Advanced Studies in Pure Mathematics 21, 1992
Zeta Functions in Geometry
pp. 21–31

On Adelic Zeta Functions of Prehomogeneous Vector Spaces with a Finitely Many Adelic Open Orbits

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Introduction

The two adelic zeta functions $Z_a(\omega, \Phi)$ and $Z_m(\omega, \Phi)$ for a prehomogeneous vector space (abbrev. P.V.) (G, ρ, V) have no relation in general. For an irreducible case, Professor J. Igusa showed that $Z_a = \tau Z_m$ with some constant τ when $\#(G_A \backslash Y_A) < \infty$ under the condition (HW) where Y is the open G-orbit in V (see Igusa [4]).

In this paper, we shall show that the condition (HW) is not necessary. Moreover, we shall show that the theorem of the same type holds even for simple P.V's and 2-simple P.V.'s of type I. It is known that when $Z_a = \tau Z_m$ holds, we can generalize Iwasawa-Tate Theory for such P.V.'s and we can have many informations (see T. Kimura [11]).

Contents

§1. Basic definitions

Let G be a connected reductive linear algebraic group and $\rho : G \longrightarrow GL(V)$ a rational representation of G with the open dense G-orbit Y. In this case, we call a triplet (G, ρ, V) a prehomogeneous vector space (abbrev. P.V.). The complement S of Y is a Zariski-closed set which is called the singular set of (G, ρ, V). We assume that the isotropy subgroup H of $\rho(G)$ at a point in Y is connected and semisimple. The irreducible components S_i of codimension one

Received January 20, 1991.

are the zeros of some irreducible polynomials $f_i(x)$ $(i = 1, 2, \ldots, r)$. Then $f_1(x), f_2(x), \ldots, f_r(x)$ are algebraically independent relative invariants, i.e., $f_i(\rho(g)x) = \chi_i(g)f_i(x)$ for $g \in G$, $x \in V$ with some rational characters χ_i of G. Moreover any relative invariant rational function $f(x)$ is of the form $f(x) = c \cdot f_1(x)^{m_1} f_2(x)^{m_2} \cdots f_r(x)^{m_r}$ with $(m_1, m_2, \ldots, m_r) \in \mathbf{Z}^r$ and some constant c (see p.60 in (M. Sato and T. Kimura [5])).

Let k be an algebraic number field. We assume that (G, ρ, V) is defined over k and all coefficients of $f_i(x)$ are in k. We denote by G_A, V_A, etc. the adelization of G, V, etc. with respect to k. Let $\Omega(k_A^\times/k^\times)$ be the space of quasicharacters of the idele class group $k_A^\times/k^\times$ and $\mathcal{S}(V_A)$ the Schwartz-Bruhat space on V_A.

For $\omega = (\omega_1, \ldots, \omega_r) \in \Omega(k_A^\times/k^\times)^r$, we write $\omega(\chi(g)) = \omega_1(\chi_1(g)) \cdots \omega_r(\chi_r(g))$ and $\omega(f(x)) = \omega_1(f_1(x)) \cdots \omega_r(f_r(x))$ $(g \in G_A, x \in Y_A = (V - S)_A)$ for simplicity. Now we define the two adelic zeta-functions $Z_a(\omega, \Phi)$ and $Z_m(\omega, \Phi)$ of (G, ρ, V).

$$Z_a(\omega, \Phi) = \int_{G_A/G_k} \omega(\chi(g)) \sum_{\xi \in Y_k} \Phi(\rho(g) \cdot \xi) d_{G_A}(g)$$

$$Z_m(\omega, \Phi) = \int_{Y_A} \omega(f(x)) \Phi(x) d_{Y_A}(x)$$

$$(\Phi \in \mathcal{S}(V_A))$$

Here d_{G_A} is a Haar measure on G_A and d_{Y_A} is a G_A-invariant measure on Y_A (see the beginning of § 2). We take the same convergence factor for d_{G_A} and d_{Y_A}. The role of $Z_a(\omega, \Phi)$ is a functional equation based on the adelic Poisson summation formula while $Z_m(\omega, \Phi)$ has an Euler product $Z_m(\omega, \Phi) = \Pi_{v \in \Sigma} Z_v(\omega_v, \Phi_v)$ when $\Phi = \otimes_{v \in \Sigma} \Phi_v$ where Σ denotes the set of places of k.

For the absolute convergence of $Z_m(\omega, \Phi)$, see p.90 in (T. Ono [13] and F. Sato [9]).

§2. Some sufficient conditions for $Z_a = \tau Z_m$

For simplicity, we assume that $G \subset GL(V)$ and (G, V) is defined over an algebraic number field k. Take a k-rational generic point $\eta \in Y_k = (V - S)_k$ and we denote by H the isotropy subgroup of G at η. Since we assume that H is semisimple, we have $\mathrm{vol}(H_A/H_k) < +\infty$ (see A. Borel and Harish-Chandra [12]), and there exists a G_A-invariant measure d_{Y_A} on Y_A. Since H is connected, $G_A \cdot \eta$ is open in Y_A. We normalize measures d_{G_A}, d_{H_A} and d_{Y_A} on G_A, H_A and Y_A by

$$\int_{G_A} \phi(g)d_{G_A}(g) = \int_{G_A\cdot\eta} d_{Y_A}(x)(\int_{H_A} \phi(gh)d_{H_A}(h)) \quad (\text{with } x = gH_A)$$

for any $\phi \in L^1(G_A)$.

Proposition 1-1. *We have*

$$(1.1) \quad \begin{aligned} &\int_{G_A/G_k} \omega(\chi(g)) \sum_{\xi\in G_k\cdot\eta} \Phi(g\xi)d_{G_A}(g) \\ &= \tau \int_{G_A\cdot\eta} \omega(f(x))\Phi(x)d_{Y_A}(x) \end{aligned}$$

for $\Phi \in \mathcal{S}(V_A)$ *where* $\tau = \int_{H_A/H_k} d_{H_A}(= \mathrm{vol}(H_A/H_k) < +\infty)$.

Proof. First we observe that $\omega(\chi(\gamma)) = 1$ for $\gamma \in G_k$ and $\omega(f(\eta)) = 1$, i.e., $\omega(\chi(g\gamma)) = \omega(f(g\eta))$.

Since G is reductive,the Haar measure d_{G_A} is right-invariant, i.e., $d_{G_A}(g\gamma) = d_{G_A}(g)$.

Now

$$\begin{aligned} \text{L.H.S.} &= \int_{G_A/G_k} \omega(\chi(g)) \sum_{\gamma\in G_k/H_k} \Phi(g\gamma\eta)d_{G_A}(g) \\ &= \int_{G_A/H_k} \omega(\chi(g))\Phi(g\eta)d_{G_A}(g) \\ &= \int_{G_A\cdot\eta} d_{Y_A}(x)(\int_{H_A/H_k} \omega(f(gh\cdot\eta))\Phi(gh\cdot\eta)d_{H_A}(h)) \\ &\qquad (\text{with } x = gH_A) \\ &= \tau \int_{G_A\cdot\eta} \omega(f(x))\Phi(x)d_{Y_A}(x) \end{aligned}$$

where $\tau = \int_{H_A/H_k} d_{H_A}$ is a finite number by assumption.

Q.E.D.

Now the following proposition is obvious.

Proposition 1-2. *Assume that* $Y_k = G_k \cdot \eta$ *and* $Y_A = G_A \cdot \eta$. *Then we have* $Z_a = \tau Z_m$.

Proposition 1-3. *Let* (G, V) *and* (G', V) *be P.V.'s satisfying* $G \subset G' \subset GL(V)$ *and* $Y' = Y$. *If* $Y_k = G_k \cdot \eta$ *and* $Y_A = G_A \cdot \eta$, *then we have* $Z'_a = \tau' Z'_m = \frac{\tau'}{\tau} Z_a$ *and* $Z'_m = Z_m$.

Proof. Since $Y' = Y$, we have $Z'_m = Z_m$. Since $G \subset G'$, we have $Y_k = G'_k \cdot \eta$ and $Y_A = G'_A \cdot \eta$, hence $Z'_a = \tau' Z_m$ by Proposition 1-2. Since $Z_a = \tau Z_m$, we have $Z'_a = \frac{\tau'}{\tau} Z_a$. Q.E.D.

Proposition 1-4. *For (GL_d, M_d) with a k-form $(GL_d(k), M_d(k))$, we have $Y_k = G_k \cdot I_d$ and $Y_A = G_A \cdot I_d$ (hence we have $Z_a = \tau Z_m$).*

Proof. Since $Y_k = GL_d(k) = G_k = G_k \cdot I_d$, and $Y_A = (GL_d)_A = G_A = G_A \cdot I_d$, we have our assertion by Proposition 1-2. Q.E.D.

Proposition 1-5. *Let G_o be a connected k-split algebraic subgroup of SL_d acting on M_d as $\rho(g_o, g_1) \cdot x = g_o x^t g_1$ $(g_o \in G_o, g_1 \in GL_d, x \in M_d)$. Then for a P.V. $(G_o \times GL_d, \rho, M_d)$ with the k-form*

$$((G_o)_k \times GL_d(k), \rho, M_d(k)),$$

we have $Z_a = \tau Z_m$.

Proof. It is clear by Proposition 1-3 and Proposition 1-4.
Q.E.D.

Theorem 1-6 (Igusa [4] with the above Proposition 1-5). *Let (G, ρ, V) be an irreducible regular P.V. defined over k such that Y_A decomposes into a finitely many G_A-orbits. Then with a suitable k-form, we have*

$$Z_a = \tau Z_m.$$

Remark. The point of Theorem 1-6 is that the condition (HW) is not necessary (see p.16 Remark in (Igusa [4])).

More explicitly, we can express Theorem 1-6 as follows.

Theorem 1-7. *We have $Z_a(\omega, \Phi) = \tau Z_m(\omega, \Phi)$ for an irreducible regular P.V. which is castling-equivalent to one of the following reduced P.V.'s with the split k-form.*

(1) *$(H \times GL_m, \rho_m, M_m)$ where H is any k-split connected semisimple algebraic subgroup of SL_m with $\rho_m(h, g)x = hx^t g$ for $(g, h) \in H \times GL_m$ and $x \in M_m$. We take a k-form $(H_k \times GL_m(k), \rho_m, M_m(k))$. The relative invariant $f(x) = \det x$.*

(2) *$(GL_{2m}, \rho, \mathrm{Alt}_{2m})$ where $\rho(g)x = gx^t g$ for $g \in GL_{2m}$ and $x = -{}^t x \in \mathrm{Alt}_{2m}$. We take a k-form $(GL_{2m}(k), \rho, \mathrm{Alt}_{2m}(k))$. The relative invariant $f(x) = \mathrm{Pf}(x)$ (= the Pfaffian of x).*

(3) $(GL_1 \times SO_{2m}, \Lambda_1 \otimes \Lambda_1, Aff^{2m})$ *with* $m \geq 2$.

Here $SO_{2m} = \{A \in SL_{2m}; {}^tAKA = K\}$ *with* $K = \begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}$ *so that* $f(x) = x_1x_{m+1} + \cdots + x_mx_{2m}$ *is the relative invariant. Let* G *be the image of* $GL_1 \times SO_{2m}$ *by* $\rho = \Lambda_1 \otimes \Lambda_1$ *in* GL_{2m}, *and put* $G_k = G \cap GL_{2m}(k)$. *We take a k-form* (G_k, k^{2m}). *For any* $\lambda \in k^\times$, *put*

$$g(\lambda) = \rho(\sqrt{\lambda}, \begin{pmatrix} \sqrt{\lambda}I_m & 0 \\ 0 & \frac{1}{\sqrt{\lambda}}I_m \end{pmatrix}).$$

Then $g(\lambda)$ *is in* G_k *and* $f(g(\lambda)x) = \lambda f(x)$.

Hence, with the $SO_{2m}(k)$*-homogeneity of* $f^{-1}(1)$, *we have* $Y_k = G_k \cdot \xi$ *with* $\xi = e_1 + e_{m+1}$. *The isotropy subgroup* G_ξ *of* G *at* ξ *is* SO_{2m-1}. *Note that* $\rho(-1, -I_{2m}) = 1$. *Since* SO_{2m-1} *is connected, we have* $Y_A = G_A \cdot \xi$ (*cf.* *Theorem* 1-8).

(3)$'$ $(GL_1 \times \mathrm{Spin}_7, \Lambda_1 \otimes (\textit{the spin rep.}), V(8))$
We identify $V(8)$ *with* Aff^8 *by the standard base*

$$\{1, e_ie_j (1 \leqq i < j \leqq 4), e_1e_2e_3e_4\}.$$

Let G *be the image of* $GL_1 \times \mathrm{Spin}_7$ *in* GL_8 *by* $\Lambda_1 \otimes$ (*the spin rep.*) *and put* $G_k = G \cap GL_8(k)$. *We take a k-form* (G_k, k^8). *Since the relative invariant is a quadratic form, we have* $G \subset GO(8)$. *By p.13 in* (*Igusa* [2]), *one sees that* $Y_k = G_k \cdot \xi$. *We have* $Y_A = G_A \cdot \xi$ (*see Igusa* [4]).

(3)$''$ $(GL_1 \times \mathrm{Spin}_9, \Lambda_1 \otimes \Lambda_1, V(16))$
Everything is similar as (3)$'$. *In this case,we have* $G \subset GO(16)$.

(4) $(Sp_m \times GL_{2r}, \Lambda_1 \otimes \Lambda_1, M_{2m,2r})$ $(m \geq 2r)$
We take k-form

$$(Sp_m(k) \times GL_{2r}(k), \Lambda_1 \otimes \Lambda_1, M_{2m,2r}(k)).$$

The relative invariant $f(x) = \mathrm{Pf}({}^txJx)$.

(5) $(GL_1 \times E_6, \Lambda_1, \mathcal{J}(27))$

$\mathcal{J}(27)$ *is the totality of* 3×3 *hermitian matrices over the octonion algebra, and the relative invariant* $f(x)$ *is their determinant. The image* G *of* $GL_1 \times E_6$ *by* $\Lambda_1 \otimes \Lambda_1$ *is* $\mathrm{Sim}(f)$ *and* G_k *is transitive on* Y_k (*see p.15 in* (*Igusa* [2]).

(6) $(\mathrm{Spin}_{10} \times GL_2$, (*a half-spin rep.*) $\otimes \Lambda_1, V(16) \otimes V(2))$

Let G be the image of $\mathrm{Spin}_{10} \times GL_2$ *in* GL_{32}. *We identify* $V(16)$ *with* Aff^{16} *by the standard basis*

$$\{1, e_ie_j(1 \leqq i < j \leqq 5), e_k^*(1 \leqq k \leqq 5)\},$$

and put $G_k = GL_{32}(k) \cap G$. *Put* $S_i(\lambda) = \lambda^{-1} + (\lambda - \lambda^{-1})e_ie_{i+5}$ *(see p.1002 in (Igusa [1])). For any* $\alpha \in k^\times$, *put* $g(\alpha) = (s_1(\lambda) \cdots s_5(\lambda), \lambda I_2)$ *with* $\lambda =^4 \sqrt{\alpha}$. *Then we have* $g(\alpha) \in G_k$ *and* $f(g(\alpha)x) = \alpha f(x)$. *Since* $f^{-1}(1)$ *is* Spin_{10}*-homogeneous, we can say that* $Y_\xi = G_k \cdot \xi$ *and* $Y_A = G_A \cdot \xi$ *for*

(6)′ $(GL_1 \times \mathrm{Spin}_{10}, \Lambda_1 \otimes (\Lambda + \Lambda), V(16) \oplus V(16))$ *where* Λ *is the (even) half-spin representation.*

In particular, we have $Y_k = G_k \cdot \xi$ *and* $Y_A = G_A \cdot \xi$ *for* (6).

(7) $(GL_7, \Lambda_3, V(35))$

Let G be the image of GL_7 *under* Λ_3 *in* GL_{35}. *For any local field* $k \neq \mathbf{R}$, Y_k *is* G_k*-homogeneous and* $Y_{\mathbf{R}} = G_{\mathbf{R}} \cdot \xi_1 \sqcup G_{\mathbf{R}} \cdot \xi_2$. *However we have* $\#(G_A \backslash Y_A) < +\infty$. *The relative invariant* $f(x)$ *is of degree* 7.

In this case, we have $Z_a = \tau Z_m$ *by (Igusa [4]).*

Theorem 1-8. *Assume that a universally transitive regular P.V.* (G, V) *defined over k satisfies the two conditions:*

(1) $Y_k = G_k \cdot \eta$

(2) *the isotropy subgroup* G_η *is connected.*

Then we have $Z_a = \tau Z_m$.

Proof. By (2), every G_A-orbit in Y_A contains a point of Y_k (see (p.14 in Igusa [4]). Then, by (1), we have $Y_A = G_A \cdot \eta$. Hence by Proposition 1-2, we obtain our result. Q.E.D.

§3. Simple P.V.'s with $\#(G_A \backslash Y_A) < +\infty$

Assume that $\#(G_A \backslash Y_A) < +\infty$ for a simple P.V. $(\tilde{G}, \rho, V)$ with $G = \rho(\tilde{G})$. Then for almost all places v of k, Y_v must be G_v-transitive. Such non-irreducible regular simple P.V.'s with a semisimple generic isotropy subgroup $H = \rho(G_\xi)$ ($\xi \in Y_k$) are given as follows. (see (T. Kimura, S. Kasai and H. Hosokawa [8])).

(1) $(GL_1 \times SL_n, \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*)$ with $H = SL_n$.

(2) $(GL_n, \overbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}^{n})$ with $H = \{1\}$.

(3) $(GL_1^n \times GL_n, \rho_n + 1 \otimes \Lambda_1)$ with $H = \{1\}$, where $\rho_n(g)x = Ax(\mathrm{diag}(\alpha_1, \cdots, \alpha_n))$ for $g = (\alpha_1, \cdots \alpha_n, A) \in GL_1^n \times GL_n$ and $x \in M_n$.

(4) $(GL_1 \times Sp_n, \Lambda_1 \oplus (\Lambda_1 + \Lambda_1))$ with $H = Sp_{n-1}$

(5) $(GL_1 \times GL_{2m}, 1 \otimes \Lambda_2 + \Lambda_1 \otimes (\Lambda_1^{(*)} + \Lambda_1^{(*)}))$ with $H = Sp_{m-1}$.

(6) $(GL_{2m+1}, \Lambda_2 \oplus \Lambda_1)$ with $H = Sp_m$.

(7) $(GL_1^3 \times GL_{2m+1}, \Lambda_2 \oplus \Lambda_1 \oplus (\Lambda_1 \oplus \Lambda_1)^{(*)})$ with $H = Sp_{m-1}$, where GL_1^3 acts on $\Lambda_1 \oplus (\Lambda_1 \oplus \Lambda_1)^{(*)}$ as scalar multiplications. Here $(\Lambda_1 \oplus \Lambda_1)^{(*)}$ stands for $\Lambda_1 \oplus \Lambda_1$ or its dual $(\Lambda_1 \oplus \Lambda_1)^*$).

(8) $(GL_1^2 \times \mathrm{Spin}_n$, (a half-spin rep.) $\oplus$(vector rep.)) $(n = 8, 10)$ with $H = (G_2)$ for $n = 8$ and $H = \mathrm{Spin}_7$ for $n = 10$.

(9) $(GL_1 \times \mathrm{Spin}_{10}, \Lambda_1 \otimes (\Lambda + \Lambda))$ with $H = (G_2)$, where Λ is the even half-spin representation.

We shall check each of them.

(1) We take a k-form $(GL_1(k) \times SL_n(k), \Lambda_1 \otimes \Lambda_1 + 1 \otimes \Lambda_1^*, k^n \oplus k^n)$. Then $Y_k = G_k \cdot \xi$ with $\xi = (e_1, e_1)$ and $G_\xi \cong SL_{n-1}(k)$.

(2) For $(GL_n(k), \overbrace{\Lambda_1 \oplus \cdots \oplus \Lambda_1}^{n}, M_n(k))$, we have $Y_k = G_k \cdot I_n$ and $H = \{1\}$.

(3) We take a k-form

$$(GL_1^n(k) \times GL_n(k), M_n(k) \oplus k^n),$$

and put $\xi = (I_n, {}^t(1, \cdots, 1))$. Then $Y_k = G_k \cdot \xi$ and $G_\xi = \{1\}$.

(4) Let G be the image of $GL_1 \times Sp_n$ in GL_{4n} by $\rho = \Lambda_1 \otimes (\Lambda_1 + \Lambda_1)$, and put $G_k = G \cap GL_{4n}(k)$. For any $\alpha \in k^\times$, put

$$g(\alpha) = (\sqrt{\alpha}, \begin{pmatrix} \sqrt{\alpha} I_n & 0 \\ 0 & \frac{1}{\sqrt{\alpha}} I_n \end{pmatrix}).$$

Then $g(\alpha) \in G_k$ and $f(g(\alpha)x) = \alpha f(x)$. Since $f^{-1}(1)$ is $Sp_n(k)$-transitive, we have $Y_k = G_k \cdot \xi$ with $\xi = (e_1, e_{n+1})$ and $G_\xi = Sp_{n-1}$ (see p.16 in [8]).

(5) We take a k-form

$$(GL_1(k) \times GL_{2m}(k), 1 \otimes \Lambda_2 + \Lambda_1 \otimes (\Lambda_1^{(*)} + \Lambda_1^{(*)}), \mathrm{Alt}_{2n}(k) \oplus k^{2m} \oplus k^{2m}),$$

where $\Lambda_1^{(*)}$ implies Λ_1 or its dual Λ_1^*. Since the generic isotropy subgroup of (GL_{2m}, Λ_2) is exactly Sp_m, we have $Y_k = G_k \cdot \xi$ and $Y_A = G_A \cdot \xi$ by (4).

(6) Consider $(GL_{2m+1}(k), \Lambda_2 \oplus \Lambda_1, \mathrm{Alt}_{2m+1}(k) \oplus k^{2m+1})$. Then $Y_k = GL_{2m+1}(k) \cdot \xi$ with

$$\xi = (\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}, {}^t(0, \cdots, 0, 1))$$

where $J = \begin{bmatrix} 0 & I_m \\ -I_m & 0 \end{bmatrix}$ and $G_\xi = Sp_m(k)$. By Theorem 1-8, we have $Y_k = G_k \cdot \xi$ and $Y_A = G_A \cdot \xi$ for (6).

(7) We take a k-form

$$(GL_{2m+1}(k) \times GL_1^3(k), \Lambda_2 + \Lambda_1 + (\Lambda_1 + \Lambda_1)^{(*)},$$
$$\mathrm{Alt}_{2m+1}(k) \oplus k^{2m+1} \oplus k^{2m+1} \oplus k^{2m+1})$$

where $GL_1^3(k)$ acts on $k^{2m+1} \oplus k^{2m+1} \oplus k^{2m+1}$ as scalar multiplications. Then we have $Y_k = G_k \cdot \xi$ with

$$\xi = (\begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix}, e_{2m+1}, e_1 + e_{2m+1}, e_{m+1} + e_{2m+1})$$

and the image of the isotropy subgroup is connected. Note that $(-I_{2m+1}, -1, -1, -1)$ is in the kernel of

$$\rho = \Lambda_2 \oplus \Lambda_1 \oplus (\Lambda_1 \oplus \Lambda_1)^{(*)}.$$

(8) Since the generic isotropy subgroup of $(\Lambda_1 \otimes \phi)(GL_1 \times \mathrm{Spin}_{2n})$ is $\phi(\mathrm{Spin}_{2n-1})$ where ϕ is the vector representation, we have $Y_k = G_k \cdot \xi$ and $Y_A = G_A \cdot \xi$ by using the results of irreducible case.

(9) In p.14 of (Igusa [2]), it is proved that $Y_k = G_k \cdot \xi$. One can see easily from p.11 of (Igusa [4]) that G_ξ is connected so that $Y_A = G_A \cdot \xi$.

From the above observation,we obtain the following theorem.

Theorem 2-1. *For a simple regular P.V. with* $\#(G_A \backslash Y_A) < +\infty$, *we have* $Z_a = \tau Z_m$.

Theorem 2-2. *For a simple regular P.V. with* $\#(G_A \backslash Y_A) < +\infty$, *we have* $\#(G_k \backslash Y_k) = \#(G_A \backslash Y_A) = 1$ *for a suitable k-form.*

§4. 2-Simple P.V.'s of Type I with $\#(G_A\backslash Y_A) < +\infty$

By (T. Kimura, S. Kasai and H. Hosokawa [8]), all non-irreducible regular 2-simple P.V.'s of Type I with $\#(G_A\backslash Y_A) < +\infty$ are given as follows. Here we adjust the scalar multiplications so that the generic isotropy subgroup $H = \rho(G_\xi)$ is semisimple.

(1) $(GL_1 \times GL_5 \times GL_2, 1 \otimes \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^* \otimes 1 + \Lambda_1 \otimes \Lambda_1^* \otimes 1)$ with $H = \{1\}$.

(2) $(GL_1 \times Sp_n \times GL_{2m}, 1 \otimes \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 \otimes (\Lambda_1^{(*)} + \Lambda_1^{(*)}))$ $(n > m)$ with $H = Sp_{n-m} \times Sp_{m-1}$.

(3) $(Sp_n \times GL_{2m+1}, \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1)$ $(n > m)$ with $H = Sp_m \times Sp_{n-m-1}$.

(4) $(GL_1^3 \times Sp_n \times GL_{2m+1}, \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1)^{(*)})$ $(n > m)$ with $H = Sp_{m-1} \times Sp_{n-m-1}$, where GL_1^3 acts on $\Lambda_1 \otimes 1 + 1 \otimes (\Lambda_1 + \Lambda_1)^{(*)}$ as scalar multiplications.

(5) $(GL_1 \times \mathrm{Spin}_{10} \times GL_2, 1 \otimes$ (a half-spin rep.) $\otimes \Lambda_1 + \Lambda_1 \otimes 1 \otimes (\Lambda_1 + \Lambda_1))$ with $H = G_2$.

(6) $(GL_1 \times \mathrm{Spin}_{10} \times GL_1^2 \times GL_2, \Lambda_1 \otimes$ (a half-spin rep.) $\otimes 1 \otimes \Lambda_1 + 1 \otimes 1 \otimes (\rho_2 + 1 \otimes \Lambda_1))$ with $H = G_2$. (See (3) in §3 for ρ_2).

We shall check each of them.

(1) We take a k-form

$$(GL_1(k) \times GL_5(k) \times GL_2(k), 1 \otimes \Lambda_2 \otimes \Lambda_1 + 1 \otimes \Lambda_1^* \otimes 1 + \Lambda_1 \otimes \Lambda_1^* \otimes 1).$$

Its generic isotropy subgroup is exactly $\{1\}$ (see (p.26 – p.27 in T. Kimura, S. Kasai and H. Hosokawa [8]). We have $Y_k = G_k \cdot \xi$.

(2) We shall consider $(GL_1 \times Sp_n \times GL_{2m}, 1 \otimes \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 \otimes (\Lambda_1^{(*)} + \Lambda_1^{(*)}))$

We take a k-form of the image of $\rho = 1 \otimes \Lambda_1 \otimes \Lambda_1 + \Lambda_1 \otimes 1 \otimes (\Lambda_1^{(*)} + \Lambda_1^{(*)})$.

Since GL_{2m}-part of the generic isotropy subgroup of $(Sp_n \times GL_{2m}, \Lambda_1 \otimes \Lambda_1)$ is Sp_m , it reduces to (4) in §3.

(3) In this case, we have $G_\xi = Sp_m \times Sp_{n-m-1}$ and $Y_k = G_k \cdot \xi$ (cf. p.102 in (M. Sato and T. Kimura [5])).

(4) $((GL_1^3\times)Sp_n\times GL_{2m+1},\Lambda_1\otimes\Lambda_1+(\Lambda_1\otimes 1+1\otimes(\Lambda_1+\Lambda_1)^{(*)})(n>m)$ where GL_1^3 acts on $\Lambda_1\otimes 1+1\otimes(\Lambda_1+\Lambda_1)^{(*)}$ as scalar multiplications. Since GL_{2m+1}-part of the generic isotropy subgroup of $(GL_1\times Sp_n\times GL_{2m+1}, 1\otimes\Lambda_1\otimes\Lambda_1+\Lambda_1\otimes\Lambda_1\otimes 1)$ is $\{\begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}; A\in Sp_m; \alpha\in GL_1\}$, it reduces to (4) of §3. We have $Y_k=G_k\cdot\xi$ and $G_\xi=Sp_{m-1}\times Sp_{n-m-1}$. Note that $(-1)^3\times(-I_{2n})\times(-I_{2m+1})$ is in the kernel of $\rho=\Lambda_1\otimes\Lambda_1+\Lambda_1\otimes 1+1\otimes(\Lambda_1+\Lambda_1)^{(*)}$.

(5) $(GL_1\times \mathrm{Spin}_{10}\times GL_2, 1\otimes\Lambda\otimes\Lambda_1+\Lambda_1\otimes 1\otimes(\Lambda_1+\Lambda_1))$ with $\Lambda=$ (a half-spin representation). Since the generic isotropy subgroup of $(GL_1\times GL_2,\Lambda_1\otimes(\Lambda_1+\Lambda_1))$ is $\{(\alpha^{-1},\alpha I_2);\alpha\in GL_1\}$, (5) reduces to (9) in §3.

(6) Since GL_2-part of the generic isotropy subgroup of $(GL_1^2\times GL_2,\rho_2+1\otimes\Lambda_1)$ is 1 (see (3) in §3) , reduces to (9) in §3.

Theorem 3-1. *For a regular 2-simple P.V. of type* I *with* $\#(G_A\backslash Y_A)<+\infty$, *we have* $Z_a=\tau Z_m$.

Theorem 3-2. *For a regular 2-simple P.V.'s of type* I *with* $\#(G_A\backslash Y_A)<+\infty$, *we have* $\#(G_k\backslash Y_k)=\#(G_A\backslash Y_A)=1$ *for a suitable k-form.*

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