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QUANTUM INTEGRABILITY AND COMPLETE SEPARATION OF VARIABLES FOR PROJECTIVELY EQUIVALENT METRICS ON THE TORUS

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Abstract. Let two Riemannian metrics g and $\bar{g}$ on the torus T^n have the same geodesics (considered as unparameterized curves). Then we can construct invariantly n commuting differential operators of second order. The Laplacian Δ_g of the metric g is one of these operators. For any $x \in T^n$, consider the linear transformation G of T_xT^n given by the tensor $g^{i\alpha}\bar{g}_{\alpha j}$. If all eigenvalues of G are different at one point of the torus then they are different at every point; the operators are linearly independent and we can globally separate the variables in the equation $\Delta_g f = \mu f$ on this torus.

1. Commuting Operators for Projectively Equivalent Metrics

Let g and $\bar{g}$ are two C^2-smooth Riemannian metrics on some manifold M^n. They are **projectively equivalent** if they have the same geodesics considered as unparameterized curves.

The problem of describing projectively equivalent metrics was stated by Beltrami in [1]. Locally, in the neighborhood of so-called sable points, it was essentially solved by Dini [3] for surfaces and by Levi-Civita [4] for manifolds of arbitrary dimension. Denote by G the tensor $g^{i\alpha}\bar{g}_{\alpha j}$. In invariant terms, G is the fiberwise-linear mapping $G\colon TM^n \to TM^n$ such that its restriction to any tangent space $T_{x_0}M^n$ is the linear transformation of $T_{x_0}M^n$ satisfying the following condition: for any vectors $\xi, \nu \in T_{x_0}M^n$, the scalar product $g(G(\xi), \nu)$ of the vectors $G(\xi)$ and ν in g is equal to the scalar product $\bar{g}(\xi, \nu)$ of the vectors ξ and ν in $\bar{g}$.

The trivial example of projectively equivalent metrics is g, $\bar{g} \stackrel{\text{def}}{=} Cg$, where C is a positive constant. For this case, all eigenvalues of G are equal to C.

Definition 1. *The metrics g, $\bar{g}$ are* ***strictly non-proportional*** *at $x_0 \in M^n$, if the eigenvalues of G are all different at x_0.*

Suppose that at the point $x \in M^n$ the metrics are strictly non-proportional. Under this assumption, Levi-Civita theorem reads as follows.

Theorem 1. (Levi-Civita [4]) *Let g and $\bar{g}$ are Riemannian metrics on M^n. Suppose that at the point $x \in M^n$ the eigenvalues of G are all different and equal to $\rho_0(x) > \rho_1(x) > \cdots > \rho_{n-1}(x)$. Then the metrics are projectively equivalent in some sufficiently small neighborhood U^n of the point x, if and only if there exists a coordinate system $x_0, \ldots, x_{n-1}$ in the neighborhood U^n such that the quadratic forms of the metrics g and $\bar{g}$ have the following form:*

$$\mathrm{d}s_g^2 = \Pi_0\,\mathrm{d}(x^0)^2 + \Pi_1\,\mathrm{d}(x^1)^2 + \cdots + \Pi_{n-1}\,\mathrm{d}(x^{n-1})^2\,, \tag{1}$$

$$\mathrm{d}s_{\bar{g}}^2 = \rho_0\Pi_0\,\mathrm{d}(x^0)^2 + \rho_1\Pi_1\,\mathrm{d}(x^1)^2 + \cdots + \rho_{n-1}\Pi_{n-1}\,\mathrm{d}(x^{n-1})^2 \tag{2}$$

where the functions $\Pi_i, \rho_i \colon U^n \to \mathbb{R}$ are given by

$$\Pi_i \stackrel{\text{def}}{=} (\lambda_i - \lambda_0)(\lambda_i - \lambda_1)\cdots(\lambda_i - \lambda_{i-1})(\lambda_{i+1} - \lambda_i)\cdots(\lambda_{n-1} - \lambda_i)\,,$$

$$\rho_i \stackrel{\text{def}}{=} \frac{1}{\lambda_0\lambda_1\cdots\lambda_{n-1}}\,\frac{1}{\lambda_i}$$

where $\lambda_0 < \lambda_1 < \cdots < \lambda_{n-1}$ are smooth functions on U^n such that for any i the function λ_i depends on the variable x^i only.

Levi-Civita theorem gives us the following series of examples of projectively equivalent metrics on the torus T^n.

Consider the n-dimensional lattice L^n in $\mathbb{R}^n$. Let $\lambda_0, \lambda_1, \ldots, \lambda_{n-1} \colon \mathbb{R}^n \to \mathbb{R}$ are smooth positive functions on $\mathbb{R}^n$ such that for any i the function λ_i depends only on the coordinate x^i, $\lambda_i < \lambda_{i+1}$ and λ_i is invariant modulo the lattice L^n. Then the metrics (1, 2) are well-defined on $\mathbb{R}^n$; by Levi-Civita theorem, they are geodesically equivalent. By definition, they are invariant modulo the lattice and therefore define two geodesically equivalent metrics on the torus $\mathbb{R}^n/L^n$.

We will call such metrics **model metrics** on the torus. Each pair of model metrics is given by the lattice L^n and by the functions $\lambda_i(x^i)$ which are invariant modulo the lattice.

The main result of this paper is the following theorem.

Theorem 2. *Let g and $\bar{g}$ are projectively equivalent metrics on the torus T^n. Suppose that they are strictly non-proportional at least at one point of T^n. Then there exists a lattice L^n, a pair of model metrics g_{model}, $\bar{g}_{\text{model}}$ on the torus $\mathbb{R}^n/L^n$ and a diffeomorphism $\phi\colon \mathbb{R}^n/L^n \to T^n$ such that $\phi^* g_{\text{model}} = g$, $\phi^* \bar{g}_{\text{model}} = \bar{g}$.*

In other words, model metrics give us all possible examples of projectively equivalent strictly non-proportional (at least at one point) metrics on the torus.

The simplest example of the lattice is the so-called **normal lattice** which is generated by the vectors $e_0 = (l_0, 0, \ldots, 0)$, $e_1 = (0, l_1, 0, \ldots, 0), \ldots, e_{n-1} = (0, 0, \ldots, 0, l_{n-1})$. Here l_i are positive numbers. We will call the pair of model metrics **normal model**, if the corresponding lattice is normal.

For normal model metrics, every function λ_i can be viewed as the function on the circle $S^1_i \stackrel{\text{def}}{=} (\mathbb{R} \mod l_i)$ and the torus $\mathbb{R}^n/L^n$ is the product $S^1_0 \times S^1_1 \times \cdots \times S^1_{n-1}$.

Theorem 3. *Let g and $\bar{g}$ are projectively equivalent metrics on the torus T^n. Suppose that there is no vector field which is Killing for both metrics and that the metrics g, $\bar{g}$ are strictly non-proportional at least at one point of the torus. Then the following statements are true.*

1. *There exists a normal lattice L^n_{big}, a covering $\phi_{\text{big}}\colon \mathbb{R}^n/L^n_{\text{big}} \to T^n$ and a pair g_{normal}, $\bar{g}_{\text{normal}}$ of normal model metrics on the torus $\mathbb{R}^n/L^n_{\text{big}}$ such that $\phi^*_{\text{big}} g = g_{\text{normal}}$, $\phi^*_{\text{big}} \bar{g} = \bar{g}_{\text{normal}}$.*
2. *There exists a normal lattice L^n_{small}, a covering $\phi_{\text{small}}\colon T^n \to R^n/L^n_{\text{small}}$ and a pair g_{normal}, $\bar{g}_{\text{normal}}$ of normal model metrics on the torus R^n/L^n_{small} such that $\phi^*_{\text{small}} g_{\text{normal}} = g$, $\phi^*_{\text{small}} \bar{g}_{\text{normal}} = \bar{g}$.*

In particular, there always exists an orthogonal sub-lattice of the lattice L^n.

If we allow the metrics to admit a Killing vector field, the lattice may do not contain an orthogonal sub-lattice (and therefore the torus neither is covered by a torus with normal model metrics nor covers a torus with normal model metrics). The example of such a situation can be easily found in the class of the flat 2-tori.

Let g and $\bar{g}$ are projectively equivalent metrics on the torus T^n; assume that they are strictly non-proportional at least at one point. Consider all vector fields which are Killing with respect to both metrics. Then it is possible to show that these vector fields commute and have no singular points. Then we have a locally-free action of the group R^k, where k is the number of independent Killing vector fields. The orbits of this action are k-tori; it is possible to make the Poisson reduction of the torus T^n which gives us some $(n-k)$-torus with two projectively equivalent metrics; these metrics admit no Killing vector field and therefore are normal model (up to a finite covering).

In other words, any torus with projectively equivalent metrics, strictly non proportional at least at one point, can be isometrically covered by the product of the torus with normal model metrics and the flat torus.

By Theorem 2, if M^n is covered by the torus and if projectively equivalent metrics are strictly non proportional at least at one point then they are strictly non proportional at each point. A kind of inverse statement is also true.

Theorem 4. *Let M^n be closed and connected. Let g, $\bar{g}$ on M^n are projectively equivalent. Suppose they are strictly non-proportional at each point of the manifold. Then the manifold can be covered by the torus.*

The main tool of the proof of Theorems 2, 3, 4 is the following construction that, given a pair of projectively equivalent metrics, produce commuting integrals (both in the classical and quantum sense) for their geodesic flows. The classical version of the construction was obtained in [16], see also [8]. The quantum version of the construction was announced in [10] and proved in [12] (see also [11]).

Let g and $\bar{g}$ are Riemannian metrics on M^n. Consider the fiberwise-linear mapping $G\colon TM^n \to TM^n$ given by the tensor $(g^{i\alpha}\, \bar{g}_{\alpha j})$. In invariant terms, for any $x_0 \in M^n$, the restriction of the mapping G to the tangent space $T_{x_0}M^n$ is the linear transformation of $T_{x_0}M^n$ such that for any vectors $\xi, \nu \in T_{x_0}M^n$ the scalar product $g(G(\xi), \nu)$ of the vectors $G(\xi)$ and ν in the metric g is equal to the scalar product $\bar{g}(\xi, \nu)$ of the vectors ξ and ν in the metric $\bar{g}$. Consider the fiberwise-linear mapping $A\colon TM^n \to TM^n$ given by

$$A \stackrel{\mathrm{def}}{=} (\det(G))^{\frac{1}{n+1}}\, G^{-1}\,.$$

Consider the characteristic polynomial

$$\det(A - \lambda\,\mathrm{Id}) = c_0\lambda^n + c_1\lambda^{n-1} + \cdots + c_n\,. \tag{3}$$

Consider the mappings $S_0, S_1, \ldots, S_{n-1}\colon TM^n \to TM^n$ given by

$$S_{n-m} \stackrel{\mathrm{def}}{=} \sum_{i=0}^{m-1} c_i A^{m-i-1}\,.$$

Here m lies in the set $\{1, 2, \ldots, n\}$ so that $k = n - m$ lies in the set $\{0, 1, \ldots, n-1\}$. Consider the linear partial-differential operators $\mathfrak{I}_0, \mathfrak{I}_1, \ldots, \mathfrak{I}_{n-1}$ given by

$$\mathfrak{I}_k(f) \stackrel{\mathrm{def}}{=} \div\,(S_k(\operatorname{grad} f))$$

where $\mathrm{grad}(f)$ denotes the gradient $g^{i\alpha}\frac{\partial f}{\partial x^\alpha}$ of the function f and $\div$ denotes the divergence with respect to the metric g.

Remark 1. *In coordinates the operators $\mathfrak{I}_k$ are given by*

$$\mathfrak{I}_k = \frac{1}{\sqrt{\det(g)}}\frac{\partial}{\partial x^i}(S_k)^i_\alpha \sqrt{\det(g)}g^{\alpha j}\frac{\partial}{\partial x^j}\,.$$

Remark 2. *The operator $(-1)^n\mathfrak{I}_{n-1}$ is exactly the Laplacian Δ_g.*

Theorem 5. *If the metrics g and $\bar{g}$ on M^n are projectively equivalent then the operators $\mathfrak{I}_k$ commute pairwise. In particular they commute with Δ_g.*

If differential operators commute then their symbols also commute (as functions on the symplectic manifold T^*M^n). In our case the symbols are the functions $I_k\colon T^*M^n \to \mathbb{R}$, $k = 0, 1, \ldots, n-1$, given by the formulas

$$\mathrm{Smbl}(\mathfrak{I}_k)(x,p) \stackrel{\text{def}}{=} I_k(x,p) = g^{\alpha j}(S_k)^i_\alpha p_i p_j\,.$$

Remark 3. *The function $\frac{(-1)^n}{2}I_{n-1}$ is equal to the Hamiltonian of the geodesic flow of the metric g.*

Theorem 6. ([16]) *If the metrics g and $\bar{g}$ on M^n are projectively equivalent then the functions I_k are commutative integrals for the geodesic flow of the metric g.*

If the metrics are strictly non-proportional at a point $x \in M^n$ then I_k are functionally independent on the cotangent bundle to some neighborhood of the point x and the operators $\mathfrak{I}_k$ are linear independent. Moreover, the following theorem shows that if the manifold is connected and geodesically complete and if the projectively equivalent metrics are strictly non-proportional at a point then they are strictly non-proportional almost everywhere and therefore the differentials of the functions I_k are linear independent at almost every point of T^*M^n.

Denote by $\lambda_0(x) \le \lambda_1(x) \le \cdots \le \lambda_{n-1}(x)$ the eigenvalues of A at $x \in M^n$.

Theorem 7. *Let g and $\bar{g}$ are projectively equivalent metrics on M^n. Suppose that M^n is geodesically complete (with respect to one of the metrics) and connected. Then for any $i \in \{0, 1, \ldots, n-2\}$ the following statements are true:*

1. *$\lambda_i(x) \le \lambda_{i+1}(y)$ for any $x, y \in M^n$.*
2. *If $\lambda_i(x) < \lambda_{i+1}(x)$ for some $x \in M^n$ then $\lambda_i(y) < \lambda_{i+1}(y)$ for almost each point $y \in M^n$.*

3. *If* $\lambda_i(x) = \lambda_{i+1}(y)$ *for some* $x, y \in M^n$ *then there exists* $z \in M^n$ *such that* $\lambda_i(z) = \lambda_{i+1}(z)$.

It is easy to see that the eigenvalues $\lambda_0(x) \le \lambda_1(x) \le \cdots \le \lambda_{n-1}(x)$ of the mapping A and the eigenvalues $\rho_0(x) \ge \rho_1(x) \ge \cdots \ge \rho_{n-1}(x)$ of the mapping G are connected by the formulas

$$\lambda_i(x) = \Big(\rho_0(x)\rho_1(x)\cdots\rho_{n-1}(x)\Big)^{\frac{1}{n+1}} \frac{1}{\rho_i(x)}\,,$$
$$\rho_i(x) = \frac{1}{\lambda_0(x)\lambda_1(x)\cdots\lambda_{n-1}(x)}\,\frac{1}{\lambda_i(x)}\,.$$

In particular, the number of different eigenvalues of G coincides with the number of different eigenvalues of A. Thus the following statement is true.

Corollary 1. *Let the metrics* g, $\bar{g}$ *are projectively equivalent on* M^n. *Suppose* M^n *is connected and geodesically complete with respect to one of these metrics. If at a point of the manifold the number of different eigenvalues of G is equal to* n_1 *then at almost every point the number of different eigenvalues of G is greater or equal than* n_1. *In particular, if the metrics are strictly non-proportional at a point then they are strictly non-proportional almost everywhere.*

If the manifold is closed then the operators $\Im_k$ are self-adjoint. If they commute, it is possible to diagonalize them simultaneously: there exists a countable basis $\Phi = \{f_1, f_2, \ldots, f_m, \ldots\}$ of the space $L_2(M^n)$ such that each f_m is an eigenfunction of each operator $\Im_k$.

In the case of normal model metrics on the torus, the equation $\Delta_g f = \mu f$ can be separated in the variables $(x^0, \ldots, x^{n-1})$. More precisely, take any function f from the basis Φ. Since f is an eigenfunction of each operator $\Im_k$, we have that f is a solution of the system of n partial differential equations

$$\Im_k f = \mu_k f\,, \qquad k = 0, 1, \ldots, n-1\,. \tag{4}$$

In coordinates $(x^0, \ldots, x^{n-1})$ the system (4) is equivalent to the system

$$\left(\frac{\partial}{\partial x^k}\right)^2 f = \left[(-1)^{k-1}\sum_{i=0}^{n-1}\Big(\lambda_k(x^k)\Big)^i \mu_i\right] f\,, \qquad k = 0, 1, \ldots, n-1\,. \tag{5}$$

We see that for each $k \in \{0, 1, \ldots, n-1\}$ the coefficients of the k-th equation depend on the variable x^k only. Then f is the product $X_0(x^0)X_1(x^1)\cdots X_{n-1}(x^{n-1})$, and for each $k \in \{0, 1, \ldots, n-1\}$ the function X_k is a solution of the k-th equation of (5) so that we reduced the system of partial differential equations (4) on the torus to the system of ordinary differential equations on the circle:

$$\left(\frac{\partial}{\partial x^k}\right)^2 X_k(x^k) = \left[(-1)^{k-1}\sum_{i=0}^{n-1}(\lambda_k(x^k))^i\mu_i\right] X_k(x^k)\,, \qquad k = 0, 1, \dots, n-1 \tag{6}$$

so that the variables in the equation $\Delta_g f = \mu f$ are completely separated.

What happens if the metrics are not strictly non-proportional everywhere? Our conjecture is that if M^n is closed and connected and if g and $\bar{g}$ are projectively equivalent and strictly non-proportional at least at one point then it is also possible to separate the variables globally: the equation $\Delta_g f = \mu f$ splits naturally into n ordinary differential equations (6) either on the interval (with von Neumann conditions on the ends) or on the circle.

The conjecture is true for two-dimensional surfaces (projectively equivalent metrics on closed surfaces were described in [2, 5, 7, 9]). It is also true for the metric of the ellipsoid and the metric of the Poisson sphere (as shown in [11–16], the metric of the ellipsoid and the metric of the Poisson sphere have projectively equivalent metrics).

2. Spectral Polynomial F_t and the Behaviour of the Eigenvalues of A

Consider the function $F\colon \mathbb{R} \times T^*M^n \to \mathbb{R}$ given by

$$F_t(x, \xi) = t^{n-1}I_{n-1}(x, \xi) + \cdots + I_0(x, \xi)\,.$$

For a fixed point $(x, \xi) \in T^*M$, the function F_t is the polynomial in t.

In the proof of Lemma 1 we will show that all the roots of the polynomial are real. Let us denote the roots of the polynomial by

$$t_0(x, \xi) \le t_1(x, \xi) \le \cdots \le t_{n-2}(x, \xi)\,.$$

Lemma 1. *Let x is a point of M^n. Then for any $i \in \{0, 1, \dots, n-2\}$ the following statements are true.*

1. *For any $\xi \in T_x^*M$,*

$$\lambda_i(x) \le t_i(x, \xi) \le \lambda_{i+1}(x)\,.$$

In particular, if $\lambda_i(x) = \lambda_{i+1}(x)$ then $t_i(x, \xi) = \lambda_i(x) = \lambda_{i+1}(x)$.

2. *If $t_i(x, \xi)$ is constant for any ξ from some subset of T^*M of non-zero measure then $\lambda_i(x) = \lambda_{i+1}(x)$.*

Proof: In the proof we assume that the point $x \in M^n$ is fixed. For simplicity, we will write λ_i, $t_i(\xi)$ instead of $\lambda_i(x)$, $t_i(x,\xi)$. Denote by σ_m the elementary symmetric polynomial of degree m of the variables $\lambda_0, \dots, \lambda_{n-1}$. Denote by $\sigma_m(\check{\lambda}_i)$ the elementary symmetric polynomial of degree m of $n-1$ variables

$$\lambda_0, \lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_{n-1} .$$

The polynomials σ_m, $\sigma_m(\check{\lambda}_i)$ and $\sigma_{m-1}(\check{\lambda}_i)$ satisfy the relation

$$\sigma_m(\check{\lambda}_i) = \sigma_m - \lambda_i \sigma_{m-1}(\check{\lambda}_i) .$$

By definition, the coefficients $c_0, \dots, c_n$ of the characteristic polynomial (3) are given by $c_m = (-1)^{n-m}\sigma_m$.

Since the metrics g, $\bar{g}$ are positive definite, there exists a basis of the space $T^*_x M^n$ such that the metric g is given by the identity matrix $\operatorname{diag}(1, 1, \dots, 1)$ and the mapping A is given by the matrix $\operatorname{diag}(\lambda_0, \lambda_1, \dots, \lambda_{n-1})$.

Let us show by induction that then the mapping S_k is given by

$$(-1)^{k-1} \operatorname{diag}\left(\sigma_{n-k-1}(\check{\lambda}_0), \sigma_{n-k-1}(\check{\lambda}_1), \dots, \sigma_{n-k-1}(\check{\lambda}_{n-1})\right) . \tag{7}$$

The base of induction is $k = n-1$. We evidently have

$$S_0 = c_0 \operatorname{Id} = (-1)^n \operatorname{diag}(1, 1, \dots, 1) = (-1)^{n-2} \operatorname{diag}\left(\sigma_0(\lambda_0), \dots, \sigma_0(\lambda_{n-1})\right) .$$

Suppose the statement is true for $k = n-m$:

$$S_{n-m} = (-1)^{n-m+1} \operatorname{diag}\left(\sigma_{m-1}(\check{\lambda}_0), \sigma_{m-1}(\check{\lambda}_1), \dots, \sigma_{m-1}(\check{\lambda}_{n-1})\right) .$$

Then for $k = n-m-1$ we have

$$\begin{aligned} S_{n-m-1} &= S_{n-m}A + c_m \operatorname{Id} = S_{n-m}A + (-1)^{n-m}\sigma_m \operatorname{Id} \\ &= \operatorname{diag}\left((-1)^{n-m+1}\lambda_0\sigma_{m-1}(\check{\lambda}_0) + (-1)^{n-m}\sigma_m, \dots\right) \\ &= (-1)^{n-m} \operatorname{diag}\left(\sigma_m(\check{\lambda}_0), \dots, \sigma_m(\check{\lambda}_{n-1})\right) . \end{aligned}$$

Thus, for any k, the mapping S_k is given by (7).

For any $\xi = (p_0, p_1, \dots, p_{n-1}) \in T^*_x M^n$, denote by P_i the polynomial

$$\begin{aligned} P_i(t) &= (t-\lambda_0)(t-\lambda_1)\cdots(t-\lambda_{i-1})(t-\lambda_{i+1})\cdots(t-\lambda_{n-1}) \\ &= \sum_{\alpha=0}^{n-1} (-1)^{n-\alpha-1} t^\alpha \sigma_{n-\alpha-1}(\check{\lambda}_i) . \end{aligned} \tag{8}$$

Then the polynomial F_t has the following form:

$$
\begin{aligned}
F_t(x,\xi) &= \sum_{i=0}^{n-1}\sum_{\alpha=0}^{n-1}(-1)^{n-i-1}p_\alpha^2\sigma_{n-i-1}(\check{\lambda}_\alpha) \\
&= (-1)^n\left(P_0(t)p_0^2 + P_1(t)p_1^2 + \cdots + P_{n-1}(t)p_{n-1}^2\right) .
\end{aligned} \tag{9}
$$

Easy to see that the coefficients of the polynomial F_t depend continuously on the eigenvalues λ_i and on the momenta p_i. Then it is sufficient to prove the first statement of the lemma assuming that the eigenvalues λ_i are all different and that the momenta p_i are non-zero. For any $\alpha \neq i$ we evidently have $P_i(\lambda_j) \equiv 0$. Then

$$
F_{\lambda_i} = (-1)^n\sum_{\alpha=0}^{n-1}P_\alpha(\lambda_i)p_\alpha^2 = (-1)^n P_i(\lambda_i)p_i^2 .
$$

Hence F_{λ_i} and $F_{\lambda_{i+1}}$ have different signs and therefore the open interval $]\lambda_i,\lambda_{i+1}[$ contains a root of the polynomial F_t. The degree of the polynomial F_t is equal $n-1$; we have $n-1$ disjoint intervals; any of these intervals contains at least one root so that all roots are real and the root number i lies between λ_i and λ_{i+1}. The first statement of the lemma is proved.

Let us prove the second statement of the lemma. Suppose that, for any ξ from some subset $U \subset T_x^*M^n$ of non-zero measure, the value $t_i(\xi)$ is constant and is equal to τ. Then, by definition of t_i, the function

$$
F_\tau(x,\xi) \stackrel{\text{def}}{=} (F_t(x,\xi))|_{t=\tau}
$$

is zero for any $\xi \in U$. The function $F_\tau(x,\xi)$ (as a function on $T_x^*M^n$) is a polynomial in ξ; since it is zero on some subset of non-zero measure, it is identically zero. Therefore, by the first statement of the lemma, for any $\xi \in T_x^*M^n$, the root $t_i(\xi)$ is equal to the constant τ.

Now let us show that, for any number τ satisfying

$$
\lambda_i \le \tau \le \lambda_{i+1} ,
$$

there exists $\xi \in T_x^*M^n$, $\xi \neq 0$ such that $t_i(\xi) = \tau$.

Indeed, consider $\xi_1,\xi_2 \in T_x^*M^n$ such that all components of ξ_1 except for the component number i are zero; all components of ξ_2 except for the component number $i+1$ are zero. In view of (9), $t_i(\xi_1) = \lambda_{i+1}$ and $t_i(\xi_2) = \lambda_i$. Let us join ξ_1 and ξ_2 by a curve that lies in $T_x^*M^n$ and that does not go through zero. Since the root $t_i(\xi)$ depends continuously on $\xi \in T_x^*M^n$, for any $\tau \in [\lambda_i,\lambda_{i+1}]$ there exists ξ lying on this curve such that $t_i(\xi) = \tau$. Thus, $\lambda_i = \lambda_{i+1}$ and the lemma is proved. □

Proof: (Theorem 7) Suppose g, $\bar{g}$ are projectively equivalent metrics on M^n. We assume that the manifold is connected and geodesically complete with respect to the metric g. Then we can joint any two points $x, y \in M^n$ by a geodesic γ. Let us identify TM^n and T^*M^n by g. By Theorem 6, we have that the functions I_k are constant on the orbits of the geodesic flow of g. Then the root t_i is also constant on each orbit of the geodesic flow of g so that

$$t_i(\gamma(0), \dot{\gamma}(0)) = t_i(\gamma(1), \dot{\gamma}(1)) .$$

Using Lemma 1, we have that

$$\lambda_i(\gamma(0)) \le t_i(\gamma(0), \dot{\gamma}(0)) , \quad \text{and} \quad t_i(\gamma(1), \dot{\gamma}(1)) \le \lambda_i(\gamma(1)) .$$

Therefore $\lambda_i(x) \le \lambda_{i+1}(y)$ and the first statement of Theorem 7 is proved.

Now suppose $\lambda_i(y) = \lambda_{i+1}(y)$ for any point y of some subset $U^n \in M^n$ of non-zero measure. Then by the first part of the corollary, the value of λ_i is a constant (independent of $y \in U^n$). Indeed, connecting any two points $y_0, y_1 \in U^n$ by a geodesic γ we obtain (we assume $y_0 = \gamma(0)$, $y_1 = \gamma(1)$)

$$\lambda_i(y_0) = \lambda_{i+1}(y_0) = t_i(y_0, \dot{\gamma}(0)) = t_i(y_1, \dot{\gamma}(1)) = \lambda_i(y_1) = \lambda_{i+1}(y_1) .$$

Denote this constant by C. Let us prove that $\lambda_i(x) = \lambda_{i+1}(x) = C$ for each point $x \in M^n$. Let us connect the point x with every point of U^n by all possible geodesics. Since the solution of an ordinary differential equation depends continuously on initial data, the initial velocity vectors (at the point x) of such geodesics form an open non-empty subset $V^n \subset T_x M^n \cong T_x^* M^n$. By the first statement of Lemma 1, for any geodesic γ passing through any point of U^n, the value $t_i(\gamma, \dot{\gamma})$ is equal to C. Hence, for any point $\xi \in V^n$, the value $t_i(\xi)$ is equal to C. Therefore, by the second statement of Lemma 1, $\lambda_i(x) = \lambda_{i+1}(x) = C$. The second statement of Theorem 7 is proved.

Let $\lambda_i(x) = \lambda_{i+1}(y) = \lambda$ for some $i \in \{0, 1, \dots, n-2\}$ (and for some constant λ). We will assume that $\lambda_i(x) < \lambda_{i+1}(x)$. Since the manifold is geodesically complete, there exists a geodesic γ (in the metric g) such that $\gamma(0) = x$, $\gamma(1) = y$. We will show that the geodesic γ has a point z such that $\lambda_i(z) = \lambda_{i+1}(z) = \lambda$; basically we will show that the geodesic γ consists of the points where either λ_i or λ_{i+1} (or both λ_i and λ_{i+1}) are equal to λ.

If t_i is a multiple root of the polynomial $F_t(\gamma(0), \dot{\gamma}(0))$, or if there exists a point $z \in M^n$ such that $\lambda_{i-1}(z) = \lambda$ then the statement obviously follows from Lemma 1 and the first statement of Theorem 7. Suppose t_i is not a multiple root and $\lambda_{i-1}(z) < \lambda$ for any z.

Consider the function $F_\lambda \colon T^*M^n \to \mathbb{R}$. Let at some point $(z, \nu) \in T^*M^n$, $\nu \neq 0$, the differential $\mathrm{d}F_\lambda$ is zero. Let us show that then either λ_i or λ_{i+1} (or both λ_i and λ_{i+1}) are equal to λ.

Indeed, consider the coordinate system such that the metric g at the point z is given by the diagonal matrix $\mathrm{diag}(1, 1, \dots, 1)$ and the mapping A is given by the diagonal matrix $\mathrm{diag}(\lambda_0, \lambda_1, \dots, \lambda_{n-1})$. Then the restriction of the function F_λ to the cotangent space $T^*_z M^n$ is given by

$$(-1)^n \sum_{\alpha=0}^{n-1} P_\alpha(\lambda) p_\alpha^2 \, ,$$

where the polynomials P_i are given by (8). Then the partial derivatives $\dfrac{\partial F_\lambda}{\partial p_\alpha}$ are given by

$$\frac{\partial F_\lambda}{\partial p_\alpha} = (-1)^n 2 P_\alpha(\lambda) p_\alpha \, .$$

Then λ is an eigenvalue of the mapping A. By the first statement of Theorem 7, either $\lambda_i(z) = \lambda$ or $\lambda_{i+1}(z) = \lambda$.

Let us show that the differential dF_λ vanishes at the point $(\gamma, \dot\gamma) \in TM^n$. Evidently the differential of any integral is preserved by the geodesic flow so that if dF_λ vanishes at one point of the geodesic orbit it vanishes at every point of the geodesic orbit and therefore for any $z \in \gamma$ either $\lambda_i(z) = \lambda$ or $\lambda_{i+1}(z) = \lambda$.

By Lemma 1, we have

$$\lambda = \lambda_i(x) \le t_i(\gamma(0), \dot\gamma(0)) = t_i(\gamma(1), \dot\gamma(1)) \le \lambda_{i+1}(y) = \lambda$$

so that λ is a root of the polynomial $F_t(\gamma(0), \dot\gamma(0))$ and therefore the geodesic orbit $(\gamma, \dot\gamma)$ lies in the topological space

$$Q \stackrel{\mathrm{def}}{=} \{(z, \eta) \in TM^n \, ; \, F_\lambda(z, \eta) = 0\} \, .$$

In order to show that the differential dF_λ vanishes at the point $(\gamma(0), \dot\gamma(0)) \in TM^n$, we show that any neighbourhood $W \subset Q \subset TM^n$ of the point $(\gamma(0), \dot\gamma(0))$ in the topological space Q is not homeomorphic to a disk.

By assumptions, the eigenspace of A corresponding to the eigenvalue λ_i is one-dimensional in some small neighborhood $U \subset M^n$ of the point x. Then there exists a smooth vector field ϕ on U such that $A\phi = \lambda_i \phi$ and $g(\phi, \phi) = 1$. In particular, the eigenvalue λ_i depends smoothly on the point of U and therefore the polynomial $P_i(t)$ from (8) depends smoothly on the point of U. Further we will write $P_i(t; z)$ instead $P_i(t)$.

Consider a coordinate system in $T_x M^n$ such that the metric g is given by the matrix $\mathrm{diag}(1, 1, \dots, 1)$ and the mapping A is given by the matrix $\mathrm{diag}(\lambda_0(x), \lambda_1(x), \dots, \lambda_{n-1}(x))$. In this coordinates, the component number i of the vector ϕ is equal ± 1 and the other components are zero; for any vector η, its component number i is equal to the scalar product $\pm g(\eta, \phi)$.

Consider the function $I\colon TM^n \to \mathbb{R}$, $I(z,\eta) \stackrel{\text{def}}{=} g(\eta,\phi)$. Evidently $I(\gamma(0),\dot\gamma(0)) = 0$ and the partial derivative $\partial I/\partial p_i$ at the point $(\gamma(0),\dot\gamma(0))$ is not zero. By implicit function theorem we have then that there exists some neighborhood V of the point $(\gamma(0),\dot\gamma(0))$ in the topological space

$$Q^* \stackrel{\text{def}}{=} \{(z,\eta) \in TM^n\,;\, I(z,\eta) = 0\}$$

such that V is homeomorphic to the direct product $U' \times D^{n-1}$, where $U' \subset U$ is a neighborhood of the point x and D^{n-1} denotes the disk of dimension $n-1$. Moreover, the restriction of the natural projection $\pi\colon TM^n \to M^n$ to V coincides with the natural projection: $U' \times D^{n-1} \to U'$.

For any point $(z,\nu) \in V \subset T^*U'$, consider the points

$$(z,\nu_+) = \left(z, \nu + \phi\sqrt{(-1)^{n-1}\frac{F_\lambda(z,\nu)}{P_i(\lambda;z)}}\right),$$

$$(z,\nu_-) = \left(z, \nu - \phi\sqrt{(-1)^{n-1}\frac{F_\lambda(z,\nu)}{P_i(\lambda;z)}}\right).$$

By assumptions, $P_i(\lambda;z)$ is not zero and $(-1)^{n-1}\dfrac{F_\lambda(z,\nu)}{P_i(\lambda;z)}$ is greater or equal zero. It vanishes if and only if $\lambda_i(z) = \lambda$. It is easy to see that if, for some points $(z^1,\nu^1),(z^2,\nu^2) \in V$, at least one of the relations

$$(z^1,\nu^1_+) = (z^2,\nu^2_+)\,, \qquad (z^1,\nu^1_-) = (z^2,\nu^2_+)\,,$$
$$(z^1,\nu^1_+) = (z^2,\nu^2_-)\,, \qquad (z^1,\nu^1_-) = (z^2,\nu^2_-)\,,$$

holds then automatically $(z^1,\nu^1) = (z^2,\nu^2)$.

It is easy to check that $F_\lambda(z,\nu_+) = F_\lambda(z,\nu_-) = 0$ and that any point (z,ξ) of some neighbourhood $W_1 \subset Q$ of the point $(\gamma(0),\dot\gamma(0))$ is either (z,ν^1_+) or (z,ν^1_-). Then some neighborhood of the point $(\gamma(0),\dot\gamma(0))$ in Q is homeomorphic to the direct product of two copies of the disk U' glued along the points z where $\lambda_i(z) = \lambda$ and the disk D^{n-1}. Then no neighborhood $W \subset Q$ of the point $(\gamma(0),\dot\gamma(0))$ is homeomorphic to $2n-1$-dimensional disk and the differential dF_λ vanishes at each point of the geodesic orbit $(\gamma,\dot\gamma)$.

Finally any point of the geodesic γ lies in one of the following sets:

$$\gamma_0 = \{z \in \gamma\,;\, \lambda_i(z) = \lambda\}\,,$$
$$\gamma_1 = \{z \in \gamma\,;\, \lambda_{i+1}(z) = \lambda\}\,.$$

The subsets γ_0, γ_1 are evidently closed and non-empty. Then they intersect; at each point z of the intersection we have $\lambda_i(z) = \lambda_{i+1}(z) = \lambda$. Theorem 7 is proved. □

Proof: (Theorem 4) Let M^n be closed connected. Let g, $\bar{g}$ are projectively equivalent metrics on M^n. Assume that they are strictly non-proportional at each point of the manifold. Then the eigenvalues λ_i are different at each point of the manifold. By Theorem 7 we have then that there exist $\tau_0, \tau_1, \dots, \tau_{n-2}$ such that for any $x \in M^n$

$$\lambda_0(x) < \tau_0 < \lambda_1(x) < \tau_1 < \cdots < \lambda_{n-2}(x) < \tau_{n-2} < \lambda_{n-1}(x)\,.$$

Consider the polynomial

$$(-1)^n(t-\tau_0)(t-\tau_1)\cdots(t-\tau_{n-1}) = C_{n-1}t^{n-1} + \cdots + C_0\,.$$

Consider the subspace

$$L^n = \{(x,\xi) \in T^*M^n \colon I_0(x,\xi) = C_0, I_1(x,\xi) = C_1, \dots, I_{n-1}(x,\xi) = C_{n-1}\}.$$

It is easy to see that at each point of L^n the matrix $W_{ij} \stackrel{\mathrm{def}}{=} \dfrac{\partial I_i}{\partial p_j}$ is non-degenerate. Let us fix the point $x \in M^n$. Denote by

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n$$

the eigenvalues of A at x. Since the functions F_{λ_i} are linear combinations of the functions I_i, it is sufficient to show that the determinant of the matrix

$$\tilde{W}_{ij} = \frac{\partial F_{\lambda_i}}{\partial p_j}$$

is not zero. Consider a basis of the space $T^*_xM^n$ such that the metric g is given by the identity matrix $\mathrm{diag}(1,1,\dots,1)$ and the mapping A is given by the matrix $\mathrm{diag}(\lambda_0, \lambda_1, \dots, \lambda_{n-1})$. In this basis, the matrix $\tilde{W}$ is the diagonal matrix

$$(-1)^n\,\mathrm{diag}(2\Pi_1(\lambda_1)p_1, 2\Pi_2(\lambda_2)p_2, \dots, 2\Pi_n(\lambda_n)p_n)\,.$$

Since $\lambda_i < \tau_i < \lambda_{i+1}$ for any $i \in \{0,1,\dots,n-1\}$ we have that $\Pi_2(\lambda_2)p_2$ is not zero and therefore the matrix W is non-degenerate.

In particular, the differentials

$$\mathrm{d}I_0,\ \mathrm{d}I_1, \dots,\ \mathrm{d}I_{n-1} \tag{10}$$

are linearly independent at each point of L^n. Then L^n is homeomorphic to the n-torus. By implicit function theorem, we have that the restriction of the natural projection $\pi \colon T^*M^n \to M^n$ to L^n has no singular points (in other words, the torus has no caustics). Then the torus L^n covers the manifold M^n. Theorem 4 is proved. □

3. The Homology Group of a Manifold Admitting Projectively Equivalent Metrics

Theorem 8. *Let M^n is a connected closed manifold of dimension n. Suppose that the metrics g, $\bar{g}$ on M^n are projectively equivalent and strictly non-proportional at least at one point of M^n. Then*

$$\dim(H_1(M^n, \mathbb{R})) \leq n\,.$$

Moreover, if there exists a point $x \in M^n$ such that

$$\lambda_i(x) = \lambda_{i+1}(x)$$

then

$$\dim(H_1(M^n, \mathbb{R})) < n\,.$$

The first homology group $H_1(T^n, \mathbb{R})$ of the torus T^n has dimension n.

Corollary 2. *Let metrics g, $\bar{g}$ on the torus T^n are projectively equivalent and strictly non-proportional at least at one point. Then they are strictly non-proportional everywhere.*

Sketch of proof of Theorem 8. For $n = 1$ the statement is trivial; For $n = 2$, in view of Theorem 6, Theorem 8 essentially follows from [7, 5], see [9]. Suppose Theorem 8 is true for all $n \leq k$. Let us explain why it is true for $n = k$. If the metrics are strictly non-proportional at each point, Theorem 8 follows from Theorem 4. Suppose there exists a point $x \in M^n$ such that

$$\lambda_i(x) = \lambda_{i+1}(x)\,.$$

Consider the set

$$\Lambda = \{x \in M^n\,;\ \lambda_i(x) = \lambda_{i+1}(x)\}.$$

By Theorem 7, the value of λ_i is constant on Λ; denote this constant by λ.

It appears that the set is a totally geodesic submanifold of co-dimension 2 and that the homology group $H_1(\Lambda, Q)$ coincides with the homology group $H_1(M^n, Q)$. The proof of this statement is quite long and will appear elsewhere; here we will try only to explain it. Actually, in view of Theorem 6, the first statement follows from the results of [6]. It also can be proved invariantly, using the same technique as in proof of the third statement of Theorem 7: the set Λ can be invariantly given as the projection of the set of the points of T^*M^n where the differential of the function F_λ is zero and the differential of the function

$$F'_\lambda \stackrel{\text{def}}{=} \left[\frac{\mathrm{d}}{\mathrm{d}t} F_t\right]_{t=\lambda}$$

is zero; this set is given in terms of integrals and therefore is preserved by the geodesic flow so that the submanifold Λ is geodesically complete.

The set Λ can have more than one connected components; it is always the case when the manifold is orientable. It appears that if we drop out one connected component then some other connected component is the deformation retract of the rest so that the homology groups $H_1(\Lambda, Q)$ and $H_1(M^n, Q)$ coincide. Since Λ is totally geodesic submanifold, the restriction of g, $\bar{g}$ to Λ are projectively equivalent. Evidently, they are strictly non-proportional at least at one point. By the inductive assumption, the homology group $H_1(\Lambda, Q)$ has dimension at most $n-2$ and the theorem is proved.

4. Global Levi-Civita's Coordinates on the Torus

Proof: (Theorem 2, 3) Suppose the metrics g, $\bar{g}$ on the torus T^n are projectively equivalent and strictly non- proportional at least at one point. Then, by Corollary 2, they are strictly non-proportional at each point of T^n. By Theorem 7, there exist constants $\tau_0, \ldots, \tau_{n-2}$ such that for any $x \in T^n$

$$\lambda_0(x) < \tau_0 < \lambda_1(x) < \tau_1 < \cdots < \lambda_{n-2}(x) < \tau_{n-2} < \lambda_{n-1}(x) .$$

Consider the polynomial

$$(-1)^n(t-\tau_0)(t-\tau_1)\cdots(t-\tau_{n-1}) = C_{n-1}t^{n-1} + \cdots + C_0 .$$

and the subset

$$\{(x,\xi) \in T^*M^n;\, I_0(x,\xi) = C_0, I_1(x,\xi) = C_1, \ldots, I_{n-1}(x,\xi) = C_{n-1}\}. \quad (11)$$

In the proof of the Theorem 4 we have shown that any connected component of the subset (11) is a Liouville torus without caustics. Then it covers T^n. Without loss of generality we can assume that the covering is a diffeomorphism (otherwise we can take the corresponding covering of the torus T^n).

Then the projection of the Hamiltonian vector field from the torus (11) to the torus T^n is a non-zero vector field on T^n. Let us denote this vector field by V.

At each point $x \in M^n$, consider the vectors v_i given by the conditions:

$$\begin{cases} G(v_i) & = \lambda_i v_i \\ g(v_i, v_i) & = \Pi_i \\ g(v_i, V) & > 0 . \end{cases} \quad (12)$$

Since at each point of T^n the eigenvalues of G are different, the first condition of (12) determines the one-dimensional linear space; the second condition determines the length and the third condition determines the direction so that the vector fields v_i are uniquely determined by the conditions (12). In Levi-Civita coordinates from Thereom 1 we evidently have $v_i = \pm\partial/\partial x^i$. Then the vector fields v_i commute and for each $j \neq i$ the vector field v_i preserves λ_j.

Therefore we have a locally-free action of the group $\mathbb{R}^n$ on M^n generated by the shifts along the integral curves of the vector fields. This action defines a covering of the torus T^n by the space $\mathbb{R}^n$. By construction, the pull-back of the metrics g, $\bar{g}$ on R^n is given by (1, 2).

The stabilizer of each point of T^n is a discrete subgroup of $\mathbb{R}^n$; since T^n is compact, the stabilizer is an n-lattice $Z^n \subset \mathbb{R}^n$ so that the metrics g, $\bar{g}$ are model metrics. Theorem 2 is proved.

Evidently each element of the stabilizer preserves each λ_i. It is easy to see that if for some i the eigenvalue λ_i is constant then the vector field v_i is Killing with respect to both metrics. If we assume that the metrics admit no Killing vector field then the functions λ_i are not constants and therefore the lattice must contain a sub-lattice generated by some vectors $e_0 = (l_0, 0, \dots, 0)$, $e_1 = (0, l_1, 0, \dots, 0)$, $\dots$, $e_{n-1} = (0, 0, \dots, 0, l_{n-1})$. Thus, up to a finite covering, the metrics g, $\bar{g}$ are normal model metrics. Theorem 3 is proved. $\square$

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