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Scattering Theory in the Energy Space for a Class of Nonlinear Wave Equations

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Dedicated to Professor ShigeToshi Kuroda on his sixtieth birthday

§1. Introduction

The purpose of this talk is to present a survey of the theory of scattering for a class of nonlinear wave equations of the form

$$\Box\varphi \equiv \partial_t^2\varphi - \Delta\varphi = -f(\varphi) \tag{1.1}$$

in a space of initial data and asymptotic states as large as the energy space associated with that equation. The exposition will follow the treatment given in [12]. Here φ is a complex valued function defined in space time $\mathbb{R}^{n+1}$, Δ is the Laplace operator in $\mathbb{R}^n$, and f is a nonlinear suitably regular complex valued function satisfying polynomial bounds at zero and at infinity. A large amount of work has been devoted to the theory of scattering for the equation (1.1) and for several other equations, and we shall devote most of this introduction to a partial review of nonlinear scattering in order to put the subsequent treatment of (1.1) into perspective.

The general setting is the following. One considers a semilinear equation

$$\partial_t u = Lu + F(u) \tag{1.2}$$

where L is a linear antiselfadjoint operator in some Hilbert space $\mathcal{H}$, and generates a one parameter unitary group $U(t) = \exp(tL)$ in $\mathcal{H}$. One is interested in situations where the global Cauchy problem for (1.2) is well understood in some space X (which may or may not coincide with $\mathcal{H}$). In particular any initial data $u_0 \in X$ should generate a unique global X valued solution of (1.2) with $u(0) = u_0$ and with suitable regularity

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in time. One is then interested in studying the asymptotic behaviour in time of the solutions of (1.2) by comparison with the solutions of the linear equation

$$\partial_t u = Lu \tag{1.3}$$

hereafter referred to as the free equation. That study gives rise to the following two questions.
(1) Given $u_\pm \in X$, does there exist a (unique) solution u of the equation (1.2) that behaves at $t \to \pm\infty$ as the solution $U(\cdot)u_\pm$ of the free equation (1.3) generated by $u_\pm$, for instance in the sense that

$$\|u(t) - U(t)u_\pm; X\| \longrightarrow 0 \qquad \text{as } t \to \pm\infty \tag{1.4}$$

or

$$\|U(-t)u(t) - u_\pm; X\| \longrightarrow 0 \qquad \text{as } t \to \pm\infty. \tag{1.4'}$$

If that is the case, one defines the wave operators $\Omega_\pm$ as the maps $u_\pm \to u(0)$ thereby obtained. This first question is referred to as that of the existence of the wave operators. Actually, one may be interested in comparing solutions of (1.2) and (1.3) in a sense different from and in fact stronger than (1.4) (1.4$'$). For instance one may require that

$$\|u - U(\cdot)u_\pm; \mathcal{X}([T, \pm\infty))\| \longrightarrow 0 \qquad \text{as } T \to \pm\infty \tag{1.5}$$

where $\mathcal{X}(I)$ is a space of X valued functions defined in a time interval I with prescribed behaviour in time. Such a convergence is in fact needed in order to develop a consistent theory of scattering.

The second question is somehow the converse of the first one.
(2) Given a generic X valued solution of (1.2) generated by initial data $u(0) = u_0 \in X$, does there exist $u_\pm \in X$ such that u behaves asymptotically as $U(\cdot)u_\pm$ as $t \to \pm\infty$ in the same sense as above. If that is the case for all $u_0 \in X$, one says that asymptotic completeness (AC) holds in X. Note that this notion of asymptotic completeness is very restrictive, since the only asymptotic evolution which is used is the free evolution. In the linear quantum mechanical many body problem, this would correspond to the case where asymptotic completeness is achieved by the completely free channel, a situation typical of purely repulsive interactions.

A general method to prove the existence of the wave operators, and the one to be used in all the examples to follow, consists in solving the Cauchy problem for (1.2) with infinite initial time. In fact the Cauchy problem for (1.2) with initial data u_0 at time t_0 is equivalent to the integral equation

$$u(t) = U(t - t_0)u_0 + \int_{t_0}^{t} d\tau U(t - \tau)F(u(\tau)). \tag{1.6}$$

The solution u expected to behave as $U(\cdot)u_\pm$ at $t \to \pm\infty$ should then be obtained by taking $u_0 = U(t_0)u_\pm$ and letting $t_0 \to \pm\infty$. Restricting one's attention to positive times for definiteness, one obtains the equation

$$u(t) = U(t)u_+ - \int_{t}^{\infty} d\tau U(t - \tau)F(u(\tau)) \tag{1.7}$$

to be solved for u for given u_+. One can then try to solve (1.7) by a contraction method in a time interval $[T, \infty)$ for T sufficiently large, and then continue the solution u thereby obtained to all times by using the known results on the Cauchy problem at finite times. The contraction step requires the use of a space $\mathcal{X}([T, \infty))$ of X valued functions of time with a suitable time decay, in order to control the integral in (1.7). That time decay has to be satisfied by the solutions $U(\cdot)u_+$ of the free equation. As a standard by product of the previous method, one obtains a proof of the existence of global solutions and of asymptotic completeness for small data. The method also requires that $F(u)$ exhibit a suitable decay in time for u in the relevant space $\mathcal{X}([T, \infty))$. This in turns requires that the function F tend to zero sufficiently fast when u tends to zero. In the case where F satisfies power bounds in u as $u \to 0$, that condition reduces to lower bounds on the associated exponents.

Asymptotic completeness for general data, once the previous results are available, reduces to proving that generic solutions of (1.2) with initial data in X exhibit the time decay that is used in the definition of the space $\mathcal{X}(\cdot)$ used to solve the Cauchy problem at infinity. The question of AC therefore reduces to the derivation of a priori estimates and depends in a specific way on the invariances and conservation laws of the equation at hand. As should be clear from a previous remark, it always requires a repulsivity condition on the interaction term F.

We now review briefly some of the available results for the most studied equations, namely the nonlinear Schrödinger (NLS) equation

$$i\partial_t \varphi = -(1/2)\Delta\varphi + f(\varphi), \tag{1.8}$$

the nonlinear wave (NLW) equation (1.1), and the nonlinear Klein Gordon (NLKG) equation

$$\square\varphi + m^2\varphi + f(\varphi) = 0 \tag{1.9}$$

which differs from (1.1) by the presence of a mass term $m^2\varphi$. For clarity we restrict our attention ot the case where the nonlinear interaction term is a single power

$$f(\varphi) = \lambda|\varphi|^{p-1}\varphi. \tag{1.10}$$

For those three equations, the global Cauchy problem is well understood in the energy space X_0, to be defined below, for $\lambda \geq 0$ and $1 \leq p < p_* \equiv 1 + 4/(n-2)$ in space dimension $n \geq 2$.

For the NLS equation, one takes $u = \varphi$ and $F(u) = -if(\varphi)$, the free evolution group is $U(t) = \exp(i(t/2)\Delta)$, the conserved energy is

$$E(\varphi) = (1/2)\|\nabla\varphi\|_2^2 + \int dx V(\varphi) \tag{1.11}$$

where $\|\cdot\|_r$ denotes the norm in $L^r \equiv L^r(\mathbb{R}^n)$ and

$$V(\varphi) = 2\lambda(p+1)^{-1}|\varphi|^{p+1} \tag{1.12}$$

in the special case (1.10). Furthermore the L^2 norm of φ is also conserved, and the energy space is the standard Sobolev space $X_0 = H^1$.

For the NLW and NLKG equations, one takes $u = (\varphi, \partial_t\varphi)$ and $F(u) = (0, -f(\varphi))$. The solution of the free equation generated by the initial data $u_0 = (\varphi_0, \psi_0)$ at time $t = 0$ is

$$\varphi^{(0)}(t) = \dot{K}(t)\varphi_0 + K(t)\psi_0 \tag{1.13}$$

where $K(t) = \omega^{-1}\sin\omega t$, $\dot{K}(t) = \cos\omega t$, $\omega = \sqrt{-\Delta}$ for NLW ($\omega = \sqrt{-\Delta + m^2}$ for NLKG), so that the free evolution group is

$$U(t) = \begin{pmatrix} \dot{K}(t) & K(t) \\ -\omega^2 K(t) & \dot{K}(t) \end{pmatrix}. \tag{1.14}$$

The energy is

$$E(\varphi, \psi) = \|\psi\|_2^2 + \|\nabla\varphi\|_2^2 (+m^2\|\varphi\|_2^2) + \int dx V(\varphi) \tag{1.15}$$

for NLW(NLKG), and is conserved in the sense that $E(\varphi, \partial_t\varphi) = $ Const. for solutions of the equation. The energy space is $X_0 = (\dot{H}^1 \cap L^{p+1}) \oplus L^2$ for NLW and $X_0 = H^1 \oplus L^2$ for NLKG, where $\dot{H}^1$ is the homogeneous Sobolev space associated with H^1.

We now summarize the main results available on the existence of the wave operators for the NLS, NLW and NLKG equations with power nonlinearity (1.10). For the NLS equation [5, 6, 7, 9, 17, 25, 42, 45, 46], the wave operators are known to exist in the energy space $X_0 = H^1$ for $4/n < p - 1 < 4/(n-2)$ [7]. In the smaller space $X = \Sigma$ defined by

$$\Sigma = H^1 \cap \mathcal{F}(H^1) \equiv \{\varphi \colon \varphi \in H^1 \text{ and } x\varphi \in L^2\} \tag{1.16}$$

the wave operators are known to exist for $4/(n+2) < p - 1 < 4/(n-2)$ [5]. Finally for $0 < p - 1 \leq 2/n$, the wave operators do no exist even in the L^2-sense, namely (1.4) with $X = L^2$ implies $u_\pm = 0$ and $u = 0$ [41].

There is a huge literature on the theory of scattering and related problems (including global existence for small data) for the NLW equation [10–12, 14–16, 18, 19, 22–24, 28, 29, 33–36, 38–42]. For that equation, the wave operators are known to exist in the space

$$X = X_0 \cap \{(\varphi, \psi) \colon (x \otimes \nabla)\varphi \in L^2, x\psi \in L^2\} \tag{1.17}$$

for $p_1(n) < p < p_*$, where $p_1(n)$ is the larger root of the equation [28, 29]

$$n(n-1)p^2 - (n^2 + 3n - 2)p + 2 = 0. \tag{1.18}$$

That lower bound on p is not expected to be optimal however. One expects the same result to hold (possibly in a smaller space) for $p_0(n) < p < p_*$, where $p_0(n)$ is the larger root of the equation

$$(n-1)p(p-1) = 2(p+1). \tag{1.19}$$

That result is proved only in dimensions $n = 2$ and 3 and on special sets of regular asymptotic states [15, 18, 36]. For $p \leq p_0(n)$, the wave operators are expected not to exist, in view of the existing finite time blow up results for small solutions [14, 18, 19, 39]. In the energy space X_0, the wave operators exist under assumptions on f which barely fail to include (1.10) with $p = p_*$, the reason being that the lower limit on p required for the existence of the wave operators turns out to be $p > p_*$ in that case and conflicts with the condition $p < p_*$ required to solve the global Cauchy problem at finite times [12]. It is one of the purposes of this talk to present that theory.

For the NLKG equation [3, 4, 9, 31], the wave operators are expected to exist for $4/n < p-1 < 4/(n-2)$ in the energy space and for $p_0(n+1) < p < p_*$ in a suitably smaller space, but the available treatments of the problem in the literature do not seem to be optimal.

We next summarize the main results available on the question of asymptotic completeness (AC) for the same equations. As mentioned above, the proof of AC requires a repulsivity condition, namely $\lambda \geq 0$ in the case (1.10), and reduces basically to the proof of a priori estimates for generic solutions of the equations. There are essentially two methods available. The first method applies to the NLS and NLW equations (but not to the NLKG equation) and exploits the approximate pseudoconformal invariance of the NLS equation and conformal invariance of the NLW equation. For the NLS equation, it yields AC in the space $X = \Sigma$ defined by (1.16) for $p_0(n+1) \leq p < p_*$ [5, 17, 45, 46]. For the NLW equation [10, 33, 40] it yields AC in the space X defined by (1.17) in a rather simple way for $4/(n-1) < p-1 < 4/(n-2)$. There are some results for lower values of p and not too high dimension ($2 \leq n \leq 4$), but they are much harder to derive and probably not optimal [10, 11]. The second method of proof of AC is based on the Morawetz inequality [30], itself a variant of the approximate dilation invariance of the equation at hand. That method has been applied first to the NLKG equation [3, 4, 9, 31] and to the NLS equation [7, 9, 25]. It is especially well suited to the proof of AC in the energy space X_0 and allows for such a proof both for the NLS and NLKG equation for $4/n < p-1 < 4/(n-2)$. Remarkably enough, that method also applies to the NLW equation in the energy space, in spite of the weakness of the time decay available in that case [12], and yields AC under conditions on f that again barely fail to include the power interaction (1.10) with $p = p_*$. It is the second purpose of this talk to present the basic steps of that method and its application to the NLW equation.

The treatment of the theory of scattering for the NLW equation to be given below is interesting for several reasons. First, it allows for a test of the power of the methods in a case where on the one hand only weak time decay is available, but where on the other hand the space time homogeneity of the free equation somewhat alleviates the algebraic complications. This situation is to be contrasted with the better behaved but more complicated case of the NLKG equation. Second, it requires the study of the NLW equation in the critical case $p = p_*$, thereby leading to a number of results of direct relevance to the Cauchy problem in that case, for which there has been a strong interest recently.

The problem of existence of the wave operators will be treated in Section 2 below, and that of asymptotic completeness in Section 3. The exposition follows closely [12] to which we refer for a more detailed treatment and in particular for all the proofs.

§2. Existence of the wave operators

In this section we shall prove the existence of the wave operators for the NLW equation (1.1) by following closely the method sketched in the introduction, namely we shall solve the associated equation (1.7) by a contraction method for large times and continue the solution thereby obtained to all times by using the known results on the Cauchy problem at finite times. An essential role in proof will be played by the space time integrability properties (STIP) associated with the free wave equation $\square\varphi = 0$, by which we mean properties of the following type. Consider a linear evolution equation

$$(2.1) \equiv (1.3) \qquad \partial_t u = Lu$$

where L is a linear antiselfadjoint operator in some Hilbert space, typicaly $\mathcal{H} = L^2$, and let again $U(t) = \exp(tL)$. Then any initial data $u_0 \in L^2$ generates a solution of (2.1)

$$(2.2) \qquad U(\cdot)u_0 \in (\mathcal{C} \cap L^\infty)(\mathbb{R}, L^2)$$

where by $\mathcal{C}(I, X)$ (resp. $L^q(I, X)$) we mean the space of continuous (resp. L^q) functions of time from some interval I to a Banach space X. Now if one is willing to give up some regularity in time, it may happen that one gains some regularity in space, namely that

$$(2.3) \qquad U(\cdot)u_0 \in L^q(\mathbb{R}, X)$$

for some q, $2 \le q < \infty$, where X may be L^r for some $r > 2$, or a Sobolev space W_r^ρ for some $r \ge 2$ and some $\rho \in \mathbb{R}$, preferably $\rho \ge 0$, or some more general space. Such properties exist for a large class of dispersive equations and have a long history [7, 13, 20, 21, 26, 27, 35, 37, 43, 44, 47]. A recent and hopefully didactic account appears in [13]. Since the wave equation is somewhat complicated in that respect, we shall first explain the basic facts on the simplest example of the Schrödinger equation $i\partial_t\varphi = -(1/2)\Delta\varphi$. In that case the unitary group $U(t)$ can be represented by the operator of convolution in space

$$(2.4) \qquad U(t) = \exp[i(t/2)\Delta] = (2\pi i t)^{-n/2} \exp[ix^2/(2t)] *_x$$

so that by the Young inequality, for any $f \in L^1$,

$$(2.5) \qquad \|U(t)f\|_\infty \le (2\pi|t|)^{-n/2}\|f\|_1$$

and by interpolation with unitarity in L^2,

$$(2.6) \qquad \|U(t)f\|_r \le (2\pi|t|)^{-\delta(r)}\|f\|_{\bar{r}}$$

for all $f \in L^{\bar{r}}$, where $2 \le r \le \infty$, r and $\bar{r}$ denote pairs of Hölder conjugate exponents, namely $1/r + 1/\bar{r} = 1$, and here and in what follows

$\delta(r) = n(1/2-1/r)$. Let now f be a function of space time and introduce the operator

$$U *_t f \equiv \int d\tau U(t-\tau) f(\tau). \tag{2.7}$$

From (2.6) and from the Hardy-Littlewood-Sobolev inequality in time, it follows that for $0 \le \delta(r) = 2/q < 1$

$$\| U *_t f; L_t^q(\mathbb{R}, L_x^r) \| \le C \| f; L_t^{\bar{q}}(\mathbb{R}, L_x^{\bar{r}}) \| \tag{2.8}$$

where the subscripts t and x serve as reminders of the variable of interest. At this point, an elementary and by now well known duality argument (see Lemma 2.1 in [13]) yields the following two results. First, the following inequalities also hold

$$\| U *_t f; L_t^{q_1}(\mathbb{R}, L_x^{r_1}) \| \le C \| f; L_t^{\bar{q}_2}(\mathbb{R}, L_x^{\bar{r}_2}) \| \tag{2.9}$$

where $0 \le \delta(r_i) = 2/q_i < 1$, $i = 1, 2$, the main point and difference with (2.8) being that now the pairs of exponents (q, r) in the left hand side and in the right hand side are completely decoupled. Second, for any $u_0 \in L^2$ and $0 \le \delta(r) = 2/q < 1$, the following estimate also holds

$$\| U(\cdot) u_0; L_t^q(\mathbb{R}, L_x^r) \| \le C \| u_0 \|_2. \tag{2.10}$$

Estimates of the type (2.9), (2.10) are especially convenient to study the Cauchy problem for semilinear equations of the type (1.2) in the form of the integral equation (1.6). In fact, one can use the estimates of the type (2.10) to control the free solution and the estimates of the type (2.9) to control the integral in the right hand side of (1.6).

We now turn to the case of the wave equation $\Box \varphi = 0$. We recall that the solution with initial data $u_0 = (\varphi_0, \psi_0)$ at time zero is given by

$$\varphi^{(0)}(t) = \dot{K}(t)\varphi_0 + K(t)\psi_0 \tag{2.11) $\equiv$ (1.13}$$

where $K(t) = \omega^{-1} \sin \omega t$, $\dot{K}(t) = \cos \omega t$ and $\omega = \sqrt{-\Delta}$. The STIP of the wave equation are best expressed in terms of homogeneous Besov spaces $\dot{B}_r^\rho \equiv \dot{B}_{r,2}^\rho(\mathbb{R}^n)$. Those spaces are to be thought of as technically more adequate substitutes for the homogeneous Sobolev spaces $\dot{W}_r^\rho$ (spaces of distributions with derivatives of order exactly ρ in L^r). In order to avoid technicalities, we refrain from giving an explicit definition. We refer for that and for a summary of basic properties to the appendix of [8] or [12], and for a more extensive treatment to [1], Chap. 6.

The basic estimate which replaces (2.6) in the case of the wave equation is the following [2, 32].

Lemma 2.1. *The following estimates hold for all r, $2 \le r \le \infty$*

$$\|\exp(i\omega t)f; \dot{B}_r^{-\beta(r)}\| \le C|t|^{-\gamma(r)}\|f; \dot{B}_{\bar{r}}^{\beta(r)}\| \tag{2.12}$$

where the loss of derivatives and the time decay exponents are given by $\beta(r) = \frac{n+1}{2}(1/2 - 1/r)$ *and* $\gamma(r) = (n-1)(1/2 - 1/r)$.

By exactly the same arguments as in the Schrödinger case, one obtains the following analogue of (2.9).

Lemma 2.2. *The following estimates hold*

$$\|K *_t f; L^{q_1}(\mathbb{R}, \dot{B}_{r_1}^{1-\beta(r_1)})\| \le C\|f; L^{\bar{q}_2}(\mathbb{R}, \dot{B}_{\bar{r}_2}^{\beta(r_2)})\| \tag{2.13}$$

for $0 \le \gamma(r_i) = 2/q_i < 1$, $i = 1, 2$.

We define the energy space for the wave equation as the space

$$X_0 = (\dot{H}^1 \cap L^{2^*}) \oplus L^2 \tag{2.14}$$

where $2^* = 2n/(n-2)$ and we restrict our attention from now on to space dimension $n \ge 3$. Finite energy initial data, namely initial data $(\varphi_0, \psi_0) \in X_0$ generate solutions of the free wave equation through (2.11). In the same way as for the Schrödinger equation, one obtains the following STIP for those solutions, in the form of inequalities similar to (2.10).

Lemma 2.3. *Let* $(\varphi_0, \psi_0) \in X_0$ *and define* $\varphi^{(0)}$ *by* (2.11). *Then*

$$\|\varphi^{(0)}; L^q(\mathbb{R}, \dot{B}_r^\rho)\| \le C\|(\varphi_0, \psi_0); X_0\| \tag{2.15}$$

for all triples (ρ, r, q) *which are admissible in the sense that*

$$0 \le \delta(r) \le n/2 \qquad (\textit{equivalently: } 2 \le r \le \infty) \tag{2.16}$$

$$0 \le 1/q = \rho + \delta(r) - 1 \equiv \sigma < 1/2 \tag{2.17}$$

$$\rho + \beta(r) \le 1 \qquad (\textit{equivalently: } 2\sigma \le \gamma(r)). \tag{2.18}$$

The STIP of Lemma 2.3 are best visualized in the (σ - ρ) plane, where σ is the variable defined by the second equality in (2.17). The variable σ characterizes the homogeneity of the relevant norms in the space variable. In particular Sobolev inequalities allow to control a given

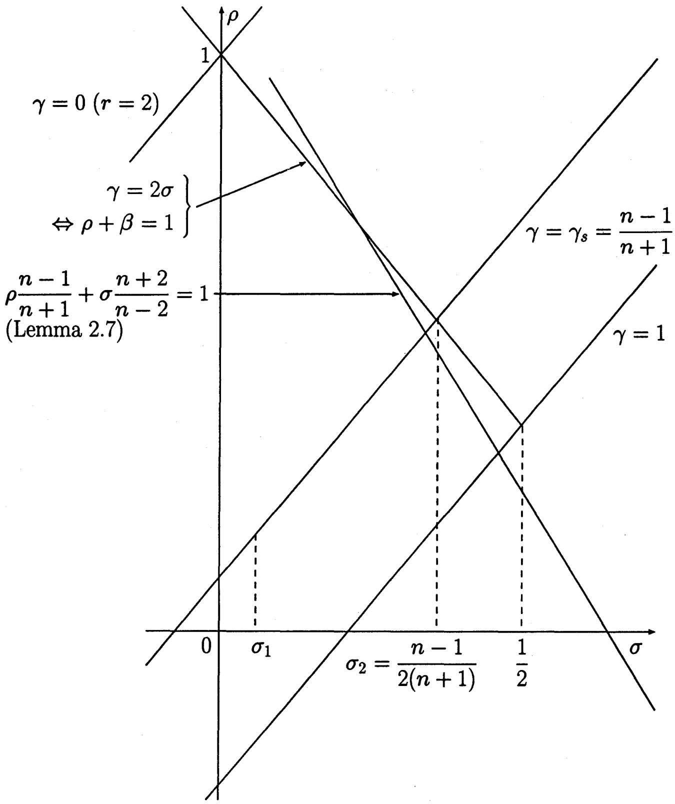

Fig. 1. STIP of $\Box\varphi = 0$

$\dot{B}^\rho_r$ norm in terms of other such norms with the same σ and higher values of ρ. The admissible region (2.16)–(2.18) is represented in Figure 1. For instance the point $(\sigma = 0, \rho = 0)$ corresponds to $L^\infty(\mathbb{R}, L^{2^*})$, the point $\sigma = 0$, $\rho = 1$ to $L^\infty(\mathbb{R}, \dot{H}^1)$, etc. Following Lemma 2.3, it is natural to introduce the following spaces of pairs of functions. For any interval $I \subset \mathbb{R}$, we define

$$(2.19)\quad \mathcal{Y}_0(I) = \{(\varphi, \psi) \colon \varphi \in L^\infty(I, L^{2^*}) \cap L^q(I, \dot{B}^\rho_r) \text{ and } \psi \in L^q(I, \dot{B}^{\rho-1}_r) \text{ for all admissible } (\rho, r, q)\}.$$

Lemma 2.3 says in particular that initial data $(\varphi_0, \psi_0) \in X_0$ generate solutions $\varphi^{(0)}$ such that $(\varphi^{(0)}, \partial_t \varphi^{(0)}) \in \mathcal{Y}_0(\mathbb{R})$.

Although we shall never need to assume faster space decay on φ than is contained in the condition $\varphi \in L^{2^*}$, it is worthwhile to remark that such a decay is preserved in time for functions in $\mathcal{Y}_0(\cdot)$ in the following sense (see Proposition 2.1 in [12]).

Lemma 2.4. *Let $(\varphi, \partial_t \varphi) \in \mathcal{Y}_0(I)$ for some interval I and let $\varphi(s) \in L^k$ for some $s \in I$ and for some k, $2 \le k \le 2^*$. Then $\varphi(t) \in L^k$ for all $t \in I$, $\varphi \in \mathcal{C}(I, L^k)$ and φ satisfies the estimate*

$$\|\varphi(t)\|_k \le C(1 + |t|)^{1-\delta(k)} \tag{2.20}$$

for all $t \in I$, where C depends on the norms of $\varphi(s)$ in L^k and of $(\varphi, \partial_t \varphi)$ in $\mathcal{Y}_0(I)$ but not otherwise on I.

We now turn to the study of finite energy solutions of the equation (1.1). We assume from now on that f satisfies the following assumption:

(H1) $f \in \mathcal{C}^1(\mathbb{C}, \mathbb{C})$ and for some p, $1 < p < \infty$,

$$|f'(z_1) - f'(z_2)| \le C \begin{cases} |z_1 - z_2| \max |z_i|^{p-2} & \text{if } p \ge 2 \\ |z_1 - z_2|^{p-1} & \text{if } p \le 2 \end{cases} \tag{2.21}$$

for all $z_1, z_2 \in \mathbb{C}$, where f' stands for any of $\partial f / \partial z, \partial f / \partial \bar{z}$.

Of special interest will be the case where $p = p_*$.

We recall that the NLW equation (1.1) can be recast in the form (1.2) with $u = (\varphi, \partial_t \varphi)$ and $U(\cdot)$ given by (1.14), so that the integral equation (1.6) reduces in that case to

$$\varphi(t) = \dot{K}(t - t_0)\varphi_0 + K(t - t_0)\psi_0 - \int_{t_0}^{t} d\tau K(t - \tau) f(\varphi(\tau)) \tag{2.22}$$

and to a second equation for $\partial_t \varphi$ which is nothing but the time derivative of (2.22) and which we shall therefore omit. Similarly the equation (1.7) which leads to the definition of the wave operators reduces to

$$\varphi(t) = \dot{K}(t)\varphi_+ + K(t)\psi_+ + \int_t^{\infty} d\tau K(t - \tau) f(\varphi(\tau)) \tag{2.23}$$

and to the time derivative thereof, which we again omit. All subsequent results in this section are derived from the equations (2.22), (2.23) by estimating the free solution and the integral in the right hand sides by Lemmas 2.3 and 2.2 respectively. That requires in addition estimates for the nonlinear interaction $f(\varphi)$ in the integrand. Besov spaces are especially convenient for that purpose since they allow for Leibniz type estimates of the following form (see Lemma 2.3 in [12] for more general results).

Lemma 2.5. *Let f satisfy* (H1) *for some $p \geq 2$. Let $1 \leq r, s, k \leq \infty$, $1/s = 1/r + 1/k$, and $0 < \rho < \min(1, n/r)$. Then*

$$\|f(\varphi); \dot{B}^\rho_s\| \leq C\|\varphi; \dot{B}^\rho_r\| \; \| |\varphi|^{p-2}\|_k. \tag{2.24}$$

We first give some preliminary results on finite energy solutions of the equation (1.1). As a preliminary to the proof of asymptotic completeness in the next section, one can easily show that solutions of (1.1) in $\mathcal{Y}_0$ have asymptotic states (see Proposition 2.3 in [12]).

Lemma 2.6. *Let f satisfy* (H1) *with $p = p_*$. Let φ be a solution of* (1.1) *such that $u = (\varphi, \partial_t\varphi) \in \mathcal{Y}_0(I)$ for some interval $I \subset \mathbb{R}$. Then $u \in \mathcal{C}(I, X_0)$. Furthermore if I is infinite, say $I = [T, \infty)$ then*

$$\exists \operatorname*{s-lim}_{t\to\infty} U(-t)u(t) = u_+ \qquad \textit{in } X_0 \tag{2.25}$$

The next result says basically that "some" of the STIP of solutions of (1.1) included in the definition of $\mathcal{Y}_0$ imply all such STIP.

Lemma 2.7. *Let f satisfy* (H1) *with $p = p_*$, let I be an interval of $\mathbb{R}$, and let φ be a solution of* (1.1) *with $\varphi \in L^q(I, \dot{B}^\rho_r)$ for one admissible triple (ρ, r, q) such that*

$$\rho(n-1)/(n+1) + \sigma(n+2)/(n-2) \geq 1. \tag{2.26}$$

Then $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(I)$.

The region defined by (2.26) in the (σ - ρ) plane is the upper right corner of the admissible region indicated on Figure 1.

We are now in a position to attack the local resolution of the equation (1.1) in the form of the integral equations (2.22) or (2.23). As mentioned earlier, we first solve that problem locally in time by a contraction method, actually by a partial contraction method whereby all the norms defining the relevant space are reproduced by the right hand side of (2.22), (2.23), but only part of them are contracted on bounded sets of that space. One could use for that purpose the space $\mathcal{Y}_0$ defined by (2.19), but it is technically more convenient to use intermediate spaces of functions φ satisfying only part of the STIP and to rely on

Lemma 2.7 to prove that the solutions thereby obtained belong to $\mathcal{Y}_0$. A convenient choice of intermediate spaces is

$$\mathcal{X}_0(I) = \bigcap_{i=1,2} L^{q_i}(I, \dot{B}^{\rho_i}_{r_s}) \tag{2.27}$$

where $r_s = 2(n+1)/(n-1)$ and (ρ_i, r_s, q_i) are two admissible triples satisfying
(2.28)

$$0 < \sigma_1 \le \min\left(\frac{n-2}{2(n+1)}, \frac{n+2}{(n+1)(n-2)}\right) < \sigma_2 = \frac{1}{2}\gamma(r_s) = \frac{n-1}{2(n+1)}.$$

The value $r = r_s$ corresponds to $\beta = 1/2$, namely to the case where there is neither gain nor loss of derivatives in (2.13). The point (ρ_2, r_s, q_2) lies on the upper boundary of the admissible region and satisfies (2.26), so that Lemma 2.7 will be applicable to solutions in $\mathcal{X}_0(I)$ (see Figure 1).

We can now state the basic local existence result (see Proposition 3.1 in [12]).

Proposition 2.1. *Let f satisfy* (H1) *with $p = p_*$.*
(1) *Let $(\varphi_0, \psi_0) \in X_0$. Then there exists $T > 0$ such that the equation* (2.22) *has a unique solution $\varphi \in \mathcal{X}_0(I)$, where $I = [t_0 - T, t_0 + T]$. Furthermore $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(I)$.*
(2) *Let $(\varphi_+, \psi_+) \in X_0$. Then there exists $T > 0$ such that the equation* (2.23) *has a unique solution $\varphi \in \mathcal{X}_0(I)$, where $I = [T, \infty)$. Furthermore $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(I)$. In particular φ satisfies* (2.25).
(3) *Let $(\varphi_0, \psi_0) \in X_0$ (resp. $(\varphi_+, \psi_+) \in X_0$) be small in X_0 norm. Then there exists a unique solution $\varphi \in \mathcal{X}_0(\mathbb{R})$ of the equation* (2.22) *(resp.* (2.23)*). Furthermore $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(\mathbb{R})$, φ satisfies* (2.25) *and its analogue as $t \to -\infty$.*

As remarked before the contraction method yields as a by product the existence of global solutions for small data (part (3) of Proposition 2.1.) and asymptotic completeness for small data.

Corollary 2.1. *Let f satisfy* (H1) *with $p = p_*$. Then the wave operators $\Omega_\pm$ exist as bijections of X_0 locally in a neighborhood of zero.*

The second step in the construction of the wave operators consists in extending the solutions obtained in Proposition 2.1 part (2) to all times. For that purpose we need to solve the global Cauchy problem at finite times. According to standard methods, this requires a priori estimates of solutions of (1.1) in the energy space, which in turn follow from energy conservation. We assume

(H2) (gauge invariance) There exists a function $V \in \mathcal{C}^1(\mathbb{C}, \mathbb{R})$ with $V(0) = 0$ such that $f(z) = \partial V/\partial \bar{z}$ and $V(z) = V(|z|) \geq -a^2|z|^2$ for all $z \in \mathbb{C}$.

The energy is then defined by (1.15) (with $m = 0$) and one can prove energy conservation in the following sense (see Proposition 3.6 in [12]).

Lemma 2.8. *Let f satisfy* (H1) *with $p = p_*$ and* (H2). *Let I be an interval of $\mathbb{R}$, let $t_0 \in I$ and $(\varphi_0, \psi_0) \in X_0$ and let φ be a solution of* (2.22) *such that $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(I)$. Then for all s and t in I*

$$E(\varphi(t), \partial_t\varphi(t)) = E(\varphi(s), \partial_t\varphi(s)). \tag{2.29}$$

The question arises at this point whether one can solve the Cauchy problem globally in time for the NLW equation in the energy space for the critical value $p = p_*$ of the exponent in f. There has been recently a strong interest for that problem, for which Proposition 3.1, especially part (1), and Lemma 2.8 are directly relevant. The answer is most probably yes but the existing proofs are restricted either to finite energy *radial* solutions, or to smooth solutions in space dimension $n \leq 7$. In order to proceed safely, we therefore assume in addition that f satisfies the assumption (H1) both for $p = p_*$ and for some $p < p_*$. It is at this point that single power nonlinearities (1.10) barely escape from the present theory. One can then derive the final result on the existence of the wave operators.

Proposition 2.2. *Let f satisfy* (H1) *both for $p = p_*$ and for $p = p_2 < p_*$, and* (H2).
(1) *Let $(\varphi_+, \psi_+) \in X_0$. Then the equation* (2.23) *has a unique solution φ such that $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0([T, \infty))$ for all $T \in \mathbb{R}$. That solution satisfies* (2.29) *for all s and t in $\mathbb{R}$ and satisfies* (2.25). *In particular the wave operator $\Omega_+ : (\varphi_+, \varphi_+) \to (\varphi(0), \partial_t\varphi(0))$ is well defined from X_0 to X_0. Similar results hold for negative times.*
(2) *Let in addition $V \geq 0$ (i.e. $a = 0$). Then $\Omega_\pm$ and $\Omega_\pm^{-1}$ are bounded operators in X_0 norm.*

Part (2) of Proposition 2.2 follows from the simple remark that for all $t \in \mathbb{R}$

$$\|u(t); X_0\|^2 \leq E \leq \|u(t); X_0\|^2 + C\|u(t); X_0\|^{2^*} \tag{2.30}$$

where E is the energy, the first inequality follows from the positivity of V and the second one from a Sobolev inequality. The same double inequality holds for $U(-t)u(t)$ instead of $u(t)$ because $U(\cdot)$ is isometric in X_0, and for u_+ by Lemma 2.6, so that $\|u(0); X_0\| \leq m(\|u_+; X_0\|)$ and $\|u_+; X_0\| \leq m(\|u(0); X_0\|)$ with $m(s) = (s^2 + Cs^{2^*})^{1/2}$.

§3. Asymptotic completeness

In this section, we sketch the proof of asymptotic completeness for the NLW equation (1.1) in the energy space by using the method originally devised in [31] for the NLKG equation, in the version given in [7, 9, 12]. It follows from Propositions 3.1 and 3.2 that the proof of AC reduces to proving that generic finite energy solutions, namely solutions of (2.22) with initial data in X_0, belong to $\mathcal{Y}_0(\mathbb{R})$. By Lemma 2.7, it suffices to prove that such solutions belong to $L^q(\mathbb{R}, \dot{B}^\rho_r)$ for one admissible triple (ρ, r, q) satisfying (2.26). The proof then reduces to a priori estimates on those solutions. We continue to restrict our attention to space dimension $n \geq 3$ to begin with. However the proof will require at some point the existence of one norm of the solutions with integrable decay in time, namely $\gamma(r) > 1$, and will therefore only apply in space dimension $n \geq 4$, since $\gamma(\infty) = 1$ for $n = 3$.

The essence of the proof consists in squeezing the given solution between two conflicting estimates which force it to decay. The first of those estimates is the Morawetz inequality [30]. For f satisfying (H2), we introduce the auxiliary potential

$$W_1(z) = \bar{z}f(z) - V(z). \tag{3.1}$$

For f a single power (1.10), W_1 reduces to $W_1(z) = (\lambda/2)(p-1)|z|^{p+1}$. We introduce also the functions $g(x) = (x^2+a^2)^{-1/2}$ and $g_1(x) = \nabla\cdot(xg)$. One checks easily that $(n-1)g \leq g_1 \leq ng$ and that $\Delta g_1 \leq 0$ for $n \geq 3$. We can now state the Morawetz inequality (see Lemma 4.3 in [12]).

Lemma 3.1. *Let f satisfy* (H1) *with $p = p_*$ and* (H2), *let I be an interval, $t_0 \in I$, $(\varphi_0, \psi_0) \in X_0$ and φ a solution of* (2.22) *with $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(I)$. Then for all s and t in I, $s \leq t$,*

$$\int_s^t d\tau \int dx g_1(x) W_1(\varphi(\tau, x)) \leq -\operatorname{Re}\langle \partial_t\varphi, (xg\cdot\nabla + \nabla\cdot xg)\varphi\rangle \Big|_t^s . \tag{3.2}$$

The Morawetz inequality is a modified version of dilation invariance. In fact the operator $x\cdot\nabla + \nabla\cdot x$ is the generator of space dilations. It fails to be defined in the energy space because of the factor x, and the function g serves to compensate for that defect. Let $A = xg\cdot\nabla + \nabla\cdot xg$.

The formal proof of (3.2) consists in computing the time derivative

$$\begin{aligned}-\partial_t \operatorname{Re}\langle \partial_t \varphi, A\varphi\rangle &= -\operatorname{Re}\langle \partial_t^2 \varphi, A\varphi\rangle \\ &= -\operatorname{Re}\langle \Delta\varphi, A\varphi\rangle + \operatorname{Re}\langle f(\varphi), A\varphi\rangle\end{aligned}$$

by using the antisymmetry of A and the equation (1.1). By elementary computations, the term with Δ is easily seen to be non negative, while

$$\operatorname{Re}\langle f(\varphi), A\varphi\rangle = \int g_1 W_1(\varphi)dx.$$

The regularity of φ provided by $\mathcal{Y}_0$ is sufficient to convert the formal proof into an actual proof. For positive V the right hand side of (3.2) is bounded by $2E$ uniformly in s, t and a. For $I = \mathbb{R}$ and $W_1 \geq 0$, and after taking the harmless limit $a \downarrow 0$, one obtains from (3.2)

$$\int dtdx|x|^{-1}W_1(\varphi(t,x)) \leq 2E/(n-1). \tag{3.3}$$

The meaning of (3.3) is best understood by seing what it forbids: it forbids in particular that φ be a localized solution travelling at finite speed. In fact, if $\varphi(t,x) = h(x - vt)$ and if f is a single power (1.10), then the left hand side of (3.3) becomes approximately for large t

$$C\int^{\infty} t^{-1}dt\ \|h\|_{p+1}^{p+1} = \infty.$$

This fact suggests that φ must either spread out in space, or recede to infinity with unbounded velocity. The second possibility is however forbidden by the second basic estimate, namely the finiteness of the propagation speed coming from the hyperbolicity of the equation. That estimate is best expressed for the present purposes in terms of the propagation of local energy. For any $\Lambda \subset \mathbb{R}^n$, we denote the complement of Λ by $\Lambda' = \mathbb{R}^n \backslash \Lambda$ and we define the energy in Λ by

$$E(\varphi, \psi; \Lambda) = \int_\Lambda dx(|\psi|^2 + |\nabla\varphi|^2 + V(\varphi)). \tag{3.4}$$

We shall also need the balls in $\mathbb{R}^n$

$$B(x_0, R) = \{x \in \mathbb{R}^n\colon |x - x_0| < R\}.$$

We can now state the local energy propagation as follows (see Lemma 4.2 in [12]).

Lemma 3.2. *Let f satisfy* (H1) *with $p = p_*$ and* (H2) *with $V \geq 0$. Let $I \subset \mathbb{R}$ with $0 \in I$, let $(\varphi_0, \psi_0) \in X_0$ and let φ be a solution of* (2.22) *with $(\varphi, \partial_t \varphi) \in \mathcal{Y}_0(I)$. Then for all $x_0 \in \mathbb{R}^n, R > 0$ and $t \in I$*

(3.5)
$$E(\varphi(t), \partial_t\varphi(t); B(x_0, R - |t|)) \leq E(\varphi(0), \partial_t\varphi(0); B(x_0, R))$$

(3.6)
$$E(\varphi(t), \partial_t\varphi(t); B'(x_0, R + |t|)) \leq E(\varphi(0), \partial_t\varphi(0); B'(x_0, R)).$$

The formal proof of Lemma 3.2 consists in noting that the energy momentum vector

$$\begin{aligned} \theta_0 &= |\partial_t\varphi|^2 + |\nabla\varphi|^2 + V(\varphi) \\ \theta &= -2\,\mathrm{Re}\,\partial_t\bar{\varphi}\nabla\varphi \end{aligned}$$

is time like and applying the Green formula to the truncated cones $0 \leq |\tau| \leq |t|$, $|x| \leq R \pm |\tau|$. Again the regularity provided by $\mathcal{Y}_0$ is sufficient to convert the formal proof into an actual proof. Lemma 3.2 will be used in the form of the following easily derived corollary (see Lemma 4.6 in [12]).

Corollary 3.1. *Under the same assumptions as in Lemma* 3.2 *with $I = \mathbb{R}$, for any $\eta > 0$*

$$\|\varphi(t); L^{2^*}(B'(0, (1+\eta)|t|)\| \longrightarrow 0 \qquad \textit{when } |t| \to \infty. \tag{3.7}$$

There remains the hard task of combining the estimates (3.3) and (3.7) to derive a priori estimates for the norm of φ in $L^q(\mathbb{R}, \dot{B}^\rho_r)$ for a suitable admissible triple (ρ, r, q). We choose such a triple with $\gamma(r) = 1 + \varepsilon$ and $\sigma = 1/2 - \varepsilon$ for some small $\varepsilon > 0$ (such a triple satisfies (2.26) for ε small enough). It is as this point that we have to restrict our attention to space dimension $n \geq 4$. For φ a solution of (2.22) with $t_0 = 0$, $(\varphi_0, \psi_0) \in X_0$ and $(\varphi, \partial_t\varphi) \in \mathcal{Y}_{0,\mathrm{loc}}(\mathbb{R})$ we define

$$k_0(t) = \|\dot{K}(t)\varphi_0 + K(t)\psi_0; \dot{B}^\rho_r\| \tag{3.8}$$

$$k(t) = \|\varphi(t); \dot{B}^\rho_r\|. \tag{3.9}$$

One of the main technical steps of the proof consists in deriving a set of integral inequalities for k by applying the estimates (2.12), (2.13), (2.15) to the integral equation (2.22). Some of these inequalities will require that f satisfy (H1) both for some $p_2 < p_*$ and for some $p_1 > p_*$. One can then prove:

Lemma 3.3. *Let $n \geq 4$ and let f satisfy* (H1) *both for $p = p_2 < p_*$ and for $p = p_1 > p_*$. Let (ρ, r, q) be an admissible triple with $\gamma(r) = 1+\varepsilon$ and $\sigma = 1/2 - \varepsilon$ for some small $\varepsilon > 0$ and let φ be a solution of* (2.22) *with $t_0 = 0$, $(\varphi_0, \psi_0) \in X_0$ and $(\varphi, \partial_t \varphi) \in \mathcal{Y}_{0,\mathrm{loc}}(\mathbb{R})$. Then for some $\eta > 0$ depending only on ε, p_1, p_2 and some $M(E)$ depending only on the energy E, φ satisfies the inequalities*

$$k(t) \leq k_0(t) + M(E) \int_0^t d\tau \min |t-\tau|^{-(1\pm\eta)} \min k(\tau)^{1\pm\eta}. \tag{3.10}$$

It is easy to see by homogeneity that $\varepsilon = 0$ and $p = p_*$ would yield (3.10) with $\eta = 0$. The combination of signs $(--)$ in (3.10) yields an information on the local regularity of k and requires only the assumption (H1) with $p_2 < p_*$ while the combination $(++)$ yields the (most important) information at infinity in time and requires $p_1 > p_*$. The crossed terms with $(+-)$ and $(-+)$ would hold with only $p = p_*$.

The next step in the proof consists in combining the estimates (3.3), (3.7) with the inequalities (3.10) in the case $(\pm, -)$ to show that $k(t)$ is small in suitable large intervals. That step requires the assumption (H1) only with $p = p_*$ and $p = p_2 < p_*$. In addition it requires the following additional repulsivity condition in order to exploit (3.3).

(H3) For some $c > 0$, for $p_2 < p_* < p_1$ and for all $z \in \mathbb{C}$

$$W_1(z) \geq c \min(|z|^{p_1+1}, |z|^{p_2+1}). \tag{3.11}$$

We introduce the auxiliary norms

$$\|\varphi; \ell^\infty(L^q(I, \dot{B}^\rho_r))\| \equiv \|k(t); \ell^\infty(L^q(I))\| = \sup_{t:\ [t,t+1]\subset I} \|k; L^q([t,t+1])\|.$$

One can then prove (see Lemma 4.5 in [12]):

Lemma 3.4. *Let $n \geq 4$, let f satisfy* (H1) *both for $p = p_*$ and for $p = p_2 < p_*$,* (H2) *with $V \geq 0$ and* (H3). *Let φ and k be as in Lemma* 3.3. *Then for any $\varepsilon_1 > 0$ and for any $\ell > 0$, there exists $a > 0$ such that*

$$\|k; \ell^\infty(L^q([a, a+\ell]))\| \leq \varepsilon_1. \tag{3.12}$$

The proof of Lemma 3.4 consists in estimating the integral in (2.22) with $t_0 = 0$ by splitting the integration region for large t in four subregions, for some small θ_1 and some large $\theta_2 < t$:
(1) In the region $t - \theta_1 \leq \tau \leq t$, one uses the estimate (3.10) with signs

$(--)$,
(2) In the region $0 \leq \tau \leq t - \theta_2$, one uses the estimate (3.10) with signs $(+-)$, thereby obtaining two estimates sublinear in k with small coefficients. In the intermediate region $t - \theta_2 \leq \tau \leq t - \theta_1$, one essentially splits the x integration in two subregions:
(3) For $t - \theta_2 \leq \tau \leq t - \theta_1$ and $|x| \leq 2\tau$, one uses a modified version of the estimate (3.10) with signs $(\pm-)$ and a consequence of the estimate (3.3).
(4) For $t - \theta_2 \leq \tau \leq t - \theta_1$ and $|x| \geq 2\tau$, one uses a modified version of the estimate (3.10) with signs $(\pm-)$ and the estimate (3.7).
Lemma 3.4 means essentially that k tends to be small in suitably located, but arbitrarily large intervals. Using that information, which is a weak form of the fact that k tends to zero at infinity, and the superlinear part of Lemma 3.3, namely the inequalities (3.10) with signs $(\pm+)$, one then proves that $k(t)$ exhibits the same time decay at infinity as $k^{(0)}$, namely belongs to $L^q(\mathbb{R})$. That last step is an elementary abstract argument based on (3.10) and (3.12) and is otherwise independent of any additional property of the equation.

Combining all previous steps yields the final result (see Proposition 4.2 in [12]).

Proposition 3.1. *Let $n \geq 4$, let f satisfy* (H1) *both for $p = p_2 < p_*$ and for $p = p_1 > p_*$,* (H2) *with $V \geq 0$ and* (H3) *(namely* (3.11)*). Let $(\varphi_0, \psi_0) \in X_0$ and let φ be a solution of* (2.22) *with $(\varphi, \partial_t\varphi) \in \mathcal{Y}_{0,\mathrm{loc}}(\mathbb{R})$. Then $(\varphi, \partial_t\varphi) \in \mathcal{Y}_0(\mathbb{R})$. In particular asymptotic completeness holds in X_0.*

We finally comment on a problem left open by the preceding proof. Although any finite energy initial data $(\varphi_0, \psi_0) \in X_0$ generates a solution in $\mathcal{Y}_0(\mathbb{R})$, no estimate is obtained for the norm of $(\varphi, \partial_t\varphi)$ in $\mathcal{Y}_0(\mathbb{R})$ in terms of the energy of the solution. This is due to the fact that the proof starts from some time translation invariant information and ends up with information of the same type, in the form of time integrals, by going at some intermediate stages through estimates which are pointwise in time. It would be interesting to know whether the map $(\varphi_0, \psi_0) \to (\varphi, \partial_t\varphi)$ is bounded from X_0 to $\mathcal{Y}_0(\mathbb{R})$. This would probably require a simplified time translation invariant version of the preceding proof.

Another challenging question would be to extend the present results — if true — to the purely critical case where f is assumed to satisfy (H1) for $p = p_*$ only.

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