beta}g_{\gamma\bar\delta} + g_{\bar\beta\gamma}g_{\alpha\bar\delta}) + h_{\bar\beta\gamma}h_{\alpha\bar\delta} - h_{\alpha\gamma}h_{\bar\beta\bar\delta} = \frac{c}{2}(\pi_1+\pi_2)_{\alpha\bar\beta\gamma\bar\delta} + h_{\bar\beta\gamma}h_{\alpha\bar\delta} - h_{\alpha\gamma}h_{\bar\beta\bar\delta}$$
2. $R_{\alpha\beta\bar\gamma\bar\delta} = h_{\beta\bar\gamma}h_{\alpha\bar\delta} - h_{\alpha\bar\gamma}h_{\beta\bar\delta}$
3. $R_{\bar\alpha\beta\gamma\delta} = h_{\beta\gamma}h_{\bar\alpha\delta}h_{\bar\alpha\gamma}h_{\beta\delta}$
4. $R_{\alpha\beta\gamma\delta} = h_{\beta\gamma}h_{\alpha\delta} - h_{\alpha\gamma}h_{\beta\delta}$
5. $R_{0\alpha\beta\gamma} = h_{\alpha\beta}h_{0\gamma} - h_{\alpha\gamma}h_{0\beta}$
6. $R_{0\bar\alpha\beta\gamma} = h_{\bar\alpha\beta}h_{0\gamma} - h_{\bar\alpha\gamma}h_{0\beta}$
7. $R_{0\alpha\beta\bar\gamma} = h_{\alpha\beta}h_{0\bar\gamma} - h_{\alpha\bar\gamma}h_{0\beta}$
8. $R_{0\bar\alpha\beta 0} = \frac{c}{4}g_{\bar\alpha\beta} + \nu h_{\bar\alpha\beta} - h_{\bar\alpha 0}h_{0\beta}$
9. $R_{0\alpha\beta 0} = \nu h_{\alpha\beta} - h_{\alpha 0}h_{0\beta}$.

Using Lemma 3.1, Lemma 3.4 and Theorems 2.1 and 2.2, we express the essential components and the decomposition of the Riemannian curvature tensor R:

Theorem 3.1. *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be in one of the basic classes W_1, W_2, W_4, W_6 of real hypersurfaces of a complex space form $\bar M^{2n+2}(c)$. Then the Riemannian curvature tensor R satisfies the following identities:*

- W_1:

$$R_{\alpha\bar\beta\gamma\bar\delta} = \frac{c}{2}(\pi_1+\pi_2)_{\alpha\bar\beta\gamma\bar\delta}\,,$$
$$R_{0\bar\alpha\beta 0} = \frac{c}{4}g_{\bar\alpha\beta} - \omega_{\bar\alpha}\omega_\beta\,,$$
$$R_{0\alpha\beta 0} = -\omega_\alpha\omega_\beta$$

and $M \in h\mathcal{R}_{11}(\tau = c\frac{n(n+1)}{16}) \oplus w\mathcal{R}$*;*

- W_2:

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{c}{2}(\pi_1 + \pi_2)_{\alpha\bar{\beta}\gamma\bar{\delta}} + \frac{f^2(\xi)}{4n^2} g_{\gamma\bar{\beta}} g_{\alpha\bar{\delta}} \,,$$

$$R_{\alpha\beta\bar{\gamma}\bar{\delta}} = \frac{f^2(\xi)}{4n^2} \begin{vmatrix} g_{\beta\bar{\gamma}} & g_{\alpha\bar{\gamma}} \\ g_{\beta\bar{\delta}} & g_{\alpha\bar{\delta}} \end{vmatrix} \,,$$

$$R_{0\bar{\alpha}\beta 0} = \left(\frac{c}{4} - \nu\frac{f(\xi)}{2n}\right) g_{\bar{\alpha}\beta}$$

and $M \in h\mathcal{R}_2 \oplus w\mathcal{R}_{11}(\tilde{\tau} = \frac{cn}{8} - \nu\frac{f(\xi)}{4})$*;*

- W_4:

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = 2(\pi_1 + \pi_2)_{\alpha\bar{\beta}\gamma\bar{\delta}} - \eta_{\gamma\bar{\beta}}\eta_{\alpha\bar{\delta}} \,,$$

$$R_{\alpha\beta\bar{\gamma}\bar{\delta}} = \begin{vmatrix} \eta_{\alpha\bar{\gamma}} & \eta_{\beta\bar{\gamma}} \\ \eta_{\alpha\bar{\delta}} & \eta_{\beta\bar{\delta}} \end{vmatrix} \,,$$

$$R_{0\bar{\alpha}\beta 0} = \frac{c}{4} g_{\bar{\alpha}\beta} + \mathrm{i}\nu\eta_{\bar{\alpha}\beta}$$

and $M \in h\mathcal{R}_2 \oplus w\mathcal{R}_1$*;*

- W_6:

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}} = \frac{c}{2}(\pi_1 + \pi_2)_{\alpha\bar{\beta}\gamma\bar{\delta}} - \frac{1}{4}(\mathcal{L}_\xi g)_{\alpha\gamma}(\mathcal{L}_\xi g)_{\bar{\beta}\bar{\delta}} \,,$$

$$R_{\alpha\beta\gamma\delta} = -\frac{1}{4} \begin{vmatrix} (\mathcal{L}_\xi g)_{\beta\gamma} & (\mathcal{L}_\xi g)_{\alpha\gamma} \\ (\mathcal{L}_\xi g)_{\beta\delta} & (\mathcal{L}_\xi g)_{\alpha\delta} \end{vmatrix} \,,$$

$$R_{0\bar{\alpha}\beta 0} = \frac{c}{4} g_{\bar{\alpha}\beta} \,,$$

$$R_{0\alpha\beta 0} = \frac{\mathrm{i}}{2}\nu(\mathcal{L}_\xi g)_{\alpha\beta}$$

and $M \in h\mathcal{R}_1 \oplus h\mathcal{R}_2^\perp \oplus w\mathcal{R}_{11}(\tilde{\tau} = \frac{cn}{8}) \oplus w\mathcal{R}_1^\perp$.

References

[1] Alexiev V. and Ganchev G., *On the Classification of Almost Contact Metric Manifolds*, Math. and Educ. in Math., Proc. of the XV Spring Conf. of UBM, Sunny Beach 1986, pp 155–161.

[2] Baikoussis Ch., *A Characterization of Real Hypersurfaces in Complex Space Forms in Terms of the Ricci Tensor*, Canadian Math. Bull. **40**(3) (1997) 257–265.

[3] Blair D., *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. **509** Springer, Berlin 1976.

[4] Chinea D. and Gonzalez C., *A Classification of Almost Contact Metric Manifolds*, Ann. Mat. Pura Appl. **156**(4) (1990) 15–36.

[5] Ganchev G. and Alexiev V., *On Some Classes of Almost Contact Metric Manifolds*, Math. and Educ. in Math., Proc. of the XV Spring Conf. of UBM, Sunny Beach 1986, pp 186–191.

[6] Ganchev G. and Hristov M., *Real Hypersurfaces of a Kähler Manifold (the Sixteen Classes)*, In: *Perspectives of Complex Analysis, Differential Geometry and Mathematical Physics*, Proc. 5th Int. Workshop on Complex Structures and Vector Fields, Varna 2000, S. Dimiev and K. Sekigawa (Eds), World Scientific, Singapore 2001, pp 147–158.

[7] Ganchev G., Ivanov S. and Michova V. *Hermitian Sectional Curvatures on Hermitian Manifold*, IC/91/40, Int. Centre for Theoretical Physics, Miramare, Trieste, May 1991.

[8] Hristov M., *On the Integrability Conditions for Almost Contact Metric Manifolds*, In: *Aspects of Complex analysis, Differential Geometry and Mathematical Physics*, Proc 4th Int. Workshop on Complex Structures and Vector Fields, Varna 1998, S. Dimiev and K. Sekigawa (Eds), World Scientific, Singapore 1999, pp 134–147.

[9] Hristov M. and Alexiev V., *Subclasses of the Conformal Almost Contact Metric Manifolds*, Math. and Educ. in Math., Proc. XIX Spring Conf. UBM, Sunny Beach 1990, pp 144–148.

[10] Tashiro Y., *On Contact Structure of Hypersurfaces in Complex Manifolds*, Tohoku Math. J., **15** (1963) 62–78.

[11] Tricerri F. and Vanhecke L., *Curvature Tensors on Almost Hermitian Manifolds*, Trans. Am. Math. Soc., **267** (1981) 365–398.

[12] Yano K. and Kon M., *CR-submanifolds of Kählerian and Sasakian Manifolds*, Progress in Mathematics, **30**, Birkhäuser, Boston, 1983.