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Zeta Functions in Geometry
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Spectrum and Geodesic Flow

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§0. Introduction

The zeta function of concern in this paper is the spectral zeta function $\zeta_\Delta(s)$ of a compact, riemannian manifold (M, g). It is defined by:

$$\zeta_\Delta(s) = \sum_{j=1}^{\infty} \lambda_j^{-s} \qquad (\mathrm{Re}\, s > \frac{1}{2}n),$$

where $n = \dim M$, and where $\{0 = \lambda_0 < \lambda_1 \leqq \cdots \leqq \uparrow \infty\}$ is the set of eigenvalues of the (positive) Laplacian Δ of (M, g). This set is called the spectrum of (M, g) and is denoted by $\mathrm{Spec}(M, g)$. ζ_Δ has a meromorphic continuation of to all $\mathbb{C}$, with simple poles at the points $s_j = \frac{n-j}{2}$ such that $s_j \neq 0, -1, -2, \ldots$. The residues $\mathrm{Res}_{s=s_j} \zeta_\Delta$ are perhaps the most classical spectral invariants. They are given by integrals of local geometric invariants of (M, g). More precisely, $\mathrm{Res}_{s=s_j} \zeta_\Delta = \int_M P_j(R, \nabla R, \ldots)\, d\,\mathrm{vol}$, where P_j is a polynomial in the curvature tensor R and its covariant derivatives [Gi]. The question naturally arises: to what extent is (M, g) determined by $\mathrm{Spec}(M, g)$?

It is of course well-known that $\mathrm{Spec}(M, g)$ does not always determine (M, g) up to isometry. Indeed, quite a variety of isospectral pairs is known at present (see [Su]). However, each known pair is quite special: for example, each isospectral pair has a common riemannian cover, and in most (if not all) cases, the manifolds have multiple length spectra. Here, the length spectrum $\mathrm{Lsp}(M, g)$ of (M, g) is the set of lengths L_{γ_j} of its closed geodesics γ_j. So one asks:

Question 1. Does $\mathrm{Spec}(M, g)$ determine (M, g) up to local isometry? Do isospectral pairs have a common riemannian cover?

Question 2. Does $\mathrm{Spec}(M, g)$ determine the generic (M, g) up to isometry? For example, if $\mathrm{Lsp}(M, g)$ is simple, is (M, g) spectrally determined?

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A somewhat more modest question is:

Question 3. Does $\mathrm{Spec}(M, g)$ determine the qualitative behavior of the geodesic flow G^t on the unit tangent $U(M, g)$ of (M, g)? For example, can one tell from $\mathrm{Spec}(M, g)$ if G^t is completely integrable, or ergodic or ... ?

The main thechniques for studying these questions come from the theory of FIO's (Fourier Integral Operatrors). Roughly speaking, an FIO is an operator $A\colon L^2(M_1) \to L^2(M_2)$ which moves the singularities of a distribution according to a symplectic transformation (or, correspondence) from T^*M_1 to T^*M_2. For example, the wave operator $U(t) = \exp it\sqrt{\Delta}$ is an FIO on $L^2(M)$ which propagates singularities along geodesics. By the singularities of a distribution $f \in \mathcal{D}'(M)$, we mean its wave front set $WF(f) \in T^*M \setminus O$. More precisely, then, the wave front set of the wave kernel $U(t, x, y) = \sum_j e^{it\sqrt{\lambda_j}} \varphi_j(x)\varphi_j(y) \in \mathcal{D}'(\mathbb{R} \times M \times M)$ is contained in $C = \{ (t, \tau, x, \xi, y, \eta) \mid \tau + |\xi| = 0,\ G^t(x, \xi) = (y, \eta) \} \subseteq T^*(\mathbb{R} \times M \times M)$. Here $\{\varphi_j\}$ denotes a normalized basis of Laplace eigenfunctions, and G^t denotes the Hamiltonian flow on $T^*M \setminus O$ generated by the norm $|\xi|$ (G^t is essentially the geodesic flow). See [D-G] or [Hö IV] for definitive expositions.

As shown in [D-G], the trace $\mathrm{tr}\, U(t) = \sum_j e^{it\sqrt{\lambda_j}}$ of the wave group is a distribution on $\mathbb{R}$ with singularities on $\mathrm{Lsp}(M, g)$. The big singularity of $\mathrm{tr}\, U(t)$ at $t = 0$ already contains all the information in the residues of ζ_Δ. The singularities at $t = L_\gamma$ determine new, non-local spectral invariants of (M, g): for example, the eigenvalues of the linear Poincaré map P_γ around γ. Little beyond the principal term at each singularity id known (but, see [D]). Weinstein has conjectured that the spectral invariants encoded in the singularity at $t = L_\gamma$ suffice to determine the Birkhoff-Moser normal form for the (non-linear) Poincaré map at γ ([F-G]). This would be decisive for Question 3.

Two more of the principal results of the wave kernel analysis are:

Theorem 0.1 ([D-G]). *Let* $N(\lambda) = \#\{ j : \sqrt{\lambda_j} \le \lambda \}$. *Then* $N(\lambda) = C_n \,\mathrm{vol}(M)\lambda^n + R(\lambda)$, *where* $R(\lambda) = O(\lambda^{n-1})$. $R(\lambda) = o(\lambda^{n-1})$ *if and only if the set* $\mathcal{C}$ *of closed geodesics of* (M, g) *has measure* 0.

Theorem 0.2 (Helton; [G.1]). *Let* $\Sigma = \{ \sqrt{\lambda_i} - \sqrt{\lambda_j} \}$ *be the difference spectrum of* $\sqrt{\Delta}$. *Then* Σ *is dense in* $\mathbb{R}$ *if and only if* G^t *is not periodic (i.e. at least one geodesic is not closed).*

Let us now turn to some new results, by ourself and others, on the questions raised above. The first result is a simpler version of Theorem 0.1:

Theorem 1.1. *Let* $\mu_{\lambda,t} = \frac{1}{N(\lambda)} \sum_{\sqrt{\lambda_j} \le \lambda} \delta_{e^{it\sqrt{\lambda_j}}}$ *be the probability measure on* S^1 *obtained by placing* δ*-functions at the points* $e^{it\sqrt{\lambda_j}}$ *for* $\sqrt{\lambda_j} \le \lambda$*. Then* $\mu_{\lambda,t}$ *has a limit* μ_t *as* $\lambda \to \infty$*.* $\mu_t = d\theta$ *(Lebesgue measure) if and only if* $\text{meas}(\mathcal{C}) = 0$.

The measure μ_t describes the asymptotic distribution of the eigenvalues of $\sqrt{\Delta}$ in intervals of size $\frac{2\pi}{t}$. They are analogous to Szegö limit measures for self-adjoint ψDO's ([G.2]), except that relevant operators here are the FIO's $U(t)$. They are also analogous to the limit measures in [LPS] for the eigenvalues of Hecke operators (sums of translations by isometries). The comparatively easy proof will be given in §1, together with a number of related results.

Having determined the level density of Spec($\sqrt{\Delta}$), we will go on in §2 to describe some recent results, due to Ya. G. Sinai, and to A. Uribe and ourself, on the distribution of level spacings for $\sqrt{\Delta}$ on certain surfaces. This work has to do with some conjectures, due to E. Wigner, F. Dyson, M. Berry, and others, on the relation between the fine structure of $\text{Spec}(M, g)$ and the qualitative dynamics of G^t.

To clarify the meaning of "fine structure," consider the eigenvalues $\sqrt{\lambda_j}$ of $\sqrt{\Delta}$ in an interval $[\lambda, \lambda+1]$ (say). By Theorems (0.1) or (1.1), these eigenvalues become uniformly distributed in $[\lambda, \lambda+1]$ as $\lambda \to \infty$, in the generic case where $\text{meas}(\mathcal{C}) = 0$. Further, $N(\lambda+1) - N(\lambda) = C_n \,\text{vol}(M)\lambda^{n-1} + o(\lambda^{n-1})$, so that the mean level spacing

$$D_\lambda \stackrel{\text{def}}{=} \frac{1}{\#(\lambda)} \sum_{\sqrt{\lambda_j} \in [\lambda, \lambda+1]} \left(\sqrt{\lambda_{j+1}} - \sqrt{\lambda_j}\right)$$

is asymptotic to $(C_n \,\text{vol}(M)\lambda^{n-1})^{-1}$ ($\#(\lambda)$ = number of terms). How do individual level spacings $\sqrt{\lambda_{j+1}} - \sqrt{\lambda_j}$ vary from the mean D_λ? To measure this, one first rescales the eigenvalues ($\sqrt{\lambda_j} \mapsto D_\lambda^{-1}\sqrt{\lambda_j}$) in $[\lambda, \lambda+1]$ to have unit mean level spacing. One then defines probability measures $d\nu_\lambda$ on $\mathbb{R}$ by placing δ-functions at the points $D_\lambda^{-1}(\sqrt{\lambda_{j+1}} - \sqrt{\lambda_j})$ for $\sqrt{\lambda_j} \in [\lambda, \lambda+1]$, and dividing by $\#(\lambda)$. The weak limits $d\nu$ of the $d\nu_\lambda$ as $\lambda \to \infty$ are by definition the level spacing distributions for $\sqrt{\Delta}$. One obviously hopes that there is a unique limit, but to our knowledge this is only previously known in trivial cases such as $\sqrt{\Delta}$ on S^1 or the harmonic oscillator on $\mathbb{R}^1$ (where $d\nu = \delta_1$). Were it unique, $\int_a^b d\nu$ would give the asymptotic probability of finding a normalized level spacing $s \in [a, b]$.

A somewhat simpler spectoral invariant is the pair correlation function $\rho^{(2)}$ of $\sqrt{\Delta}$. It is defined just like $d\nu$ except that one places δ-functions at all pairs $D_\lambda^{-1}(\sqrt{\lambda_i} - \sqrt{\lambda_j})$ with $\sqrt{\lambda_i}, \sqrt{\lambda_j} \in [\lambda, \lambda+1]$. The resulting measures $d\rho_\lambda^{(2)}$ are no longer of mass one, and their limits need not be finite measures. But $\int_a^b d\rho^{(2)}$ still seems the best definition of the asymptotic probability of finding some normalized eigenvalue gap in $[a, b]$.

The literature on "quantum chaos" contains a variety of numerical experiments and conjectures on $d\nu$, $d\rho^{(2)}$ and related spectoral invariants. Some of it will be reviewed briefly in §2. The most relevant aspect for this paper is the following widespread ([Si.1], [B-T])

Conjecture 2.1. *Let (M, g) be a surface with completely integrable geodesic flow. Then $d\nu = \rho e^{-\rho u}\, du$ for some constant ρ (determined by Weyl's law), and $d\rho^{(2)} = \delta_0 + \rho\, du$ (as measures on $\mathbb{R}$).*

This means that the normalized spacings S_j behave like independent, identically distributed Poisson random variables. The δ-function in $d\rho^{(2)}$ is due to the self-correlations of the eigenvalues.

Recently, Sinai has obtained a striking result on this conjecture for torii of revolution. Such a torus is made by revolving certain curves $y = F(x)$, $(0 \le x \le L)$ around the x-axis and gluing together the ends. Sinai defines a reasonably natural probability measure P on the family $\mathcal{F}$ of these curves. He proves ([S1], Theorem 2):

Theorem 2.2. *There exists a subsequence $\{\lambda_j\}$, $\lambda_j \to \infty$, and a set $\mathcal{F}_0 \subset \mathcal{F}$, $P(\mathcal{F}_0) = 1$, so that for any $F \in \mathcal{F}_0$, $d\nu_{\lambda_j}^F$ tends weakly to $\rho e^{-\rho u}\, du$.*

The actual wording of Sinai's theorem is somewhat different; see [S1-2] for the more precise statements.

Our own work with A. Uribe is on the pair correlation function for Zoll surfaces Z [U-Z]. These are surfaces with periodic, hence completely integrable, geodesic flows: $G^{2\pi} = \mathrm{Id}$. The round sphere is not the only smooth example: in fact, there is an infinite-dimensional family of examples [B]. Zoll Laplacians Δ_Z provide the most extreme cases where the hypotheses of Theorems 0.1 and 1.1 fail. In fact, Weinstein [W1] has shown that $\mathrm{Spec}(\Delta_Z)$ is a union of clusters $\mathcal{C}_\ell = \{\, \ell(\ell+1) + \mu_j^\ell :$

$j = 1, \dots, 2\ell + 1\}$ of fixed width around the spectrum of the round sphere. It follows that $\mathrm{Spec}(\sqrt{\Delta_Z})$ lies in intervals of widths $O(1/\ell)$ around the points $\lambda = \ell + 1/2$, $\ell \in \mathbb{N}$. He also shows that the eigenvalue clusters have strong asymptotic properties (see §2): so one would expect Conjecture 2.1 to be false in these cases. Indeed it is. To describe the actual pair correlation function, we must recall that the space of closed geodesics on Z is a symplectic 2-sphere $\mathcal{O}$. Further, we must recall that Δ_Z is unitarily equivalent to the round Laplacian Δ_{can} plus a 0th order ψDO (pseudo-differencial operator) A on $L^2(S^2)$ [W2]. The principal symbol σ_A of A defines a smooth fuction on $U(M,g)$. Avaraging it over closed geodesics (= orbits of G^t), we get a smooth function $\hat{\sigma}_A$ on $\mathcal{O}$. Let $\Xi_{\hat{\sigma}_A}$ be its Hamilton vector field on $\mathcal{O}$, let Ψ^s_A be its flow, and let P_0 be the primitive period set of Ψ^s_A (the set of least periods of its closed orbits). Finally, let $d\rho^{(2)}_\ell$ be the pair correlation measure for the eigenvalues of $\sqrt{\Delta_Z}$ in $[\ell + 1/2 - \epsilon, \ell + 1/2 + \epsilon]$, where ϵ is small enough so the intervals don't overlap. Let us also assume for simplicity that $\hat{\sigma}_A$ is a perfect Morse function on $\mathcal{O}$, and that the period $s(E)$ of the level $\hat{\sigma}_A = E$ is an increasing function of E.

Theorem 2.3 ([U-Z]). *With the above notation and hypotheses, there is a unique weak limit $d\rho^{(2)}$ of the sequence $\{d\rho^{(2)}_\ell\}$. Its Fourier transform is given by: $\frac{1}{(2\pi)^2} d\rho^{(2)}\hat{}(s) = V\delta_0(s) + \sum_k M(s/k)$, where $M(s) = 1_{P_0}(s)s^2$ (1_{P_0} = characteristic function of P_0), and where V is the volume of $\{\,(o_1, o_2) \in \mathcal{O} \times \mathcal{O} \,:\, \hat{\sigma}_A(o_1) = \hat{\sigma}_A(o_2)\,\}$.*

For the general result, see [U-Z].

Our third result gives affirmative answers to restricted versions of Questions 1 and 2. The restriction is partly on the dimension and curvature of (M, g). However, we also impose a new kind of condition on the isospectral pair: we will require that isospectral Laplacians Δ_1, resp. Δ_2 be intertwined by some unitary FIO $U\colon L^2(M_1) \to L^2(M_2)$: $U\Delta_1 U^* = \Delta_2$. For short, we will call such (M_i, g_i) or Laplacians Δ_i *Fourier-isospectral.*

Theorem 3.5. *Suppose that (M_1, g_1), resp. (M_2, g_2), are Fourier-isospectral surfaces of negative curvature. Then the (Mi, g_i) possess a common finite riemannian cover.*

Theorem 3.10. *In addition to the hypothesis of Theorem 1.1, suppose* $\mathrm{Lsp}(M_1, g_1)$ *is simple. Then the (M_i, g_i) are isometric.*

These theorems are Laplace-spectral applications of recent results of Croke and Otal (independently) on the marked length spectrum of negatively curved surfaces. We will sketch the main ideas of the proofs; complete proofs will appear elsewhere [Z2].

§1. Eigenvalues of the wave group

The wave group is the unitary group $U(t) = \exp it\sqrt{\Delta}$ of FIO's on $L^2(M)$. Its eigenvalues $\{\exp it\sqrt{\lambda_j}\}$ on S^1 can be ordered by the corresponding eigenvalues $\sqrt{\lambda_j}$ of $\sqrt{\Delta}$. The question we consider in this section is: how are the eigenvalues of $U(t)$ distributed on S^1. More precisely, what is the weak limit μ_t of the sequence of measures $\mu_{\lambda,t}$ defined above in §0?

These measures μ_t turn out to depend only on the set $\mathcal{C} \subset S^*M$ of periodic points for the geodesic flow G^t. This should be expected from the work of Duistermaat-Guillemin, Ivrii and more recently of Gureev, Safarov and Vasileev on the Weyl law with remainder for $N(\lambda)$. Indeed, Theorem 0.1 shows that if $\text{meas}(\mathcal{C}) = 0$, then the eigenvalues of the wave group become uniformly distributed on S^1. Here, meas is the Liouville measure on S^*M (induced from the symplectic volume measure). In the opposite case of Zoll manifolds, where all geodesics are closed and of least common period T, their work shows that the spectrum of $\sqrt{\Delta}$ clusters around an arithmetic progression $\{\frac{2\pi}{T}(k + \frac{1}{4}\alpha)\}$, α being the common Maslov (Morse) index of the closed geodesics.

Our calculation of the measures μ_t will give a direct and very simple proof of these eivenvalue distribution theorems. On the other hand, the result is somewhat weaker than the remainder estimate of theorem 0.1, since $o(\lambda^n)$ could fail for a sparse set of λ without affecting the asymptotic distribution.

Theorem 1.1. *With the above notation,* $\mu_t = d\theta$ *if and only if* $\text{meas}(\mathcal{C}) = 0$.

Proof. By definition, $\mu_{\lambda,t}(f) = \frac{1}{N(\lambda)} \operatorname{Tr} \pi_\lambda f(U(t))\pi_\lambda$, where π_λ is orthogonal projection onto the span of $\{\, \phi_j \colon \sqrt{\lambda_j} \leq \lambda \,\}$, where $f \in C(S^1)$ and where $f(U(t))$ is determined by the spectral theorem. The notation $\mu(f)$ of course refers to the pairing of measures and continuous functions. To determine the weak limit of $\mu_{\lambda,t}$ as $\lambda \to \infty$ it suffices to calculate the limiting moments $\lim_{\lambda\to\infty} \mu_{\lambda,t}(z^k)$ for $k \in \mathbb{Z}$. So we consider $\operatorname{Tr} \pi_\lambda U(t)^k \pi_\lambda = \operatorname{Tr} \pi_\lambda U(kt)\pi_\lambda$. Now, recall that $U(t)$ is an FIO for each fixed t whose underlying canonical relation is the graph $\Gamma_t \subset (T^*M \setminus 0) \times (T^*M \setminus 0)^-$ of the geodesic flow G^t on $(T^*M \setminus 0)$ (the minus means to reverse the sign of the symplectic form). Further, its principal symbol σ_t is a certain canonical section $\nu_t \otimes s_t$ of the bundle $\Omega^{1/2}_{\Gamma_t} \otimes M_{\Gamma_t}$, where $\Omega^{1/2}_{\Gamma_t}$ is the space of $\frac{1}{2}$-densities on Γ_t and where M_{Γ_t} is its Maslov bundle (a flat, trivializable Hermitian line bundle). In fact, ν_t is juct the pull-back of the symplectic volume $\frac{1}{2}$-density on $T^*M \setminus 0$ under the natural projection from Γ_t (either one). Further, s_t is the section of M_{Γ_t} which corresponds to the cannonical constant section s_0 of M_{Γ_0} under the natural symplectic diffeomorphism between Γ_t and Γ_0 (see [D-G] for more details).

Next, we observe that the asympotics of order λ^n of $\operatorname{Tr} \pi_\lambda U(tk)\pi_\lambda$ can be determined from the main singularity at $s = 0$ of the distributional trace $\operatorname{Tr} U(s)U(tk)$. More precisely, $\operatorname{Tr} U(s)U(tk)$, as a Lagrangean distribution in s, has only a discrete set of singularities and at each singularity s_0 it has an expansion as a sum of homogeneous distributions $(s - s_0 - io)^r$. Suppose that the most singular term at$s_0 = 0$ has the form $\alpha(kt)(s + io)^{\frac{1}{2} - \frac{e(kt)}{2}}$ (for some constants $\alpha(kt)$ and $e(kt)$). A standard kind of argument, which we only sketch, shows (with C_n a certain universal constant):

(1.2) **Lemma.** $\operatorname{Tr} \pi_\lambda U(tk)\pi_\lambda = C_n \alpha(kt)\lambda^{\frac{1}{2}(e(tk)+1)} + O(\lambda^{n-1})$.

Proof (sketch). Let $N(\lambda;\, tk) = \operatorname{Tr} \pi_\lambda U(tk)\pi_\lambda$, so that $dN(\lambda;\, tk) = \sum_j e^{itk\sqrt{\lambda_j}}\delta(\lambda - \sqrt{\lambda_j})$. Also, let ρ be a smooth function on $\mathbb{R}$, with compactly supported Fourier transform $\hat{\rho}$, such that $\operatorname{supp} \hat{\rho}$ contains only the singularity at $s = 0$ of $\operatorname{Tr} U(s)U(tk)$. We claim: the convolution $\rho * dN(\lambda;\, tk) = C_n \alpha(tk)\lambda^{\frac{1}{2}(e(tk)+1)} + O(\lambda^{\frac{1}{2}(e(kt)-1)})$. The proof of this goes very much as in [D-G, Proposition 2.1] or [G.2, Lemma 1]. To make it plausible, we note that $\rho * dN\hat{}(s) = \hat{\rho}(0)\alpha(kt)(s + i0)^{\frac{1}{2}(1-e(kt))} +$ smoother, so $\rho * dN(\lambda;\, tk)$ should be $\hat{\rho}(0)C_n\alpha(kt)\lambda^{\frac{1}{2}(1+e(kt))}$ + lower order. A more complete proof uses an oscillatory integral expression for $\rho * dN(\lambda;\, tk)$ and applies the method of stationary phase. Since $U(tk)$ is an FIO, the integral expression for $\rho * dN$ is a little more complicated than in [G.2, Lemma 1]; the details are in [Z.1].

Now, choose ρ so that additionally $\int \rho(s)\, ds = 1$. Then we claim: $N(\lambda;\, tk) = \rho * dN(\lambda;\, tk) + O(\lambda^{n-1})$. Indeed, this follows from Hormander's estimate $O(\lambda^{n-1})$ for the remainder in the Weyl law, just as in [G.1, Lemmas 2–3].

The two asymptotic formulae combine into (1.2). □

The lemma shows that $\mu_t \neq 0$ if and only if $e(kt) = 2n - 1$ and $\alpha(kt) \neq 0$. We now identify these constants.

Let $\operatorname{Fix}(G^t)$ be the set if fixed points of G^t in $T^*M \setminus 0$, and $\operatorname{SFix}(G^t)$

denote the same for S^*M (the unit co-sphere bundles). For now, we will assume these fixed point sets to be clean (cf. [D-G]).

(1.3) **Lemma.** (i) $e(kt) = \dim \mathrm{SFix}(G^{kt})$.
(ii) *If* $e(kt) = 2n-1$, $\alpha(kt) = e^{i\pi/4m(kt)} \operatorname{meas}(\mathrm{SFix}(G^{kt}))$, *where* $m(kt)$ *is the common Morse index of the periodic geodesics of period* kt.

Proof. Both (i) and (ii) follow from the clean composition theorem for FIO's [Hö IV, §25]. Briefly, $\alpha(kt)$ is the principal symbol of $\operatorname{Tr} U(s)U(tk)$ at the singularity $s=0$, and $e(kt)$ is the excess of the composition. To identify them, we recall that taking the trace corresponds to intersecting the graph composition $\Gamma_s \circ \Gamma_{tk}$ with the diagonal in $(T^*M \setminus 0) \times T^*M \setminus 0)$. Of course, this intersection is $\mathrm{Fix}(G^s \circ G^{kt})$. As with the fixed point set of any symplectic map, $\mathrm{Fix}(G^s \circ G^{kt})$ has a natural volume density [D-G, §4]. Furthermore, each component has a Maslov index (the common Morse index of the closed geodesics in the component). Under the clean intersection hypothesis, the set of s for which $\mathrm{Fix}(G^s \circ G^{kt}) \neq 0$ is discrete and equals $\operatorname{sing\,supp} \operatorname{Tr} U(s)U(kt)$. In particular, $s=0$ is in the singular support only of $\mathrm{Fix}(G^{kt}) \neq \emptyset$. The composition formula [Hö IV, 25.2.10] then shows that the principal symbol $\alpha(kt)$ is the sum over the components if $\mathrm{SFix}(G^{kt})$ of the volume times the Maslov factor of each. Here, we use that the symbol of $U(kt)$ is essentially the canonical volume density on the graph. It also shows that the order of the singularity is the excess $e(kt)$, equal to the maximal dimension of these components. These statements are more precise versions of (i)–(ii) above. For more details, see [Z1]. □

Under the clean fixed point hypothesis, $e(kt) = 2n-1$ implies that $\mathrm{SFix}(G^{kt})$ is a submanifold of full Liouville measure in S^*M, hence equals S^*M. Cancelling the common factors C_n and $\operatorname{meas}(S^*M)$ in $N(\lambda; tk)$ and $N(\lambda)$, we get:

(1.4) **Corollary.**

$$\mu_t(z^k) = \begin{cases} e^{i\pi/4m(kt)} & G^{kt} = \mathrm{id}, \\ 0 & G^{kt} \neq \mathrm{id}. \end{cases}$$

It follows that $\mu_t = d\theta$ if and only if $G^{kt} \neq \mathrm{id}$ for $k \neq 0$, proving the theorem. □

The proof shows a little more than what is stated in Theorem 1.1, since it always gives a formula for μ_t if $\operatorname{meas}(\mathcal{C}) \neq 0$. This adds little if $\mathrm{Fix}(G^{kt})$ is clean: indeed, $\operatorname{meas} \mathrm{SFix}(G^{kt}) > 0$ would then imply

$\mathrm{SFix}(G^{kt}) = S^*M$ (as noted above). It adds more, however, if we relax the cleanliness condition somewhat. For instance, we might allow it to fail along a submanifolds of positive codimension in S^*M: let us call such a flow "almost clean". A natural "almost clean" flow occurs, for example, if we generically perturb the round sphere outside an equatorial annulus, so that $\mathrm{SFix}(G^t)$ has measure 0 for $t \neq 2\pi k$ ($k \in \mathbb{N}$), and $\mathrm{SFix}(G^{2\pi k})$ consists of the great circles in the annulus. More generally, such an equatorial annulus could be designed an any surface. The unclean points are of course on the great circles touching the boundary of the annulus tangentially.

(1.5) **Theorem.** *With the above notation and terminology, suppose G^t is almost clean. Then μ_t exists and has the moments:*

$$\mu_t(z^k) = \sum_j e^{i\pi 4\, m_j(k,t)} \frac{\mathrm{meas}(\mathrm{SFix}\, G^{kt})_j}{\mathrm{meas}\, S^*M}$$

(*the sum runs over the components* $\mathrm{SFix}(G^{kt})_j$ *of* SFix, *and* m_j *is the common Morse index of all closed geodesics in the component*).

The proof is not difficult, and probably extends to much more general flows [Z1].

§2. Level spacing and pair correlation

The measure μ_t of §1 is what some physicists would call the level density of $\mathrm{Spec}(\sqrt{\Delta})$, on the length scale $\frac{2\pi}{t}$. We saw that it is determined by rather crude dynamical invariants of G^t. To get at deeper relations between spectrum and geodesic flows, a variety of physicists (see §0) suggest studying the spectrum on the increasingly fine length scale D_λ in $[\lambda, \lambda+1]$. This is technically difficult, however, because the normalized eigenvalues $D_\lambda^{-1}\sqrt{\lambda_j}$, $\sqrt{\lambda_j} \in [\lambda, \lambda+1]$, are not eigenvalues of a first order ψDO, and the unitary group they generate doesn't propagate singularities. This would seem to spoil the classical FIO methods of §1. On the other hand, the k-correlation functions for $k \geq 2$ and level spacings distribution depend only on differences $D_\lambda^{-1}(\sqrt{\lambda_i} - \lambda_j)$ of such normalized eigenvalues. In dimension $n = 2$, such differences are in fact eigenvalues of a first order operator; so it is plausible that FIO methods could apply to them.

We will now see that this approach succeeds for the rather special case of Zoll surfaces. The result is quite different from the Poisson statistics conjectured for surfaces with completely integrable geodesic flow. (Conjecture 2.1 in §0). This is due to the finer clustering of eigenvalues in the Zoll as opposed to the typical completely integrable case.

Proof of Theorem 2.3 [U-Z]. Let g_Z be a Zoll metric on S^2: i.e. a metric whose geodesic flow G^t is 2π-periodic (say). By a theorem of Weinstein [W.1, Theorem 4.1], the Zoll Laplacian Δ_Z is unitarily equivalent to $\Delta_{\text{can}} + A$ on $L^2(S^2)$, where Δ_{can} is the standard Laplacian and A is a 0th order self-adjoint ψDO. Guillemin [G.3, Lemma 1] then goes on to show that $\Delta_{\text{can}} + A$ is unitarily equivalent to $\Delta_{\text{can}} + A^\sharp$, where $A^\sharp$ is self adjoint and $[\Delta_{\text{can}}, A^\sharp] = 0$. Incidentally, these equivalences are all Fourier-isospectralities.

Let $\mathcal{H}_\ell$ be the space of spherical harmonics of degree ℓ on S^2, and let π_ℓ be the orthogonal projection from $L^2(S^2)$ to $\mathcal{H}_\ell$. Then $\pi_\ell A^\sharp \pi_\ell$ is a self-adjoint $(2\ell+1) \times (2\ell+1)$ matrix with some eigenvalues μ_j^ℓ, $(j = 1, \ldots, 2\ell+1)$. The level density for these eigenvalues was determined by Weinstein: more precisely, he showed that the weak limit of the cluster measures $\frac{1}{2\ell+1} \sum_{j=1}^{\ell} \delta(\mu - \mu_j^\ell)$ equals the pushforward $\hat{\sigma}_{A*}\, dm$ of Liouville measure by the principal symbol $\hat{\sigma}_A$ of $A^\sharp$ [W2]. Here, $\hat{\sigma}_A$ is the average of the principal symbol σ_A of A over closed geodesics of S^2.

Our problem is essentially to determine the limiting pair correlation function for the eigenvalues of $\pi_\ell A^\sharp \pi_\ell$. Up to a sparse subsequence, these eigenvalues are contained in the interval $[\min \hat{\sigma}_A, \max \hat{\sigma}_A]$, so the mean level spacing D_ℓ for the eigenvalues of $\pi_\ell A^\sharp \pi_\ell$ is $\frac{w}{2\ell+1}$, where w is the width $\max \hat{\sigma}_A - \min \hat{\sigma}_A$ of the support of $(\hat{\sigma}_A)_* dm$. Let us normalize things so that $D_\ell = \frac{1}{\ell} + O(\frac{1}{\ell^2})$. The problem is then to calculate the weak limit of $d\rho_\ell^{(2)}$, where

$$\rho_\ell^{(2)}(f) = \frac{1}{2\ell+1} \sum_{i,j=1}^{2\ell+1} f\left(\ell(\mu_i^\ell - \mu_j^\ell)\right).$$

This version of the pair correlation differs only in inessential ways from the one concerning $\Delta_Z^{1/2}$ in §0. Indeed, $\Delta_Z^{1/2}$ is unitarily equivalent to $(\Delta_{\text{can}} + A^\sharp)^{1/2}$. For simplicity, we will replace $\Delta_{\text{can}}^{1/2}$ by the ψDO $P = (\Delta_{\text{can}} + \frac{1}{4})^{1/2} - \frac{1}{2}$, which has eigenvalue ℓ in $\mathcal{H}_\ell$. The spectrum of $(P^2 + A^\sharp)^{1/2}$ then consists of clusters of eigenvalues $\mu_j^\ell = (\ell^2 + \mu_j^\ell)^{1/2} = \ell + \frac{1}{2}\ell^{-1}\mu_\ell^j + O(\ell^{-2})$ around $\ell \in \mathbb{N}$, of widths $O(1/\ell)$. The mean level density $\bar{D}_\ell$ in the ℓ-th cluster is then asymptotically $\frac{1}{2}\ell^{-2}$ (if we normalize things as before). The pair correlation measure for the ℓ-th cluster is thus

$\bar{\rho}_\ell^{(2)}(f) = \frac{1}{2\ell+1}\sum_{i,j=1}^{\ell} f(\bar{D}_\ell(\bar{\mu}_i^\ell - \bar{\mu}_j^\ell)) = \rho_\ell^{(2)}(f) + O(\ell^{-1})$ for any smooth f ($O(\ell^{-1})$ depends of course on f). It obviously follows that the limit measures of $d\bar{\rho}_\ell^{(2)}$ and $d\rho_\ell^{(2)}$ are the same.

To calculate the limit of $\rho_\ell^{(2)}(f)$ we form the Fourier series $\Upsilon_f(\theta) = \sum_{\ell=0}^{\infty}(2\ell+1)\rho_\ell^{(2)}(f)e^{i\ell\theta}$. Somewhat as in §1, $\Upsilon_f(\theta)$ will be a Lagrangean distribution and its main singularities will determine the asymptotic behavior of $\rho_\ell^{(2)}$. In fact, since Υ_f is also a Hardy series on S^1, its complete singularity expansion determines (in principle) a complete asymptotic expansion for $\rho_\ell^{(2)}$ (cf. [G.3, §3]).

We first note that

$$\Upsilon_f(\theta) = \operatorname{Tr} e^{i\theta P}\Pi f(Q)\Pi \tag{2.4}$$

where $\Pi\colon L^2(S^2\times S^2) \to \bigoplus_\ell \mathcal{H}_\ell \otimes \mathcal{H}_\ell$ is the orthogonal "diagonal ladder" projection $\bigoplus_\ell \pi_\ell \otimes \pi_\ell$, where $Q = PA^\sharp \otimes 1 - 1 \otimes PA^\sharp$, and where $e^{i\theta P}$ acts on the first factor.

The main point is that each operator in (2.4) is an FIO, and one can analyze the main singularity of Υ_f by a symbolic calculation. We have already discussed $e^{i\theta P}$ in §1. The operator Π is also an FIO, belonging to a certain $*$-algebra $\mathcal{R}_\Sigma$ introduced by Guillemin-Sternberg in their study of homogeneous quantization [G-S]. To describe it, we note that Π is the projection onto the kernel of $P \otimes 1 - 1 \otimes P$. Symbolically, this kernel corresponds to the characteristic variety $\Sigma \subset T^*(S^2 \times S^2)$ where $|\xi_1| - |\xi_2| = 0$. The Hamilton flow $G^t \times G^{-t}$ of $|\xi_1| - |\xi_2|$ determines an S^1-action on Σ (the directions where the symplectic form along Σ degenerates). Let $\Lambda \subset \Sigma \times \Sigma$ be the equivalence relation of belonging to the same orbit (or, null leaf). Then Λ is Lagrangean in $T^*(S^2 \times S^2) \times T^*(S^2 \times S^2)^-$ and $\mathcal{R}_\Sigma$ is the algebra of FIO's associated to Λ (see [G-S] for more detail).

Last, $f(Q)$ is an FIO associated to a rather similar canonical relation Λ_f to Λ. In fact, Λ_f is the equivalence relation of belonging to the same orbit of the Hamiltonian flow $\exp s\Xi_{\hat{\sigma}_A} \times \exp -s\Xi_{\hat{\sigma}_A}$ on $\Sigma_{\hat{\sigma}_A} = \{ (x_1,\xi_1,x_2,\xi_2) \in T^*(S^2 \times S^2) : |\xi_1|\hat{\sigma}_A(x_1,\xi_1) = |\xi_2|\hat{\sigma}_A(x_2,\xi_2) \}$, with $s \in \operatorname{supp}\hat{f}$. To make this plausible, we observe that $f(Q) = \int_{\mathbb{R}} \hat{f}(s)e^{isPA^\sharp} \otimes e^{-isPA^\sharp}\,ds$, and that $\Sigma_{\hat{\sigma}_A}$ is just the characteristic variety of $PA^\sharp \otimes 1 - 1 \otimes PA^\sharp$ (see [U-Z] for more details).

Under generic conditions, the composition in (2.4) is clean, and $\Upsilon_f(\theta)$ is a Lagrangean distribution on S^1. In [U-Z] we find that

$$\operatorname{sing\,supp}\Upsilon_f = \{ \theta : \exists s \in \operatorname{supp}\hat{f},\ t \in S^1 : \operatorname{Fix}(\chi^{(s,t,\theta)}) \neq \emptyset \},$$

where $\chi^{(s,t,\theta)} = (G^\theta \times \mathrm{id}) \circ (G^t \times G^{-t}) \circ (\exp s\Xi_{\hat{\sigma}_A} \times \exp -s\Xi_{\hat{\sigma}_A})$, and where Fix means the fixed points in $\Sigma \cap \Sigma_{\hat{\sigma}_A}$. As in §1, the main singularities occur at θ for which $\bigcup_{\substack{s\in \mathrm{supp}\,\hat{f} \\ t\in S^1}} \mathrm{Fix}(\chi^{(s,t,\theta)})$ is of maximal dimension.

To analyze $\mathrm{Fix}(\chi^{(s,t,\theta)})$, we use the obvious symmetry properties of the relevant flows. First $(G^\theta \times \mathrm{id}) \circ (G^t \times G^{-1})$ defines an $\mathbb{R}^+$-homogeneous $S^1 \times S^1$ action on Σ. The quotient $\Sigma/S^1 \times S^1 \times \mathbb{R}^+$ is evidently the symplectic manifold $\mathcal{O} \times \mathcal{O}$, where $\mathcal{O}$ is the space of geodesics of G^t on $S^*(S^2)$. Now $\exp s\Xi_{\hat{\sigma}_A} - \exp -s\Xi_{\hat{\sigma}_A}$ commutes with this $S^1 \times S^1$ action, so descends to a flow $\Psi^s_A \times \Psi^{-s}_A$ on $\mathcal{O} \times \mathcal{O}$. It is the Hamiltonean flow of $H(o_1, o_2) = \hat{\sigma}_A(o_1) - \hat{\sigma}_A(o_2)$ of the difference $(\hat{\sigma}_A)_1 - (\hat{\sigma}_A)_2$ on Σ, which of course descends to the quotient because the $\hat{\sigma}_A$'s are invariant. Since only pairs (o_1, o_2) with $H(o_1, o_2) = 0$ contribute to $\mathrm{Fix}(\chi^{(s,t,\theta)})$, we see that $\mathrm{Fix}(\chi^{(s,t,\theta)}) = \emptyset$ unless there are closed geodesics o_1, resp. o_2 so that:

$$(2.6) \qquad \begin{cases} \hat{\sigma}_A(o_1) = \sigma_A(o_2) \\ \Psi^s_A(o_i) = o_i, \;\; s \in \mathrm{supp}\,\hat{f} \qquad (i = 1, 2) \end{cases}$$

The solutions (s, o_1, o_2) of (2.6) may be easily determined if $\hat{\sigma}_A$ is a Morse function on $\mathcal{O}$, which we henceforth assume. A variety of cases are then possible, according to whether the o_i lie on a regular or critical level, and to whether or not they lie on the same component of the level. For brevity, we are only considering the simplest possible case where $\hat{\sigma}_A$ is a perfect Morse function (with just minimum and maximum critical points), and refer to [U-Z] for the general case. Then the levels $\hat{\sigma}_A = E$ are all given by orbits of Ψ^s_A, so the solutions (s, o_1, o_2) consist of pairs of points on levels where $s = ks_0(E)$ is a multiple of the (primitive) period $s_0(E)$ of the level $(k \in \mathbb{N})$. Of course, there are two exceptional pairs $(o^\pm, o^\pm)$ corresponding to the maximum o^+ and minimum o^- points.

Each solution corresponds to a pair of closed geodesics o_{z_i} in $S^*(S^2)$, which are carried back to themselves, possibly with rotation, by the flow $\exp \Xi_{\hat{\sigma}_A}$ on $S^*(S^2)$. Since both o_{z_i} occur on the same reduced Ψ^s_A-orbit, these rotation angles coincide. In the case at hand, where levels are orbits, the rotation angle depends only on the level; it will be denoted by $\tau(E)$.

Now consider a fixed point $(z_1, z_2) \in (S^*(S^2) \times S^*(S^2)) \cap \Sigma_{\hat{\sigma}_A}$ for $\chi^{(s,t,\theta)}$, and denote the geodesic through z_i by o_{z_i}. Then (s, o_{z_1}, o_{z_2}) is a solution of (2.6). Further, the rotation angle for o_{z_1} is evidently $-(t+\theta)$, while that for o_{z_1} is $-t$. Since the angles coincide, we must have $\theta \equiv 0$

(mod 2π). Hence, $\Upsilon_f(\theta)$ is singular only at $\theta = 0$.

The order of the singularity at $\theta = 0$ is determined by the dimension of $F_0 \stackrel{\text{def}}{=} \bigcup_{\substack{s \in \operatorname{supp} \hat{f} \\ t \in S^1}} (s, t, \operatorname{Fix}(\chi^{(s,t,0)}))$. F_0 consists of a finite number of components (depending on $\operatorname{supp} \hat{f}$). First, if $0 \in \operatorname{supp} \hat{f}$, there is the contribution from $s = 0$. Evidently, $\operatorname{Fix}(\chi^{(0,t,0)}) \neq \emptyset$ if and only if $t \equiv 0$ (mod 2π). So $s = 0$ contributes the identity component $F_0^{(\text{id})} = (0, 0, \Sigma_{\hat{\sigma}_A} \cap \Sigma)$, of dimension 6. Second, consider $s \in \operatorname{supp} \hat{f} \cap P$, where P is the period set of Ψ_A^s. By definition, P is the union of intervals $[ks_0(E_{\min}), ks_0(E_{\max})]$, with $k \in \mathbb{N}$, and with $s_0(E_{\min}) > 0$ equal to the infimum of the periods of (non-critical) orbits, and with $s_0(E_{\max}) < \infty$ equal to the supremum. For simplicity, we are assuming that only one level $\hat{\sigma}_A = E(s)$ has a given primitive period s, and that $E(s)$ increases with s. Then each $s \in \operatorname{supp} \hat{f} \cap P$ contributes the points $(s, \tau(E_j(s)), \mathbb{R}^+ \cdot (F(s) \times F(s)))$ to $F_0(s)$, where $F(s)$ consists of the $z \in S^*(S^2)$ whose G^t-orbit o_z lies in the level $E(s)$. As s runs over P, we get a second 6-dimensional (homogeneous) component $F_0^{(\text{reg})} = \bigcup_{s \in P} (s, \tau(E_j(s)), \mathbb{R}^+ \cdot (F(s) \times F(s)))$, consisting of regular fixed points.

Besides $F_0^{(\text{id})}$ and $F_0^{(\text{reg})}$, F_0 consists two additional components $F_0^{\pm}$ corresponding to the critical orbits o^+ and o^-. These orbits have well-defined rotation angles $\tau(E_+)$ and $\tau(E_-)$, and so $F_0^{\pm} = \bigcup_{s \in \mathbb{R}} (s, \tau(E_{\pm}), o^{\pm} \times o^{\pm})$ is only of dimension 4. Hence the main singularity is determined by the 6-dimensional manifolds $F_0^{(\text{id})}$ and $F_0^{(\text{reg})}$.

These fixed point sets determine the order d and principal symbol $\alpha_0(0)$ of Υ_f at its singularity at $\theta = 0$ much as in Lemma 1.3. We will only describe the results, and refer to [U-Z] for details. First, d is clearly related to $\dim F_0$, and one finds that $d = 1$. Second, $\alpha_0(0)$ is expressed in the composition formula for FIO's [Ho IV, 25.2.7] as a sum of integrals over the maximal dimensional components of $F_0 \setminus \mathbb{R}^+$. Corresponding to $F_0^{(\text{id})}$, we get $\hat{f}(0) \operatorname{vol}(\Sigma_{\hat{\sigma}_A} \cap S^*(S^2) \times S^*(S^2))$, where vol is the surface density on the hypersurface $\hat{\sigma}_A = 0$ in $S^*(S^2) \times S^*(S^2)$ (which carries its Liouville density). Corresponding to the above fixed point sets $F(s) \times F(s)$ for $\chi^{(s,t,0)}$, we get $\hat{f}(s)(2\pi)^2 \left|\hat{\sigma}_A = E(s)\right|^2$, where $|\hat{\sigma}_A = E|$ is the measure on this curve in $\mathcal{O}$, relative to the line element induced from the symplectic form on $\mathcal{O}$. Using period-energy coordinates, one easily shows that $|\hat{\sigma}_A = E(s)| = s$. The contribution from the primitive period spectrum is accordingly $(2\pi)^2 \int_{P_0} \hat{f}(s) s^2 \, ds$. A similar contribution arises from $2P_0$, $3P_0$, ... , the k-th one being $(2\pi)^2 \int_{kP_0} \hat{f}(s)(s/k)^2 \, ds$.

On the other hand, since Υ_f is a Lagrangean, Hardy distribution with singularity only at 0, we have:

$$\Upsilon_f(e^{i\theta}) \sim \sum_{\ell=0}^{\infty} \alpha_0(\ell) b_{d-\ell}(e^{i\theta}), \tag{2.8}$$

where $b_r(e^{i\theta}) = \sum_{n=0}^{\infty} n^r e^{in\theta}$ [G-U, §7]. Comparing coefficients in (2.4) and (2.8), we see that

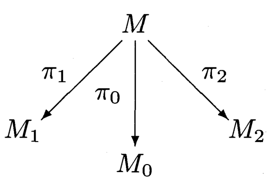
$$\rho_\ell^{(2)}(f) = \alpha_0(0) + O(\ell^{-1}). \tag{2.9}$$

Theorem (2.3) follows immediately. □

§3. Fourier-isospectrality and local isometry

To motivate Theorems 3.5 and 3.6, we begin by recalling the construction of the largest class of non-isomorphic, isospectral pairs (M_i, g_i) $(i = 1,2)$ [Su]. With Sunada, we suppose M_1 and M_2 fit into a diagram

(3.1)



of finite normal covers. We let H_i denote the covering group for π_i and G that for π_0. Further, we let p_i denote the (not necessarily normal) covering projections: $M_i \to M_0$.

(3.2) **Theorem** [Su]. *Suppose $L^2(G/H_1) \simeq L^2(G/H_2)$ are equivalent G-modules. Then, for any g_0 on M_0, $p_1^*(g_0)$ is isospectral to $p_2^*(g_0)$.*

To this theorem we add the following observation: one may explicitly construct a unitary intertwining operator $U\colon L^2(M_1) \to L^2(M_2)$ between the Laplacians Δ_i of $(M_i, p_i^*(g_0))$. Indeed, let $S_A\colon L^2(G/H_1) \to L^2(G/H_2)$ be the intertwining (convolution) operator

$$S_A f(x) = \sum_{y \in G/H_1} A(xy^{-1}) f(y) \tag{3.3}$$

corresponding to a complex-valued function A on the double coset space $H_2\backslash G/H_1$ [Ma, p. 365]. Then set:

$$U_A = \sum_{g\in G} A(g)\pi_{2*}T_g\pi_1{}^* \tag{3.4}$$

where T_g is translation by g on $L^2(M)$. It is not hard to show that S_A unitary implies U_A unitary [Z2]. U_A obviously intertwines the Laplacians since each term in it is a composition of local isometries.

The unitary operator in (3.4) is a very simple kind of FIO: it is a sum of finite Radon transforms, with canonical relation C equal to the union of the co-normal bundles $N^*(\text{graph}(\pi_2 \circ T_g\pi_1^{-1}))$ of the graphs of the correspondences determined by (3.1).

Thus, the Sunada examples are Fourier-isospectral in the sense of §0: the Laplacian are intertwinable by a unitary FIO $U: L^2(M_1) \to L^2(M_2)$. This observation raises two obvious questions: First, how "generically" are isospectral pairs Fourier? Second, must all Fourier-isospectral pairs have a common riemannian cover?

We will sketch below a proof of Theorem 3.5, to the effect that negatively curved Fourier-isospectral surface have a common riemannian cover. Most likely, the same theorem is true in higher dimensions and with weaker hypotheses on the curvature (see below). We therefore expect the answer to the second question to be "yes" for broad classes of metrics. This gives some evidence that the answer to Question 2 of §0 is also "yes" for these metrics.

The first question above is therefore fundamental, since one of the principal isospectral problem can be reduced to it. Unfortunately, we know little about it at present, beyond testing known examples for Fourier-isospectrality. Besides the Sunada examples,the only ones tested so far are the De Turck-Gordon-Wilson nilmanifold ones: F. Marhuenga has shown that (some of) these are Fourier-isospectral, at least via the singular FIO's of Melose-Uhlmann [M]. More general types of FIO's may be needed in other cases. However, it would not be surprising if some (hopefully rare) examples failed to be Fourier.

(3.5) *Proof of Theorem* (3.5) ([Z2]). We suppose there is a unitary FIO U, as above, so that $U\Delta_1 U^* = \Delta_2$.

Let (C, σ_U) be the symbolic data associated to U: $C \subset (T^*M_1 \setminus 0) \times (T^*M_2 \setminus 0)$ is its canonical relation and σ_U is its principal symbol (a homogeneous section of the bundle $\Omega_C^{1/2} \otimes M_C$ of 1/2-densities times Maslov factor over C ([Hö IV])).

The isospectral equations:

$$\begin{cases} (\Delta_x - \Delta_y)U(x,y) = 0 \\ U^*U = UU^* = \mathrm{id} \end{cases} \tag{3.6}$$

lead to the following equations for (C, σ_U), which we call the equation of "symbolic isospectrality":

$$\begin{cases} G_1^t \times G_2^{-t}(C) = C, \ \ (G_1^t \times G_2^{-t})^* \sigma_U = \sigma_U \\ \sigma_U \circ \sigma_U^* = \sigma_U^* \circ \sigma_U = \sigma_i^0 \end{cases} \tag{3.7}$$

Here, G_i^t is (as usual) the geodesic flow on $T^*M_i \setminus 0$, $\circ$ denotes symbol composition, σ^* is the adjoint, and σ_i^0 denotes the canonical 1/2-density on the diagonal in $(T^*M_i \setminus 0) \times (T^*M_i \setminus 0)$. Implicit in the second equation is that the diagonal is a component of $C \circ C^t$ and $C^t \circ C$ (C^t is the transposed canonical relation and $\circ$ is composition of relations).

The equations (3.7) say that C is invariant under the product flow $G_1^t \times G_2^{-t}$ and that σ_U is an invariant, unitary 1/2-density on C. Let us first consider the implications of this unitarity. Few canonical relations are unitarizable in the sense that there exists a unitary 1/2-density on them. In fact:

(3.8) **Lemma** ([Z.2]). *Suppose $C \subset (T^*M_1 \setminus 0)(T^*M_2 \setminus 0)^-$ is a unitarizable canonical relation. Then the projections $\pi_i \colon C \to T^*M_i \setminus 0$ are finite homogeneous covers.*

Thus, we may define a symplectic correspondence $\chi \colon T^*M_1 \setminus 0 \to T^*M_2 \setminus 0$ by $\chi = \pi_2 \circ \pi_1$; χ is finitely multivalued, with finitely multivalued inverse. C is in an obvious sense the graph of χ. The invariance of C under $G_1^t \times G_2^{-t}$ then implies that χ intertwines the flows: $G_2^t \circ \chi = \chi \circ G_1^t$. Thus, up to finite ambiguity, the geodesic flows are (symplectically) isomorphic.

This brings us to the following well-known isometry problem for isomorphic geodesic flows: suppose there exists a diffeomorphism $\phi \colon S^*M \to S^*N$ between unit sphere bundles of compact riemannian manifolds (M, g) and (N, h) so that $\phi G_g^t \phi^{-1} = G_h^t$ (G_g^t e.g., is the geodesic flow for g). Are (M, g) and (N, h) isometric? The answer is known to be "no" for Zoll manifolds [W.1], but is widely conjectured to be "yes" for negatively curved manifolds (for example). Recently, Croke [Cr] and Otal [O] have solved the problem for negatively curved surfaces. Otal's version involves the notion of the marked length spectrum $\mathcal{L}_g$ of the metric g: $\mathcal{L}_g \colon \hat{\pi}_1(M) \to \mathbb{R}^+$ is the map on the set $\hat{\pi}_1(M)$ of free homotopy classes of loops on M which takes a class $\hat{\gamma}$ to the length $L(\gamma)$ of the unique closed geodesic relative to g in its class.

(3.8) **Theorem** ([O], [Cr]). *Suppose g_1 and g_2 are negatively curved metrics on a surface M and $\mathcal{L}_{g_1} = \mathcal{L}_{g_2}$. Then g_1 is isometric to g_2.*

The hypothesis $\mathcal{L}_{g_1} = \mathcal{L}_{g_2}$ here is equivalent to the conjugacy of the geodesic flows (see the references in [Cr]). To apply this Theorem (3.8) of Croke-Otal, we need to resolve the ambiguity in the χ by passing to approproate covers. Since (3.8) is so far only proved in dimension 2, we also restrict ourselves to surfaces.

(3.9) **Lemma** ([Z.2]). *Let $\pi_i \colon C \to T^*M_i \backslash 0$ be the coveres in (3.8), and let* $\dim M_i = 2$. *Then:* (a) *the manifolds M_i possess a common finite cover M;* (b) *the projection π_i factors as a composition of covers $C \xrightarrow{q_i} T^*M \setminus 0 \to T^*M_i \setminus 0$, where $C \xrightarrow{q_i} T^*M \setminus 0$ is a fiber preserving cyclic cover of $\mathbb{R}^2 \setminus 0$ bundles over M, and $T^*M \setminus 0 \to T^*M_i \setminus 0$ is induced from the cover $M \to M_i$;* (c) *$q_2 = q_1 \circ \tilde{\chi}$, where $\tilde{\chi}$ is a homogeneous symplectic diffeomorphism of C (with respect to the pull back of the symplectic form on $T^*M \setminus 0$).*

Now lift the metrics g_i on $T^*M \setminus 0$ to metrics $\tilde{g}_i$ on $T^*M_i \setminus 0$ and further to homogeneous functions H_i on C. From the fact that χ conjugates the geodesic flow of the g_i, one can show that $\tilde{\chi}$ conjugates the Hamiltonean flows of the H_i. Due to the cyclicity of the covers $q_i \colon C \to T^*M \setminus 0$, such a conjugation is enough to show that $\mathcal{L}_{\tilde{g}_1} = \mathcal{L}_{\tilde{g}_2}$. The Theorem of Croke-Otal then implies $\tilde{g}_1$ is isometric to $\tilde{g}_1$, proving Theorem 3.5. □

Finally, a brief word on Theorem 3.10. From Theorem 3.5, one sees that M is a common riemannian cover of the M_i. Therefore, it defines an isometric correspondence form M_1 to M_2. The key point is that simplicity of the length spectra forces it to be a bijection on geodesics. Now suppose the correspondence takes a point $m_1 \in M_1$ to at least two distinct points m_2 and m_2' of M_2. Then the radial geodesics emanating from m_1 would go to radial geodesics enamating from m_2, and also to those of m_2'. Since the correspondence is 1–1 on geodesics, all the rays from m_2 would have to go through m_2'. Thus, m_2 and m_2' would be conjugate, contradicting the assumption of negative curvature.

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