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On fibered links of singularities of polar weighted homogeneous mixed polynomials

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Abstract.

Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a polar weighted homogeneous mixed polynomial. If $f(\mathbf{z},\bar{\mathbf{z}})$ has an isolated singularity at the origin $\mathbf{o}$, then $f(\mathbf{z},\bar{\mathbf{z}})$ gives a fibered link in a sphere centered at $\mathbf{o}$. In this paper, we study fibered links which are determined by polar weighted homogeneous mixed polynomials and show the existence of mixed polynomials whose Milnor fibers cannot be obtained from a disk by plumbings of Hopf bands.

§1. Introduction

Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a polynomial expanded in a convergent power series of variables $\mathbf{z}=(z_1,\dots,z_n)$ and $\bar{\mathbf{z}}=(\bar{z}_1,\dots,\bar{z}_n)$

$$f(\mathbf{z},\bar{\mathbf{z}}) := \sum_{\nu,\mu} c_{\nu,\mu}\mathbf{z}^{\nu}\bar{\mathbf{z}}^{\mu},$$

where $\mathbf{z}^{\nu}=z_1^{\nu_1}\cdots z_n^{\nu_n}$ for $\nu=(\nu_1,\dots,\nu_n)$ (respectively $\bar{\mathbf{z}}^{\mu}=\bar{z}_1^{\mu_1}\cdots\bar{z}_n^{\mu_n}$ for $\mu=(\mu_1,\dots,\mu_n)$). $\bar{z}_j$ represents the complex conjugate of z_j. A polynomial $f(\mathbf{z},\bar{\mathbf{z}})$ of this form is called a *mixed polynomial* [13], [14].

Let $\mathbf{o}$ be the origin of $\mathbb{C}^n$. Assume that $f(\mathbf{o})=0$ and $\mathbf{o}$ is an isolated singularity of $f(\mathbf{z},\bar{\mathbf{z}})$. Then $K_f := S_{\varepsilon}^{2n-1}\cap f^{-1}(0)$ is a *link* i.e., K_f is an oriented codimension-two closed smooth submanifold in the $(2n-1)$-sphere S^{2n-1} [9, Corollary 2.9]. A link K is said to be *fibered* if there exists a trivialization $K\times D^2\to N(K)$ of a tubular neighborhood $N(K)$ of K in S^{2n-1} and a fibration of the link exterior $E(K)=S^{2n-1}\setminus \mathrm{Int}(N(K))$, $\phi_1 : E(K)\to S^1$ such that $\phi_0|\partial N(K)=\phi_1|\partial N(K)$, where $\phi_0 : N(K)\to D^2$ is a trivialization $K\times D^2\to N(K)$ composed with the

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second factor. This fibration is also called an *open book decomposition* of S^{2n-1}. A fiber of ϕ_1 is called a *fiber surface of the fibration of* K.

It is well-known that a complex polynomial $f(\mathbf{z})$ has a locally trivial fibration

$$\frac{f}{|f|} : S_\varepsilon^{2n-1} \setminus K_f \to S^1,$$

where $S_\varepsilon^{2n-1} := \{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid \sum_{i=1}^n |z_i|^2 = \varepsilon\}$. This fibration is called the *Milnor fibration of* $f(\mathbf{z})$ *at* $\mathbf{o}$ and its fiber surface the *Milnor fiber of* $f(\mathbf{z}, \bar{\mathbf{z}})$. If the origin $\mathbf{o}$ is an isolated singularity of $f(\mathbf{z})$, then the link K_f is fibered.

We consider the class of mixed polynomials which was first introduced by Ruas–Seade–Verjovsky [17] and J. L. Cisneros-Molina [1]. Let $p_1, \ldots, p_n$ be integers such that $\gcd(p_1, \ldots, p_n) = 1$. We define an S^1-action on $\mathbb{C}^n$ as follows:

$$s \circ \mathbf{z} = (s^{p_1} z_1, \ldots, s^{p_n} z_n), \quad s \in S^1.$$

If there exists a positive integer d_p such that the mixed polynomial $f(\mathbf{z}, \bar{\mathbf{z}})$ satisfies

$$f(s^{p_1} z_1, \ldots, s^{p_n} z_n, \bar{s}^{p_1} \bar{z}_1, \ldots, \bar{s}^{p_1} \bar{z}_n) = s^{d_p} f(\mathbf{z}, \bar{\mathbf{z}}), \quad s \in S^1,$$

we say that $f(\mathbf{z}, \bar{\mathbf{z}})$ is *polar weighted homogeneous.* The weight vector $(p_1, \ldots, p_n)$ is called the *polar weights* and d_p is called the *polar degree* respectively. In this case, K_f is fibered and its monodromy is given by

$$(z_1, \ldots, z_n) \mapsto \left(\exp\Big(\frac{2p_1\pi i}{d_p}\Big) z_1, \ldots, \exp\Big(\frac{2p_n\pi i}{d_p}\Big) z_n\right),$$

see [13], [14]. Oka introduced the notation of strongly non-degeneracy for mixed polynomials and proved that those polynomilas guarantee the existence of the Milnor fibration [14].

In the present paper, we study the topology of the Milnor fibers of some polar weighted homogeneous mixed polynomials in two variables $f(z_1, z_2, \bar{z}_1, \bar{z}_2)$. In S^3, if a surface F is a plumbing of another surface F' and a Hopf band, we call F *is obtained from* F' *by plumbing a Hopf band.* A surface F is called a *Hopf plumbing* if it can be obtained from a disk by a plumbing of a finite number of Hopf bands. A plumbing operation is useful for the study of fibered links, for instance used by D. Gabai [4] and by E. Giroux [5]. It is known that the fiber surface of the Milnor fibration of a complex polynomial is a Hopf plumbing (cf. [7]). Plumbings can also be defined in high dimensional case. D. Lines studied high dimensional fibered knots by using plumbings [8]. We study fiber surfaces of polar weighted homogeneous mixed polynomials. The main theorems in this paper are the followings.

Theorem 1. *Let $f(\mathbf{z}, \bar{\mathbf{z}})$ be a polar weighted homogeneous mixed polynomial given by $\prod_{j=1}^{m+1}(z_1 + \alpha_j z_2) \prod_{j=m+2}^{2m+1} \overline{(z_1 + \alpha_j z_2)}$, where $\alpha_j \neq \alpha_{j'}$ $(j \neq j')$. Then the Milnor fiber of $f(\mathbf{z}, \bar{\mathbf{z}})$ has genus 0 and cannot be obtained by plumbing Hopf band on a surface. In particular it is not obtained from a disk by plumbing Hopf bands.*

Theorem 2. *Let $g(\mathbf{z}, \bar{\mathbf{z}})$ and $f(\mathbf{z}, \bar{\mathbf{z}})$ be polar weighted homogeneous mixed polynomials given by $g(\mathbf{z}, \bar{\mathbf{z}}) = z_1(z_1^3 + z_2^5)\overline{(z_1^3 - z_2^5)z_2}$ and $f(\mathbf{z}, \bar{\mathbf{z}}) = \prod_{j=1}^{k+1}(z_1 + \alpha_j z_2^2) \prod_{j=k+2}^{2k+1} \overline{(z_1 + \alpha_j z_2^2)}$, where $k \geq 2$ and $\alpha_j \neq \alpha_{j'}$ $(j \neq j')$. Then the Milnor fibers of $g(\mathbf{z}, \bar{\mathbf{z}})$ and $f(\mathbf{z}, \bar{\mathbf{z}})$ have the following properties:*

- *the genus of the Milnor fiber of $g(\mathbf{z}, \bar{\mathbf{z}})$ and that of $f(\mathbf{z}, \bar{\mathbf{z}})$ are 1 and k respectively,*
- *the Milnor fibers of $g(\mathbf{z}, \bar{\mathbf{z}})$ and $f(\mathbf{z}, \bar{\mathbf{z}})$ cannot be obtained from a disk by plumbing Hopf bands.*

Corollary 1. *The Milnor fiber in Theorems 1 and 2 cannot appear as Milnor fibers of holomorphic functions.*

This paper is organized as follows. In Section 2 we give the definitions of plumbings and Seifert forms of links. In Section 3 we calculate the Seifert forms of fibered links which are determined by a class of mixed polynomials and prove Theorem 1. In Section 4 we introduce the enhancement to the Milnor number and prove Theorem 2.

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§2. Preliminaries

2.1. Plumbings

We give the definition of plumbings and its fundamental properties.

Let F be a compact oriented surface embedded in S^3. A surface F is a *plumbing* of two compact oriented surfaces F_1 and F_2 if they satisfy the following properties:

- $F = F_1 \cup F_2$ such that $F_1 \cap F_2$ is a square disk with edges a_1, b_1, a_2, b_2 where a_i is contained in ∂F_1 and is a proper arc in F_2 for all i, and b_i is contained in ∂F_2 and is a proper arc in F_1 for all i.

- There exist 3-balls B_1 and B_2 in S^3 such that
  (1) $B_1 \cup B_2 = S^3$ and $B_1 \cap B_2 = \partial B_1 = \partial B_2 = S^2$,
  (2) $B_i \supset F_i$ for $i = 1, 2$,
  (3) $\partial B_1 \cap F_1 = \partial B_2 \cap F_2 = F_1 \cap F_2$.

D. Gabai showed that the plumbing operation has the following property [4].

Theorem 3. *F is a fiber surface if and only if both F_1 and F_2 are fiber surfaces.*

Note that this theorem is often used to decide that a surface is a fiber surface or not in knot theory.

A *Hopf band* is an unknotted annulus, embedded in S^3 with one full twist. If a Hopf band is isotopic to the fiber surface of the Milnor fibration of $f(\mathbf{z}) = z_1^2 + z_2^2$, the Hopf band is called *positive* otherwise it is called *negative.* If a surface F is a plumbing of a surface F_1 and a

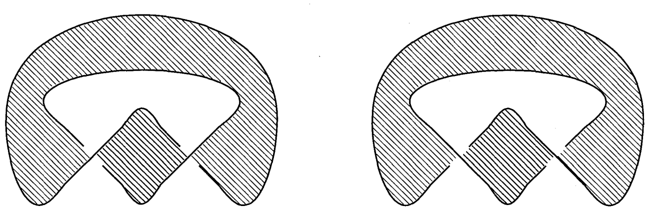

Fig. 1. Positive and negative Hopf bands

Hopf band, we say that F *is obtained from F_1 by plumbing a Hopf band,* or F' *is obtained from F by deplumbing a Hopf band.* If a fiber surface F is obtained from a disk by plumbing a finite number of Hopf bands, F is called a *Hopf plumbing.* As mentioned in [7], the link of an isolated singularity of 2-variables complex polynomial has a closed positive braid presentation, and hence the fiber surface is a Hopf plumbing.

Note also that it is known by Giroux in [5] that any fiber surface in S^3 can be obtained from a disk by a combination of plumbings and deplumbings of Hopf bands (cf. [3]).

2.2. Seifert forms

A fibered link K is *simple* if K is $(n-3)$-connected and its fiber surface, which by definition is a fiber of ϕ_1, is $(n-2)$-connected. Let (S^{2n-1}, K) be a simple fibered link and F a fiber surface of the fibration of K. We set $\alpha, \beta \in \tilde{H}_{n-1}(F; \mathbb{Z})$ and a and b to be cycles on F representing α and β respectively. We define

$$L_K(\alpha, \beta) := \text{link}(a^+, b)$$

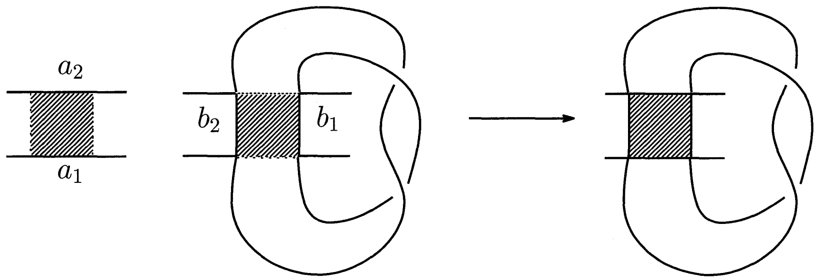

Fig. 2. A plumbing of a surface and a Hopf band

where a^+ is a pushed off of a to the positive side of F by a transverse vector field and $\mathrm{link}(a^+, b)$ is the linking number of a^+ and b. The *Seifert form* L_K of K is the non-singular bilinear form

$$L_K : \tilde{H}_{n-1}(F; \mathbb{Z}) \times \tilde{H}_{n-1}(F; \mathbb{Z}) \to \mathbb{Z}$$

on the $(n-1)$-th homology group $\tilde{H}_{n-1}(F; \mathbb{Z})$ of the fiber of the fibration, with respect to a choice of basis of $\tilde{H}_{n-1}(F; \mathbb{Z})$. Note that L_K becomes an invertible integer matrix.

Let $A = (a_{i,j})$ and A' be integral unimodular matrices. We say that A' is an *extension of* A if A' is congruent to

$$\left(\begin{array}{ccc|c} a_{1,1} & \dots & a_{1,n} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} & 0 \\ \hline b_1 & \dots & b_n & \varepsilon \end{array}\right)$$

where n is the rank of A, $b_i \in \mathbb{Z}, i = 1, \dots, n$ and $\varepsilon = \pm 1$.

If a fiber surface F is obtained from F_1 by a plumbing of a Hopf band, the Seifert form of F is an extension of the Seifert form of F_1. In particular, if a fiber surface is obtained from a disk by successive plumbings of Hopf bands then its Seifert form becomes a unimodular lower triangular matrix for a suitable choice of the basis.

§3. Proof of Theorem 1

We focus on the following type of mixed polynomials

$$f(\mathbf{z}, \bar{\mathbf{z}}) := \prod_{j=1}^{m+1} (z_1 + \alpha_j z_2) \prod_{j=m+2}^{2m+1} \overline{(z_1 + \alpha_j z_2)},$$

where $\alpha_j \neq \alpha_{j'}$ $(j \neq j')$, $\overline{z_1 + \alpha_j z_2}$ represents the complex-conjugate of $z_1 + \alpha_j z_2$. Remark that $f(\mathbf{z}, \bar{\mathbf{z}})$ has $m+1$ holomorphic factors and m complex-conjugate factors. Such a type of mixed polynomials is a special case of polynomials of forms $(f\bar{g}, \mathbf{o})$ studied by A. Pichon and J. Seade in [15], [16], where $(f, \mathbf{o})$ and $(g, \mathbf{o})$ are complex polynomials with isolated singularities at $\mathbf{o}$ and with no common branches. The origin $\mathbf{o}$ is an isolated singularity of $f(\mathbf{z}, \bar{\mathbf{z}})$ and $K_f := S^3_\varepsilon \cap f^{-1}(0)$ is an oriented fibered link in the 3-sphere S^3_ε. The S^1-action on S^3_ε is

$$s \circ (z_1, z_2) = (sz_1, sz_2), \quad s \in S^1$$

and $f(\mathbf{z}, \bar{\mathbf{z}})$ satisfies

$$f(s \circ \mathbf{z}, \overline{s \circ \mathbf{z}}) = s f_m(\mathbf{z}, \bar{\mathbf{z}}).$$

So $f(\mathbf{z}, \bar{\mathbf{z}})$ is polar weighted homogeneous with polar degree 1. The monodromy map $h : F \to F$ is given by this S^1-action.

We calculate the Seifert form L_{K_f} in order to prove Theorem 1.

Lemma 1. *The homology group $H_1(F; \mathbb{Z})$ has rank $2m$ and there exists a basis of $H_1(F; \mathbb{Z})$ in which the matrix L_f is the following $2m \times 2m$ matrix:*

$$L_{K_f} = \begin{pmatrix} 0 & 1 & \dots & 1 & -1 & \dots & \dots & -1 \\ 1 & 0 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \vdots & \vdots & \vdots \\ 1 & \dots & 1 & 0 & -1 & \dots & \dots & -1 \\ -1 & \dots & \dots & -1 & 2 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & 1 & 2 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 1 \\ -1 & \dots & \dots & -1 & 1 & \dots & 1 & 2 \end{pmatrix}.$$

Proof. Since K_f is the invariant set for the S^1-action, the Euler characteristic of the fiber surface F of the fibration $f/|f| : S^3_\varepsilon \setminus K_f \to S^1$ is equal to $-2m+1$, which can be calculated from the splice diagram of Eisenbud and Neumann [2]. Since the number of link components of K_f is $2m+1$, the genus of the fiber surface F of $f/|f|$ is 0. Thus a basis of $H_1(F; \mathbb{Z})$ is represented by $2m$ connected components of the link K_f. Let a_i be the link component of K_f whose orientation is the same as that of the S^1-action for $i = 1, \dots, m+1$ and b_i be the link component of K_f which has the orientation opposite to the S^1-action for $i = 1, \dots, m$. Then the cycles $\{a_i, b_i \mid i = 1, \dots, m\}$ constitute a basis of $H_1(F;\mathbb{Z})$. We may choose the orientation of each a_i and b_i such that it coincides with the orientation of the corresponding component of K_f. By easy calculus, we have $\mathrm{link}(a_i, a_j) = \mathrm{link}(b_i, b_j) = 1$ and $\mathrm{link}(a_i, b_j) = \mathrm{link}(b_i, a_j) = -1$, where $1 \leq i, j \leq m$ and $i \neq j$.

We now calculate the diagonal components of L_{K_f}. The fiber surface of $f/|f|$ is a union of a disk which has m holes and m bands with 1-full twist as shown in Fig. 3. The figure 3 represents the fiber surface for $m = 2$. The dotted line and the dashed line represent a_i and b_i respectively.

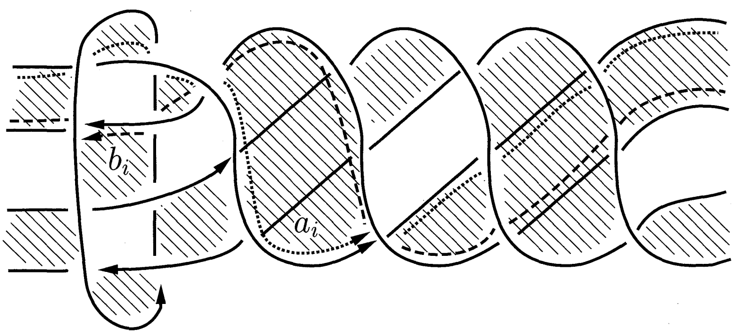

Fig. 3. The fiber surface is obtained by closely the above surface canonically as closed braids. The orientations of a_i and b_i are a right direction and a left direction respectively.

Each loop representing a_i or b_i passes through four half twist bands. We can easily check that $\mathrm{link}(a_i^+, a_i) = 0$ and $\mathrm{link}(b_i^+, b_i) = 2$ for $i = 1, \ldots, m$. This completes the proof. Q.E.D.

Proof of Theorem 1. By applying change of the basis of $H_1(F;\mathbb{Z})$, the Seifert form L_{K_f} is represented by

$$L'_{K_f} = {}^tPL_{K_f}P = \begin{pmatrix} I & 0 \\ I & I \end{pmatrix} L_{K_f} \begin{pmatrix} I & I \\ 0 & I \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & \dots & 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & \ddots & \vdots & 0 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & 0 \\ 1 & \dots & 1 & 0 & 0 & \dots & 0 & -1 \\ -1 & 0 & \dots & 0 & 0 & \dots & \dots & 0 \\ 0 & -1 & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 0 & -1 & 0 & \dots & \dots & 0 \end{pmatrix}$$

where I is the $m \times m$ unit matrix.

Let $Q = (q_{i,j})$ be a $2m \times 2m$ integral unimodular matrix. For a contradiction, we compute the diagonal coefficient of ${}^tQL'_{K_f}Q$:

the i-th diagonal coefficient of ${}^tQL'_{K_f}Q$

$$\begin{aligned} &= \sum_{k=1}^{m}\Big(\sum_{j=1,j\neq k}^{m} q_{j,i} - q_{m+k,i}\Big)q_{k,i} - \sum_{k=m+1}^{2m} q_{k-m,i}q_{k,i} \\ &= \sum_{k=1}^{m}\Big(\sum_{j=1,j\neq k}^{m} q_{j,i} - 2q_{m+k,i}\Big)q_{k,i} \\ &= \sum_{k=1}^{m}\Big(\sum_{j=1,j\neq k}^{m} q_{j,i}\Big)q_{k,i} - 2\sum_{k=1}^{m} q_{m+k,i}q_{k,i} \\ &= \sum_{k=1}^{m}\Big(\sum_{j=1}^{m} q_{j,i} - q_{k,i}\Big)q_{k,i} - 2\sum_{k=1}^{m} q_{m+k,i}q_{k,i}. \end{aligned}$$

Let N be the cardinal of $\{k \mid q_{k,i} \text{ is odd}\}$. If N is an even integer, $(\sum_{j=1}^{m} q_{j,i})\sum_{k=1}^{m} q_{k,i}$ and $\sum_{k=1}^{m} q_{k,i}^2$ are even integers. If N is an odd integer, $(\sum_{j=1}^{m} q_{j,i})\sum_{k=1}^{m} q_{k,i}$ and $\sum_{k=1}^{m} q_{k,i}^2$ are odd integers. Thus each diagonal component of ${}^tQL'_{K_f}Q$ is an even integer. This means that the diagonal coefficients are even for any choice of basis of $H_1(F;\mathbb{Z})$. If a Hopf band can be deplumbed from the fiber surface of K_f, the Seifert form of K_f is represented by

$$\left(\begin{array}{ccc|c} a_{1,1} & \cdots & a_{1,2m-1} & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{2m-1,1} & \cdots & a_{2m-1,2m-1} & 0 \\ \hline a_{2m,1} & \cdots & a_{2m,2m-1} & \pm 1 \end{array}\right).$$

Since the $(2m, 2m)$-component of the Seifert form of K_f is even, it is a contradiction. Hence the fiber surface of K_f cannot admit a deplumbing of a Hopf band. Q.E.D.

§4. Enhancements to the Milnor numbers

We will show Theorem 2 by studying *the enhancement to the Milnor number*. Let K be a fibered link in S^3. We introduce the definition of the enhancement to the Milnor number $\lambda(K)$. To define the enhanced Milnor number, we first construct a nowhere zero vector field $\xi(K)$ on S^3. On $E(K) = S^3 \setminus \mathrm{Int}N(K)$, $\xi(K)$ is a transverse field to the fiber surfaces of the fibration, in the same direction of the monodromy of the fibration; on K, $\xi(K)$ is the tangent field of K; on the rest of $N(K)$, $\xi(K)$ can be taken as $r(\partial/\partial\theta) + (1-r^2)(\partial/\partial\phi)$ on $N(K)$, where (r, θ) are the coordinates of the meridian disk of $N(K) \cong D^2 \times S^1$ and ϕ is the coordinate of the longitude of $N(K)$. The homotopy class of $\xi(K)$ only depends on K.

Next we set ψ to be a vector field which is homotopic to the field of tangent vectors to the fibers of the Hopf fibration, and we define two subsets $\Delta^+(K)$ and $\Delta^-(K)$ in S^3 by

$$\Delta^{\pm}(K) := \{x \in S^3 \mid \psi(x) = \pm t\, \xi(K)(x) \;\text{ for some } t > 0\}.$$

If $\xi(K)$ and ψ are nowhere-zero vector fields on S^3 in general position, $\Delta^{\pm}(K)$ are compact oriented 1-manifolds in S^3. Since $\Delta^{\pm}(K)$ are disjoint, we can consider their linking number $\mathrm{link}(\Delta^+(K), \Delta^-(K))$. It is called the *enhancement* to the Milnor number and denote it by

$$\lambda(K) := \mathrm{link}(\Delta^+(K), \Delta^-(K)) \in \mathbb{Z}.$$

If the fiber surface F is obtained from another fiber surface by a plumbing of a Hopf band, the enhancements of two fibered links have the following relation.

Theorem 4 ([10]). *Let F, F_1 and F_2 be the fiber surfaces of fibered links K, K_1 and K_2 respectively. If F_1 (resp. F_2) is obtained from F by* *plumbing a positive (resp. negative) Hopf band, then*

$$\lambda(K_1) = \lambda(K),$$
$$\lambda(K_2) = \lambda(K) + 1.$$

Corollary 2. *If K is a fibered link whose fiber surface is a Hopf plumbing, then $\lambda(K)$ is a non-negative integer.*

If $f(0,\ldots,0,z_j,0,\ldots,0)$ is non-zero for each $j = 1,\ldots,n$, then we say that $f(\mathbf{z},\bar{\mathbf{z}})$ is *convenient*. In [6], the author studied the enhancements to the Milnor numbers of convenient polar weighted homogeneous mixed polynomials for 2 variables.

Theorem 5 ([6]). *Let $f(\mathbf{z},\bar{\mathbf{z}})$ be a convenient polar weighted homogeneous mixed polynomial for 2 variables with an isolated singularity at the origin. Then the enhancement to the Milnor number $\lambda(K_f)$ is $(-pqm_- + p + q)m_-$, where m_- is the number of link components of K_f whose orientations are opposite to the S^1-action and (p,q) is the polar weights of $f(\mathbf{z},\bar{\mathbf{z}})$.*

We calculate the enhancement to the Milnor number of K which is determined by the polar weighted homogeneous mixed polynomial $f(\mathbf{z},\bar{\mathbf{z}})$ to prove Theorem 2.

Proof of Theorem 2. We set mixed polynomials $f(\mathbf{z},\bar{\mathbf{z}})$ as

$$f(\mathbf{z},\bar{\mathbf{z}}) := \prod_{j=1}^{k+1}(z_1 + \alpha_j z_2^2) \prod_{j=k+2}^{2k+1} \overline{(z_1 + \alpha_j z_2^2)},$$

where $\alpha_j \neq \alpha_{j'}$ $(j \neq j')$ and define the S^1-action on the 3-sphere

$$s \circ (z_1, z_2) = (s^2 z_1, s z_2), \quad s \in S^1.$$

Then $f(\mathbf{z},\bar{\mathbf{z}})$ is a convenient polar weighted homogeneous polynomial. By using the splice diagram of K_f [2] and Theorem 5, we can show that the genus of the fiber surface of K_f is equal to k and the enhancement $\lambda(K_f)$ is equal to $(-2k+3)k$. If $k \geq 2$, $\lambda(K_f)$ is a negative integer. Thus the fiber surface of K_f cannot be obtained from a disk by plumbings of Hopf bands.

We consider the case of genus 1. Set

$$g(\mathbf{z},\bar{\mathbf{z}}) = z_1(z_1^3 + z_2^5)\overline{(z_1^3 - z_2^5)z_2}.$$

We can easily check that $g(\mathbf{z},\bar{\mathbf{z}})$ is a polar weighted homogeneous mixed polynomial with the polar weights $(5,3)$ and the genus of the fiber surface of the fibered link K_g is 1. $g(\mathbf{z},\bar{\mathbf{z}})$ is not convenient, but the enhancement

$\lambda(K_g)$ can be calculated as the same way in [6]. The result is $\lambda(K_g) = -12$. Thus the fiber surface of K_g cannot be obtained from a disk by plumbings of Hopf bands. Q.E.D.

Example 1. *Consider the following mixed polynomial:*

$$f(\mathbf{z}, \bar{\mathbf{z}}) = (z_1 + z_2^2)(z_1 + 2z_2^2)(z_1 + 3z_2^2)\overline{(z_1 + 4z_2^2)(z_1 + 5z_2^2)}.$$

Then the genus of the fiber surface of K_f and the enhancement $\lambda(K_f)$ to the Milnor number of K_f are equal to 2 and -2 respectively. Thus the fiber surface of K_f cannot be obtained from a disk by plumbing Hopf bands.

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