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Some Observations concerning the Distribution of the Zeros of the Zeta Functions (I)

Akio Fujii

§1. Introduction

Let $Z(s)$ be a zeta function which has nice properties like the Riemann zeta function $\zeta(s)$. Let σ_0 be the critical point of $Z(s)$ and suppose that the Riemann Hypothesis (R.H.) holds for $Z(s)$, namely, all the non-trivial zeros of $Z(s)$ are of the form $\sigma_0 + i\gamma$ with a real number γ. The purpose of the present article is to find some of the characteristic properties of the distribution of the zeros of $\zeta(s)$. We shall approach this problem by comparing it with the distribution of the zeros of $Z(s)$ from the following three points of views.

(A) To study the pair correlation of the zeros of $Z(s)$. Namely, to find an asymptotic law for the quantity

$$\sum_{\substack{0<\gamma,\gamma'\leq T \\ 0<\gamma-\gamma'\leq \frac{2\pi\alpha}{\log\frac{T}{2\pi}}}} \cdot 1 \qquad \text{as} \quad T\to\infty,$$

where γ and γ' run over the imaginary parts of the non-trivial zeros of $Z(s)$ and α is a positive number.

(B) To find an asymptotic law for the mean value

$$\int_0^T \left(S_Z(t+\Delta) - S_Z(t)\right)^2 dt \qquad \text{as} \quad T\to\infty,$$

where $S_Z(t) = \frac{1}{\pi}\arg Z(\sigma_0 + it)$ as usual and $\Delta > 0$.

(C) To find an asymptotic law for the sum

$$\sum_{0<\gamma\leq T} \left|Z(\sigma_0 + i(\gamma+\Delta))\right|^2 \qquad \text{as} \quad T\to\infty.$$

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As we shall see below (A),(B) and (C) are related each other. In particular, to study (A) and (B), it is inevitable to evaluate the following Montgomery's sum

$$F_Z(a) \equiv F_Z(a,T) \equiv \frac{1}{\frac{T}{2\pi}\log T} \sum_{0<\gamma,\gamma'\le T} \left(\frac{T}{2\pi}\right)^{ia(\gamma-\gamma')} w(\gamma-\gamma'),$$

where a is real, $T > T_0$ and $w(t) = \dfrac{4}{4+t^2}$.
As a result, we shall realize that the distribution of the zeros of $\zeta(s)$ represent the most primitive, or genuine, feature. And that the distribution of the zeros of $Z(s)$ is of the same type as that of $\zeta(s)$ if $Z(s)$ is primitive, namely, it is not decomposed of the products of the ordinal zeta functions which have the Euler product expansion and the functional equation like $\zeta(s)$. Dirichlet L-functions and Ramanujan τ-zeta function belong to this category.

In general, roughly speaking, it might be that

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ 0<\gamma-\gamma'\le \frac{2\pi\alpha}{\log\frac{T}{2\pi}}}} \cdot 1$$
$$\sim \frac{T}{2\pi}\log T\cdot\{\text{a linear combination of “Uniform distributions”}$$
$$+\text{ a linear combination of “GUE distributions”}\},$$

where “Uniform distribution” is of the form

$$U(\alpha) = \int_0^\alpha dt$$

and “GUE distribution” is of the form

$$G(\alpha) = \int_0^\alpha \left(1 - \left(\frac{\sin \pi t}{\pi t}\right)^2\right) dt.$$

It might also be that

$$\int_0^T \left(S_Z\left(t + \frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right) - S_Z(t)\right)^2 dt$$
$$\sim \frac{T}{\pi^2} \times \text{a linear combination of “GUE parts”},$$

where "GUE part" is of the form

$$g(\alpha) = \log(2\pi\alpha) + C_0 - Ci(2\pi\alpha) + 1 - \cos(2\pi\alpha) + \pi^2\alpha - 2\pi\alpha Si(2\pi\alpha)$$

with the Euler constant C_0.
In particular, it might happen that even when the pair correlation of the zeros of $Z(s)$ is not of GUE type, the mean value

$$\int_0^T \left(S_Z \left(t + \frac{2\pi\alpha}{\log \frac{T}{2\pi}} \right) - S_Z(t) \right)^2 dt$$

is of GUE type. The simplest example might be the Dedekind zeta function of a quadratic number field.

The contents of the rest of the present article is as follows.

We always assume R.H. for $Z(s)$ in each section if we do not mention anything about it.

The proofs of the other new results will appear in the subsequent articles.

§2. Statements of the general conjectures

We start with stating some conjectures concerning our problems (A) and (B). In the subsequent sections we shall see the background of the conjectures by describing the results and the conjectures for some special cases.

Now suppose that $Z(s)$ is of the form

$$Z(s) = Z_1(s)Z_2(s)\cdots Z_R(s)$$

with the primitive zeta functions $Z_j(s)$ of b_j-type for $j = 1, 2, \ldots, R$, where we say that $Z_j(s)$ is of b_j-type if

$$\frac{Z_j'}{Z_j}(2\sigma_0 - \sigma + it) \sim -b_j \log t \qquad \text{as} \quad t \to \infty \quad \text{for any} \quad \sigma > 2\sigma_0.$$

We rewrite $Z(s)$ as follows.

$$Z(s) = G_1(s)G_2(s)\cdots G_K(s),$$

where we put

$$G_h(s) = \prod_{Z_j(s) \text{ is of } b_h\text{-type}} Z_j(s)$$

and we suppose that $0 < b_1 \leq b_2 \leq \ldots \leq b_K$ and that

$$\lim_{T\to\infty} \frac{1}{T \log\log T} \int_0^T S_{Z_j}(t) S_{Z_n}(t) dt = 0$$

if and only if $Z_j(s)$ and $Z_n(s)$ belong to different G_h's.
Let m_h be the number, with multiplicity, of $Z_j(s)$ in $G_h(S)$. We put further

$$M = m_1 b_1 + m_2 b_2 + \cdots + m_K b_K$$

and

$$A = m_1^2 + m_2^2 + \cdots + m_K^2.$$

Using these notations, our first conjecture on Montgomery's sum $F_Z(a)$ may be stated as follows.

Conjecture 1.

$$F_Z(a) = \begin{cases} Aa + (M^2 + o(1))(\frac{T}{2\pi})^{-2a} \log T + o(1) & \textit{for } \ 0 \leq a \leq b_1 \\ m_1^2 b_1 + (A - m_1^2)a + o(1) & \textit{for } \ b_1 \leq a \leq b_2 \\ m_1^2 b_1 + m_2^2 b_2 + (A - m_1^2 - m_2^2)a + o(1) & \textit{for } \ b_2 \leq a \leq b_3 \\ \cdots\cdots\cdots & \cdots \\ m_1^2 b_1 + m_2^2 b_2 + \cdots + m_K^2 b_K + o(1) & \textit{for } \ b_K \leq a, \end{cases}$$

uniformly in bounded intervals.

Our second conjecture is concerned with the pair correlation of the zeros of $Z(s)$.

Conjecture 2. *For any* $\alpha > 0$,

$$\sum_{\substack{0<\gamma,\gamma'\leq T \\ 0<\gamma-\gamma'\leq \frac{2\pi\alpha}{\log\frac{T}{2\pi}}}} \cdot 1$$

$$= \frac{T}{2\pi}\log T\cdot\{(M^2 - m_1^2 b_1^2 - \cdots - m_K^2 b_K^2)U(\alpha)$$

$$+ m_1^2 b_1 G(b_1\alpha) + \cdots + m_K^2 b_K G(b_K\alpha) + o(1)\}.$$

Our third conjecture is concerned with our problem (B).

Conjecture 3. *For any* $\alpha > 0$,

$$\int_0^T \left(S_Z\left(t + \frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right) - S_Z(t)\right)^2 dt$$

$$= \frac{T}{\pi^2}\cdot\{m_1^2 g(b_1\alpha) + \cdots + m_K^2 g(b_K\alpha) + o(1)\}.$$

Conjectures 1 and 2 for $\zeta(s)$ were proposed by Montgomery [17] and Conjecture 3 for $\zeta(s)$ is the GUE part of the conjecture (19) proposed by Berry [1].

We do not dare to describe any conjecture concerning our problem (C) since only the mean values for $\zeta(s)$ and Dirichlet L-functions have been evaluated.

§3. Riemann zeta function

$\zeta(s)$ is the zeta function from which we start and to which we return. We describe some of the known results and conjectures concerning this. In this section we omit writing suffix ζ.

Montgomery [17] and Goldston-Montgomery [14] have shown the following theorem.

Theorem 1. *For any* a *in* $0 \leq a \leq 1$,

$$F(a) = a + O\left(\sqrt{\frac{\log\log T}{\log T}}\right) + \left(1 + O\left(\sqrt{\frac{\log\log T}{\log T}}\right)\right)\left(\frac{T}{2\pi}\right)^{-2a}\log\frac{T}{2\pi}.$$

For $a \geq 1$, Montgomery [17] has conjectured the following.

Conjecture 4.

$$F(a) = 1 + o(1) \qquad \textit{for} \quad a \geq 1,$$

uniformly in bounded intervals.

Theorem 1 and Conjecture 4 imply the following Montgomery's pair correlation conjecture.

Conjecture 5. *For any $\alpha > 0$,*

$$\sum_{\substack{0<\gamma,\gamma'\leq T \\ 0<\gamma-\gamma'\leq \frac{2\pi\alpha}{\log\frac{T}{2\pi}}}} \cdot 1 = \frac{T}{2\pi}\log T \cdot \{G(\alpha) + o(1)\}.$$

As is noticed by Dyson, the density function

$$1 - \left(\frac{\sin \pi t}{\pi t}\right)^2$$

in $G(\alpha)$ is exactly the density function of the pair correlation of the eigenvalues of Gaussian Unitary Ensembles (GUE).

By the Riemann-von Mangoldt formula, we see that

$$\sum_{\substack{0<\gamma,\gamma'\leq T \\ 0<\gamma-\gamma'\leq \frac{2\pi\alpha}{\log\frac{T}{2\pi}}}} \cdot 1 = \sum_{0<\gamma\leq T} \sum_{\gamma - \frac{2\pi\alpha}{\log\frac{T}{2\pi}} < \gamma' < \gamma} \cdot 1$$

$$= \frac{T}{2\pi}\log\frac{T}{2\pi}\int_0^\alpha dt - \sum_{0<\gamma\leq T} S\left(\gamma - \frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right) + O(T),$$

because

$$\sum_{0<\gamma\leq T} S(\gamma) = \frac{1}{2\pi}\int_C^T S(t)\log t\, dt + O\left(\left(\frac{\log T}{\log\log T}\right)^2\right)$$

$$= \frac{1}{2\pi}[S_1(t)\log t]_C^T - \frac{1}{2\pi}\int_C^T S_1(t)\frac{1}{t}\, dt + O\left(\left(\frac{\log T}{\log\log T}\right)^2\right)$$

$$= O\left(\left(\frac{\log T}{\log\log T}\right)^2\right),$$

$S_1(T)$ being the integral $\int_0^T S(t)\, dt$ which is $O\left(\frac{\log T}{(\log\log T)^2}\right)$.
Hence, Conjecture 5 is equivalent to the following one.

Conjecture 5′. *For any $\alpha > 0$,*

$$\sum_{0<\gamma\leq T} S\left(\gamma - \frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right) = \frac{T}{2\pi}\log\frac{T}{2\pi}\cdot\left\{\int_0^{\alpha}\left(\frac{\sin\pi t}{\pi t}\right)^2 dt + o(1)\right\}.$$

Concerning this kind of sum, we have shown in [9] and [10] the following theorem.

Theorem 2. *Suppose that $a \ll T^A$ with some positive constant A. Then we have*

$$\sum_{0<\gamma\leq T,\,\gamma+a>0} S(\gamma+a) \ll T\log T.$$

If we use this in the above argument, then we get the following consequence.

Corollary 1. *For any $\alpha > 0$,*

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ 0<\gamma-\gamma'\leq\frac{2\pi\alpha}{\log\frac{T}{2\pi}}}} \cdot 1 = \frac{T}{2\pi}\log\frac{T}{2\pi}\left(\int_0^{\alpha} dt + O(1)\right).$$

We turn our attentions to our problem (B). We recall some of the basic results concerning the mean value of $S(t)$. First, Selberg [20] has shown the following theorem.

Theorem 3. *For each integer $k \geq 1$,*

$$\int_0^T S(t)^{2k}\,dt = \frac{(2k)!}{(2\pi)^{2k}k!}T(\log\log T)^k + O(T(\log\log T)^{k-\frac{1}{2}}).$$

A short interval version of Theorem 3 has been shown by the author in [2] and [4].

Theorem 4. *For $0 < \Delta \ll 1$ and for each integer $k \geq 1$,*

$$\int_0^T \big(S(t+\Delta) - S(t)\big)^{2k}\,dt$$

$$= \frac{(2k)!}{(2\pi)^{2k}k!}T\big(2\log(2+\Delta\log T)\big)^k + O(T\big(\log(2+\Delta\log T)\big)^{k-\frac{1}{2}}).$$

Theorems 3 and 4 were proved without assuming R.H. Under R.H., Goldston [13] has refined Theorem 3 for $k = 1$ as follows.

Theorem 5.

$$\int_0^T S(t)^2\,dt = \frac{T}{2\pi^2}\log\log T + \frac{T}{2\pi^2}\Big(\int_1^\infty \frac{F(a)}{a^2}\,da + C_0 + \sum_{m=2}^{\infty}\sum_p \left(-\frac{1}{m}+\frac{1}{m^2}\right)\frac{1}{p^m}\Big) + o(T),$$

where p runs over the prime numbers.

In fact, Goldston [13] has shown, under R.H., that

$$\int_1^\infty \frac{F(a)}{a^2}\,da \qquad \text{is bounded.}$$

A short interval version of Theorem 5 has been shown under R.H. by the author in [9] and [10] (cf. also Odlyzko [18]).

Theorem 6. *Suppose that $0 < \Delta = o(1)$. Then we have*

$$\int_0^T \big(S(t+\Delta) - S(t)\big)^2\,dt = \frac{T}{\pi^2}\Big\{\int_0^{\Delta\log\frac{T}{2\pi}} \frac{1-\cos(a)}{a}\,da + \int_1^\infty \frac{F(a)}{a^2}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\Big\} + o(T).$$

We notice that

$$\left|\int_1^\infty \frac{F(a)}{a^2}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\right| \le 2\int_1^\infty \frac{F(a)}{a^2}\,da \ll 1.$$

If we assume Conjecture 4, then we get the following result.

Corollary 2 (Under Conjecture 4). *For $0 < \alpha = o(\log T)$, we have*

$$\int_0^T \left(S\left(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right) - S(t)\right)^2 dt = \frac{T}{\pi^2}(g(\alpha)+o(1)).$$

The right hand side is nothing but the GUE part of Berry's formula (19) conjectured in [1].

Finally, it is noteworthy that the density function

$$1-\left(\frac{\sin \pi t}{\pi t}\right)^2$$

appears also in the coefficient of the main term of the following mean value theorem on the problem (C) due to Gonek [15].

Theorem 7. *For* $|\alpha| \le \frac{1}{4\pi}\log\frac{T}{2\pi}$,

$$\sum_{0<\gamma\le T}\left|\zeta\left(\frac{1}{2}+i\left(\gamma+\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)\right)\right|^2 = \left(1-\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log^2\frac{T}{2\pi}+O(T\log T).$$

The author has refined this and evaluated the coefficients of the lower main terms as follows (cf. [6] and [7]).

Theorem 7′. *For* $0<\alpha\ll\log T$,

$$\sum_{0<\gamma\le T}\left|\zeta\left(\frac{1}{2}+i(\gamma+\frac{2\pi\alpha}{\log\frac{T}{2\pi}})\right)\right|^2$$
$$=\left(1-\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2\right)\frac{T}{2\pi}\log^2\frac{T}{2\pi}$$
$$+2\left(-1+C_0+(1-2C_0)\frac{\sin 2\pi\alpha}{2\pi\alpha}+\Re\left(\frac{\zeta'}{\zeta}(1+i\frac{2\pi\alpha}{\log\frac{T}{2\pi}})\right)\right)\frac{T}{2\pi}\log\frac{T}{2\pi}$$
$$+G(T,\alpha)+O(T^{\frac{9}{10}}\log^2 T),$$

where $G(T,\alpha)(=O(T))$ *can be described explicitly.*

In closing this section, we notice that all these results on $\zeta(s)$ can be generalized to Dirichlet L-functions. We notice only the following theorem which corresponds to Theorem 7′. Let $L(s,\chi)$ be the Dirichlet L-function with a primitive Dirichlet character χ mod $q\ge 1$. Let $\gamma(\chi)$ run over the imaginary parts of the zeros of $L(s,\chi)$. When $q=1$, we suppose that $L(s,\chi)=\zeta(s)$.

Theorem 8. *For* $0 < \Delta \ll 1$,

$$\sum_{0<\gamma(\chi)\leq T} \left| L(\frac{1}{2} + i\,(\gamma(\chi) + \Delta)\,, \chi) \right|^2$$

$$= \left(1 - \left(\frac{\sin(\frac{\Delta}{2}\log\frac{qT}{2\pi})}{\frac{\Delta}{2}\log\frac{qT}{2\pi}}\right)^2\right)\frac{\varphi(q)}{q}\frac{T}{2\pi}\log^2\left(\frac{qT}{2\pi}\right)$$

$$+ \frac{T}{\pi}\log\frac{qT}{2\pi}[\frac{\varphi(q)}{q}\left\{-1 + C_0 - C_1(q)\frac{q}{\varphi(q)}\right\}$$

$$+ (1 - 2C_0(\chi_0))\frac{\sin(\Delta\log\frac{qT}{2\pi})}{\Delta\log\frac{qT}{2\pi}}$$

$$+ \frac{\varphi(q)}{q}\Re\left(\frac{L'}{L}(1 - i\Delta, \chi_0) - \frac{1}{i\Delta}\right)]$$

$$+ G(T, \Delta, \chi) + O(T^{\frac{9}{10}}\log^2 T),$$

where χ_0 *is the principal character* mod$q$, $C_0(\chi_0)$ *is the constant term of the Laurent expansion of* $L(s, \chi_0)$ *at* $s = 1$, $\varphi(q)$ *is the Euler function,*

$$C_1(q) = \sum_{d|q}\frac{\mu(d)}{d}\log d$$

and $G(T, \Delta, \chi)(= O(T))$ *can be described explicitly.*

We notice that the remainder term $O(T^{\frac{9}{10}}\log^2 T)$ in the above theorem is obtained under G.R.H. (Generalized Riemann Hypothesis) for all $L(s, \nu)$, ν being a Dirichlet character mod$q$.

In particular, we see that for $0 < \alpha \ll \log T$ and for any $q \geq 1$,

$$\sum_{0<\gamma(\chi)\leq T} \left| L\left(\frac{1}{2} + i\left(\gamma(\chi) + \frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right), \chi\right) \right|^2$$

$$\sim \left(1 - \left(\frac{\sin(\pi\alpha)}{\pi\alpha}\right)^2\right)\frac{\varphi(q)}{q}\frac{T}{2\pi}\log^2 T.$$

§4. Dedekind zeta functions

We shall treat the simplest case, the Dedekind zeta functions of quadratic number fields. However, we shall treat a slightly more general zeta function

$$Z(s) = L(s, \chi)L(s, \psi),$$

where χ is a primitive Dirichlet character mod$q \geq 1$, ψ is a primitive Dirichlet character mod$k \geq 1$, $L(s,\chi)$ and $L(s,\psi)$ are the corresponding Dirichlet L-functions. We may extend our results to the Dedekind zeta functions of the cyclotomic number fields, for example.

We may notice that we have previously studied the distribution of the zeros of $Z(s)$ in a comparative study of the zeros of Dirichlet L-functions [3] and [5]. Our primary problems have been:

(i) Is there a coincident zero of $L(s,\chi)$ and $L(s,\psi)$ if $\chi \neq \psi$?

(ii) To get a quantitative expression of the independence of the distribution of $\gamma(\chi)$ and that of $\gamma(\psi)$ if $\chi \neq \psi$, where $\gamma(\chi)$ or $\gamma(\psi)$ run over the imaginary parts of the zeros of $L(s,\chi)$ or $L(s,\psi)$, respectively.

Concerning these problems we have shown the following theorems.

Theorem 9. *If $\chi \neq \psi$, then a positive proportion of the zeros of $L(s,\chi)$ and $L(s,\psi)$ are non-coincident.*

Theorem 10. *Suppose that $\chi \neq \psi$. Let $\Phi(n)$ be a positive increasing function which tends to ∞ as $n \to \infty$. We put*

$$\Delta_n(\chi,\psi) = n - m \qquad \textit{if} \qquad \gamma_m(\chi) \leq \gamma_n(\psi) \leq \gamma_{m+1}(\chi),$$

where $\gamma_m(\chi)$ denotes the m-th non-negative imaginary part of the zeros of $L(s,\chi)$. Then we have

$$\left|\Delta_n(\chi,\psi)\right| > \frac{2\pi\sqrt{\log\log n}}{\Phi(n)}$$

for almost all $n > n_0$.

We have obtained these as the consequences of the following mean value theorem in [5]. We put $S(t,\chi) = S_{L(s,\chi)}(t)$ and $S(t,\psi) = S_{L(s,\psi)}(t)$.

Theorem 11. *Suppose that $\chi \neq \psi$. For $0 < \Delta \ll 1$ and for each integer $k \geq 1$,*

$$\int_0^T (S(t+\Delta,\chi) - S(t,\chi) - \big(S(t+\Delta,\psi) - S(t,\psi)\big)\big)^{2k}\,dt$$
$$= \frac{(2k)!}{(2\pi)^{2k}k!}T\big(4\log(2+\Delta\log T)\big)^k$$
$$+ O(T\big(\log(2+\Delta\log T)\big)^{k-\frac{1}{2}}).$$

Theorems 9,10 and 11 were proved without assuming any unproved hypothesis.

We now describe our results on our present problems. We first notice the following theorem which expresses an orthogonality relation of $S(t,\chi)$ and $S(t,\psi)$.

Theorem 12.

$$\begin{aligned}
&\int_0^T S(t,\chi)S(t,\psi)\,dt \\
=&\delta_{\chi,\psi}\frac{T}{2\pi^2}\log\log T \\
&+\frac{T}{2\pi^2}\left\{\int_1^\infty \frac{F_{\chi,\psi}(a)}{a^2}\,da\right. \\
&\qquad + \delta_{\chi,\psi}\left(C_0+\sum_{m=2}^{\infty}\sum_p \frac{1}{p^m}\left(\frac{1}{m^2}-\frac{1}{m}\right)-b_1(q)-b(q)\right) \\
&\qquad \left. + (1-\delta_{\chi,\psi})\Re\left(\sum_{m=1}^{\infty}\sum_p \frac{(\chi\overline{\psi})(p^m)}{p^m m^2}\right)\right\} \\
&+O\left(\frac{T}{\sqrt{\log T}}\right),
\end{aligned}$$

where p runs over the prime numbers and we put

$$\delta_{\chi,\psi}=\begin{cases}1 & \text{if } \chi=\psi\\ 0 & \text{otherwise,}\end{cases}$$

$$b_r(q)=\sum_{p|q}\frac{1}{p^r}, \qquad b(q)=\sum_{r=2}^{\infty}\frac{b_r(q)}{r^2},$$

and

$$\begin{aligned}
&F_{\chi,\psi}(a)\equiv F_{\chi,\psi}(a,T) \\
\equiv&\frac{1}{\frac{T}{2\pi}\log\frac{T}{2\pi}}\Re\{\sum_{0<\gamma(\chi),\gamma(\psi)\le T}\left(\frac{T}{2\pi}\right)^{ia(\gamma(\chi)-\gamma(\psi))} w(\gamma(\chi)-\gamma(\psi))\}.
\end{aligned}$$

This is a generalization of Theorem 5, where $\chi=\psi$ and $k=q=1$. Concerning $F_{\chi,\psi}$, we have the following result.

Theorem 13. *For $0 \le a \le 1$, we have*

$$F_{\chi,\psi}(a,T) = \delta_{\chi,\psi} \cdot a + \left(\frac{T}{2\pi}\right)^{-2a} \log T + O(T^{-2a}) + O\left(\frac{1}{\sqrt{\log T}}\right).$$

This implies the following corollary which makes Theorem 12 meaningful.

Corollary 3.

$$\left|\int_1^\infty \frac{1}{a^2} F_{\chi,\psi}(a)\, da\right| \ll 1.$$

According to Conjectures 1,2 and 3 described in the section 2, Theorem 12 implies the following conjectures.

Conjecture 6. *Suppose that $\chi \neq \psi$. Then*

$$F_Z(a) = \begin{cases} 2a + (4+o(1))(\frac{T}{2\pi})^{-2a} \log T + o(1) & \text{for } 0 \le a \le 1 \\ 2 + o(1) & \text{for } 1 \le a, \end{cases}$$

uniformly in bounded intervals.

Conjecture 7. *For $\chi \neq \psi$ and for any $\alpha > 0$,*

$$\sum_{\substack{0<\gamma,\gamma' \le T \\ 0<\gamma-\gamma' \le \frac{2\pi\alpha}{\log \frac{T}{2\pi}}}} \cdot 1 = \frac{T}{2\pi} \log T \{2U(\alpha) + 2G(\alpha) + o(1)\}.$$

This says that the distribution of the zeros of $Z(s)$ for $\chi \neq \psi$ might not be of genuine GUE type.

Conjecture 8. *For $\chi \neq \psi$ and for any $\alpha > 0$, we have*

$$\int_0^T \left(S_Z(t + \frac{2\pi\alpha}{\log \frac{T}{2\pi}}) - S_Z(t)\right)^2 dt = \frac{T}{\pi^2} \cdot \{2g(\alpha) + o(1)\}.$$

In view of Theorem 13, Conjecture 6 is essentially the following.

Conjecture 6′.

$$F_{\chi,\psi}(a) = \delta_{\chi,\psi} + o(1) \qquad \textit{for} \ \ a \geq 1,$$

uniformly in bounded intervals.

Similarly, Conjecture 7 may be stated as follows.

Conjecture 7′. *For any $\alpha > 0$,*

$$\sum_{\substack{0<\gamma(\chi),\gamma'(\psi)\leq T \\ 0<\gamma(\chi)-\gamma'(\psi)\leq \frac{2\pi\alpha}{\log T}}} \cdot 1$$
$$= \left\{ \int_0^{\alpha} \left(1 - \delta_{\chi,\psi} \cdot \left(\frac{\sin(\pi t)}{\pi t} \right)^2 \right) dt + o(1) \right\} \cdot \frac{T}{2\pi} \log \frac{T}{2\pi}.$$

Concerning Conjecture 8, we may state first the following theorem.

Theorem 14. *Suppose that $0 < \Delta = o(1)$. Then we have*

$$\int_0^T \big(S(t+\Delta,\chi) - S(t,\chi)\big)\big(S(t+\Delta,\psi) - S(t,\psi)\big)\, dt$$
$$= \frac{T}{\pi^2} \left\{ \delta_{\chi,\psi} \int_0^{\Delta \log \frac{T}{2\pi}} \frac{1-\cos(a)}{a}\, da \right.$$
$$\left. + \int_1^{\infty} \frac{F_{\chi,\psi}(a)}{a^2} \left(1 - \cos\left(a\Delta \log \frac{T}{2\pi} \right) \right) da \right\}$$
$$+ O\left(T \left(\Delta^2 + \frac{1}{\sqrt{\log T}} \right) \right).$$

This is a generalization of Theorem 6, where $\chi = \psi$ and $k = q = 1$. Thanks to Theorem 13, we have

$$\left| \int_1^{\infty} \frac{1}{a^2} \left(1 - \cos\left(a\Delta \log \frac{T}{2\pi}\right) \right) F_{\chi,\psi}(a)\, da \right| \ll 1.$$

As a consequence of Theorem 14, we get the following.

Corollary 4. *For $\chi \neq \psi$ and for $0 < \Delta = o(1)$, we have*

$$\int_0^T \big(S_Z(t+\Delta) - S_Z(t)\big)^2\, dt$$
$$= \frac{T}{\pi^2}\left\{2\int_0^{\Delta \log \frac{T}{2\pi}} \frac{1-\cos(a)}{a}\, da \right.$$
$$+ \int_1^{\infty} \frac{1}{a^2}\left(1 - \cos\left(a\Delta \log \frac{T}{2\pi}\right)\right)$$
$$\left. \cdot \big(F_{\chi,\chi}(a) + F_{\psi,\psi}(a) + 2F_{\chi,\psi}(a)\big)\, da\right\}$$
$$+ O\left(T\left(\Delta^2 + \frac{1}{\sqrt{\log T}}\right)\right).$$

If we combine Corollary 4 with Conjecture 6′, we get

Corollary 5 (Under Conjecture 6′). *For $0 < \alpha = o(\log T)$, we have*

$$\int_0^T \left(S_Z\left(t + \frac{2\pi\alpha}{\log \frac{T}{2\pi}}\right) - S_Z(t)\right)^2 dt = \frac{T}{\pi^2}\{2(1+\delta_{\chi,\psi})g(\alpha) + o(1)\}.$$

Thus we see that Conjecture 8 is valid for $\alpha = o(\log T)$ under Conjecture 6′.

Similarly, we obtain the following results which give some improvements of Theorem 11 for $k = 1$ either under G.R.H. or under G.R.H. with Conjecture 6′.

Corollary 6. *For $\chi \neq \psi$ and for $0 < \Delta = o(1)$, we have*

$$\int_0^T \big(S(t+\Delta,\chi) - S(t,\chi) - \big(S(t+\Delta,\psi) - S(t,\psi)\big)\big)^2\, dt$$
$$= \frac{T}{\pi^2}\left\{2\int_0^{\Delta \log \frac{T}{2\pi}} \frac{1-\cos(a)}{a}\, da \right.$$

$$+\int_1^\infty \frac{1}{a^2}\left(1-\cos(a\Delta\log\frac{T}{2\pi})\right)$$

$$\cdot\left(F_{\chi,\chi}(a)+F_{\psi,\psi}(a)-2F_{\chi,\psi}(a)\right)da\Bigg\}$$

$$+O\left(T\left(\Delta^2+\frac{1}{\sqrt{\log T}}\right)\right).$$

Corollary 7 (Under Conjecture 6′). *For $\chi\neq\psi$ and for $0<\alpha=o(\log T)$, we have*

$$\int_0^T (S(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}},\chi)-S(t,\chi)-(S(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}},\psi)-S(t,\psi)))^2\,dt$$

$$=\frac{T}{\pi^2}(2g(\alpha)+o(1)).$$

Finally, we notice that we can show the following theorem which supplements Theorem 8 stated in the section 3.

Theorem 15. *For $\chi\neq\psi$ and for $0<\Delta\ll 1$, we have*

$$\sum_{0<\gamma(\psi)\leq T}\left|L(\frac{1}{2}+i(\gamma(\psi)+\Delta),\chi)\right|^2$$

$$=\frac{T}{2\pi}\log^2\left(\frac{qT}{2\pi}\right)\frac{\varphi(q)}{q}$$

$$+\frac{T}{\pi}\log\frac{qT}{2\pi}\frac{\varphi(q)}{q}\left\{-1+\frac{1}{2}\log\left(\frac{k}{q}\right)+C_0-C_1(q)\frac{q}{\varphi(q)}\right.$$

$$\left.+\Re\frac{L'}{L}(1-i\Delta,\overline{\chi}\psi)\right\}$$

$$+G(T,\Delta,\chi,\psi)+O(T^{\frac{9}{10}}\log^2 T),$$

where $G(T,\Delta,\chi,\psi)(=O(T))$ can be described explicitly.

We notice that the remainder term $O(T^{\frac{9}{10}}\log^2 T)$ in the above theorem is obtained under G.R.H. for all $L(s,\psi\nu)$, ν being a Dirichlet character mod$q$.

§5. Ramanujan τ-zeta function

Let $\tau(n)$ be the Ramanujan τ-function defined by

$$\sum_{n=1}^{\infty} \tau(n) z^n = z \prod_{n=1}^{\infty} (1 - z^n)^{24} \qquad \text{for } |z| < 1.$$

Let $Z(s)$ be the Ramanujan τ-zeta function defined by

$$Z(s) = \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \qquad \text{for } \Re s > \frac{13}{2}.$$

It is well known that $Z(s)$ can be continued analytically to the complex plane and is entire. It has a functional equation of the form

$$(2\pi)^{-s}\Gamma(s)Z(s) = (2\pi)^{-(12-s)}\Gamma(12-s)Z(12-s).$$

It has an Euler product expansion of the form

$$\begin{aligned} Z(s) &= \prod_p \left(1 - \tau(p)p^{-s} + p^{11-2s}\right)^{-1} \\ &= \prod_p \left(1 - \alpha_p \cdot p^{-s}\right)^{-1} \cdot \left(1 - \overline{\alpha_p} \cdot p^{-s}\right)^{-1}, \end{aligned}$$

where p runs over the prime numbers, $|\alpha_p| = p^{\frac{11}{2}}$ and $\Re s > \frac{13}{2}$. $Z(s)$ has and is expected to have other nice properties like $\zeta(s)$. However, as we shall see below, the precise statements of theorems and conjectures must be slightly modified.

Concerning Montgomery's sum, we can show the following theorem.

Theorem 16. *For any a in $0 \le a \le 1$, we have*

$$F_Z(a) = a + \left(4 + o(1)\right) \left(\frac{T}{2\pi}\right)^{-2a} \log T + O\left(\frac{1}{\sqrt{\log T}}\right).$$

For $a \ge 1$, the following conjecture might be expected.

Conjecture 9.

$$F_Z(a) = \begin{cases} a + o(1) & \text{for } 1 \le a \le 2 \\ 2 + o(1) & \text{for } a \ge 2 \end{cases},$$

uniformly in bounded intervals.

Conjecture 8 and Theorem 16 might suggest the following conjecture.

Conjecture 10. *For any $\alpha > 0$,*

$$\sum_{\substack{0<\gamma,\gamma'\le T \\ 0<\gamma-\gamma'\le \frac{2\pi\alpha}{\log \frac{T}{2\pi}}}} \cdot 1 = \frac{T}{2\pi}\log T \cdot \{2G(2\alpha)+o(1)\}.$$

For the mean value theorem on $S_Z(t+\Delta)-S_Z(t)$, we can show the following theorem.

Theorem 17. *For $0<\Delta = o(1)$, we have*

$$\begin{aligned}
&\int_0^T \left(S_Z(t+\Delta)-S_Z(t)\right)^2 dt \\
&= \frac{T}{\pi^2}\{\int_0^1 \frac{1-\cos(a\Delta\log\frac{T}{2\pi})}{a}\,da \\
&\qquad + \int_1^\infty \frac{1}{a^2}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right) F_Z(a)\,da\} \\
&\quad + O\left(T\left(\Delta^2+\frac{1}{\sqrt{\log T}}\right)\right).
\end{aligned}$$

If we assume Conjecture 9 on $F_Z(a)$, then we get a more precise result on $S_Z(t+\Delta)-S_Z(t)$ as follows.

Corollary 8 (Under Conjecture 9). *For $0<\Delta = \frac{2\pi\alpha}{\log\frac{T}{2\pi}} = o(1)$, we have*

$$\begin{aligned}
&\int_0^T \left(S_Z\left(t+\frac{2\pi\alpha}{\log\frac{T}{2\pi}}\right)-S_Z(t)\right)^2 dt \\
&= \frac{T}{\pi^2}\left\{\int_0^2 \frac{1}{a}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right) da \right. \\
&\qquad \left. + 2\cdot\int_2^\infty \frac{1}{a^2}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right) da + o(1)\right\} \\
&= \frac{T}{\pi^2}\cdot\{g(2\alpha)+o(1)\}.
\end{aligned}$$

§6. Some other examples

Example 1. $Z(s) = \zeta(s+\frac{1}{4})\zeta(2s)$.

In this case we have $\sigma_0 = \dfrac{1}{4}$,

$$M = 1\cdot 1 + 1\cdot 2 = 3$$

and

$$A = 1^2 + 1^2 = 2.$$

We may express an orthogonality relation of $\arg\zeta(\frac{1}{2}+it)$ and $\arg\zeta(\frac{1}{2}+i2t)$ in the following form.

$$\int_0^T S(2t)S(t)\,dt = -\frac{T}{2\pi^2}\left\{\frac{3}{2}\sum_{m=1}^{\infty}\sum_p \frac{1}{m^2 p^{\frac{3}{2}m}} - \int_1^{\infty}\frac{F_1(a)}{a^2}\,da + o(1)\right\},$$

where we put

$$F_1(a) \equiv F_1(a,T) \equiv \frac{1}{\frac{T}{2\pi}\log T}\Re\{\sum_{0<\frac{\gamma}{2},\gamma'\le T}\left(\frac{T}{2\pi}\right)^{ia(\frac{\gamma}{2}-\gamma')} w\left(\frac{\gamma}{2}-\gamma'\right)\},$$

γ and γ' running over the imaginary parts of the zeros of $\zeta(s)$ and

$$\int_1^{\infty}\frac{F_1(a)}{a^2}\,da \ll 1.$$

In particular, this implies that

$$\begin{aligned}
&\int_0^T \left(S(t+t) - S(t)\right)^2 dt \\
&= \frac{T}{\pi^2}\log\log T \\
&\quad + \frac{T}{2\pi^2}\{2C_0 + 2\sum_{m=2}^{\infty}\sum_p\left(-\frac{1}{m}+\frac{1}{m^2}\right)\frac{1}{p^m} + 3\sum_{m=1}^{\infty}\sum_p \frac{1}{m^2 p^{\frac{3}{2}m}} \\
&\qquad + \int_1^{\infty}\frac{F(a,2T)+F(a,T)-2F_1(a)}{a^2}\,da + o(1)\},
\end{aligned}$$

where $F(a,2T)$ and $F(a,T)$ are $F(a,2T) = F_\zeta(a,2T)$ and $F(a,T) = F_\zeta(a,T)$, respectively. This should be compared with Theorem 6 in the section 3, where the shorter intervals are considered. We can evaluate $F_1(a)$ for $0 \le a \le 1$ and we conjecture that

$$F_1(a) = o(1) \quad \text{for} \quad a \ge 1.$$

Now Conjecture 1 in the section 1 suggests that

$$F_Z(a) = \begin{cases} 2a + \big(9 + o(1)\big)\left(\frac{T}{2\pi}\right)^{-2a} \log T + o(1) & \text{for } 0 \le a \le 1 \\ 1 + a + o(1) & \text{for } 1 \le a \le 2 \\ 3 + o(1) & \text{for } 2 \le a, \end{cases}$$

uniformly in bounded intervals, while we can show that

$$F_Z(a) = 2a + (9 + o(1)) \left(\frac{T}{2\pi}\right)^{-2a} \log T + o(1) \quad \text{for } 0 \le a \le 1.$$

Conjecture 2 suggests that

$$\sum_{\substack{0<\gamma,\gamma'\le T \\ 0<\gamma-\gamma'\le \frac{2\pi\alpha}{\log \frac{T}{2\pi}}}} \cdot 1 = \frac{T}{2\pi} \log T \cdot \{4U(\alpha) + G(\alpha) + 2G(2\alpha) + o(1)\},$$

where γ and γ' run over the imaginary parts of the zeros of $Z(s)$ as in the statement of Conjecture 2. Thus if γ and γ' run over the imaginary parts of the zeros of $\zeta(s)$, there might be no pair correlation of γ and $\frac{\gamma'}{2}$ in such a sence that the distribution of the pairs is uniformly distributed.

Example 2. $Z(s) = \zeta^k(s)$.

In this case $A = k^2$ and $M = k$. Our conjectures stated in the section 2 coincide with the trivial consequences of the conjectures on $\zeta(s)$.

Example 3. The zeta function attached to the Maas wave forms can be treated in a similar manner.

Example 4. Selberg zeta functions do not belong to the category in the section 2 unless we shall generalize the framework. Since the eigenvalues of the Laplace-Beltrami operator are the zeros of Selberg zeta function, we cannot conclude anything on their distribution from the present context. Nevertheless, as we have noticed in [8], the eigenvalues of the Laplace-Beltrami operator on L^2(the complex upper half plane $/\Gamma$) for any principal congruence subgroup $\Gamma = \Gamma_p$ of level p a prime > 2 are not GUE distributed in the context of [8].

§7. Proofs of some theorems stated in the section 4

7-1. Proof of Theorem 14

In this subsection we shall prove Theorem 14 and suppose that χ and ψ are primitive characters mod$q \geq 1$ and $k \geq 1$, respectively. We shall use the following lemma which is a generalization of Goldston's explicit formula [13] for $S(t)$.

Lemma 1. *Suppose that $x \geq 4$, $t \geq 1$ and $t \neq \gamma(\chi)$. Then we have*

$$S(t,\chi) = \Im\left(\frac{1}{\pi}\sum_{n\leq x}\frac{\Lambda(n)\chi(n)}{\sqrt{n}\log n\, n^{it}} f\left(\frac{\log n}{\log x}\right)\right)$$

$$+\frac{1}{\pi}\sum_{\gamma(\chi)} h\big((t-\gamma(\chi))\log x\big) - \delta(\chi)x^{\frac{1}{2}}g(x,t) + O\left(\frac{1}{t\log^2 x}\right)$$

$$= A(t,\chi) + B(t,\chi) - \delta(\chi)x^{\frac{1}{2}}g(x,t) + O\left(\frac{1}{t\log^2 x}\right), \qquad say,$$

where we put

$$f(u) = \frac{\pi u}{2}\cot\left(\frac{\pi u}{2}\right),$$

$$g(x,t) = \frac{1}{\pi}\Im\left\{\int_0^\infty \frac{x^{-it}}{((\frac{1}{2}-it)\log x)^2 - u^2)}\frac{u}{\sinh u}\,du\right\},$$

$$h(v) = \sin v\int_0^\infty \frac{u}{u^2+v^2}\frac{du}{\sinh u}$$

and

$$\delta(\chi) = \begin{cases} 1 & \textit{if} \quad q = 1 \\ 0 & \textit{if} \quad q \neq 1. \end{cases}$$

Using this lemma with $x = \left(\frac{T}{2\pi}\right)^\beta$ with $0 < \beta \leq 1$, we get

$$
\begin{aligned}
S \equiv & \int_0^T \big(S(t+\Delta,\chi) - S(t,\chi)\big)\big(S(t+\Delta,\psi) - S(t,\psi)\big)\,dt \\
= & \int_1^T \big(B(t+\Delta,\chi) - B(t,\chi)\big)\big(B(t+\Delta,\psi) - B(t,\psi)\big)\,dt \\
& - \int_1^T \big(A(t+\Delta,\chi) - A(t,\chi)\big)\big(A(t+\Delta,\psi) - A(t,\psi)\big)\,dt \\
& + \int_1^T \big(A(t+\Delta,\chi) - A(t,\chi)\big)\big(S(t+\Delta,\psi) - S(t,\psi)\big)\,dt \\
& + \int_1^T \big(S(t+\Delta,\chi) - S(t,\chi)\big)\big(A(t+\Delta,\psi) - A(t,\psi)\big)\,dt \\
& + O\left(\frac{\log^2 T}{\log^2 x} + \delta(\chi)\delta(\psi)\frac{x}{\log^4 x} + \big(\delta(\chi)+\delta(\psi)\big)\frac{\sqrt{x}}{\log x}\right) \\
= & S_1 + S_2 + S_3 + S_4 \\
& + O\left(\frac{\log^2 T}{\log^2 x} + \delta(\chi)\delta(\psi)\frac{x}{\log^4 x} + \big(\delta(\chi)+\delta(\psi)\big)\frac{\sqrt{x}}{\log x}\right), \quad \text{say.}
\end{aligned}
$$

$$
\begin{aligned}
S_4 = & \int_1^T S(t,\chi)\{2A(t,\psi) - A(t-\Delta,\psi) - A(t+\Delta,\psi)\}\,dt \\
& - \int_1^{1+\Delta} S(t,\chi)\{A(t,\psi) - A(t-\Delta,\psi)\}\,dt \\
& + \int_T^{T+\Delta} S(t,\chi)\{A(t,\psi) - A(t-\Delta,\psi)\}\,dt \\
= & \int_1^T S(t,\chi)\{2A(t,\psi) - A(t-\Delta,\psi) - A(t+\Delta,\psi)\}\,dt \\
& + O\left(\Delta \frac{\log T}{\log\log T}\frac{\sqrt{x}}{\log x}\right) \\
= & \frac{2}{\pi}\Im\left\{\sum_{n\le x}\frac{\Lambda(n)\psi(n)}{\sqrt{n}\log n} f\left(\frac{\log n}{\log x}\right)\big(1-\cos(\Delta\log n)\big)\right. \\
& \qquad \left.\cdot \int_1^T S(t,\chi) e^{-it\log n}\,dt\right\} \\
& + O\left(\Delta \frac{\log T}{\log\log T}\frac{\sqrt{x}}{\log x}\right).
\end{aligned}
$$

Here we use the following lemma whose proof will be given in the subsequent paper.

Lemma 2. *For an integer $n \geq 1$, we have*

$$\int_0^T \cos(t\log n)S(t,\chi)\,dt$$
$$=\frac{1}{2\pi}\frac{\Lambda(n)}{\sqrt{n}\log n}T\Im(\chi(n))$$
$$+O(\sqrt{n}\log\log(3n)+n^{\frac{1}{\log\log T}}\frac{\log T}{\log\log T}+\frac{\sqrt{n}}{\log n}\sqrt{\frac{\log T}{\log\log T}})$$

and

$$\int_0^T \sin(t\log n)S(t,\chi)\,dt$$
$$=\frac{1}{2\pi}\frac{\Lambda(n)}{\sqrt{n}\log n}T\Im(\frac{1}{i}\chi(n))$$
$$+O(\sqrt{n}\log\log(3n)+n^{\frac{1}{\log\log T}}\frac{\log T}{\log\log T}+\frac{\sqrt{n}}{\log n}\sqrt{\frac{\log T}{\log\log T}}).$$

Using this lemma, we get

$$S_4=\frac{2}{\pi}\Im\{\sum_{n\leq x}\frac{\Lambda(n)\psi(n)}{\sqrt{n}\log n}f\left(\frac{\log n}{\log x}\right)(1-\cos(\Delta\log n))\}$$
$$\cdot[\frac{1}{2\pi}\frac{\Lambda(n)}{\sqrt{n}\log n}T(\Im(\chi(n))-i\Im(\frac{1}{i}\chi(n)))$$
$$+O(\sqrt{n}\log\log(3n)+n^{\frac{1}{\log\log T}}\frac{\log T}{\log\log T}+\frac{\sqrt{n}}{\log n}\sqrt{\frac{\log T}{\log\log T}})]$$
$$+O\left(\Delta\frac{\log T}{\log\log T}\frac{\sqrt{x}}{\log x}\right)$$
$$=\frac{T}{\pi^2}\Im\{i\sum_{n\leq x}\frac{\Lambda^2(n)\psi(n)\overline{\chi(n)}}{n\log^2 n}f\left(\frac{\log n}{\log x}\right)(1-\cos(\Delta\log n))\}$$
$$+O(\frac{x}{\log(3x)}\log\log(3x)+\frac{x}{\log^2(3x)}\sqrt{\frac{\log T}{\log\log T}}$$
$$+\frac{\sqrt{x}}{\log x}\frac{\log T}{\log\log T}(x^{\frac{1}{\log\log T}}+\Delta)).$$

Similarly, we get

$$
\begin{aligned}
S_3 =& \frac{T}{\pi^2}\Im\{i\sum_{n\le x}\frac{\Lambda^2(n)\chi(n)\overline{\psi(n)}}{n\log^2 n}f\left(\frac{\log n}{\log x}\right)\left(1-\cos(\Delta\log n)\right)\} \\
&+O(\frac{x}{\log(3x)}\log\log(3x)+\frac{x}{\log^2(3x)}\sqrt{\frac{\log T}{\log\log T}} \\
&+\frac{\sqrt{x}}{\log x}\frac{\log T}{\log\log T}\left(x^{\frac{1}{\log\log T}}+\Delta\right)).
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
S_3+S_4 =& \frac{T}{\pi^2}2\Re\{\sum_{p\le x}\frac{\chi(p)\overline{\psi(p)}}{p}f(\frac{\log p}{\log x})\left(1-\cos(\Delta\log p)\right)\} \\
&+O(T\sum_{p^r\le x,\, r\ge 2}\frac{1}{p^r r^2}\sin^2\left(\frac{\Delta}{2}\log p^r\right)) \\
&+O(\frac{x}{\log(3x)}\log\log(3x)+\frac{x}{\log^2(3x)}\sqrt{\frac{\log T}{\log\log T}} \\
&+\frac{\sqrt{x}}{\log x}\frac{\log T}{\log\log T}\left(x^{\frac{1}{\log\log T}}+\Delta\right)).
\end{aligned}
$$

We next evaluate S_2.

$$
\begin{aligned}
S_2 =& -\frac{1}{\pi^2}\{\int_1^T\Im(\sum_{m\le x}\frac{\Lambda(m)\chi(m)}{\sqrt{m}\log m\, m^{it}}(m^{-i\Delta}-1)f\left(\frac{\log m}{\log x}\right) \\
&\cdot\Im(\sum_{n\le x}\frac{\Lambda(n)\psi(n)}{\sqrt{n}\log n\, n^{it}}(n^{-i\Delta}-1)f\left(\frac{\log n}{\log x}\right)dt\} \\
=& \frac{1}{2\pi^2}\Re\{\sum_{m\le x}\sum_{n\le x}\frac{\Lambda(m)\Lambda(n)\chi(m)\psi(n)}{\sqrt{mn}\log m\log n} \\
&\cdot(m^{-i\Delta}-1)(n^{-i\Delta}-1)f\left(\frac{\log m}{\log x}\right)f\left(\frac{\log n}{\log x}\right)\int_1^T\frac{1}{(mn)^{it}}dt\}
\end{aligned}
$$

$$- \frac{1}{2\pi^2}\Re\{ \sum_{m\le x}\sum_{n\le x} \frac{\Lambda(m)\Lambda(n)\chi(m)\overline{\psi}(n)}{\sqrt{mn}\log m \log n}$$

$$\cdot (m^{-i\Delta}-1)(n^{-i\Delta}-1) f\left(\frac{\log m}{\log x}\right) f\left(\frac{\log n}{\log x}\right)\int_1^T (\frac{n}{m})^{it}\,dt\}$$

$$= -\frac{T}{\pi^2}\Re\{\sum_{p\le x}\frac{1}{p}f^2\left(\frac{\log p}{\log x}\right)(\chi\overline{\psi})(p)\big(1-\cos(\Delta\log p)\big)\}$$

$$+ O(T\sum_{p^r\le x,\, r\ge 2}\frac{1}{p^r r^2}\sin^2(\frac{\Delta}{2}\log p^r)) + O(x\frac{\log\log(3x)}{\log(3x)}).$$

Since

$$\sum_{p^r\le x,\, r\ge 2}\frac{1}{p^r r^2}\sin^2(\frac{\Delta}{2}\log p^r) \ll T\Delta^2,$$

we get

$$S_2+S_3+S_4$$

$$=\frac{T}{\pi^2}\Re\{\sum_{p\le x}\frac{\chi(p)\overline{\psi(p)}}{p}\left(2f\left(\frac{\log p}{\log x}\right)-f^2\left(\frac{\log p}{\log x}\right)\right)\big(1-\cos(\Delta\log p)\big)\}$$

$$+O(\frac{x}{\log(3x)}\log\log(3x)+\frac{x}{\log^2(3x)}\sqrt{\frac{\log T}{\log\log T}}$$

$$+\frac{\sqrt{x}}{\log x}\frac{\log T}{\log\log T}\big(x^{\frac{1}{\log\log T}}+\Delta\big)+T\Delta^2).$$

If we put

$$\sum_{p\le \gamma}\frac{(\chi\overline{\psi})(p)}{p} = \delta_{\chi,\psi}\cdot\log\log Y + C(\chi\overline{\psi}) + r_{\chi,\psi}(Y)$$

with some constant $C(\chi\overline{\psi})$, then for any positive constant A

$$r_{\chi,\psi}(Y) = O\left(\frac{1}{(\log Y)^A}\right) \quad \text{for} \quad Y > Y_0.$$

Using this we see that

$$\sum_{p\le x} \frac{\chi(p)\overline{\psi(p)}}{p} \left(2f\left(\frac{\log p}{\log x}\right) - f^2\left(\frac{\log p}{\log x}\right)\right)\left(1-\cos(\Delta \log p)\right)$$

$$=\delta_{\chi,\psi}\cdot\left\{-\int_0^\beta \frac{1}{a}\left(1-f(\frac{a}{\beta})\right)^2\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\right.$$

$$\left.+\int_0^\beta \frac{1}{a}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\right\}$$

$$+O\left(\Delta^2+\frac{1}{\log x}\right).$$

Thus we get

$$S_2+S_3+S_4$$

$$=\frac{T}{\pi^2}\delta_{\chi,\psi}\cdot\left\{-\int_0^\beta \frac{1}{a}\left(1-f(\frac{a}{\beta})\right)^2\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\right.$$

$$\left.+\int_0^\beta \frac{1}{a}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\right\}$$

$$+O\left(T\Delta^2+\frac{T}{\log T}\log\log T\right).$$

We are left to evaluate S_1.

$$S_1 = 2\int_1^T B(t,\chi)B(t,\psi)\,dt - \int_1^T B(t+\Delta,\chi)B(t,\psi)\,dt$$

$$-\int_1^T B(t+\Delta,\psi)B(t,\chi)\,dt + O(\Delta\log^2 T).$$

As in pp.158–160 of [13], we get an evaluation of a typical integral as follows.

$$\int_1^T B(t+\Delta,\chi)B(t,\psi)\,dt$$

$$=\frac{1}{\pi^2}\sum_{0<\gamma(\chi),\gamma(\psi)\le T}\int_{-\infty}^{+\infty} h\big(\big(t+\Delta-\gamma(\chi)\big)\log x\big)h\big(\big(t-\gamma(\psi)\big)\log x\big)\,dt$$

$$+O(\log^3 T)$$

$$=\frac{1}{\pi^2\log x}\sum_{0<\gamma(\chi),\gamma(\psi)\le T}\hat{k}\big(\big(\gamma(\chi)-\gamma(\psi)-\Delta\big)\log x\big)+O(\log^3 T),$$

where we put

$$k(u)=\begin{cases}\left(\frac{1}{2u}-\frac{\pi^2}{2}\cot(\pi^2 u)\right)^2 & \text{for } |u|\le\frac{1}{2\pi}\\ \frac{1}{4u^2} & \text{for } |u|>\frac{1}{2\pi}\end{cases}$$

and $\hat{k}(u)$ is the Fourier transpose of $k(u)$.

Thus we get

$$S_1=\frac{2}{\pi^2\log x}\sum_{0<\gamma(\chi),\gamma(\psi)\le T}\hat{k}\big(\big(\gamma(\chi)-\gamma(\psi)\big)\log x\big)$$

$$-\frac{1}{\pi^2\log x}\sum_{0<\gamma(\chi),\gamma(\psi)\le T}\hat{k}\big(\big(\gamma(\chi)-\gamma(\psi)-\Delta\big)\log x\big)$$

$$-\frac{1}{\pi^2\log x}\sum_{0<\gamma(\chi),\gamma(\psi)\le T}\hat{k}\big(\big(\gamma(\psi)-\gamma(\chi)-\Delta\big)\log x\big)$$

$$+O(\log^3 T).$$

We notice first that

$$\sum_{0<\gamma(\chi),\gamma(\psi)\le T} \hat{k}(\big(\gamma(\chi)-\gamma(\psi)-\Delta\big)\log x)(1-w(\gamma(\chi)-\gamma(\psi)))$$

$$= \sum_{0<\gamma(\chi),\gamma(\psi)\le T} \hat{k}(\big(\gamma(\chi)-\gamma(\psi)-\Delta\big)\log x)\frac{(\gamma(\chi)-\gamma(\psi))^2}{4+(\gamma(\chi)-\gamma(\psi))^2}$$

$$\ll \sum_{0<\gamma(\chi),\gamma(\psi)\le T} \mathrm{Min}(1, \frac{1}{((\gamma(\chi)-\gamma(\psi)-\Delta)\log x)^2}) \cdot \frac{(\gamma(\chi)-\gamma(\psi))^2}{4+(\gamma(\chi)-\gamma(\psi))^2}$$

$$\ll \Delta^2 T\log T,$$

where we have used the following lemma which can be proved as in pp.99–100 of [10].

Lemma 3. *Suppose that $a \ll T^A$ with some positive constant A. Then we have*

$$\sum_{0<\gamma(\chi)\le T,\,\gamma(\chi)+a>0} S(\gamma(\chi)+a,\psi) \ll T\log T.$$

We notice second that

$$\sum_{0<\gamma(\chi),\gamma(\psi)\le T} \hat{k}(\big(\gamma(\chi)-\gamma(\psi)-\Delta\big)\log x)w(\gamma(\chi)-\gamma(\psi))$$

$$= \int_{-\infty}^{+\infty} k(u) \sum_{0<\gamma(\chi),\gamma(\psi)\le T} e(-u(\gamma(\chi)-\gamma(\psi)-\Delta)\log x) \cdot w(\gamma(\chi)-\gamma(\psi))\,du$$

$$= \int_{-\infty}^{+\infty} k(u) \sum_{0<\gamma(\chi),\gamma(\psi)\le T} e(-u(\gamma(\chi)-\gamma(\psi))\log x) \cdot w(\gamma(\chi)-\gamma(\psi))e(u\Delta\log x)\,du.$$

Hence, we get

$$\begin{aligned}
S_1 =&\frac{2}{\pi^2 \log x}\int_{-\infty}^{+\infty} k(u) \sum_{0<\gamma(\chi),\gamma(\psi)\le T} e(-u(\gamma(\chi)-\gamma(\psi))\log x) \\
&\qquad\qquad \cdot w(\gamma(\chi)-\gamma(\psi))\,du \\
&-\frac{1}{\pi^2 \log x}\int_{-\infty}^{+\infty} k(u) \sum_{0<\gamma(\chi),\gamma(\psi)\le T} e(-u(\gamma(\chi)-\gamma(\psi))\log x) \\
&\qquad\qquad \cdot w(\gamma(\chi)-\gamma(\psi))e(u\Delta\log x)\,du \\
&-\frac{1}{\pi^2 \log x}\int_{-\infty}^{+\infty} k(u) \sum_{0<\gamma(\chi),\gamma(\psi)\le T} e(-u(\gamma(\psi)-\gamma(\chi))\log x) \\
&\qquad\qquad \cdot w(\gamma(\psi)-\gamma(\chi))e(u\Delta\log x)\,du \\
&+O\left(\Delta^2 T\frac{\log T}{\log x}\right) \\
=&\frac{1}{\pi^2 \log x}\int_{-\infty}^{+\infty} k(u) \sum_{0<\gamma(\chi),\gamma(\psi)\le T} e(-u(\gamma(\chi)-\gamma(\psi))\log x) \\
&\qquad\qquad \cdot w(\gamma(\chi)-\gamma(\psi))(1-e(u\Delta\log x))\,du \\
&-\frac{1}{\pi^2 \log x}\int_{-\infty}^{+\infty} k(u) \sum_{0<\gamma(\chi),\gamma(\psi)\le T} e(-u(\gamma(\psi)-\gamma(\chi))\log x) \\
&\qquad\qquad \cdot w(\gamma(\psi)-\gamma(\chi))(1-e(u\Delta\log x))\,du \\
&+O\left(\Delta^2 T\frac{\log T}{\log x}\right) \\
=&\frac{T\log\frac{T}{2\pi}}{\pi^4\beta\log x}\int_0^{\infty} k(\frac{a}{2\pi\beta})F_{\chi,\psi}(a)\left(1-\cos(a\Delta\log\frac{T}{2\pi})\right)\,da \\
&+O\left(\Delta^2 T\frac{\log T}{\log x}\right)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{T}{\pi^2}\left\{\int_0^\beta \frac{1}{a^2}\left(1-\frac{\pi a}{2\beta}\cot\left(\frac{\pi a}{2\beta}\right)\right)^2\right.\\
&\qquad\qquad \cdot F_{\chi,\psi}(a)\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\\
&\qquad + \int_\beta^1 \frac{1}{a^2}F_{\chi,\psi}(a)\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\\
&\qquad \left. + \int_1^\infty \frac{1}{a^2}F_{\chi,\psi}(a)\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\right\}\\
&\quad + O\left(\Delta^2 T\frac{1}{\beta}\right).
\end{aligned}
$$

Using Theorem 13, we get

$$
\begin{aligned}
S_1 &= \frac{T}{\pi^2}\delta_{\chi,\psi}\int_0^\beta \frac{1}{a}\left(1-f\left(\frac{a}{\beta}\right)\right)^2\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\\
&\quad + \frac{T}{\pi^2}\delta_{\chi,\psi}\int_\beta^1 \frac{1}{a}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\\
&\quad + \frac{T}{\pi^2}\int_1^\infty \frac{1}{a^2}F_{\chi,\psi}(a)\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)da\\
&\quad + O\left(\Delta^2 T\frac{1}{\beta}\right) + O\left(\frac{T}{\sqrt{\log T}}\right).
\end{aligned}
$$

Combining all of our evaluations and taking $\beta = \frac{1}{2}$, we get

$$
\begin{aligned}
S &= \frac{T}{\pi^2}\{\delta_{\chi,\psi}\int_0^{\Delta\log\frac{T}{2\pi}} \frac{1-\cos(a)}{a}\,da\\
&\qquad + \int_1^\infty \frac{1}{a^2}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)F_{\chi,\psi}(a)\,da\}\\
&\quad + O\left(T\left(\Delta^2+\frac{1}{\sqrt{\log T}}\right)\right).
\end{aligned}
$$

This proves our Theorem 14 as described in the section 4. For completeness, we shall give a proof of Lemma 1 below. By evaluating the integral

$$
I = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{L'}{L}(s,\chi)\frac{x^{z-s}}{z-s}\,dz
$$

for $c > \mathrm{Max}(\sigma, 1)$, in two ways, we get for $x > 1$, $x \neq p^n$ and for $s \neq -a-2n$, $s \neq \rho$,

$$\frac{L'}{L}(s,\chi) = -\sum_{n\leq x}\frac{\Lambda(n)\chi(n)}{n^s} + \delta(\chi)\frac{x^{1-s}}{1-s} - \sum_{\rho}\frac{x^{\rho-s}}{\rho-s} + \sum_{n=0}^{+\infty}\frac{x^{-a-2n-s}}{a+2n+s},$$

where $a = a(\chi) = \frac{1}{2}(1-\chi(-1))$.

Thus we get

$$\begin{aligned}
&x^{\sigma-\frac{1}{2}}\frac{L'}{L}(\sigma+it,\chi) - x^{\frac{1}{2}-\sigma}\frac{L'}{L}(1-\sigma+it,\chi)\\
=&-x^{-\frac{1}{2}}\sum_{n\leq x}\frac{\Lambda(n)\chi(n)}{n^{it}}\left(\left(\frac{x}{n}\right)^{\sigma} - \left(\frac{x}{n}\right)^{1-\sigma}\right)\\
&-(1-2\sigma)\sum_{\gamma}\frac{x^{i(\gamma(\chi)-t)}}{(\frac{1}{2}-\sigma)^2+(\gamma(\chi)-t)^2} - \delta(\chi)x^{\frac{1}{2}}\frac{x^{-it}(1-2\sigma)}{(\sigma-it)(1-\sigma-it)}\\
&+x^{-\frac{1}{2}-it}(1-2\sigma)\sum_{n=0}^{+\infty}\frac{x^{-a-2n}}{(a+2n+\sigma+it)(a+2n+1-\sigma+it)}.
\end{aligned}$$

By the functional equation of $L(s,\chi)$, we get for $t \geq 1$,

$$\begin{aligned}
&\Im(\frac{L'}{L}(1-\sigma+it,\chi))\\
=&\Im(\frac{L'}{L}(\sigma+it,\chi))\\
&+\frac{1}{2}\Im(\frac{\Gamma'}{\Gamma}(\frac{1}{2}(\sigma+it+a))) + \frac{1}{2}\Im(\frac{\Gamma'}{\Gamma}(\frac{1}{2}(1-\sigma+it+a)))\\
=&\Im(\frac{L'}{L}(\sigma+it,\chi)) + O\left(\frac{|\sigma-\frac{1}{2}|}{t}\right).
\end{aligned}$$

We notice also that

$$\left| x^{-\frac{1}{2}-it}(1-2\sigma)\sum_{n=0}^{+\infty}\frac{x^{-a-2n}}{(a+2n+\sigma+it)(a+2n+1-\sigma+it)} \right|$$

$$\ll \frac{x^{-\frac{1}{2}}}{t}\left|\sigma-\frac{1}{2}\right|\left(\frac{x^{-a}}{t}+x^{-2}\right).$$

Hence, we get

$$\left(x^{\sigma-\frac{1}{2}}-x^{\frac{1}{2}-\sigma}\right)\Im(\frac{L'}{L}(\sigma+it,\chi))$$

$$=-x^{-\frac{1}{2}}\Im\left(\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{n^{it}}\left(\left(\frac{x}{n}\right)^{\sigma}-\left(\frac{x}{n}\right)^{1-\sigma}\right)\right)$$

$$-(1-2\sigma)\Im\left(\sum_{\gamma}\frac{x^{i(\gamma(\chi)-t)}}{(\frac{1}{2}-\sigma)^2+(\gamma(\chi)-t)^2}\right)$$

$$-\delta(\chi)x^{\frac{1}{2}}\Im\left(\frac{x^{-it}(1-2\sigma)}{(\sigma-it)(1-\sigma-it)}\right)$$

$$+O\left(\frac{x^{\frac{1}{2}-\sigma}}{t}\left|\sigma-\frac{1}{2}\right|\right)+O\left(\frac{x^{-\frac{1}{2}}}{t}\left|\sigma-\frac{1}{2}\right|\left(\frac{x^{-a}}{t}+x^{-2}\right)\right).$$

Now for $t\ge 1$ and $t\ne\gamma$,

$$S(t,\chi)=-\frac{1}{\pi}\Im\left(\int_{\frac{1}{2}}^{\infty}\frac{L'}{L}(\sigma+it,\chi)\,d\sigma\right)$$

$$=\frac{1}{\pi}x^{-\frac{1}{2}}\Im\left(\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{n^{it}}\int_{\frac{1}{2}}^{\infty}\frac{(\frac{x}{n})^{\sigma}-(\frac{x}{n})^{1-\sigma}}{x^{\sigma-\frac{1}{2}}-x^{\frac{1}{2}-\sigma}}\,d\sigma\right)$$

$$+\frac{1}{\pi}\Im(\sum_{\gamma} x^{i(\gamma(\chi)-t)}$$

$$\cdot\int_{\frac{1}{2}}^{\infty}\frac{(1-2\sigma)}{((\frac{1}{2}-\sigma)^2+(\gamma(\chi)-t)^2)(x^{\sigma-\frac{1}{2}}-x^{\frac{1}{2}-\sigma})}\,d\sigma)$$

$$+\,\delta(\chi)\frac{1}{\pi}x^{\frac{1}{2}}\Im(x^{-it}$$

$$\cdot\int_{\frac{1}{2}}^{\infty}\frac{(1-2\sigma)}{(\sigma-it)(1-\sigma-it)}\frac{d\sigma}{2\sinh((\sigma-\frac{1}{2})\log x)})$$

$$+\,O\left(\frac{1}{t\log^2 x}\right)$$

$$=\Im\left(\frac{1}{\pi}\sum_{n\le x}\frac{\Lambda(n)\chi(n)}{\sqrt{n}\log n\,n^{it}}f(\frac{\log n}{\log x})\right)$$

$$+\,\frac{1}{\pi}\sum_{\gamma(\chi)}h(\big(t-\gamma(\chi)\big)\log x)-\delta(\chi)x^{\frac{1}{2}}g(x,t)+O\left(\frac{1}{t\log^2 x}\right)$$

$$=A(t,\chi)+B(t,\chi)-\delta(\chi)x^{\frac{1}{2}}g(x,t)+O\left(\frac{1}{t\log^2 x}\right),\quad \text{say,}$$

where the functions f, g and h are introduced in the statement of Lemma 1. This proves Lemma 1.

7-2. Proof of Theorem 13

We start with the following formula which can be proved as above.

For $1 < \sigma < 2$ and $x \geq 1$,

$$
\begin{aligned}
C(x,\chi,t,\sigma) \equiv & \sum_{\gamma(\chi)} \frac{x^{i\gamma(\chi)}}{(\frac{1}{2}-\sigma)^2 + (\gamma(\chi)-t)^2} \\
= & -\frac{x^{-\frac{1}{2}}}{2\sigma-1} \left\{ \sum_{n\leq x} \Lambda(n)\chi(n) \left(\frac{x}{n}\right)^{1-\sigma+it} + \sum_{n>x} \Lambda(n)\chi(n) \left(\frac{x}{n}\right)^{\sigma+it} \right\} \\
& + \frac{x^{\frac{1}{2}-\sigma+it}}{2\sigma-1} \log\tau \\
& + O\left(\frac{x^{\frac{1}{2}-\sigma}}{2\sigma-1} + \frac{x^{-\frac{1}{2}}}{\tau} \left(\frac{x^{-a}}{\tau} + x^{-2} \right) + \delta(\chi) x^{\frac{1}{2}} \tau^{-2} \right) \\
= & D(x,\chi,t,\sigma) + E_1(t,\sigma) + E_2(\chi,t,\sigma), \qquad \text{say,}
\end{aligned}
$$

where we put $\tau = |t| + 2$.

We put $x = \left(\frac{T}{2\pi}\right)^\alpha$ with $0 \leq \alpha \leq 1$. We notice first that

$$
\begin{aligned}
& \int_0^T C(x,\chi,t,\frac{3}{2}) \overline{C(x,\psi,t,\frac{3}{2})}\, dt \\
= & \sum_{0<\gamma(\chi),\gamma(\psi)\leq T} x^{i(\gamma(\chi)-\gamma(\psi))} \int_{-\infty}^{+\infty} \frac{dt}{(1+(\gamma(\chi)-t)^2)(1+(\gamma(\psi)-t)^2)} \\
& + O(\log^3 T) \\
= & \sum_{0<\gamma(\chi),\gamma(\psi)\leq T} x^{i(\gamma(\chi)-\gamma(\psi))} \frac{2\pi}{4+(\gamma(\chi)-\gamma(\psi))^2} \\
& + O(\log^3 T).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
& \Re\{ \sum_{0<\gamma(\chi),\gamma(\psi)\leq T} x^{i(\gamma(\chi)-\gamma(\psi))} w(\gamma(\chi)-\gamma(\psi)) \} \\
= & \frac{1}{\pi} 2\Re\{ \int_0^T \left(D(x,\chi,t,\frac{3}{2}) + E_1(t,\frac{3}{2}) + E_2(\chi,t,\frac{3}{2}) \right) \\
& \cdot \left(\overline{D(x,\psi,t,\frac{3}{2})} + \overline{E_1(t,\frac{3}{2})} + E_2(\psi,t,\frac{3}{2}) \right) dt \}
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{\pi}2\Re\{\int_0^T D(x,\chi,t,\frac{3}{2})\overline{D(x,\psi,t,\frac{3}{2})}\,dt\}\\
&+\frac{1}{\pi}2\Re\{\int_0^T D(x,\chi,t,\frac{3}{2})\overline{E_1(t,\frac{3}{2})}\,dt\}\\
&+\frac{1}{\pi}2\Re\{\int_0^T E_1(t,\frac{3}{2})\overline{D(x,\psi,t,\frac{3}{2})}\,dt\}\\
&+\frac{1}{\pi}2\Re\{\int_0^T D(x,\chi,t,\frac{3}{2})E_2(\psi,t,\frac{3}{2})\,dt\}\\
&+\frac{1}{\pi}2\Re\{\int_0^T \overline{D(x,\psi,t,\frac{3}{2})}E_2(\chi,t,\frac{3}{2})\,dt\}\\
&+\frac{1}{\pi}2\int_0^T \left|E_1(t,\frac{3}{2})\right|^2 dt+\frac{1}{\pi}2\int_0^T E_2(\chi,t,\frac{3}{2})E_2(\psi,t,\frac{3}{2})\,dt\\
&+O(\int_0^T \left|E_1(t,\frac{3}{2})\right|\left|E_2(\chi,t,\frac{3}{2})+E_2(\psi,t,\frac{3}{2})\right|dt)\\
=&U_1+U_2+U_3+\cdots+U_8,\quad \text{say.}
\end{aligned}
$$

$$
\begin{aligned}
U_1=&\frac{1}{\pi}\left\{\int_0^T\left|D(x,\chi,t,\frac{3}{2})+D(x,\psi,t,\frac{3}{2})\right|^2 dt\right.\\
&\left.-\int_0^T\left|D(x,\chi,t,\frac{3}{2})\right|^2 dt-\int_0^T\left|D(x,\psi,t,\frac{3}{2})\right|^2 dt\right\}
\end{aligned}
$$

Here we use the following lemma which can be proved by modifying the proof of Lemma 7 in Goldston-Montgomery [14].

Lemma 4. *Suppose that* $\sum_{n=2}^{\infty}\Lambda(n)|a(n)|<\infty$, $\frac{3}{T}\le\delta\le 1$ *and* $A(n)>0$ *is a continuous function such that* $A(u)\approx A(u')$ *whenever* $u\approx u'$ *and* $a(n)\ll A(u)$ *for integers* $n\ge 1$. *Then we have*

$$
\begin{aligned}
&\int_0^T\left|\sum_{n=1}^{\infty}\Lambda(n)a(n)n^{it}\right|^2 dt\\
&=\left(T+O(\delta^{-1})\right)\sum_{n=1}^{\infty}\Lambda^2(n)|a(n)|^2+O(\delta T\int_{\delta^{-1}}^{\infty}A^2(u)u\,du).
\end{aligned}
$$

Using this lemma with $A(u)=\mathrm{Min}((\frac{x}{u})^{-\frac{1}{2}},(\frac{x}{u})^{\frac{3}{2}})$, we get

$$\begin{aligned}
U_1 =&\frac{1}{4x\pi}T\sum_{n=1}^{\infty}\Lambda^2(n)\left(|\chi(n)+\psi(n)|^2-|\chi(n)|^2-|\psi(n)|^2\right)\\
&\cdot \mathrm{Min}\left(\left(\frac{x}{n}\right)^{-1},\left(\frac{x}{n}\right)^{3}\right)\\
&+O\left(x^{-1}\delta^{-1}\sum_{n=1}^{\infty}\Lambda^2(n)\cdot\mathrm{Min}\left(\left(\frac{x}{n}\right)^{-1},\left(\frac{x}{n}\right)^{3}\right)\right)\\
&+O\left(x^{-1}\delta T\int_{\delta^{-1}}^{\mathrm{Max}(x,\delta^{-1})}\frac{u^2}{x}\,du\right)+O\left(x^{-1}\delta T\int_{\mathrm{Max}(x,\delta^{-1})}^{\infty}\frac{x^3}{u^2}\,du\right)\\
=&\frac{1}{4x\pi}T\frac{1}{x}\sum_{n\le x}\Lambda^2(n)(\chi\overline{\psi}+\overline{\chi}\psi)(n)n\\
&+\frac{1}{4x\pi}Tx^3\sum_{n>x}\Lambda^2(n)(\chi\overline{\psi}+\overline{\chi}\psi)(n)\frac{1}{n^3}\\
&+O\left(x^{-1}\delta^{-1}\left(\frac{1}{x}\sum_{n\le x}\Lambda^2(n)n+x^3\sum_{n>x}\Lambda^2(n)\frac{1}{n^3}\right)\right)\\
&+O(\mathrm{Min}(\delta Tx,\delta^2Tx^2))\\
=&\frac{T}{2\pi}\delta_{\chi,\psi}\log x+O\left(T\exp\left(-A\sqrt{\log(3x)}\right)\right)\\
&+O(\mathrm{Min}(\delta Tx,\delta^2Tx^2))+O(\delta^{-1}\log(3x)),
\end{aligned}$$

where A is some positive constant.

Here we choose

$$\delta=T^{-\frac{1}{3}}x^{-\frac{2}{3}}\log^{\frac{1}{3}}(3x)\qquad\text{if}\quad x\le\delta^{-1}$$

and

$$\delta=T^{-\frac{1}{2}}x^{-\frac{1}{2}}\log^{\frac{1}{2}}(3x)\qquad\text{if}\quad x\ge\delta^{-1}.$$

Then we get

$$\begin{aligned}
U_1 =&\frac{T}{2\pi}\delta_{\chi,\psi}\log x+O\left(T\exp\left(-A\sqrt{\log(3x)}\right)\right)\\
&+O(\mathrm{Min}(T^{\frac{1}{2}}x^{\frac{1}{2}}\log^{\frac{1}{2}}(3x),T^{\frac{1}{3}}x^{\frac{1}{3}}\log^{\frac{2}{3}}(3x))).
\end{aligned}$$

$$U_2, U_3 \ll x^{-2} \sum_{n\le x} \Lambda(n)\sqrt{n}\frac{\log T}{\log n} + \sum_{n>x} \frac{\Lambda(n)}{n^{\frac{3}{2}}}\frac{\log T}{\log n}$$
$$\ll x^{-\frac{1}{2}} \log T.$$

By Schwartz inequality, we get

$$U_4 \ll \frac{T}{x}\sqrt{\log x} + \delta(\psi)T\sqrt{\log T}.$$

Similarly, we get

$$U_5 \ll \frac{T}{x}\sqrt{\log x} + \delta(\chi)T\sqrt{\log T}.$$
$$U_6 = \frac{T}{2\pi x^2}\left(\log^2 T + O(\log T)\right).$$
$$U_7 \ll \frac{T}{x^2} + \big(\delta(\chi) + \delta(\psi)\big)\sqrt{\frac{T}{x}} + \delta(\chi)\delta(\psi)x.$$
$$U_8 \ll \frac{T}{x^2}\log T + \big(\delta(\chi) + \delta(\psi)\big)x^{-\frac{1}{2}}.$$

Combining all of our evaluations we get

$$\Re\{ \sum_{0<\gamma(\chi),\gamma(\psi)\le T} x^{i(\gamma(\chi)-\gamma(\psi))} w(\gamma(\chi)-\gamma(\psi))\}$$
$$= \frac{T}{2\pi}\delta_{\chi,\psi}\log x + \frac{T}{2\pi x^2}\log^2 T + O(\frac{T}{x^2}\log T) + O(\frac{T}{x}\sqrt{\log x})$$
$$+ O(T\exp\left(-A\sqrt{\log(3x)}\right))$$
$$+ O(\mathrm{Min}(T^{\frac{1}{2}}x^{\frac{1}{2}}\log^{\frac{1}{2}}(3x), T^{\frac{1}{3}}x^{\frac{2}{3}}\log^{\frac{2}{3}}(3x)))$$
$$+ O(\delta(\chi)\delta(\psi)x) + O(\big(\delta(\chi)+\delta(\psi)\big)T\sqrt{\log T}).$$

This proves our theorem as described in the section 4.

7-3. Proof of Theorem 12

Our proof of Theorem 12 goes pararel to the proof of Theorem 14 given in the subsection 7-1. We shall use some of the notations used there.

Using Lemma 1 with $x = \left(\frac{T}{2\pi}\right)^{\beta}$ with $0 < \beta \le 1$, we get

$$S \equiv \int_0^T S(t,\chi)S(t,\psi)\,dt$$

$$= \int_1^T B(t,\chi)B(t,\psi)\,dt - \int_1^T A(t,\chi)A(t,\psi)\,dt$$

$$+ \int_1^T A(t,\chi)S(t,\psi)\,dt + \int_1^T A(t,\psi)S(t,\chi)\,dt$$

$$+ O(\frac{\log^2 T}{\log^2 x} + \delta(\chi)\delta(\psi)\frac{x}{\log^4 x} + \big(\delta(\chi)+\delta(\psi)\big)\frac{\sqrt{x}}{\log^2 x})$$

$$= S_1 + S_2 + S_3 + S_4$$

$$+ O(\frac{\log^2 T}{\log^2 x} + \delta(\chi)\delta(\psi)\frac{x}{\log^4 x} + \big(\delta(\chi)+\delta(\psi)\big)\frac{\sqrt{x}}{\log^2 x}), \quad \text{say.}$$

$$S_4 = \frac{1}{\pi}\Im\{\sum_{n\le x}\frac{\Lambda(n)\psi(n)}{\sqrt{n}\log n}f(\frac{\log n}{\log x})\cdot\int_1^T S(t,\chi)e^{-it\log n}\,dt\}$$

$$= \frac{T}{2\pi^2}\Im\{i\sum_{n\le x}\frac{\Lambda^2(n)\psi(n)\overline{\chi(n)}}{n\log^2 n}f(\frac{\log n}{\log x})\}$$

$$+ O(\frac{x}{\log(3x)}\log\log(3x) + \frac{x}{\log^2(3x)}\sqrt{\frac{\log T}{\log\log T}}$$

$$+ \frac{\sqrt{x}}{\log x}\frac{\log T}{\log\log T}x^{\frac{1}{\log\log T}}).$$

Similarly, we get

$$S_3 = \frac{T}{2\pi^2}\Im\{i\sum_{n\le x}\frac{\Lambda^2(n)\chi(n)\overline{\psi(n)}}{n\log^2 n}f(\frac{\log n}{\log x})\}$$

$$+ O(\frac{x}{\log(3x)}\log\log(3x) + \frac{x}{\log^2(3x)}\sqrt{\frac{\log T}{\log\log T}}$$

$$+ \frac{\sqrt{x}}{\log x}\frac{\log T}{\log\log T}x^{\frac{1}{\log\log T}}).$$

We next evaluate S_2.

$$S_2 = \frac{1}{2\pi^2}\Re\{\sum_{m\le x}\sum_{n\le x}\frac{\Lambda(m)\Lambda(n)\chi(m)\psi(n)}{\sqrt{mn}\log m\log n} \cdot f(\frac{\log m}{\log x})f(\frac{\log n}{\log x})\int_1^T \frac{1}{(mn)^{it}}\,dt\}$$

$$-\frac{1}{2\pi^2}\Re\{\sum_{m\le x}\sum_{n\le x}\frac{\Lambda(m)\Lambda(n)\chi(m)\overline{\psi}(n)}{\sqrt{mn}\log m\log n} \cdot f(\frac{\log m}{\log x})f(\frac{\log n}{\log x})\int_1^T (\frac{n}{m})^{it}\,dt\}$$

$$= -\frac{T}{2\pi^2}\Re\{\sum_{n\le x}\frac{\Lambda^2(n)(\chi\overline{\psi})(n)}{n\log^2 n}f^2(\frac{\log n}{\log x})\} + O(x\frac{\log\log(3x)}{\log(3x)}).$$

Hence, we get

$$S_2+S_3+S_4 = \frac{T}{2\pi^2}\Re\{\sum_{n\le x}\frac{\Lambda^2(n)(\chi\overline{\psi})(n)}{n\log^2 n}\left(2f(\frac{\log n}{\log x}) - f^2(\frac{\log n}{\log x})\right)\}$$
$$+O(\frac{x}{\log(3x)}\log\log(3x) + \frac{x}{\log^2(3x)}\sqrt{\frac{\log T}{\log\log T}} + \frac{\sqrt{x}}{\log x}\frac{\log T}{\log\log T}x^{\frac{1}{\log\log T}}).$$

Since

$$\sum_{n\le x}\frac{\Lambda^2(n)(\chi\overline{\psi})(n)}{n\log^2 n}\left(2f(\frac{\log n}{\log x}) - f^2(\frac{\log n}{\log x})\right)$$
$$= \delta_{\chi,\psi}\{\log\log x + C_0 + \sum_p\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right) - \int_0^1\frac{(f(u)-1)^2}{u}\,du - \sum_{p|q}\frac{1}{p} + \sum_{p,r\ge 2}\frac{(\chi\overline{\psi})(p^r)}{p^r r^2}\}$$
$$+(1-\delta_{\chi,\psi})\sum_{n=2}^{\infty}\frac{\Lambda^2(n)(\chi\overline{\psi})(n)}{n\log^2 n} + O(\frac{\log\log x}{\log x}),$$

we get

$$
\begin{aligned}
&S_2+S_3+S_4\\
=&\delta_{\chi,\psi}\frac{T}{2\pi^2}\{\log\log x+C_0+\sum_p\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)\\
&\qquad -\int_0^1\frac{(f(u)-1)^2}{u}\,du-\sum_{p|q}\frac{1}{p}+\sum_{p,r\ge 2}\frac{(\chi\overline{\psi})(p^r)}{p^r r^2}\}\\
&+(1-\delta_{\chi,\psi})\frac{T}{2\pi^2}\Re\{\sum_{n=2}^{\infty}\frac{\Lambda^2(n)(\chi\overline{\psi})(n)}{n\log^2 n}\}+O(\frac{T}{\log x}\log\log x)\\
&+O(\frac{x}{\log(3x)}\log\log(3x)+\frac{x}{\log^2(3x)}\sqrt{\frac{\log T}{\log\log T}}\\
&\qquad +\frac{\sqrt{x}}{\log x}\frac{\log T}{\log\log T}x^{\frac{1}{\log\log T}}).
\end{aligned}
$$

We shall finally evaluate S_1.

$$
\begin{aligned}
S_1=&\frac{1}{\pi^2\log x}\sum_{0<\gamma(\chi),\gamma(\psi)\le T}\hat{k}((\gamma(\chi)-\gamma(\psi))\log x)\\
&+O(\log^3 T)\\
=&\frac{T}{2\pi^4\beta^2}\int_0^\infty k(\frac{a}{2\pi\beta})F_{\chi,\psi}(a)\,da+O(\frac{T}{\log x})\\
=&\frac{T}{2\pi^2}\delta_{\chi,\psi}\int_0^\beta\frac{1}{a}\left(1-f(\frac{a}{\beta})\right)^2 da+\frac{T}{2\pi^2}\delta_{\chi,\psi}\int_\beta^1\frac{1}{a}\,da\\
&+\frac{T}{2\pi^2}\int_1^\infty\frac{1}{a^2}F_{\chi,\psi}(a)\,da+O(\frac{T}{\sqrt{\log T}}).
\end{aligned}
$$

Combining all of our evaluations and taking $\beta=\frac{1}{2}$, we get Theorem 12 as described in the section 4.

7-4. Proof of Corollary 3

We shall prove that

$$
\left|\int_1^\infty\frac{1}{a^2}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right)F_{\chi,\psi}(a)\,da\right|\ll 1.
$$

In fact, the argument below proves at the same time that

$$\left| \int_1^\infty \frac{1}{a^2} F_{\chi,\psi}(a)\, da \right| \ll 1.$$

We put $G(a,\chi,t) = C((\frac{T}{2\pi})^a, \chi, t, \frac{3}{2})$. Then

$$\begin{aligned}
F_{\chi,\psi}(a) =& \frac{2}{\pi} \frac{1}{\frac{T}{2\pi}\log\frac{T}{2\pi}} \Re\{ \int_0^T G(a,\chi,t)\overline{G(a,\psi,t)}\, dt \} \\
&+ O(\frac{\log^2 T}{T}) \\
=& \frac{1}{\pi} \frac{1}{\frac{T}{2\pi}\log\frac{T}{2\pi}} \int_0^T |G(a,\chi,t) + G(a,\psi,t)|^2\ dt \\
&- \frac{1}{\pi} \frac{1}{\frac{T}{2\pi}\log\frac{T}{2\pi}} \left\{ \int_0^T |G(a,\chi,t)|^2\ dt + \int_0^T |G(a,\psi,t)|^2\ dt \right\} \\
&+ O(\frac{\log^2 T}{T}) \\
=& \frac{1}{\pi} \frac{1}{\frac{T}{2\pi}\log\frac{T}{2\pi}} \int_0^T |G(a,\chi,t) + G(a,\psi,t)|^2\ dt \\
&- \frac{1}{2}\{F_{\chi,\chi}(a) + F_{\psi,\psi}(a)\} + O(\frac{\log^2 T}{T}).
\end{aligned}$$

Thus we get

$$\begin{aligned}
&\int_1^\infty \frac{1}{a^2} \left(1 - \cos(a\Delta \log \frac{T}{2\pi})\right) F_{\chi,\psi}(a)\, da \\
&= -\frac{1}{2} \int_1^\infty \frac{1}{a^2} \left(1 - \cos(a\Delta \log \frac{T}{2\pi})\right) F_{\chi,\chi}(a)\, da \\
&\quad - \frac{1}{2} \int_1^\infty \frac{1}{a^2} \left(1 - \cos(a\Delta \log \frac{T}{2\pi})\right) F_{\psi,\psi}(a)\, da \\
&\quad + \frac{1}{\pi} \frac{1}{\frac{T}{2\pi}\log\frac{T}{2\pi}} \int_1^\infty \frac{1}{a^2} \left(1 - \cos(a\Delta \log \frac{T}{2\pi})\right) \\
&\qquad\qquad \cdot \int_0^T |G(a,\chi,t) + G(a,\psi,t)|^2\ dt da \\
&\quad + O(\frac{\log^2 T}{T}).
\end{aligned}$$

Hence we have only to prove that

$$I = \frac{1}{\pi}\frac{1}{\frac{T}{2\pi}\log\frac{T}{2\pi}}\int_1^\infty \frac{1}{a^2}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right) \cdot \int_0^T |G(a,\chi,t)+G(a,\psi,t)|^2\, dtda$$

is bounded, since we see by Goldston's argument in [13] that

$$\left|\int_1^\infty \frac{1}{a^2}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right) F_{\chi,\chi}(a)\, da\right| \leq \int_1^\infty \frac{1}{a^2} F_{\chi,\chi}(a)\, da \ll 1$$

and

$$\left|\int_1^\infty \frac{1}{a^2}\left(1-\cos\left(a\Delta\log\frac{T}{2\pi}\right)\right) F_{\psi,\psi}(a)\, da\right| \leq \int_1^\infty \frac{1}{a^2} F_{\psi,\psi}(a)\, da \ll 1.$$

Now

$$\begin{aligned}
I \ll& \frac{1}{T\log T}\sum_{r=1}^\infty \frac{1}{r^2}\int_{r-\frac{1}{2}}^{r+\frac{1}{2}} (1-|a-r|)\int_0^T |G(a,\chi,t)+G(a,\psi,t)|^2\, dtda \\
\ll& \sum_{r=1}^\infty \frac{1}{r^2}\int_{r-1}^{r+1}(1-|a-r|)\left(F_{\chi,\psi}(a)+\frac{1}{2}\{F_\chi(a)+F_\psi(a)\}\right) da \\
&+ \frac{\log^2 T}{T} \\
=& \frac{1}{\frac{T}{2\pi}\log\frac{T}{2\pi}}\sum_{r=1}^\infty \frac{1}{r^2} \\
&\cdot \{\frac{1}{2}\sum_{0<\gamma(\chi),\gamma(\psi)\leq T}\left(\left(\frac{T}{2\pi}\right)^{ir(\gamma(\chi)-\gamma(\psi))}+\left(\frac{T}{2\pi}\right)^{-ir(\gamma(\chi)-\gamma(\psi))}\right) \\
&\cdot \left(\frac{\sin\left(\frac{\gamma(\chi)-\gamma(\psi)}{2}\log\frac{T}{2\pi}\right)}{\frac{\gamma(\chi)-\gamma(\psi)}{2}\log\frac{T}{2\pi}}\right)^2 w(\gamma(\chi)-\gamma(\psi))
\end{aligned}$$

$$+\frac{1}{2}\sum_{0<\gamma(\chi),\gamma'(\chi)\le T}\left(\frac{T}{2\pi}\right)^{ir(\gamma(\chi)-\gamma'(\chi))}$$

$$\cdot\left(\frac{\sin(\frac{\gamma(\chi)-\gamma'(\chi)}{2}\log\frac{T}{2\pi})}{\frac{\gamma(\chi)-\gamma'(\chi)}{2}\log\frac{T}{2\pi}}\right)^2 w(\gamma(\chi)-\gamma'(\chi))$$

$$+\frac{1}{2}\sum_{0<\gamma(\psi),\gamma'(\psi)\le T}\left(\frac{T}{2\pi}\right)^{ir(\gamma(\psi)-\gamma'(\psi))}$$

$$\cdot\left(\frac{\sin(\frac{\gamma(\psi)-\gamma'(\psi)}{2}\log\frac{T}{2\pi})}{\frac{\gamma(\psi)-\gamma'(\psi)}{2}\log\frac{T}{2\pi}}\right)^2 w(\gamma(\psi)-\gamma'(\psi))$$

$$+O(1)$$

$$\ll\sum_{r=1}^{\infty}\frac{1}{r^2}\ll 1,$$

since

$$\sum_{0<\gamma(\chi),\gamma(\psi)\le T}\left(\frac{\sin(\frac{\gamma(\chi)-\gamma(\psi)}{2}\log\frac{T}{2\pi})}{\frac{\gamma(\chi)-\gamma(\psi)}{2}\log\frac{T}{2\pi}}\right)^2 w(\gamma(\chi)-\gamma(\psi))\ll T\log T$$

and

$$\sum_{0<\gamma(\chi),\gamma'(\chi)\le T}\left(\frac{\sin(\frac{\gamma(\chi)-\gamma'(\chi)}{2}\log\frac{T}{2\pi})}{\frac{\gamma(\chi)-\gamma'(\chi)}{2}\log\frac{T}{2\pi}}\right)^2 w(\gamma(\chi)-\gamma'(\chi))\ll T\log T.$$

We remark that the last two inequalities can be derived using Theorem 13 and the formula (3) of p.182 of Montgomery [17].

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Department of Mathematics
Rikkyo University
Tokyo 171
Japan