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On Special Values of Selberg Zeta Functions

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§0. Introduction

Let X be an arithmetic object (or "motif"). Typical examples of X are

1) algebraic varieties defined over algebraic number fields,
2) complex or ℓ-adic Galois representations of algebraic number fields,
3) prehomogeneous vector spaces defined over algebraic number fields, or
4) automorphic forms of "type A_0".

Associated with X, we have a zeta (or L-) function $\zeta_X(s)$. The function $\zeta_X(s)$ is defined for $\mathrm{Re}(s) >> 0$ by a Euler product or by a Dirichlet series. It is believed, or partly proved, that $\zeta_X(s)$ has a meromorphic continuation to the whole s-plane and has a functional equation with respect to $s \mapsto 1-s$ (we will take such a normalization of $\zeta_X(s)$).

For an integer n, the special value $\zeta_X(n)$ is very mysterious, but we have a widely believed dogma;

Dogma. *The leading coefficient of the Laurent expansion of $\zeta_X(s)$ at $s = n$ is a product of three numbers A, P and R;*

$$\zeta_X(s) = A \cdot P \cdot R \cdot (s-n)^e + \cdots$$

where A is an algebraic number (over $\mathbf{Q}$), P is a transcendental number (over $\mathbf{Q}$) and $R = \mathrm{vol}(L\backslash\mathbf{R}^r)$ with a lattice L in $\mathbf{R}^r$ with $r = \mathrm{ord}_{s=1-n}\,\zeta_X(s)$.

These three numbers A, P and R in Dogma are called the algebraic part, the period and the regulator of $\zeta_X(s)$ at $s = n$.

A typical example is the residue formula of Dedekind zeta function; let K be a finite algebraic number field and $\zeta_K(s)$ the Dedekind zeta function of K. Then $\zeta_K(s)$ has a simple pole at $s = 1$ and

$$\mathrm{Res}_{s=1}\zeta_K(s) = h \cdot (w\sqrt{|D|})^{-1} \cdot 2^{r_1}(2\pi)^{r_2} \cdot R(K)$$

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where h is the class number of K, w is the number of root of unity contained in K, D is the absolute discriminant of K, r_1 (resp. r_2) is the number on real (resp. complex) places of K and $R(K) = \mathrm{vol}(U_K\backslash\mathbf{R}^{r_1+r_2-1})$ is the regulator of K. Here U_K, the unit group of K, is a lattice in $\mathbf{R}^{r_1+r_2-1}$ with $r_1+r_2-1 = \mathrm{ord}_{s=0}\zeta_K(s)$. Then $A = h\cdot(w\sqrt{|D|})^{-1}$, $P = 2^{r_1}(2\pi)^{r_2}$ and $R = R(K)$ with the notation in Dogma.

Special values of Selberg zeta function are considered by [DH] or by [F]. Let Γ be a discrete torsion-free co-compact subgroup of $SL_2(\mathbf{R})$ and $Z_\Gamma(s)$ the Selberg zeta function associated with Γ. Then they give a formula

$$Z'(1) = C^{g-1}\cdot\det\Delta$$

where $\det\Delta$ (resp. g) is a functional determinant of the Laplacian Δ on (resp. the genus of) the Riemann surface $\Gamma\backslash\mathcal{H}$ and C is a constant. The purpose of this paper is to give an interpretation of the formula according to the Dogma on special values of arithmetic zeta functions. My conclusion is that C^{g-1} is the period and $\det\Delta$ is the regulator. See §4 for detailed discussion.

Notation. We will use the following notations;

$$G = SL_2(\mathbf{R}) = \{2\times 2-\text{matrices of determinant }1\}$$
$$K = SO(2,\mathbf{R}) = \{g\in G \mid {}^tg\cdot g = 1\}$$

where K is a maximal compact subgroup of G. An element of K is denoted by $r(\theta) = \begin{pmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{pmatrix}$ $(\theta\in\mathbf{R})$. For all $n\in\mathbf{Z}$, let δ_n be a unitary representation of K defined by $\delta_n(r(\theta)) = \exp(\sqrt{-1}n\theta)$. We will denote by $C_c(G)$ the complex vector space consisting of the complex-valued continuous functions with compact support.

§1. Harmonic analysis on $SL_2(\mathbf{R})$

1) Spherical functions

We denote by $\widehat{G}$ the unitary equivalence classes of the irreducible unitary representations of G, and

$$\widehat{G}(n) = \{\ \pi\in\widehat{G} \mid m(\delta_n,\pi|_K) > 0\ \}$$

where $m(\delta_n,\pi|_K)$ is the multiplicity of δ_n in the unitary representation $\pi|_K$ of K. For a $(\pi,H)\in\widehat{G}(n)$, we have $m(\delta_n,\pi|_K)=1$ and put

$$\varphi_{n,\pi}(x) = (\pi(x)u,u)\quad (x\in G)$$

with $u \in H$ such that $|u| = 1$ and $\pi(k)u = \delta_n(k)u$ for all $k \in K$. The function $\varphi_{n,\pi}$ is a spherical function of type δ_n in the sense of [W]. Two representations π and π' in $\widehat{G}(n)$ are unitarily equivalent if and only if $\varphi_{n,\pi} = \varphi_{n,\pi'}$.

The Lie algebra $\mathcal{G} = \mathrm{Lie}(G)$ of G has an Iwasawa decomposition $\mathcal{G} = \mathcal{K} \oplus \mathcal{A} \oplus \mathcal{N}$ where $\mathcal{K} = \mathrm{Lie}(K)$ and

$$\mathcal{A} = \{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mid a \in \mathbf{R}\}, \quad \mathcal{N} = \{\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbf{R}\}.$$

Corresponding Iwasawa decomposition of G is $G = K \cdot A \cdot N$ where

$$A = \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid 0 < a \in \mathbf{R}\}, \quad N = \{\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbf{R}\}.$$

Any element $x \in G$ is uniquely expressed by $x = k(x) \cdot \exp H(x) \cdot n(x)$ with $k(x) \in K$, $H(x) \in \mathcal{A}$ and $n(x) \in N$. A Haar measure $d_K(k)$, $d_A(a)$ and $d_N(n)$ on K, A and N are defined respectively by

$$d_K(r(\theta)) = (2\pi)^{-1}dr,$$
$$d_A\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} = 2y^{-1}dy,$$
$$d_N\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \pi^{-1}dx.$$

Then

$$\int_K d_K(k) = 1, \qquad \int_N \exp(-2\rho(H(\overline{n}))d_N(n) = 1$$

where $\overline{n} = \theta(n^{-1})$ with a Cartan involution $\theta(x) =^t x^{-1}$ of G and ρ is the half-sum of positive roots of $\mathcal{G}$, in our case $\rho\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} = a$. A Haar measure $d_G(x)$ on G is normalized so that

$$\int_G f(x)d_G(x) = \int_K d_K(k)\int_A d_A(a)\int_N d_N(n)f(kan) \cdot \exp(-2\rho(H(\overline{n})).$$

Put, for all $n \in \mathbf{Z}$ and $s \in \mathbf{C}$,

$$\varphi_{n,s}(x) = \int_K \alpha_{n,s}(kxk^{-1})d_K(k) \qquad (\ x \in G\ )$$

where

$$\alpha_{n,s}(x) = \delta_n(k(x)) \cdot \exp\{-(s+1)\rho(H(x))\}.$$

Then we have

$$\varphi_{n,s}(1) = 1$$
$$\varphi_{n,s}(kxk') = \delta_n(k) \cdot \varphi_{n,s}(x) \cdot \delta_n(k')$$

for all k, $k' \in K$, and

$$\varphi_{n,s}\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} = (\cosh t)^{-(s+1)} F((s+1+n)/2, (s+1-n)/2; 1; \tanh^2 t)$$

where

$$F(a,b;1;z) = \sum_{n=0}^{\infty} (a)_n (b)_n z^n/(n!)^2 \qquad (|z| < 1)$$
$$((a)_n = a(a+1)\cdots(a+n-1), \quad (a)_0 = 1)$$

is the hypergeometric function. We have

$$\varphi_{n,s} = \varphi_{m,t}$$

if and only if $n = m$ and $s = \pm t$. If part is due to Euler's formula

$$F(a,b;1;z) = (1-z)^{1-a-b} F(1-a, 1-b; 1; z)$$

and only if part is due to the formula

$$\Omega \varphi_{n,s} = \frac{1}{8}(s^2 - 1) \cdot \varphi_{n,s}$$

where

$$\Omega = \frac{1}{8}(H^2 + 2X_+X_- + 2X_-X_+) \in U(\mathcal{G} \otimes_{\mathbf{R}} \mathbf{C})$$
$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is the Casimir operator on G. All spherical functions of type δ_n on G are equal to $\varphi_{n,s}$ for some $s \in \mathbf{C}$ [Wr: vol.II, p.43, Example].

Let $L^1(G, \delta_n)$ be a complex vector space consisting of the complex-valued measurable functions f on G such that

1) $\int_G |f(x)| d_G(x) < \infty$
2) $f(kxk') = \delta_n(k) f(x) \delta_n(k')$ for all k, $k' \in K$.

Put $C_c(G, \delta_n) = C_c(G) \cap L^1(G, \delta_n)$ etc. The spherical Fourier transform of $f \in C_c(G, \delta_n)$ is defined by

$$\widehat{\varphi}_{n,s}(f) = \int_G \varphi_{n,s}(x) f(x) d_G(x)$$
$$= \int_A F_f(a) \exp\{-s\rho(H(a))\} d_A(a)$$

where

$$F_f(a) = \int_N f(an)d_N(n) \cdot \exp\{\rho(H(a))\} \qquad (a \in A)$$

is the Abel transform of f.

2) Unitary dual of $SL_2(\mathbf{R})$

The unitary dual $\widehat{G}$ of G is consisting of the following series of representations;

a) Principal series

Let

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbf{R}^\times, \quad b \in \mathbf{R} \right\}$$

be a parabolic subgroup of G. The modular function Δ_P of P is defined by

$$\Delta_P(x) = \exp\{-2\rho(H(x))\} = a^2 \quad (x = \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in P).$$

For $s \in \sqrt{-1}\,\mathbf{R}$ and $j = 0, 1$, define a unitary character $\chi_{j,s}$ of P by

$$\chi_{j,s} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = (a/|a|)^j \cdot |a|^s.$$

Let $\pi^{j,s} = \mathrm{Ind}_P^G \chi_{j,s}$ be the induced representation of G. The representation space $E^{j,s}$ of $\pi_{j,s}$ consists of the complex-valued measurable functions f on G such that

1) $f(xp) = \Delta_P(p)^{1/2}\chi_{j,s}(p)^{-1}f(x)$ for all $p \in P$,
2) $\int_K |f(k)|^2 d_K(k) < \infty$,

with an inner product

$$(f, g) = \int_K f(k)\overline{g(k)}d_K(k)$$

and the operation of $\pi_{j,s}$ is defined by

$$(\pi_{j,s}(x)f)(y) = f(x^{-1}y) \qquad (f \in E_{j,s}, \quad x, y \in G).$$

Define an element $f_n^{j,s} \in E_{j,s}$ by

$$f_n^{j,s}(kp) = \Delta_P(p)^{1/2}\chi_{j,s}(p^{-1})\delta_n(k^{-1}) \qquad (k \in K, p \in P).$$

Then $\{f_n^{j,s} \mid n \in \mathbf{Z}, \quad n \equiv j \quad (2)\}$ is a complete ortho-normal system of $E^{j,s}$. We have

$$(\pi^{j,s}(x) f_n^{j,s}, f_n^{j,s}) = \varphi_{n,s}(x) \qquad (x \in G)$$

for all $n \in \mathbf{Z}$ such that $n \equiv j \quad (2)$.

Unitary representation $\pi^{0,s}$ is irreducible for all $s \in \sqrt{-1}\mathbf{R}$ [Wr: Th.5.5.2.3]. Unitary representation $\pi^{1,s}$ is irreducible for all $0 \neq s \in \sqrt{-1}\mathbf{R}$ [Wr: Th.5.5.2.1]. They are the members of the principal series. We will denote by $\widehat{G}_p$ the subset of $\widehat{G}$ consisting of the principal series representation. Put $\widehat{G}(n)_p = \widehat{G}_p \cap \widehat{G}(n)$.

b) Discrete series

Upper half plane $\mathbf{H} = \{z \in \mathbf{C} \mid \operatorname{Im} z > 0\}$ is identified with a quotient space G/K via $x \mapsto x(\sqrt{-1})$ where $x(z) = (az+b)/(cz+d)$ for $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z \in \mathbf{H}$. Let $d_{G/K}(\dot{x})$ be a G-invariant measure on $G/K = \mathbf{H}$ such that

$$\int_G f(x) d_G(x) = \int_{G/K} \left(\int_K f(xk) d_K(k) \right) d_{G/K}(\dot{x}).$$

Then

$$d_{G/K}(z) = \frac{1}{\pi} \cdot \frac{dxdy}{y^2} \qquad (z = x + \sqrt{-1}\,y \in \mathbf{H}).$$

Take an integer $n > 1$. Let $L^2(\mathbf{H}, n)$ be a complex Hilbert space consisting of complex-valued measurable f on $\mathbf{H}$ such that

$$\int_{\mathbf{H}} |f(z)|^2 (\operatorname{Im} z)^n d_{G/K}(z) < \infty$$

with inner product

$$(f, g) = \int_{\mathbf{H}} f(z) \overline{g(z)} (\operatorname{Im} z)^n d_{G/K}(z).$$

Define a unitary representation π^n of G on $L^2(\mathbf{H}, n)$ by

$$(\pi^n(x) f)(z) = f(x^{-1}(z)) J(x^{-1}, z)^{-n} \qquad (x \in G, \quad f \in L^2(\mathbf{H}, n))$$

where $J(x, z) = cz + d$ for $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $z \in \mathbf{H}$. Let E_n (resp. E_{-n}) be the subspace of $L^2(\mathbf{H}, n)$ consisting of holomorphic (resp. anti-holomorphic) functions on $\mathbf{H}$. Then $E_{\pm n}$ is a G-stable closed subspace of $L^2(\mathbf{H}, n)$, and put $\pi_{\pm n} = \pi^n|_{E_{\pm n}}$. Put for $0 \le \ell \in \mathbf{Z}$

$$f_{n,\ell}(z) = \left(\frac{\sqrt{-1}-z}{z+\sqrt{-1}}\right)^{\ell}\left(\frac{\sqrt{-1}+z}{2}\right)^{-n} \qquad (z \in \mathbf{H})$$

$$f_{-n,\ell}(z) = \overline{f}_{n,\ell}(z) = \left(\frac{\sqrt{-1}+\overline{z}}{\sqrt{-1}-\overline{z}}\right)^{\ell}\left(\frac{\overline{z}-\sqrt{-1}}{2}\right)^{-n}.$$

Then $\{f_{\pm n,\ell} |\ 0 \le \ell \in \mathbf{Z}\}$ is a complete orthogonal system of $E_{\pm n}$ such that

$$|f_{\pm n,\ell}|^2 = 4 \cdot \frac{\ell!(n-2)!}{(\ell+n-1)!}.$$

For all integers $\ell \ge 0$, we have

$$\pi_{\pm n}(k) f_{\pm n,\ell} = \delta_{\pm(n+2\ell)}(k) \cdot f_{\pm n,\ell} \qquad \text{for all} \quad k \in K$$

$$|f_{\pm n,\ell}|^{-2}(\pi_{\pm n}(x) f_{\pm n,\ell}, f_{\pm n,\ell}) = \varphi_{\pm(n+2\ell),n-1}(x).$$

The unitary representations $\pi_{\pm n}$ $(1 \le \ell \in \mathbf{Z})$ are irreducible and square-integrable of formal degree $(n-1)/4$. They are the members of the discrete series representations. We have

$$\varphi_{n,n-1}(x) = \left(\frac{\sqrt{-1} - \overline{x(\sqrt{-1})}}{2\sqrt{-1}}\right)^{-n} \overline{J(x, \sqrt{-1})}^{-n}$$

$$\varphi_{-n,n-1}(x) = \left(\frac{\sqrt{-1} + x(\sqrt{-1})}{2\sqrt{-1}}\right)^{-n} J(x, \sqrt{-1})^{-n} \qquad (x \in G).$$

If $n > 2$, then $\pi_{\pm n}$ is integrable. We will denote by $\widehat{G}_d$ the subset of $\widehat{G}$ consisting of the discrete series representations. Put $\widehat{G}_d(n) = \widehat{G}_d \cap \widehat{G}(n)$.

c) Complementary series

Take a real number $0 < \sigma < 1$. Let E^σ be a complex vector space consisting of complex-valued measurable functions f on G such that
1) $f(xp) = \Delta_P(p)^{(1+\sigma)/2} f(x)$ for all $p \in P$
2) $\int_K |f(k)|^2 d_K(k) < \infty$.

A positive definite inner product $(,)_\sigma$ on E^σ is defined by

$$(f, g)_\sigma = C_\sigma^{-1} \int_K \int_K H(kl^{-1})^{(\sigma-1)/2} f(k) \overline{g(l)} d_K(k) d_K(l)$$

where $H(k) = \det(1 - k^2)$ and

$$C_\sigma = \int_K H(k)^{(\sigma-1)/2} d_K(k) = \frac{2^{\sigma-1}}{\pi} \cdot \Gamma(\frac{\sigma}{2})/\Gamma(\frac{\sigma+1}{2}).$$

A completion of E^σ with respect to $(,)_\sigma$ is denoted by $\widehat{E}^\sigma$. A unitary representation π^σ of G on $\widehat{E}^\sigma$ is defined by

$$(\pi^\sigma(x)f)(y) = f(x^{-1}y) \qquad (f \in \widehat{E}^\sigma, \quad x, y \in G).$$

Then $(\pi^\sigma, \widehat{E}^\sigma)$ is an irreducible unitary representation of G. Let f_n^σ be a function on G defined by

$$f_n^\sigma(kp) = \delta_n(k^{-1}) \cdot \Delta_P(p)^{(\sigma+1)/2} \qquad (k \in K, \quad p \in P).$$

Then $\{f_n^\sigma \mid n \in 2\mathbf{Z}\}$ is a complete orthogonal system of $\widehat{E}^\sigma$ such that

$$|f_0^\sigma| = 1, \qquad |f_{2m}^\sigma|^2 = \prod_{j=1}^{|m|} \frac{2j-1-\sigma}{2j-1+\sigma} \qquad (0 \neq m \in \mathbf{Z})$$

$$\pi^\sigma(k) f_n^\sigma = \delta_n(k) \cdot f_n^\sigma \qquad (k \in K).$$

We have

$$|f_n^\sigma|^{-2}(\pi^\sigma(x) f_n^\sigma, f_n^\sigma) = \varphi_{n,\sigma}(x) \qquad (x \in G).$$

d) Limit of discrete series

We will use the notations of a). Let $E_+^{1,0}$ (resp. $E_-^{1,0}$) be a closure of the complex linear span of $\{f_n^{1,0} \mid 0 < n \in \mathbf{Z}, \quad n = \text{odd}\}$ (resp. $\{f_n^{1,0} \mid 0 > n \in \mathbf{Z}, \quad n = \text{odd}\}$). Then $E_\pm^{1,0}$ is a G-stable closed subspace of $E^{1,0}$. Put $\pi_\pm^{1,0} = \pi^{1,0}|_{E_\pm^{1,0}}$. Then $(\pi_\pm^{1,0}, E_\pm^{1,0})$ is an irreducible unitary representation of G.

Using the notations defined above, the set $\widehat{G}(n)$ is described by the following table;

i) $n = 0$

π	$\varphi_{n,\pi}$
$(\pi^{0,s}, E^{0,s})$ for $s \in \sqrt{-1}\mathbf{R}$	$\varphi_{0,s}$
$(\pi^\sigma, \widehat{E}^\sigma)$ for $0 < \sigma < 1$	$\varphi_{0,\sigma}$
$\mathbf{1}_G$	$\varphi_{0,\pm 1} = 1$

,

ii) $n = \text{even} \neq 0$

π	$\varphi_{n,\pi}$
$(\pi^{0,s}, E^{0,s})$ for $s \in \sqrt{-1}\,\mathbf{R}$	$\varphi_{n,s}$
(π_m, E_m) for $1 < \lvert m\rvert \leq \lvert n\rvert$, $m = \text{even}$ $mn > 0$	$\varphi_{n,\lvert m\rvert-1}$
$(\pi^\sigma, \widehat{E}^\sigma)$ for $0 < \sigma < 1$	$\varphi_{n,\sigma}$

,

iii) $n = \text{odd}$

π	$\varphi_{n,\pi}$
$(\pi^{0,s}, E^{0,s})$ for $s \in \sqrt{-1}\,\mathbf{R}$	$\varphi_{n,s}$
(π_m, E_m) for $1 < \lvert m\rvert \leq \lvert n\rvert$, $m = \text{even}$ $mn > 0$	$\varphi_{n,\lvert m\rvert-1}$
$(\pi_\varepsilon^{1,0}, E_\varepsilon^{1,0})$ $\varepsilon = \text{sign}(n)$	$\varphi_{n,0}$

.

For any $\pi \in \widehat{G}(n)$, there exists uniquely a $s \in \mathbf{C}/\{\pm 1\}$ such that $\varphi_{n,\pi} = \varphi_{n,s}$. Then the set $\widehat{G}(n)$ is identified with $\mathcal{D}(n)/\{\pm 1\}$ via the mapping $\pi \mapsto \varphi_{n,\pi} = \varphi_{n,s} \mapsto s$, where

$$\mathcal{D}(n) = \begin{cases} \sqrt{-1}\,\mathbf{R} \cup \{\sigma \in \mathbf{R} \mid 0 < |\sigma| < 1\} \cup \{\pm 1\} & \text{if} \quad n = 0 \\ \sqrt{-1}\,\mathbf{R} \cup \{\sigma \in \mathbf{R} \mid 0 < |\sigma| < 1\} \cup \{\pm 1\} \cup \{m \in \mathbf{Z} : \text{odd}\} & \\ & \text{if} \quad n = \text{even} \neq 0 \\ \sqrt{-1}\,\mathbf{R} \cup \{0 \neq m \in \mathbf{Z} : \text{even}\} & \text{if} \quad n = \text{odd}. \end{cases}$$

3) Paley-Wiener theorem

Let $\mathcal{G} = \mathcal{K} \oplus \mathcal{V}$ be a Cartan decomposition with respect to a Cartan involution $\theta(X) = -{}^tX$ ($\mathcal{K} = \text{Lie}(K)$). Then all elements $x \in G$ are expressed uniquely $x = k \cdot \exp X(x)$ with $k \in K$ and $X(x) \in \mathcal{V}$. Define an inner product $\langle\ ,\ \rangle$ on $\mathcal{G}$ by $\langle X, Y\rangle = -B_{\mathcal{G}}(X, \theta Y)$ where $B_{\mathcal{G}}(X,Y) = 4 \cdot \text{tr}(XY)$ is the Killing form of $\mathcal{G}$. Then $\mathcal{G}$ is an Euclidean space with norm $|X| = \langle X, X\rangle^{1/2}$. Put $\sigma(x) = |X(x)|$ for $x \in G$.

For all $0 \leq r \in \mathbf{Z}$, $0 < p \in \mathbf{R}$ and invariant differential operator $D \in U(\mathcal{G} \otimes_{\mathbf{R}} \mathbf{C})$ on G (here $U(\mathcal{G} \otimes_{\mathbf{R}} \mathbf{C})$ is the universal enveloping algebra), put

$$\nu^p_{D,r}(f) = \sup_{x\in G}(1+\sigma(x))^r\omega_0(x)^{-2/p}|Df(x)|$$

for $f \in C^\infty(G)$, where

$$\omega_0(x) = \varphi_{0,0}(x) = \int_K \exp\{-\rho(H(xk))\}d_K(k).$$

Let $\mathcal{C}^p(G)$ be a complex vector space consisting of the complex-valued C^∞-functions f on G such that $\nu^p_{D,r}(f) < \infty$ for all $0 \le r \in \mathbf{Z}$ and $D \in U(\mathcal{G} \otimes_{\mathbf{R}} \mathbf{C})$. we have $\mathcal{C}^q(G) \subset \mathcal{C}^p(G) \subset L^p(G)$ for $0 < q \le p$.

Take a $f \in \mathcal{C}^2 \cap L^1(G)$. Then $\pi(f)$ is a trace class operator for all $\pi \in \widehat{G}_d \cup \widehat{G}_p$ [Wr: Th.10.2.1.1, vol.II, p.174, Example(1)]. We have the following Planchrel formula [M, Wk1,2];

$$f(1) = \sum_{\pi\in\widehat{G}_d} d(\pi)\cdot \mathrm{tr}(\pi(f)) + \frac{1}{4\pi}\sum_{j=0,1}\int_{-\infty}^{\infty} \mathrm{tr}(\pi^{j,2\sqrt{-1}r}(f))\mu_j(r)dr$$

where $d(\pi)$ is the formal degree of $\pi \in \widehat{G}_d$ and

$$\mu_j(r) = \begin{cases} \pi r\cdot\tanh(\pi r) & \text{if} \quad j=0 \\ \pi r\cdot\coth(\pi r) & \text{if} \quad j=1. \end{cases}$$

If $f \in \mathcal{C}^2(G) \cap L^1(G,\delta_n)$ with $n \equiv j \quad (2)$, we have

$$\begin{aligned} f(x) = &\sum_{\substack{1<m\le|n| \\ m\equiv n \quad (2)}} \frac{m-1}{4}\widehat{\varphi}_{n,m-1}(f)\cdot\varphi_{n,m-1}(x^{-1}) \\ &+ \frac{1}{4\pi}\int_{-\infty}^{\infty}\widehat{\varphi}_{n,2\sqrt{-1}r}(f)\cdot\varphi_{n,2\sqrt{-1}r}(x^{-1})\mu_j(r)dr. \end{aligned}$$

For $f \in \mathcal{C}^1(G,\delta_n) = \mathcal{C}^1(G)\cap L^1(G,\delta_n)$, the spherical Fourier transform

$$\widehat{\varphi}_{n,s}(f) = \int_G \varphi_{n,s}(x)f(x)d_G(x) \qquad (s\in\mathbf{C})$$

converges absolutely for $s \in \mathcal{D}_1 \cup \mathcal{D}(n)$ where $\mathcal{D}_1 = \{s\in\mathbf{C} \mid |\mathrm{Re}(s)| \le 1\}$ [Tr, TV]. A characterization of the function $s \mapsto \widehat{\varphi}_{n,s}(f)$ is given by [Tr]. Let $\mathcal{C}^1(\widehat{G},\delta_n)$ be a complex vector space consisting of the complex-valued continuous functions F on $\mathcal{D}_1 \cup \mathcal{D}(n)$ such that

1) F is holomorphic on $|\mathrm{Re}(s)| < 1$,
2) $F(-s) = F(s)$,
3) $\sup_{|\mathrm{Re}(s)|<1}\left|\frac{d^m}{ds^m}F(s)\right|(1+|s|)^\alpha < \infty$ for all $0 < m \in \mathbf{Z}$ and $\alpha \in \mathbf{R}$.
Then we have

Theorem 1.1. *The space $\mathcal{C}^1(G,\delta_n)$ is* **C***-linear isomorphic to $\mathcal{C}^1(\widehat{G},\delta_n)$ by a* **C***-linear mapping $f \mapsto \widehat{\varphi}_{n,s}(f)$.*

Proof. [Tr] considered an operator-valued function $\pi \mapsto \pi(f)$ on $\widehat{G}(n)$. But, for a $f \in \mathcal{C}^1(G,\delta_n)$ and $(\pi,H) \in \widehat{G}(n)$, we have $\pi(f)u \neq 0$ only for $u \in H$ such that $\pi(k)u = \delta_n(k)u$ for all $k \in K$, and in this case we have $\pi(f)u = \widehat{\varphi}_{n,\pi}(f)u$. So, in our case, the operator-valued function $\pi \mapsto \pi(f)$ can be identified with a function $\pi \mapsto \widehat{\varphi}_{n,\pi}(f)$. The weight function $\mu_j(r)$ of the Planchrel measure is given by

$$\mu_j(r) = (c_n(r)c_n(-r))^{-1}$$

for $n \in \mathbf{Z}$ such that $n \equiv j \quad (2)$, where

$$c_n(r) = \pi^{-1/2}\Gamma\left(\frac{\sqrt{-1}\,r}{2}\right)\Gamma\left(\frac{\sqrt{-1}\,r+1}{2}\right) \times\left\{\Gamma\left(\frac{\sqrt{-1}\,r+1+n}{2}\right)\Gamma\left(\frac{\sqrt{-1}\,r+1-n}{2}\right)\right\}^{-1}$$

with Euler's gamma function $\Gamma(s)$. Put

$$\begin{aligned} V_n &= \{s \in \sqrt{-1}\,\mathbf{R} \mid \mathrm{Im}(s) \leq 0, \quad c_n(s) = 0\} \\ &= \begin{cases} \emptyset & \text{if} \quad |n| \leq 1 \\ \{-\sqrt{-1}\,k \mid 0 < k < |n|, \quad k \equiv n+1(2)\} & \text{if} \quad |n| > 1. \end{cases} \end{aligned}$$

Then

$$V_n \cap \mathcal{D}_1 = \begin{cases} \emptyset & \text{if} \quad |n| \leq 1 \quad \text{or} \quad n = \text{odd} \\ \{-\sqrt{-1}\} & \text{if} \quad |n| > 1, \quad n = \text{even} \end{cases}$$

corresponds, by the identification given in 2), exactly to $\pi \in \widehat{G}_d(G)$ which is not integrable. With a characterization of the integrable representations by [HS], the linear relations appearing in Definition 3 of [Tr] becomes trivial. Then our $\mathcal{C}^1(\widehat{G},\delta_n)$ coincides with the space $\mathcal{C}^1(\widehat{G},F)$ with $F = \{\delta_n\}$ defined in [Tr]. Then [Tr: §11,Th.1] gives our result.

Q.E.D.

4) Trace formula

Let Γ be a discrete subgroup of G such that $\Gamma\backslash G$ is compact, and (χ, V) a finite dimensional unitary representation of Γ. Then, for all $\gamma \in \Gamma$, the centralizer G_γ of γ in G is unimodular and $\Gamma_\gamma\backslash G_\gamma$ is compact where $\Gamma_\gamma = \Gamma \cap G_\gamma$. Fix a Haar measure $d_{G_\gamma}(x)$ on G_γ. A G-invariant measures $d_{G_\gamma\backslash G}(\dot{x})$ on $G_\gamma\backslash G$ and $d_{\Gamma_\gamma\backslash G_\gamma}(\dot{x})$ on $\Gamma_\gamma\backslash G_\gamma$ are normalized so that

$$\int_G f(x)d_G(x) = \int_{G_\gamma\backslash G}\left(\int_{G_\gamma} f(yx)d_{G_\gamma}(y)\right)d_{G_\gamma\backslash G}(\dot{x})$$

and

$$\int_{G_\gamma} f(x)d_{G_\gamma}(x) = \int_{\Gamma_\gamma\backslash G_\gamma}\sum_{\sigma\in\Gamma_\gamma} f(\sigma x)d_{\Gamma_\gamma\backslash G_\gamma}(\dot{x})$$

respectively.

Let $\pi^\chi = \mathrm{Ind}_\Gamma^G\chi$ be a unitarily induced representation of G. Then, for all $f\in\mathcal{C}^1(G,\delta_n)$, we have a trace formula

$$\sum_{\pi\in\widehat{G}(n)} m(\pi,\pi^\chi)\widehat{\varphi}_{n,\pi}(f)$$

(1.4.1)

$$= \sum_{\{\gamma\}_\Gamma\in\mathrm{Conj}(\Gamma)} \mathrm{tr}\,\chi(\gamma)\cdot\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)\cdot\int_{G_\gamma\backslash G} f(x^{-1}\gamma x)d_{G_\gamma\backslash G}(\dot{x})$$

where $m(\pi,\pi^\chi)$ is the discrete multiplicity of π in π^χ, $\mathrm{Conj}(\Gamma)$ is the Γ-conjugacy classes of Γ and $\mathrm{vol}(\Gamma_\gamma\backslash G_\gamma)$ is the volume of $\Gamma_\gamma\backslash G_\gamma$ with respect to the G_γ-invariant measure $d_{\Gamma_\gamma\backslash G_\gamma}(\dot{x})$ [M: Cor.2.16].

Suppose that Γ is torsion-free. Then any $1\neq\gamma\in\Gamma$ is hyperbolic, that is, it is G-conjugate to an element $h(\gamma) = \begin{pmatrix} a(\gamma) & 0 \\ 0 & a(\gamma)^{-1}\end{pmatrix}\in G$ such that $|a(\gamma)|>1$. Then, for all $1\neq\gamma\in\Gamma$ and $f\in\mathcal{C}^1(G,\delta_n)$, we have

(1.4.2)

$$\begin{aligned}
&\int_{G_\gamma\backslash G} f(x^{-1}\gamma x)d_{G_\gamma\backslash G}(\dot{x}) \\
&= D(h(\gamma))\left(\frac{a(\gamma)}{|a(\gamma)|}\right)^n\cdot F_f\begin{pmatrix}|a(\gamma)| & 0\\ 0 & |a(\gamma)|^{-1}\end{pmatrix} \\
&= D(h(\gamma))\left(\frac{a(\gamma)}{|a(\gamma)|}\right)^n\frac{1}{4\pi}\int_{-\infty}^{\infty}\widehat{\varphi}_{n,\sqrt{-1}\,r}(f)\exp\sqrt{-1}\,r\log|a(\gamma)|)dr
\end{aligned}$$

where $D(h) = |a-a^{-1}|^{-1}$ for $1\neq h = \begin{pmatrix} a & 0\\ 0 & a^{-1}\end{pmatrix}\in G$. In this case, the trace formula is written in the form

$$
\begin{aligned}
&\sum_{s\in\mathcal{D}(n)/\{\pm1\}} m(\pi^s,\pi^\chi)\widehat{\varphi}_{n,s}(f)\\
=&\dim V\cdot\mathrm{vol}(\Gamma\backslash G)\sum_{\substack{1<m\le|n|\\ m\equiv n(2)}}\frac{m-1}{4}\widehat{\varphi}_{n,m-1}(f)\\
&+\dim V\cdot\mathrm{vol}(\Gamma\backslash G)\frac{1}{4\pi}\int_{-\infty}^{\infty}\widehat{\varphi}_{n,2\sqrt{-1}\,r}(f)\mu_j(r)dr\\
&+\sum_{1\ne\{\gamma\}_\Gamma\in\mathrm{Conj}(\Gamma)}\mathrm{tr}\,\chi(\gamma)\cdot\frac{2\log|a(\gamma)|}{(\Gamma_\gamma:\langle\gamma\rangle)}\cdot D(h(\gamma))\left(\frac{a(\gamma)}{|a(\gamma)|}\right)^n\\
&\qquad\times\frac{1}{4\pi}\int_{-\infty}^{\infty}\widehat{\varphi}_{n,\sqrt{-1}\,r}(f)\exp(\sqrt{-1}\,r\log|a(\gamma)|)dr
\end{aligned}
$$

for all $f\in\mathcal{C}^1(G,\delta_n)$. Here $\pi^s\in\widehat{G}(n)$ is the unitary representation corresponding to $s\in\mathcal{D}(n)$ (see 2)), and $j=0,1$ such that $n\equiv j\quad(2)$. In other word, by Theorem 1.1, we have

$$
\begin{aligned}
&\sum_{s\in\mathcal{D}(n)/\{\pm1\}} m(\pi^s,\pi^\chi)g(s)\\
&=\dim V\cdot\mathrm{vol}(\Gamma\backslash G)\cdot\sum_{\substack{1<m<|n|\\ m\equiv n\quad(2)}}\frac{m-1}{4}g(m-1)\\
(1.4.5)\qquad&+\dim V\cdot\mathrm{vol}(\Gamma\backslash G)\cdot\frac{1}{4\pi}\int_{-\infty}^{\infty}g(2\sqrt{-1}\,r)\mu_j(r)dr\\
&+\sum_{1\ne\{\gamma\}_\Gamma\in\mathrm{Conj}(\Gamma)}\mathrm{tr}\chi(\gamma)\cdot\frac{2\log|a(\gamma)|}{(\Gamma_\gamma:\langle\gamma\rangle)}\cdot D(h(\gamma))\left(\frac{a(\gamma)}{|a(\gamma)|}\right)^n\\
&\qquad\times\frac{1}{4\pi}\int_{-\infty}^{\infty}g(\sqrt{-1}\,r)\exp(\sqrt{-1}\,r\log|a(\gamma)|)dr
\end{aligned}
$$

for all $g\in\mathcal{C}^1(\widehat{G},\delta_n)$. By [HP], we have

$$m(\pi,\pi^\chi)=\dim V\cdot\mathrm{vol}(\Gamma\backslash G)\cdot d(\pi)$$

for all integrable $\pi \in \widehat{G}_d$. Then we have

$$\begin{aligned}
&\sum_{s\in\mathcal{D}'(n)/\{\pm 1\}} m(\pi^s,\pi^\chi)g(s)+C(n)g(1)\\
&=\dim V\cdot \mathrm{vol}(\Gamma\backslash G)\cdot\frac{1}{4\pi}\int_{-\infty}^{\infty} g(2\sqrt{-1}\,r)\mu_j(r)dr\\
&\quad+\sum_{1\neq\{\gamma\}_\Gamma\in\mathrm{Conj}(\Gamma)} \mathrm{tr}\,\chi(\gamma)\cdot\frac{2\log|a(\gamma)|}{(\Gamma_\gamma:\langle\gamma\rangle)}\cdot D(h(\gamma))\left(\frac{a(\gamma)}{|a(\gamma)|}\right)^n\\
&\qquad\qquad\times\frac{1}{4\pi}\int_{-\infty}^{\infty}\widehat{\varphi}_{n,\sqrt{-1}\,r}(f)\exp(\sqrt{-1}\,r\log|a(\gamma)|)dr
\end{aligned}$$

for all $g \in \mathcal{C}^1(\widehat{G},\delta_n)$. Here

$$\mathcal{D}'(n)=\begin{cases}\sqrt{-1}\,\mathbf{R}\cup\{\sigma\in\mathbf{R}\mid 0<|\sigma|<1\}\cup\{\pm1\} & \text{if}\quad n=0\\ \sqrt{-1}\,\mathbf{R}\cup\{\sigma\in\mathbf{R}\mid 0<|\sigma|<1\} & \text{if}\quad n=\text{even}\neq 0\\ \sqrt{-1}\,\mathbf{R} & \text{if}\quad n=\text{odd}\end{cases}$$

and

$$C(n)=\begin{cases} m(\pi_{\pm2},\pi^\chi)-\dim V\cdot\mathrm{vol}(\Gamma\backslash G) & \text{if}\quad n=\text{even}\neq 0\\ 0 & \text{if}\quad n=\text{odd, or}\quad n=0.\end{cases}$$

5) Selberg zeta functions

Let Γ be a discrete torsion-free subgroup of G such that $\Gamma\backslash G$ is compact. In this case, we have $\mathrm{vol}(\Gamma\backslash G)=2g-2$ where g is the genus of the Riemann surface $\Gamma\backslash G/K$. Let (χ,V) be a finite dimensional unitary representation of Γ. Any element $1\neq\gamma\in\Gamma$ is hyperbolic, and the centralizer Γ_γ is a cyclic group. A non-trivial Γ-conjugacy class $\{\gamma\}_\Gamma\in\mathrm{Conj}(\Gamma)$ is called primitive if the cyclic group Γ_γ is generated by γ. We will denote by $P(\Gamma)$ the subset of $\mathrm{Conj}(\Gamma)$ consisting of the primitive Γ-conjugacy classes.

For $j=0,1$, a Selberg zeta function is defined by

$$Z_{\Gamma,j}(\chi,s)=\prod_{\{\gamma\}\in P(\Gamma)}\prod_{\ell=0}^{\infty}\det\left(1-\chi(\gamma)\left(\frac{a(\gamma)}{|a(\gamma)|}\right)^j a(\gamma)^{-2(s+\ell)}\right).$$

The infinite product converges absolutely for $\mathrm{Re}(s)>1$, and has a meromorphic continuation to the whole $s\in\mathbf{C}$. It has a functional equation

$$\begin{aligned}&Z_{\Gamma,j}(\chi,1-s)\\ &=Z_{\Gamma,j}(\chi,s)\cdot\exp\left\{2\cdot\dim V\cdot(2g-2)\int_0^{s-1/2}\mu_j(\sqrt{-1}\,r)dr\right\}.\end{aligned}$$

This functional equation can be written in a "symmetric form" [V]; let $\Gamma_2(s)$ be the double Γ-function of Bernes defined by

$$\begin{aligned}&\Gamma_2(s+1)^{-1}\\ =&(2\pi)^{s/2}\cdot\exp[-\{s(s+1)-\gamma s^2\}/2]\\ &\times\prod_{n=1}^{\infty}(1+s/n)^n\cdot\exp\{-s+s^2/(2n)\}\end{aligned}$$

where γ is Euler's constant ($\gamma = 0.577215\cdots$). The double Γ-function has the following properties;

$$\log\frac{\Gamma_2(1-s)}{\Gamma_2(1+s)} = s\cdot\log(2\pi) - \int_0^s \pi x\cdot\cot(\pi x)dx$$

$$\Gamma_2(s+1) = \Gamma(s)^{-1}\Gamma_2(s).$$

Define the "Γ-factor" $Z_{\Gamma,j}^{\infty}(\chi,s)$ by

$$Z_{\Gamma,j}^{\infty}(\chi,s) = \begin{cases}\{(2\pi)^s\Gamma_2(s)\Gamma_2(s+1)\}^{\dim V(2g-2)} & \text{if} \quad j=0\\ \{(2\pi)^s\Gamma_2(s+1/2)^2\}^{\dim V(2g-2)} & \text{if} \quad j=1\end{cases}$$

and put

$$Z_{\Gamma,j}^{*}(\chi,s) = Z_{\Gamma,j}^{\infty}(\chi,s)\cdot Z_{\Gamma,j}(\chi,s).$$

Then we have a functional equation

$$Z_{\Gamma,j}^{*}(\chi,1-s) = Z_{\Gamma,j}^{*}(\chi,s).$$

The zeros of $Z_{\Gamma,j}(\chi,s)$ and its orders are described by the following tables;

i) $j=0$

Table 1.1

zero	order
1	$m(\mathbf{1}_\Gamma,\chi)$
0	$m(\mathbf{1}_\Gamma,\chi)+\dim V\cdot(g-1)/2$
$-n \quad (0<n\in\mathbf{Z})$	$2\cdot m(\pi_{2n+2},\pi^\chi)$
$0<s<1$	$m(\pi^{2s-1},\pi^\chi)$
$\mathrm{Re}(s)=\frac{1}{2}$	$m(\pi^{2s-1},\pi^\chi)$

ii) $j = 1$

Table 1.1

zero	order
$\frac{1}{2}$	$2 \cdot m(\pi_3, \pi^\chi)$
$\frac{1}{2} - n \quad (0 < n \in \mathbf{Z})$	$2 \cdot m(\pi_{2n+1}, \pi^\chi)$
$\mathrm{Re}(s) = \frac{1}{2}$	$m(\pi^{2s-1}, \pi^\chi)$

§2. Special values of Selberg zeta functions

Special values of Selberg zeta function are considered by [DH, F]. We will recall their methods and results.

Let Γ be a discrete torsion-free subgroup of G such that $\Gamma\backslash G$ is compact. Let (χ, V) be a finite dimensional unitary representation of Γ. Fix an integer $n \in \mathbf{Z}$ such that $n \equiv j \quad (2)$ with $j = 0, 1$. Now consider a heat equation

$$\Delta f = \frac{\partial f}{\partial t} \tag{2.1}$$

for $f(*, t) \in \mathcal{C}^1(G, \delta_n)$ $(t > 0)$. Here $\Delta = 8 \cdot \Omega$ with the Casimir operator $\Omega \in U(\mathcal{G} \otimes_{\mathbf{R}} \mathbf{C})$ on G. Because $\Delta\varphi_{n,s} = (s^2 - 1)\varphi_{n,s}$ for all $s \in \mathbf{C}$, the spherical Fourier transform $\widehat{f}(s, t) = \widehat{\varphi}n, s(f(*, t))$ satisfies an equation

$$\frac{\partial \widehat{f}}{\partial t} = (s^2 - 1)\widehat{f}. \tag{2.2}$$

Then $\widehat{f}(s, t) = C(s) \exp\{(s^2 - 1)t\}$ with a function $C(s)$ of s. Put $g_t(s) = \exp\{(s^2 - 1)t\}$. Then $g_t \in \mathcal{C}^1(\widehat{G}, \delta_n)$ for $t > 0$ and, by Theorem 1.1, there exists a $f(*, t) \in \mathcal{C}^1(G, \delta_n)$ such that $\widehat{f}(s, t) = g_t(s)$. This function f is a fundamental solution of the heat equation (2.1).

We will consider the case of $n = 0$. Applying trace formula (1.4.6) to $g_t \in \mathcal{C}^1(\widehat{G}, \mathbf{1}_K) \quad (t > 0)$, we have an identity

$$H(t) + m(\mathbf{1}_\Gamma, \chi) = I(t) + G(t) \quad (t > 0) \tag{2.3}$$

where

$$H(t) = \sum_{s \in \mathcal{D}\text{"}(0)/\{\pm\}} m(\pi^s, \pi^\chi) \cdot g_t(s)$$

with $\mathcal{D}"(0) = \sqrt{-1}\,\mathbf{R} \cup \{\sigma \in \mathbf{R}\ | 0 < |\sigma| < 1\}$, and

$$I(t) = \dim V \cdot \mathrm{vol}(\Gamma\backslash G) \cdot \frac{1}{4\pi} \int_{-\infty}^{\infty} g_t(2\sqrt{-1}\,r)dr$$

$$G(t) = (4\pi)^{-1/2} \sum_{1 \neq \{\gamma\}_\Gamma \in \mathrm{Conj}(\Gamma)} \mathrm{tr}\chi(\gamma) \cdot \frac{\log|a(\gamma)|}{(\Gamma_\gamma :< \gamma >)} D(h(\gamma))$$
$$\times\, t^{-1/2} \exp\{-t - \frac{4}{t}(\log|a(\gamma)|)^2\}.$$

The term $m(\mathbf{1}_\Gamma, \chi)$ comes from Frobenius reciprocity law; $m(\mathbf{1}_G, \pi^\chi) = m(\mathbf{1}_\Gamma, \chi)$. The Mellin transforms of $H(t)$, $I(t)$ and $G(t)$ are denoted by $\widetilde{H}(s)$, $\widetilde{I}(s)$ and $\widetilde{G}(s)$ respectively, that is,

$$\widetilde{H}(s) = \int_0^\infty H(t)t^{s-1}dt \quad \text{etc.}$$

Let $\chi_{(0,1]}$ (resp. $\chi_{(1,\infty)}$) be the characteristic function of the interval $(0,1]$ (resp $(1,\infty)$), and put

$$\begin{aligned} A(t) &= H(t) + m(\mathbf{1}_\Gamma, \chi) \cdot \chi_{(0,1]}(t) - I(t) \\ &= G(t) - m(\mathbf{1}_\Gamma, \chi) \cdot \chi_{(1,\infty]}(t). \end{aligned} \tag{2.4}$$

Then $A(t)$ decays exponentially as $t \to +0$ or $t \to +\infty$, and the Mellin transform $\widetilde{A}(s)$ is an entire function of s. We have

$$\widetilde{I}(s) = \dim V \cdot \mathrm{vol}(\Gamma\backslash G)\Gamma(s) \cdot J(s)$$

where

$$J(s) = (4\pi)^{-1} \int_0^\infty (r^2+1)^{-s} \mu_0(r/2)dr \quad (\mathrm{Re}(s) > 1)$$

which has a meromorphic continuation to the whole s-plane whose singularities are simple poles at $s = 1, 0, -1, -2, \cdots$. By (2.4), we have

$$\widetilde{G}(s) = \widetilde{A}(s) - m(\mathbf{1}_\Gamma, \chi) \cdot s^{-1}$$

and $\widetilde{G}(s)$ has a meromorphic continuation to the whole s-plane. On the other hand we have

$$\widetilde{G}(s) = \Gamma(1-s)^{-1} \int_0^\infty \{r(r+2)\}^{-s} \frac{d}{dr} \log Z_{\Gamma,0}(\chi, \frac{r}{2} + 1)dr$$

for $\mathrm{Re}(s) < 1$ by the formula

$$\int_0^\infty (4\pi t)^{-1/2} \exp\{-\left(\frac{c}{2}\right)^2 t - \left(\frac{\ell}{2}\right)^2 \frac{1}{t}\} t^{s-1} dt$$
$$= \Gamma(1-s)^{-1} \int_0^\infty \{x(x+c)\}^{-s} \exp\{-\ell(x+\frac{c}{2})\} dx$$
$$(\mathrm{Re}(s) < 1, \ell > 0, c > 0)$$

cited in [F]. Then

$$\widetilde{H}(s) = \widetilde{G}(s) + \widetilde{I}(s) \tag{2.5}$$

by (2.4). On the other hand, we have

$$\widetilde{H}(s) = \Gamma(s)\zeta_{\Gamma,\chi}(s,\Delta)$$
$$\zeta_{\Gamma,\chi}(s,\Delta) = \sum_{r \in \mathcal{D}"(0)/\{\pm\}} m(\pi^r, \pi^\chi)(1-r^2)^{-s}.$$

Then $\zeta_{\Gamma,\chi}(s,\Delta)$ is a meromorphic function of s, holomorphic at $s=0$. By the Table 1.5.1, the Dirichlet series $\zeta_{\Gamma,\chi}(s,\Delta)$ is

$$4^s \zeta_{\Gamma,\chi}(s,\Delta) = \frac{1}{2} \sum_{\omega=\frac{1}{2}+\sqrt{-1}\,u} (\frac{1}{4}+u^2)^{-s}$$

where $\sum_{\omega=\frac{1}{2}+\sqrt{-1}\,u}$ is the summation with multiplicity over the zeros $\omega = \frac{1}{2}+\sqrt{-1}\,u$ of $Z_{\Gamma,0}(\chi,s)$ such that $0 < \mathrm{Re}(\omega) < 1$. Let

$$Z_{\Gamma,0}(\chi,s) = A \cdot (s-1)^{m(\mathbf{1}_\Gamma,\chi)} + [\text{terms of degree} > m(\mathbf{1}_\Gamma,\chi)]$$

be the Laurent expansion of $Z_{\Gamma,0}(\chi,s)$ at $s=1$. Then, by (2.5), we have

$$\zeta'_{\Gamma,\chi}(0,\Delta) = \lim_{s\to 0} \Gamma(s)(\zeta_{\Gamma,\chi}(s,\Delta) - \zeta_{\Gamma,\chi}(0,\Delta))$$
$$= 2\cdot m(\mathbf{1}_\Gamma,\chi)\cdot \log 2 - \log A - \dim V \cdot (2g-2)J'(0).$$

Now put

$$\det\Delta_{\Gamma,\chi} = \exp\{-\zeta'(0,\Delta)\}$$

which is the functional determinant of Δ operating on the space

$$E^\chi(\mathbf{1}_K) = \{u \in E^\chi | \pi^\chi(k)u = u \quad \text{for all} \quad k \in K\}.$$

Here E^χ is the representation space of $\pi^\chi = \mathrm{Ind}_\Gamma^G \chi$. Then we have

Theorem 2.1. *The Laurent expansion at $s=1$ of $Z_{\Gamma,0}(\chi,s)$ is*

$$Z_{\Gamma,0}(\chi,s) = R\cdot P\cdot (s-1)^{m(\mathbf{1}_\Gamma,\chi)} + [\textit{terms of degree} > m(\mathbf{1}_\Gamma,\chi)]$$

where

$$R = \det \Delta_{\Gamma,\chi},$$
$$P = 4^{m(\mathbf{1}_\Gamma,\chi)}\cdot C^{\dim V\cdot(2g-2)}$$

with $C = \exp J'(0)$.

§3. Dedekind zeta functions

Now we will consider Dedekind zeta functions. Detailed discussion will be given in [Ta].

Let K be a finite algebraic number field and

$$\zeta_K(s) = \prod_{v<\infty}(1-N(v)^{-s})^{-1} \qquad (\mathrm{Re}(s)>1)$$

the Dedekind zeta function of K. Then we have the explicit formula (Weil [15])

(3.1)
$$\begin{aligned}
&\sum_\omega \Phi(\omega)\\
=&F(0)\log|D| + 2\int_{-\infty}^{\infty} F(x)\cosh(x/2)dx\\
&-\sum_{v<\infty}\sum_{n=1}^{\infty} N(v)^{-n/2}\log N(v)\cdot\{F(\log N(v^n)) + F(-\log N(v^n))\}\\
&+\sum_{v|\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\Phi(\frac{1}{2}+\sqrt{-1}\,x)\cdot \mathrm{Re}(\tfrac{\Gamma_v'}{\Gamma_v}(\tfrac{1}{2}+\sqrt{-1}\,x))dx
\end{aligned}$$

where F is a suitable test function and

$$\Phi(s) = \int_{-\infty}^{\infty} F(x)\exp((s-1/2)x)dx, \tag{3.2}$$

$\sum_\omega$ is the summation with multiplicity over the zeros ω of $\zeta_K(s)$ such that $0<\mathrm{Re}(\omega)<1$, D is the absolute discriminant of K, $\sum_{v<\infty}$ (resp. $\sum_{v|\infty}$) is the summation over the finite places (resp. infinite places) of K, and $\Gamma_v(s) = \pi^{-s/2}\Gamma(s/2)$ if v is real, $\Gamma_v = (2\pi)^{1-s}\Gamma(s)$ if v is complex.

The heat equation

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial F}{\partial t}$$

has a fundamental solution

$$F_t(x) = (4\pi \cdot t)^{-1/2} \exp\{-(x/2)^2 t^{-1}\} \qquad (x \in \mathbf{R}, t > 0). \tag{3.3}$$

Applying the explicit formula to the function $F_t(x)$, we have

$$H(t) = I(t) + 2 + G(t) + \sum_{v|\infty} I_v(t) \tag{3.4}$$

where

$$H(t) = \sum_{\omega=\frac{1}{2}+\sqrt{-1}u} \exp\{-(\frac{1}{4}+u^2)t\}$$

$$G(t) = -2\sum_{v<\infty}\sum_{n=1}^{\infty} N(v)^{-n/2}\log N(v)$$
$$\times (4\pi \cdot t)^{-12}\exp[-\frac{1}{4}\{t + (\log N(v))^2 t^{-1}\}]$$

$$I(t) = \log|D| \cdot (4\pi \cdot t)^{-1/2}\exp(-t/4)$$

$$I_v(t) = \frac{2}{\pi}\int_0^\infty \exp\{-(\frac{1}{4}+x^2)t\} \cdot \mathrm{Re}(\frac{\Gamma_v'}{\Gamma_v}(\frac{1}{2}+\sqrt{-1}\,x))dx.$$

Let $\chi_{(0,1]}$ (resp. $\chi_{(1,\infty)}$) be the characteristic function of $(0,1]$ (resp. $(1,\infty)$), and put

$$\begin{aligned} A(t) &= H(t) - 2\chi_{(0,1]}(t) - I(t) - \sum_{v|\infty} I_v(t) \\ &= G(t) + 2\chi_{(1,\infty)}(t). \end{aligned} \tag{3.5}$$

The Mellin transform of $H(t)$ etc. is denoted by $\widetilde{H}(s)$ etc., that is

$$\widetilde{H}(s) = \int_0^\infty H(t)t^{s-1}dt \qquad \text{etc.}$$

Now $A(t)$ decays exponentially as $t \to +0$ or $t \to +\infty$, and the Mellin transform $\widetilde{A}(s)$ is an entire function of s. Then, by calculating $\widetilde{A}(s)$ in two ways, we have

$$\widetilde{H}(s) = \widetilde{G}(s) + \widetilde{I}(s) + 2\Gamma(s)\sum_{v|\infty} J_v(s). \tag{3.6}$$

Here

$$\widetilde{H}(s) = 2\Gamma(s)\zeta(s, \Delta_K)$$

with

$$\zeta(s, \Delta_K) = \frac{1}{2}\sum_{\omega=\frac{1}{2}+\sqrt{-1}\,u}(\frac{1}{4}+u^2)^{-s}$$

where $\sum_{\omega=\frac{1}{2}+\sqrt{-1}\,u}$ is the summation with multiplicity over the zeros $\omega = \frac{1}{2}+\sqrt{-1}\,u$ of $\zeta_K(s)$ such that $0 < \mathrm{Re}(\omega) < 1$. The infinite series $\zeta(s, \Delta_K)$ converges absolutely for $\mathrm{Re}(s) > 1/2$ and has a meromorphic continuation to the whole s-plane which is holomorphic except for the possible double poles at $s = \frac{1}{2} - j (0 \le j \in \mathbf{Z})$. We have

$$\widetilde{G}(s) = 2\Gamma(1-s)^{-1}\int_0^{\infty}\frac{\zeta_K'}{\zeta_K}(x+1)\cdot\{x(x+1)\}dx$$

for $\mathrm{Re}(s) < 0$, and $\widetilde{G}(s)$ has a meromorphic continuation to the whole s-plane with unique simple pole at $s = 0$ with residue 2 because we have $\widetilde{G}(s) = \widetilde{A}(s) + 2s^{-1}$ by (3.5). We have also

$$\widetilde{I}(s) = (4\sqrt{\pi})^{-1}\log|D|\cdot\Gamma(s-1/2)$$

and

$$J_v(s) = \frac{1}{\pi}\int_0^{\infty}(\frac{1}{4}+x^2)^{-s}\mathrm{Re}(\frac{\Gamma_v'}{\Gamma_v}(\frac{1}{2}+\sqrt{-1}\,x))dx$$

converges absolutely for $\mathrm{Re}(s) > 1/2$, and has a meromorphic continuation to the whole s-plane which is holomorphic except for the possible double poles at $s = \frac{1}{2} - j (0 \le j \in \mathbf{Z})$.

Now (3.6) with some calculations gives an identity

$$\zeta'(0, \Delta_K) = -\log(\mathrm{Res}_{s=1}\zeta_K(s)) - \frac{1}{4}\log|D| + \sum_{v|\infty} J_v'(0).$$

Then we have the following

Theorem 3.1. *We have a residue formula*

$$\mathrm{Res}_{s=1}\,\zeta_K(s) = |D|^{-1/4}(\exp J_1'(0))^{r_1}(\exp J_2'(0)))^{r_2}\det\Delta_K$$

where $J_1(s) = J_{\mathrm{real}}(s)$, $J_2(s) = J_{\mathrm{complex}}(s)$ *and*

$$\det \Delta_K = \exp(-\zeta'(0, \Delta_K))$$

is the "functional determinant" of the "Laplacian" Δ_K *associated with* K.

§4. Concluding discussion

There exist a lot of analogies between Dedekind zeta functions and Selberg zeta functions. They are both defined by Euler products, and so they have no zero in the region of absolute convergence of the Euler products. They have also symmetric functional equations with respect to $s \to 1 - s$. Such a functional equation for Dedekind zeta function is well-known, and for Selberg zeta function is given in §1, 5). Because of these functional equations, Dedekind zeta functions or Selberg zeta functions have zeros on the negative real line which are called trivial zeros. All other zeros are in the critical strip $0 < \mathrm{Re}(s) < 1$, and they are called non-trivial zeros. The generalized Riemann hypothesis states that the non-trivial zeros are on the line $\mathrm{Re}(s) = 1/2$. Table 1.5.1 or 1.5.2 shows that the generalized Riemann hypothesis is almost valid for Selberg zeta functions, that is, except for the finite number of zeros on real line $0 < s < 1$, the non-trivial zeros are on the line $\mathrm{Re}(s) = 1/2$ (there is an example such that $Z_{\Gamma,0}(\chi, s)$ does have a zero on $0 < s < 1$).

Selberg [Se] defined "Selberg zeta function" in order to solve the following crossword puzzle:

Crossword Puzzle

Dedekind zeta function	?
explicit formula	Selberg's trace formula

So the relation between Dedekind zeta functions and Selberg zeta functions are clear from its origin.

The arguments in §1 and §2 are quite parallel. We can find a tight correspondence between these two arguments

Table 4.1

Dedekind zeta function	Selberg zeta function
explicit formula	trace formula
heat eq. $\frac{\partial^2 F}{\partial x^2} = \frac{\partial F}{\partial t}$	heat eq. $\Delta f = \frac{\partial f}{\partial t}$
$\zeta(s, \Delta_K)$ $= \frac{1}{2}\sum_{\omega=\frac{1}{2}+\sqrt{-1}u}(\frac{1}{4}+u^2)^{-s}$	$\zeta_{\Gamma,\chi}(s, \Delta)$ $=2^{1-2s}\sum_{\omega=\frac{1}{2}+\sqrt{-1}u}(\frac{1}{4}+u^2)^{-s}$
$\det\Delta_K = \exp\{-\zeta'(0, \Delta_K)\}$	$\det\Delta_{\Gamma,\chi} = \exp\{-\zeta'(0, \Delta)\}$
$\zeta_K(s) = \frac{a}{s-1} + b + c(s-1) + \cdots$	$Z_{\Gamma,0}(\chi, s) = A\cdot(s-1)^{m(\mathbf{1}_\Gamma,\chi)} + \cdots$
$a = C_1^{r_1}C_2^{r_2}\det\Delta_K\|D\|^{-1/4}$	$A = C^{\dim V(2g-2)}\det\Delta_{\Gamma,\chi}4^{m(\mathbf{1}_\Gamma,\chi)}$
$\log(w\sqrt{\|D\|}2^{-r_1}(2\pi)^{-r_2})$	$g =$ genus of $\Gamma\backslash G/K$

Here $\zeta(s, \Delta_K)$ (resp. $\zeta_{\Gamma,\chi}(s, \Delta)$) is the summation with multiplicity over the non-trivial zeros of $\zeta_K(s)$ (resp. $Z_{\Gamma,0}(\chi, s)$). Weil [Wel] shows that the number $\log(w\sqrt{|D|}2^{-r_1}(2\pi)^{-r_2})$ plays the role of "genus" for the finite algebraic number field K. In other word, the numbers r_1 and r_2 are linearly related with the "genus" of K.

Now the residue of $\zeta_K(s)$ at $s = 1$ is $a = C_1^{r_1}C_2^{r_2}\det\Delta_K \cdot |D|^{-1/4}$ (Theorem 3.1), and the leading coefficient of $Z_{\Gamma,0}(\chi, s)$ at $s = 1$ is $A = C^{\dim V(2g-2)}\det\Delta_{\Gamma,\chi}$ (Theorem 2.1). The factors $C_1^{r_1}C_2^{r_2}$ for $\zeta_K(s)$ and $C^{\dim V(2g-2)}$ for $Z_{\Gamma,0}(\chi, s)$ are depending only on the genus of algebraic number field K or Riemann surface $\Gamma\backslash G/K$. On the other hand, the factors $\det\Delta_K$ for $\zeta_K(s)$ and $\det\Delta_{\Gamma,\chi}$ for $Z_{\Gamma,0}(\chi, s)$ depend deeply on the non-trivial zeros of $\zeta_K(s)$ and $Z_{\Gamma,0}(\chi, s)$ respectively. So the correspondence of these factors is

Table 4.2

Dedekind zeta function	Selberg zeta function
$C_1^{r_1}C_2^{r_2}$	$C^{\dim V(2g-2)}$
$\|D\|^{-1/4}\det\Delta_K$	$4^{m(\mathbf{1}_\Gamma,\chi)}\det\Delta_{\Gamma,\chi}$

Now compare the formula in Theorem 3.1 and the classical residue formula of $\zeta_K(s)$ given in §0. Then the factors $C_1^{r_1}C_2^{r_2}$ and $2^{r_1}(2\pi)^{r_2}$

depend only on the structure of $K \otimes_{\mathbf{Q}} \mathbf{R}$. On the other hand, the factors $h\cdot(w\sqrt{|D|})^{-1}\cdot R(K)$ and $|D|^{-1/4}\det\Delta_K$ depend deeply on the arithmetic of number field K. So the correspondence of the factors is

Table 4.3

classical formula	Theorem 3.1
$2^{r_1}(2\pi)^{r_2}$	$C_1^{r_1}C_2^{r_2}$
$h\cdot(w\sqrt{\lvert D\rvert})^{-1}\cdot R(K)$	$\lvert D\rvert^{-1/4}\det\Delta_K$

.

As is pointed out in §0, the factors $2^{r_1}(2\pi)^{r_2}$ and $h\cdot(w\sqrt{|D|})^{-1}\cdot R(K)$ are the period and the regulator respectively of the special value of $\zeta_K(s)$ at $s=1$ in the sense of Dogma given in §0. So, by Table 4.3, the factor $C_1^{r_1}C_2^{r_2}$ is the period and the factor $|D|^{-1/4}\det\Delta_K$ is the regulator. Then Table 4.2 shows that the factors $C^{\dim V(2g-2)}$ and $4^{m(\mathbf{1}_\Gamma,\chi)}\det\Delta_{\Gamma,\chi}$ play the role of the period and the regulator respectively of the special value of $Z_{\Gamma,0}(\chi,s)$ at $s=1$.

Problem. For the Dedekind zeta function, the factor $h\cdot(w\sqrt{|D|})^{-1}\cdot R(K)$ contains the algebraic part $h\cdot(w\sqrt{|D|})^{-1}$ of the special value of $\zeta_K(s)$ at $s=1$. Does the factor $4^{m(\mathbf{1}_\Gamma,\chi)}\det\Delta_{\Gamma,\chi}$ for Selberg zeta function contain the "algebraic part" of the special value of $Z_{\Gamma,0}(\chi,s)$ at $s=1$?

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