Advanced Studies in Pure Mathematics 21, 1992
Zeta Functions in Geometry
pp. 141–152

Zeta Function, Class Number and Cyclotomic Units of Cyclotomic Function Fields

Keqin Feng

Abstract.

Cyclotomic function fields was introduced and used by D. Hayes [15] in 1974 to construct the maximal abelian extension of rational function fields $F_q(T)$. Since then many basic properties on cyclotomic fields have been researched by S. Galovich and M. Rosen [9, 10], D. Goss [13, 14], etc. In this paper we survey several results on class number and cyclotomic units of cyclotomic function fields.

§1. Cyclotomic function fields and its Zeta function

Our notations follow [9, 10, 13, 14]. Let $k = F_q(T)$ be the rational function field over finite field F_q, $R_T = F_q[T]$, k^{ac} the algebraic closure of k. There are two endomorphisms on the F_q-module k^{ac}:

$$\begin{array}{lll} \phi : k^{\mathrm{ac}} \to k^{\mathrm{ac}} & \text{(Frobenius)} & u \mapsto u^q, \\ \mu : k^{\mathrm{ac}} \to k^{\mathrm{ac}} & \text{(multiplication by } T) & u \mapsto Tu. \end{array}$$

For $M = M(T) \in R_T$, $u \in k^{\mathrm{ac}}$, the action of M on u is defined by

$$u^M = M(\phi + \mu)(u).$$

Then k^{ac} becomes a R_T-module. Let

$$\Lambda_M = \{\lambda \in k^{\mathrm{ac}} | \lambda^M = 0\}$$

be the submodule of the M-tosion elements. $K = k(\Lambda_M) = k(\lambda)$ is called the cyclotomic field with conductor M where λ is any primitive M-tosion element. K/k is an algebraic extension and the galois group $G = \mathrm{Gal}(K/k)$ is isomorphic to $\left(R_T/(M)\right)^*$ canonically:

$$G \simeq \left(R_T/(M)\right)^*, \quad \sigma_A \mapsto A \pmod{M}$$

Received October 27, 1990.

where σ_A is determined by $\sigma_A(\lambda) = \lambda^A$.

For a finite prime P of R_T, where $P = P(T)$ is a monic irreducible polynomial in $R_T = F_q[T]$, the decomposition law of P in $K = k(\Lambda_M)$ is very like the case as the decomposition law of a prime number p in the cyclotomic number field $\mathbf{Q}(\zeta_m)$. On the other hand, the decomposition of the infinite prime $\infty = \left(\frac{1}{T}\right)$ of k in K is

$$\infty = (p_1, \dots, p_g)^e, \quad e = q-1, \quad g = \Phi(M)/(q-1)$$

where $\Phi(M) = \sharp\big(R_T/(M)\big)^* = [K:k]$. Therefore the maximal unramified subfield of ∞ in K is $K^+ = k(\lambda^{q-1})$ and K^+ is called the maximal "real" subfield of K, $\mathrm{Gal}(K/K^+) = \{\sigma_a | a \in F_q^*\}$.

$$\begin{array}{lcl} K = k(\Lambda_M) = k(\lambda) & & \{1\} \\ \quad | & & \;| \\ K^+ = k(\lambda^{q-1}) & & F_q^* \\ \quad | & & \;| \\ k = F_q(T) & & G = \big(R_T/(M)\big)^* \end{array}$$

From the decomposition law of all primes of k in $K = k(\Lambda_M)$ we obtain the Zeta function of K as follows $(U = q^{-s})$

$$Z_k(U) = Z_{k,\infty}(U) \cdot Z_{k,\mathrm{finite}}(U) = (1-U)^{-\frac{\phi(M)}{q-1}} \prod_{\chi} L(U, \chi^*)$$

where the product is taken over the Dirichlet characters χ of the finite abelian group $\big(R_T/(M)\big)^*$, χ^* is the primitive character associated with χ, and the L-function $L(U, \chi^*)$ is

$$L(U, \chi^*) = \sum_{n=0}^{\infty} S_n(\chi^*) U^n, \quad S_n(\chi) = \sum_{A} \chi^*(A)$$

where A is taken over the monic polynomials of degree n in R_T.

Similarly, the Zeta function of K^+ is

$$Z_{K^+}(U) = (1-U)^{-\frac{\phi(M)}{q-1}} \prod_{\substack{\chi \\ \chi(F_q^*)=1}} L(U, \chi^*).$$

For $\chi = \chi_0$ (trivial character), $L(U, \chi_0) = (1-qU)^{-1}$. Therefore

$$Z_{K^+}(U) = \frac{L_+(U)}{(1-U)(1-qU)}$$

where $(d = \deg M)$

$$L_+(U) = \prod_{\substack{\chi \neq \chi_0 \\ \chi(F_q^*)=1}} \frac{L(U, \chi^*)}{1-U}$$

and the class number h_M^+ of divisors of degree zero of K^+ is

$$h_M^+ = L_+(1) = \prod_{\substack{\chi \neq \chi_0 \\ \chi(F_q^*)=1}} (-\sum_{n=1}^{d-1} n \cdot S_n(\chi^*)).$$

Similarly, the class number h_M of divisors of K is

$$h_M = h_M^+ \cdot \prod_{\substack{\chi \neq \chi_0 \\ \chi(F_q^*)=1}} (-\sum_{n=0}^{d-1} S_n(\chi^*))$$

$h_M^- = h_M / h_M^+ \in \mathbf{Z}$ is called the relative class number of K.

§2. Class number of cyclotomic function field

For the number field case, let h_l and h_l^+ be the class number of cyclotomic field $\mathbf{Q}(\zeta_l)$ and its maximal real subfield $\mathbf{Q}(\zeta_l + \bar{\zeta}_l)$ respectively ($\zeta_l = e^{\frac{2\pi i}{l}}$, l is an odd prime number), $h_l^- = h_l / h_l^+$. The well-known Kummer's results say that

(1) $\quad l \mid h_l^+ \Rightarrow l \mid h_l^- \Leftrightarrow l \mid h_l$

(2) $\quad l \mid h_l^- \Leftrightarrow l \mid B_n \quad$ for some n, $n = 2, 4, \ldots, l-3$,

where B_n is the Bernoulli number. And the Vandiver Conjecture says that

(3) $\quad l \nmid h_l^+ \quad$ for any odd prime number l.

The prime number l satisfying $l \mid h_l$ is called irregular.

For the cyclotomic function field cases, the things are quite different. Let $q = p^m$, P be a monic irreducible polynomial in R_T with degree d. Then $R_T/(P) = F_{q^d}$. The character group $\hat{G}$ of $G = \mathrm{Gal}\big(k(\Lambda_p)/k\big) \cong \big(R_T/(P)\big)^*$ is a cyclic group of order $\Phi(P) = q^d = 1$ generated by the

Teichmuller character ω. And $\omega^i(F_q^*) = 1 \Leftrightarrow (q-1) \mid i$. Let $\xi = e^{\frac{2\pi i}{q^d-1}}$, $\mathfrak{p}$ is a prime ideal of $\mathbf{Z}[\xi]$ over p. We have the canonical isomorphism

$$\mathbf{Z}[\xi]/\mathfrak{p} \ (\cong F_{q^d}) \simeq R_T/(P)$$
$$\omega^i(A) \pmod{\mathfrak{p}} \mapsto A^i (\bmod P) \ (A \in R_T, 0 \le i \le \Phi(P)-1).$$

By this isomorphism,

$$\sum_{n=1}^{d-1} nS_n(\chi) \ \ (\text{for } \chi = \omega^i,\ q-1 \mid i) \text{ and} \sum_{n=0}^{d-1} S_n(\chi) \ \ (\text{for } \chi = \omega^i,\ q-1 \nmid i)$$

become

$$\sum_{n=1}^{d-1} nS_n^i(T) \quad \text{and} \quad \sum_{n=0}^{d-1} S_n^i(T)$$

respectively where

$$S_n^i(T) = \sum_{\substack{A \in R_T \\ \text{monic} \\ \deg A = n}} A^i \in R_T.$$

Let

$$\beta_i(T) = \begin{cases} \sum\limits_{n=0}^{d-1} S_n^i(T), & (1 \le i \le \Phi(P)-1,\ q-1 \nmid i) \\ \sum\limits_{n=1}^{d-1} nS_n^i(T), & (1 \le i \le \Phi(P)-1,\ q-1 \mid i) \end{cases}$$

we obtain the following results as an analogy of Kummer's result (2):

$$\begin{aligned} p \mid h_P^+ &\Leftrightarrow p \mid L(1,\chi) && (\text{for some } \chi \ne \chi_0,\ \chi(F_q^*) = 1) \\ &\Leftrightarrow P(T) \mid \beta_i(T) && (\text{for some } i,\ 1 \le i \le \Phi(P)-1,\ q-1 \mid i) \end{aligned}$$

$$\begin{aligned} p \mid h_P^- &\Leftrightarrow p \mid L(1,\chi) && (\text{for some } \chi,\ \chi(F_q^*) \ne 1) \\ &\Leftrightarrow P \mid \beta_i(T) && (\text{for some } i,\ 1 \le i \le \Phi(P)-1,\ q-1 \nmid i) \end{aligned}$$

The $\beta_i(T)$ is called the Bernoulli-Goss polynomial in R_T since it is an analogy of classical Bernoulli number and introduced first by D. Goss.

Contrasting with Kummer's result (1) and Vandiver Conjecture (3), the following data calculated by Ireland and Small [17] show that each possibility on divisibility of h_P and h_P^+ by p can occur.

($q = p = 3$, $P(T)$ is an irreducible polynomial in $F_3[T]$)

cases	$P(T)$	h^+	h^-
$3 \nmid h_P^+ h_P^-$	$2+T^2+T^3$	$53 \cdot 313$	$2^{12} \cdot 5 \cdot 79$
$3 \mid h_P^+,\ 3 \mid h_P^-$	$1+2T+T^3$	3^9	$2^{12} \cdot 3^6$
$3 \nmid h_P^+,\ 3 \mid h_P^-$	$1+2T^2+T^3$	$53 \cdot 313$	$2^{12} \cdot 3 \cdot 131$
$3 \mid h_P^+,\ 3 \nmid h_P^-$	$2+T^2+T^4$	$2^7 \cdot 3 \cdot 11^2 \cdot 17$	$2^{39} \cdot 241 \cdot 3329$
		$\cdot 29^2 \cdot 421 \cdot 191969^2$	$\cdot 65521 \cdot 1322641$

Thus we introduce the following

Detinition. An irreducible polynomial $P = P(T)$ in $R_T = F_q[T]$ is called regular (irregular) if $p \nmid h_P$ $(p \mid h_P = h_P^+ h_P^-)$. P is called irregular of first (second) type if $p \mid h_P^-$ $(p \mid h_P^+)$ where $q = p^m$.

Goss [13] and Feng [3] proved that there exist infintely many irregular P of second type in $F_q[T]$ for each q, and there exist infinitely many irregular P of first type in $F_q[T]$ for each $q \geq 3$ (for $q = 2$, $h_P^- = 1$, there is no irregular P of first type in $F_2[T]$).

The above result was improved in Feng and Gao [7] by following

Theorem 2.1. *For each $q \geq 3$, $q = p^m$, there exist infinitely many irreducible P in $F_q[T]$ such that $p \mid h_P^+$ and $p^{q-2} \mid h_P^-$. Particularly, there exist infinitely many irreducible polynomials in $F_q[T]$ which are irregular both in first and second type.*

On the other hand, concerning to regular irreducible polynomials, the result of Ireland and Small [16] shows that regular irreducible polynomials are rare at least for the case $q = p$ and $\deg P = 2$. In [7] we generalized their result, determined all regular irreducible quadratic $P(T)$ in $F_q[T]$ for all $q = p^m$, $2 \leq p \leq 269$.

We call that $P(T)$ and $Q(T)$ in $F_q[T]$ is equivalent if there exist $a \in F_q$ such that $P(T) = Q(T+a)$. It is easy to see that equivalent polynomials have same regularity.

Theorem 2.2 ([7]). *For $q = p^m$, $2 \leq p \leq 269$, an irreducible polynomial $P(T)$ in $F_q[T]$ is regular if $P(T)$ is equivalent to one of the following polynomial $P_0(T)$.*

(a) $q = 2^m$, $P_0(T) = T^2 + cT + c^2 d$, *$c$ is a primitive element of F_q, d is a fixed element in the set* $F_q - \{\alpha^2 + \alpha | \alpha \in F_q\}$.
(b) $q = 3^m$, $P_0(T) = T^2 - d$, *d is a primitive element of $F_q(T)$.*
(c) $q = 5$, $P_0(T) = T^2 + 3$. $q = 25$, $P_0(T) = T^2 \pm (1 \pm 2\sqrt{2})$.
(d) $q = 7$, $P_0(T) = T^2 + 1$.

(e) $q = 13,\ P_0(T) = T^2 + 5.$
(f) $q = 31,\ P_0(T) = T^2 + 5, T^2 + 25.$

Recently Gekeler [11] discussed the regularity of irreducible polynomials with degree ≥ 3.

§3. Cyclotomic units in cyclotomic function fields

We start with cyclotomic number field $K_m = \mathbf{Q}(\zeta_m)$ ($\zeta_m = e^{\frac{2\pi i}{m}}$, $m \geq 5$, $m \not\equiv 2 \pmod 4$). For each a, $(a, m) = 1$, let

$$\epsilon_a = \zeta_m^{\frac{1-a}{2}} \frac{1 - \zeta_m^a}{1 - \zeta_m}$$

which is an unit (of the ring of integers) in the real cyclotomic field $K_m^+ = \mathbf{Q}(\zeta_m + \zeta_m^-)$. It is well-known since Kummer that for $m = p^a$ (p is an odd prime number) the system of cyclotomic units

$$C_m = \{\epsilon_a | 2 \leq a < \frac{m}{2},\ (a, m) = 1\}$$

is independent, and

$$(1) \qquad [E_m^+ : \langle \pm C_m \rangle] = h_m^+$$

where E_m^+ and h_m^+ are the whole unit group and the class number of K_m^+ respectively. For general m, C_m may not be independent. The first such example has been presented explicitly by Ramachandra in 1966. On the other hand, let $m = \prod_{i=1}^{s} p_i^{e_i}$, and for each subset I of $S = \{1, 2, \ldots, s\}$, let

$$m_I = \prod_{i \in I} p_i^{e_i}.$$

For $2 \leq a < \frac{m}{2}$, $(a, m) = 1$, let

$$\epsilon_a' = \zeta_m^{d_a} \prod_I \frac{1 - \zeta_m^{a m_I}}{1 - \zeta_m^{m_I}}, \quad d_a = \frac{1-a}{2} \sum_I m_I$$

where I run through all proper subsets of S. In the same paper Ramachandra proved that (also see Washington [21, Theorem 8.3]) for any $m \geq 5$, $m \not\equiv 2 \pmod 4$, the system of cyclotomic units

$$C_m' = \{\epsilon_a' | 2 \leq a < \frac{m}{2},\ (a, m) = 1\}$$

is always independent, and

$$(2) \qquad [E_m^+ : \langle \pm C_m' \rangle] = h_m^+ \prod_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \prod_{p_i \nmid f_\chi} \left(\phi(p_i^{e_i}) + 1 - \chi^*(p_i) \right) \ (\neq 0)$$

where f_χ is the conductor of χ, $\phi(n)$ is the Euler function. Feng and Pei [2, 8] determined all m such that C_m is independent. levesque [18] constructed a series of new systems of cyclotomic units which are independent in many cases and have smaller index to E_m^+ than Ramachandra's one. Let $\mathcal{D}$ be a fixed subset of the set $\{d | 1 \leq d < m,\ d \mid m\}$ (proper divisors of m). For $2 \leq a < \frac{m}{2}$, $(a, m) = 1$, we defined the following real cyclotomic unit

$$\lambda_a(\mathcal{D}) = \zeta_m^{b_a} \prod_{d \in \mathcal{D}} \frac{1 - \zeta_m^{ad}}{1 - \zeta_m^d}, \quad b_a = \frac{1-a}{2} \sum_{d \in \mathcal{D}} d.$$

Let

$$\bar{C}_m^+(\mathcal{D}) = \{\lambda_a(\mathcal{D}) | 2 \leq a < \frac{m}{2},\ (a, m) = 1\}.$$

levesque [18] prove that

$$(3) \qquad [E_m^+ : \langle \pm \bar{C}_m(\mathcal{D}) \rangle] = h_m^+ i(\mathcal{D})$$

where

$$(4) \qquad i(\mathcal{D}) = \prod_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \sum_{\substack{d \in \mathcal{D} \\ f_\chi | \frac{m}{d}}} \frac{\phi(m)}{\phi(\frac{m}{d})} \prod_{p | \frac{m}{d}} (1 - \chi^*(p)).$$

The condition $i(\mathcal{D}) \neq 0$ exactly means that $\bar{C}_m^+$ is a (maximal) independent system of cyclotomic units in K_m^+. Furthermore, let

$$m = \prod_{i=1}^{s} p_i^{e_i}, \quad S = \{1, 2, \ldots, s\}.$$
$$T_0 = \{i \in S | \exists \text{even} \chi \not\equiv \chi_0 \bmod m,\ \chi^*(p_i) = 1\}.$$
$$T_0 \subseteq T \subseteq S, \quad \mathcal{D}(T) = \{m_I | I \subseteq T\}.$$

Then

$$(5)\ i(\mathcal{D}(T)) = \prod_{\substack{\chi \neq \chi_0 \\ \chi(-1)=1}} \prod_{\substack{i \in T \\ p_i \nmid f_\chi}} \left(\phi(p_i^{e_i}) + 1 - \chi^*(p_i) \right) \prod_{j \in S \setminus T} (1 - \chi^*(p_j)) \ (\neq 0)$$

Particularly, it derives the Ramachandra's result (formula(2)) by taking $T = S$. For smaller set T we have smaller value of $i(\mathcal{D}(T))$.

Based several computation data, Levesque [18] raised the following:

(A) If $\mathcal{D}^* = \mathcal{D} \cup \{d^*\}$, $m \neq d^* \mid m$, $d^* \notin \mathcal{D}$, is $i(\mathcal{D}) < i(\mathcal{D}^*)$?
(B) If $d_1 \in \mathcal{D}_1$, $d_2 \in \mathcal{D}_2$, $\mathcal{D}_1 - \{d_1\} = \mathcal{D}_2 - \{d_2\}$, is $i(\mathcal{D}_2) < i(\mathcal{D}_1)$?

Feng [4] proved that the question (A) has a affirmative answer for $m = p^\alpha$ and $p^\alpha q^\beta$, the question (B) has affirmative answer for $m = p^\alpha$, but has negative answer in general (the simplest example is $m = 3^2 5^3$, $\mathcal{D}_1 = \{1, 3, 15\}$, $\mathcal{D}_2 = \{1, 3, 25\}$, $i(\mathcal{D}_2) \geq i(\mathcal{D}_1)$).

Sinnott [20] consider the subgroup P_m of K_m^* generated by $(\pm\zeta_m)$ and the elements

$$a_x = 1 - \zeta_m^x \quad (1 \leq x \leq m-1)$$

Then $\widetilde{C_m^+} = P_m \cap E_m^+$ is the subgroup of E_m^+ generated by the cyclotomic units of all level d $(d \mid m)$. Sinnott proved that

$$\left[E_m^+ : \widetilde{C_m^+}\right] = h_m^+ \cdot 2^b$$

where

$$b = \begin{cases} 0, & \text{if } \omega(m) = 1 \\ 2^{\omega(m)-2} + 1 - \omega(m), & \text{if } \omega(m) \geq 2 \end{cases}$$

and $\omega(m)$ is the number of distinct prime factors of m.

Milnor conjectured that every multiplicative relation between the numbers a_x $(\mathrm{mod}\, \pm\zeta_m)$ is a consequence of the following relations:

$$\text{(conjugate relation)} \qquad a_x = a_{m-x}\left(-\zeta_m^x\right) \quad (1 \leq x \leq (m-1))$$

$$\text{(distribution relation)} \quad a_{dx} = \prod_{\nu=0}^{d-1} a_{x+\nu\frac{m}{d}} \qquad (d \mid m,\ 1 \leq x \leq \frac{m}{d} - 1)$$

But Ennola [22] first showed that this is not true if $\omega(m) \geq 3$. Let $V = \mathbf{Z}^{m-1}$, R is the subgroup of V generated by all possible relations between cyclotomic units, namely,

$$R = \{\underline{c} = (c_1, \ldots, c_{m-1}) \in V \mid \sum_{x=1}^{m-1} c_x ln|a_x| = 0\}$$

Let R_1 be the subgroup of V generated by

$$e_x - e_{m-x} \qquad (1 \le x \le (m-1))$$
$$e_{dx} - \sum_{\nu=0}^{d-1} e_{x+\nu\frac{m}{d}} \quad (d \mid m,\ 1 \le x \le \frac{m}{d} - 1)$$

where $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, \ldots, 0), \ldots, e_{m-1} = (0, \ldots, 0, 1)$. It is obvious that R_1 is a subgroup of R. C.-G. Schmidt [19] proved that

$$R/R_1 = (\mathbf{Z}/2\mathbf{Z})^{2^{\omega(m)-1} - \omega(m)}.$$

Sinnott and Schmidt's results based on the Mazur's distribution theory and some cohomological calculation. Recently, Gold and Kim [12] obtained a bases for the Sinnott's full cyclotomic unit group.

Now we are back to cyclotomic function field $K = k(\Lambda_m)$, $k = F_q(T)$, M is a monic polynomial in $R_T = F_q[T]$. Let O_K be the integral closure of R_T in K, U_M the unit group of O_K, then

$$U_M = F_q^* \times V_M, \quad V_M \cong \mathbf{Z}^r, \quad r = \frac{\Phi(M)}{q-1} - 1$$

Choosing a primitive M-torsion element $\lambda \in \Lambda_M$, then $K = k(\lambda)$, $K^+ = k(\lambda^{q-1})$. Let $A_0 = 1, A_1, \ldots, A_r \in R_T$ be a representive system of $\mathrm{Gal}(K^+/k) = \left(R_T/(M)\right)^*/F_q^*$. Galovich and Rosen [9] proved that for $M = P^n$ (P is a monic irreducible polynomial in R_T), the "cyclotomic" unit system

$$C_M^+ = \{\epsilon_i = \lambda^{A_i}/\lambda | 1 \le i \le r\}$$

is independent and

$$\left[U_M^+ : F_q^* \langle C_M^+ \rangle\right] = h(O_{K^+})$$

where $h(O_{K^+})$ is the ideal class number of K^+. For general case of M, Cheng and Feng [1] determined all M such that C_M^+ is independent. Feng and Yin [23] proved the following theorems which are analogies of levesque's cyclotomic units system (formula (3), (4) and (5)).

Theorem 3.1. *Let $\mathcal{D}$ be a subset of the proper monic factor of M,*

$$C_M^+(\mathcal{D}) = \{\lambda_i(\mathcal{D}) \prod_{D \in \mathcal{D}} \frac{\lambda^{DA_i}}{\lambda^D} | 1 \le i \le r\}.$$

Then

$$\left[U_M^+ : F_q^* \langle C_M^+(\mathcal{D}) \rangle\right] = h(O_{K^+}) \cdot i(\mathcal{D})$$

where

$$i(\mathcal{D}) = \prod_{\substack{\chi\neq\chi_0 \\ \chi(F_q^*)=1}} \sum_{\substack{D\in\mathcal{D} \\ F_\chi | \frac{M}{D}}} \frac{\Phi(M)}{\Phi(\frac{M}{D})} \prod_{P|\frac{M}{D}} (1-\chi^*(P)).$$

F_χ = conductor of χ and the condition $i(\mathcal{D}) \neq 0$ exactly means that $C_M^+(\mathcal{D})$ is independent.

Theorem 3.2. *Let*

$$M = \prod_{i=1}^{s} P_i^{e_i}, \quad S = \{1,2,\dots,s\}.$$
$$T_0 = \{i \in S | \exists\chi \not\equiv \chi_0 \bmod M, \ \chi^*(F_q^*) = \chi^*(P_i) = 1\}.$$
$$T_0 \subseteq T \subseteq S, \quad \mathcal{D}(T) = \{M_I = \prod_{i\in I} P_i^{e_i} | I \subseteq T\}.$$

Then

$$i(\mathcal{D}(T)) = \prod_{\substack{\chi\neq\chi_0 \\ \chi(F_q^*)=1}} \prod_{\substack{i\in T \\ P_i\nmid F_\chi}} \left(\Phi(P_i^{e_i}) + 1 - \chi^*(P_i)\right) \prod_{j\in S\setminus T} (1-\chi^*(Pj)) \ (\neq 0)$$

Therefore $C_M^+(\mathcal{D}(T))$ is (maximal) independent cyclotomic system in $K = k(\Lambda_M)$.

By chosing $T = S$, we get the following independent cyclotomic unit system of $K = k(\Lambda_M)$ which is an analogy of Ramachandra's system (formula (2)).

Theorem 3.3. *Let* $\mathcal{D} = \{M_I = \prod_{i\in I} P_i^{e_i} | I \subseteq S\}$. *Then* $C_M^+(\mathcal{D})$ *is a maximal independent unit system and*

$$\left[U_M^+ : F_q^*\langle C_M^+(\mathcal{D})\rangle\right] = h(O_{K^+}) \cdot i(\mathcal{D})$$
$$i(\mathcal{D}(T)) = \prod_{\substack{\chi\neq\chi_0 \\ \chi(F_q^*)=1}} \prod_{P_i\nmid F_\chi} \left(\Phi(P_i^{e_i}) + 1 - \chi^*(P_i)\right) \ (\neq 0)$$

We also can raise the following two questions for $K = k(\Lambda_M)$ as levesque did for number field cases:

(A) If $\mathcal{D}^* = \mathcal{D} \cup \{D^*\}$, $M \neq D^* \mid M$, $D^* \notin \mathcal{D}$, is $i(\mathcal{D}) < i(\mathcal{D}^*)$?
(B) If $D_1 \in \mathcal{D}_1$, $D_2 \in \mathcal{D}_2$, $\deg D_1 < \deg D_2$, $\mathcal{D}_1 - \{D_1\} = \mathcal{D}_2 - \{D_2\}$, is $i(\mathcal{D}_2) < i(\mathcal{D}_1)$?

We have the following partial answer for these questions:

Theorem 3.4 ([23]). *Let P and Q be distinct monic irreducible polynomials in $R_T = F_q[T]$. For $M = P^\alpha$, both questions* (A) *and* (B) *have affirmative answer. For $M = P^\alpha Q^\beta$, the queation* (A) *has affirmative answer.*

Galovich and Rosen [10] researched an analogy of Sinnott's full cyclotomic unit group in cyclotomic function field. Let P_M be the subgroup of $K^* = k(\Lambda_M)^*$ generated by F_q^* and

$$\Lambda_M^* = \Lambda_M - \{0\} = \{\lambda^D | D \in R_T,\ \ 0 \leq \deg D < \deg M\}$$

Then $\widetilde{C_M^+} = U_M^+ \cap P_M$ is the full cyclotomic unit group in K^+. By using Sinnott's method Galovich and Rosen proved that

$$[U_M^+ : \widetilde{C_M^+}] = h(O_{K^+})(q-1)^b$$

where

$$b = \begin{cases} 0, & \text{if } \omega(M) = 1 \\ (q-1)^{\omega(M)-2} + 1 - \omega(M), & \text{if } \omega(M) \geq 2 \end{cases}$$

and $\omega(M)$ is the number of distinct irreducible monic polynomial factors of M.

Feng [5] proved an analogy of Ennola and C.-G. Schmidt's results, obtained that

$$R/R_1 = (\mathbf{Z}/(q-1)\mathbf{Z})^{2^{\omega(M)-1} - \omega(M)}$$

where R and R_1 are the suitable version of the Schmidt's definition. From these Feng [6] constructed a bases for the full cyclotomic unit group $\widetilde{C_M^+}$.

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Chinese University of Science and Technology
Hefei 230026, China