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Mirror symmetry between orbifold projective lines and cusp singularities

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Abstract.

We report on our recent study on the mirror symmetry between orbifold projective lines and cusp singularities. Both of their homological mirror symmetry, an equivalence of triangulated categories, and their classical mirror symmetry, an isomorphism of Frobenius manifolds, are discussed.

§1. Introduction

Let A be a triple of positive integers (a_1, a_2, a_3). We can naturally associate two mathematical objects from completely different geometric origins; an orbifold projective line $\mathbb{P}^1_A := \mathbb{P}^1_{a_1,a_2,a_3}$, an orbifold $\mathbb{P}^1$ with at most three isotropic points of orders a_1, a_2, a_3, and a polynomial $f_A(x_1, x_2, x_3) := x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - q^{-1}x_1x_2x_3$, $q \in \mathbb{C}\backslash\{0\}$. At first glance, $\mathbb{P}^1_A$ and f_A are unrelated. However, it turns out by mirror symmetry that they can be considered as two different geometric realizations of more intrinsic objects behind them.

Mirror symmetry is a categorical duality between algebraic geometry and symplectic geometry. One of our motivations is to apply some ideas of mirror symmetry to singularity theory in order to understand various mysterious correspondences among isolated singularities, root systems, Weyl groups, Lie algebras, discrete groups, finite dimensional algebras and so on. In this paper, we explain some of these correspondences by taking the case of the mirror symmetry between $\mathbb{P}^1_A$ and f_A.

After preparing some necessary notations in Section 2, we discuss in Section 3 the homological mirror symmetry between $\mathbb{P}^1_A$ and f_A, an

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equivalence of triangulated categories associated to $\mathbb{P}^1_A$ and f_A. In Section 4, we study the classical mirror symmetry between $\mathbb{P}^1_A$ and f_A, an isomorphism of Frobenius manifolds associated to Gromov–Witten theory for $\mathbb{P}^1_A$ and the deformation theory of f_A.

§2. Notations and useful elementary facts

Throughout this paper, we denote by A a triple of positive integers (a_1, a_2, a_3). For the later use, we set

$$\mu_A := 2 + \sum_{i=1}^{3} (a_i - 1)\,, \tag{1}$$

$$\chi_A := 2 + \sum_{i=1}^{3} \left(\frac{1}{a_i} - 1\right). \tag{2}$$

The numbers μ_A and χ_A will appear, for example, as the orbifold Euler number of $\mathbb{P}^1_A$, the total dimension of orbifold cohomology groups of $\mathbb{P}^1_A$, and the orbifold Euler characteristics of $\mathbb{P}^1_A$, respectively. Note that one has

$$\mu_A = \begin{cases} \dim_{\mathbb{C}} \mathbb{C}[x_1, x_2, x_3] \Big/ \left(\frac{\partial f_A}{\partial x_1}, \frac{\partial f_A}{\partial x_2}, \frac{\partial f_A}{\partial x_3}\right) & \text{if} \quad \chi_A > 0, \\ \dim_{\mathbb{C}} \mathbb{C}[[x_1, x_2, x_3]] \Big/ \left(\frac{\partial f_A}{\partial x_1}, \frac{\partial f_A}{\partial x_2}, \frac{\partial f_A}{\partial x_3}\right) & \text{if} \quad \chi_A \leq 0. \end{cases} \tag{3}$$

In the case $\chi_A > 0$ we regard the polynomial f_A as a globally defined function on the affine variety $\mathbb{C}^3$ with several critical points, whereas in the case $\chi_A \leq 0$, we regard it as a germ of holomorphic function on a suitable small neighborhood of the origin of $\mathbb{C}^3$ defining an isolated singularity only at the origin. Therefore, μ_A can be interpreted as the Milnor number for f_A.

Consider the classification of A with $\chi_A > 0$ and $\chi_A = 0$. It is obviously given by Table 1 and Table 2, respectively. If $\chi_A \geq 0$, we shall mean by *type* of A the corresponding name given in the "Type" row of Table 1 and Table 2. It is a name of the weighted homogeneous singularity (and its root system) related by Orlov's semi-orthogonal decomposition of the triangulated category associated to A discussed in the next section.

$A = (a_1, a_2, a_3)$	Type
$(p, q, 1)$	$A_{p,q}$ $(p, q \geq 1)$
$(r, 2, 2)$	D_{r+2} $(r \geq 2)$
$(3, 3, 2)$	E_6
$(4, 3, 2)$	E_7
$(5, 3, 2)$	E_8

Table 1. The classification of A with $\chi_A > 0$

$A = (a_1, a_2, a_3)$	Type
$(3, 3, 3)$	$E_6^{(1,1)}$
$(4, 4, 2)$	$E_7^{(1,1)}$
$(6, 3, 2)$	$E_8^{(1,1)}$

Table 2. The classification of A with $\chi_A = 0$

§3. Homological mirror symmetry

We associate two triangulated categories to A. On the algebraic geometry side, the triangulated category to consider is the derived category $D^b\mathrm{coh}(\mathbb{P}^1_A)$ of bounded complexes of coherent sheaves on $\mathbb{P}^1_A$. On the symplectic geometry side, we regard f_A as a globally defined tame polynomial on $\mathbb{C}^3$ if $\chi_A > 0$ and as a germ of a holomorphic function defined on a small neighborhood of $0 \in \mathbb{C}^3$ if $\chi_A \leq 0$ and then consider the derived category $D^b\mathrm{Fuk}^{\to}(f_A)$ of the directed Fukaya category $\mathrm{Fuk}^{\to}(f_A)$. Here, $\mathrm{Fuk}^{\to}(f_A)$ is a directed A_∞-category which can be thought of as a "categorification" of a distinguished basis of vanishing cycles in the Milnor fiber of f_A. In this paper, we omit the details about Fukaya categories and refer the interested reader to [28], [29] for Fukaya categories associated to singularities and to [12] for generality. For the convenience of the reader, we give a rough definition of $\mathrm{Fuk}^{\to}(f_A)$:

Definition 3.1. *The directed Fukaya category* $\mathrm{Fuk}^{\to}(f_A)$ *is a strictly unital* A_∞*-category consists of*

- μ_A *vanishing graded Lagrangian submanifolds* $\mathcal{L}_1, \ldots, \mathcal{L}_{\mu_A}$ *in the Milnor fiber of* f_A *together with an ordering of these objects as* $(\mathcal{L}_1, \ldots, \mathcal{L}_{\mu_A})$ *such that*

$$\mathrm{Fuk}^{\to}(f_A)(\mathcal{L}_i, \mathcal{L}_j) = \begin{cases} 0 & \text{if } i > j, \\ \mathbb{C} \cdot \mathrm{id}_{\mathcal{L}_i} & \text{if } i = j, \\ \bigoplus_{p \in \mathcal{L}_i \cap \mathcal{L}_j} \mathbb{C}[\deg(p)] & \text{if } i < j, \end{cases} \tag{4}$$

where $[-]$ *denotes the translation of the complex,* $\deg(p)$ *is defined by the gradings* $\mathrm{gr}_{\mathcal{L}_i} : \mathcal{L}_i \longrightarrow \mathbb{R}$ *and* $\mathrm{gr}_{\mathcal{L}_j} : \mathcal{L}_j \longrightarrow \mathbb{R}$ *as the largest integer less than or equal to* $\mathrm{gr}(\mathcal{L}_j)|_p - \mathrm{gr}(\mathcal{L}_i)|_p$*,*

- *the (non-trivial) composition maps*

$$\begin{aligned}(5)\quad \mathfrak{m}_A^n : \mathrm{Fuk}^{\rightarrow}(f_A)(\mathcal{L}_{i_{n-1}}, \mathcal{L}_{i_n}) \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathrm{Fuk}^{\rightarrow}(f_A)(\mathcal{L}_{i_1}, \mathcal{L}_{i_2}) \\ \longrightarrow \mathrm{Fuk}^{\rightarrow}(f_A)(\mathcal{L}_{i_1}, \mathcal{L}_{i_n})[2-n], \quad i_1 < \cdots < i_n,\end{aligned}$$

defined by the "numbers of pseudo-holomorphic polygons" with boundaries on $\mathcal{L}_1, \ldots, \mathcal{L}_{\mu_A}$ *and corners on intersection points.*

Although $\mathrm{Fuk}^{\rightarrow}(f_A)$ depends on many choices other than the initial data f_A, it turns out that after taking the derived category it will be an invariant of the polynomial f_A as a triangulated category.

3.1. Homological mirror symmetry conjecture

In order to formulate our homological mirror symmetry conjecture, we recall some terminologies for a quiver and its path algebra.

Definition 3.2. *A quiver* Q *is an oriented graph. More precisely, a quiver is a quadruple* $(Q_0, Q_1; s, t)$ *where* Q_0 *is a set called the set of vertices,* Q_1 *is a set called the set of arrows and* s, t *are maps from* Q_1 *to* Q_0 *which associate the source vertex and the target vertex for each arrow. An arrow with the source* $s(a)$ *and the target* $t(a)$ *is often written as* $s(a) \xrightarrow{a} t(a)$*.*

Definition 3.3. *A path of length* $l \geq 1$ *from the vertex* v *to the vertex* v' *in a quiver* Q *is a symbol* $(v|\alpha_1 \cdots \alpha_l|v')$ *with arrows* α_i*,* $i = 1, \ldots, l$ *such that* $s(v_1) = v$*,* $t(v_l) = v'$ *and* $s(\alpha_{i+1}) = t(\alpha_i)$*,* $i = 1, \ldots, l-1$*. A path of length* 0 *is a symbol* $(v|v)$ *defined for each vertex* $v \in Q_0$*. For a path* $p = (v|\alpha_1 \cdots \alpha_l|v')$*, set* $s(p) := v$ *and* $t(p) := v'$*. An ordered pair of paths* (p_1, p_2) *is composable if* $t(p_1) = s(p_2)$*. The composition of composable paths* $((v_1|\alpha_1 \cdots \alpha_l|v_1'), (v_2|\beta_1 \cdots \beta_m|v_2'))$ *is a path* $(v_1|\alpha_1 \cdots \alpha_l \beta_1 \cdots \beta_m|v_2')$*.*

The path algebra $\mathbb{C}Q$ *of a quiver* Q *is then defined as the* $\mathbb{C}$*-vector space generated by all paths in* Q *together with the associative product structure defined by the composition of paths, where the product of two non-composable paths is set to be zero. A quiver with relations is a pair* (Q, I) *where* Q *is a quiver and* I *is an ideal of* $\mathbb{C}Q$*.*

The homological mirror symmetry conjecture between $\mathbb{P}^1_A$ and f_A can be formulated as follows:

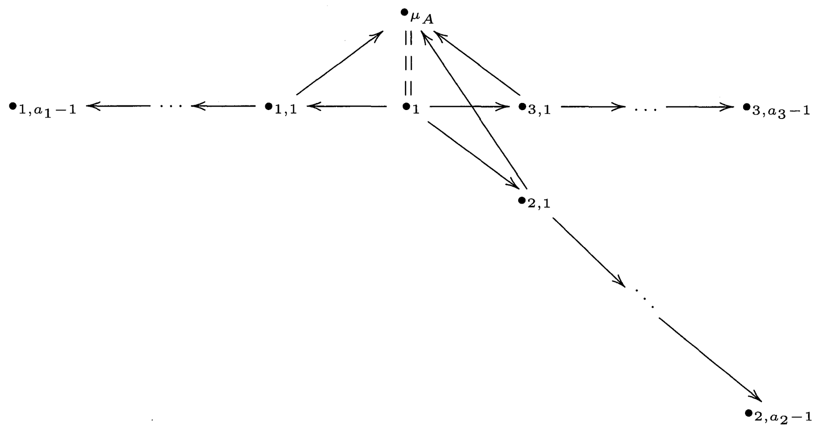


Fig. 1. The quiver with relations (Q_A, I). The double dotted line denotes two linear combinations $p_1 + p_3$ and $p_2 + p_3$ of three paths p_1, p_2, p_3 from $\bullet_1$ to $\bullet_{\mu_A}$, which generate the ideal I.

Conjecture 1 (cf. [11], [32]). *There should exist triangulated equivalences*

$$D^b\mathrm{coh}(\mathbb{P}^1_A) \simeq D^b(\text{mod-}\mathbb{C}Q_A/I) \simeq D^b\mathrm{Fuk}^{\to}(f_A), \tag{6}$$

where (Q_A, I) is the quiver with relations given in Fig. 1.

The first equivalence $D^b\mathrm{coh}(\mathbb{P}^1_A) \simeq D^b(\text{mod-}\mathbb{C}Q_A/I)$ is given by Geigle–Lenzing [16] more than twenty years ago. The author observed that $K_0(D^b\mathrm{coh}(\mathbb{P}^1_A))$ together with the symmetrized Euler form is isomorphic as a lattice to the Milnor lattice of f_A computed by Ebeling [10] and Gabrielov [15], which leads him to the above conjecture.

If $a_i = 1$ for some $i = 1, 2, 3$, Conjecture 1 has been already known in different contexts to some experts including Auroux–Katzarkov–Orlov [5], Seidel [28], van Straten [31] and Ueda. Also the cases when $\chi_A = 0$, which correspond to simple elliptic hypersurface singularities, are known (cf. [5], [13], [33]).

3.2. Our result

The following theorem was first stated by the author in 2008. Together with the above known results, it turns out that Conjecture 1 holds for $\chi_A \geq 0$.

Theorem 2. *If $a_3 = 2$, we have a triangulated equivalence*

$$D^b\mathrm{Fuk}^{\to}(f_A) \simeq D^b(\text{mod-}\mathbb{C}Q_A/I). \tag{7}$$

The author's contribution to this theorem is the idea to relate the polynomial f_A with the quiver with relations (Q_A, I). He learned from Seidel that the stable equivalence of the Fukaya categories (Lemma 3) follows from results in Section 18 of his book [29]. It is also known by Seidel [28] how to calculate Fukaya categories for curve singularities based on A'Campo's divide [3], [4] (see Proof of Lemma 4 below).

Now, we shall give a proof of Theorem 2.

Proof. For simplicity, we set $q = 1$ here since $D^b\mathrm{Fuk}^{\to}(g_A)$ does not depend on q. Consider the reduction of surface singularities to curve singularities.

Lemma 3. *Assume that $a_3 = 2$. Then, we have a triangulated equivalence*

$$D^b\mathrm{Fuk}^{\to}(f_A) \simeq D^b\mathrm{Fuk}^{\to}(g_A), \tag{8}$$

where $g_A := g_A(x_1, x_2) := x_1^{a_1} + x_2^{a_2} - \frac{1}{4}x_1^2x_2^2$.

Proof. Note that $f_A = g_A + (x_3 - \frac{1}{2}x_1x_2)^2$. The statement follows from the stable equivalence of Fukaya categories. See Theorem 2.1 of [13] and Section 18 of [29] for example. Q.E.D.

The problem is reduced to calculate the Fukaya categories for curve singularities. The following statement holds:

Lemma 4. *Let f' be a curve singularity and μ be its Milnor number. There exists a distinguished basis of vanishing graded Lagrangian submanifolds $\mathcal{L}_1, \ldots, \mathcal{L}_\mu$ in the Milnor fiber of g such that $\mathrm{Fuk}^{\to}(f')(\mathcal{L}_i, \mathcal{L}_j)$ is at most one dimensional complex concentrated in degree 0.*

Proof. See Conjecture 8.1 of [28] proven for $n = 1$ which states an equivalence between $\mathrm{Fuk}^{\to}(f')$ and the Morse category associated to f', Section $(7A)$ and the first half of Section $(7B)$ of [28] on the description of the Morse category for $n = 1$. We shall explain how to calculate morphisms and compositions more in detail after the next Corollary. Q.E.D.

This yields that the derived category $D^b\mathrm{Fuk}^{\to}(f')$ of $\mathrm{Fuk}^{\to}(f')$ is described by a finite dimensional algebra.

Corollary 5. *There exists a quiver Q and relations I such that*

$$D^b\mathrm{Fuk}^{\to}(f') \simeq D^b(\text{mod-}\mathbb{C}Q/I). \tag{9}$$

Proof. It is easy to see that $\mathcal{L}_1, \dots, \mathcal{L}_\mu$ forms a strongly exceptional collection of $D^b\mathrm{Fuk}^{\rightarrow}(f')$. Then, the functor $D^b\mathrm{Fuk}^{\rightarrow}(f')(\oplus_{i=1}^{\mu}\mathcal{L}_i, -)$ gives the triangulated equivalence between $D^b\mathrm{Fuk}^{\rightarrow}(f')$ and $D^b(\text{mod-}E)$ where E is the endomorphism algebra $\oplus_{i,j=1}^{\mu}\mathrm{Fuk}^{\rightarrow}(f')(\mathcal{L}_i, \mathcal{L}_j)$. Since E is a basic finite dimensional $\mathbb{C}$-algebra, by Gabriel's theorem (see [14]), one has a unique quiver Q such that for some relations I the $\mathbb{C}$-algebra $\mathbb{C}Q/I \simeq E$. Q.E.D.

Here, we give a general recipe to construct a quiver with relations from an A'Campo's divide [3], [4], which can be obtained by setting $n = 1$ in Section $(7A)$ of [28].

The quiver Q and relations I in Corollary 5 can be described as follows:

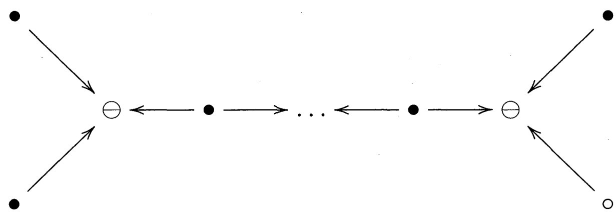
(i) Choose a real Morsification h of f'.
(ii) Draw a picture of $h^{-1}(0)$ in $\mathbb{R}^2$.
(iii) Put a vertex $\bullet$ to each ordinary double point.
(iv) Put a vertex with a sign $\oplus$ ($\ominus$) into each compact connected component of $\mathbb{R}^2 \backslash h^{-1}(0)$ if h is positive (resp. negative) on the component.
(v) Draw 1 arrow $\rightarrow$ from $\oplus$ to $\bullet$ (from $\bullet$ to $\ominus$) if $\bullet$ is on the boundary of the component for $\oplus$ (resp. $\ominus$).
(vi) Draw 1 dotted line from $\oplus$ to $\ominus$ if there are 2 paths from $\oplus$ to $\ominus$, which means a commutative relation between them.

Note that the pair (Q, I) depends on the choice of a real Morsification h of f'. However, it is known that the derived category $D^b(\text{mod-}\mathbb{C}Q/I)$, as a triangulated category, becomes an invariant of g. Indeed, two different choices of pairs (Q_1, I_1) and (Q_2, I_2) are connected by a sequence of mutations, the braid group action on the set of distinguished basis of vanishing cycles, which gives the desired triangulated equivalence. See the first half of Section $(7B)$ of [28] for a typical example of this mutation.

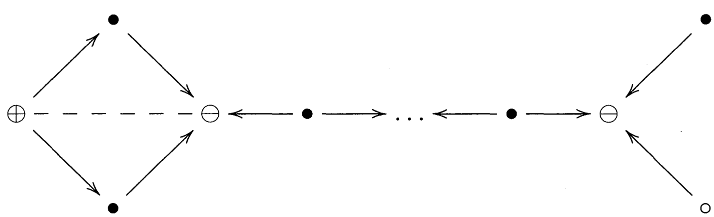
Now, we give a quiver with relations (Q'_A, I') for g_A. First, we consider the case when $\chi_A > 0$. After applying suitable mutations to (Q'_A, I'), we obtain the *extended Dynkin quiver* (the extended Dynkin diagram with an arbitrary orientation) of type A (cf. Fig. 28 and Fig. 30 in [1]). In the pictures below, $\circ$ denotes the vertex to remove in order to get the finite Dynkin quiver of type A. It is also known by [16] that $D^b\mathrm{coh}(\mathbb{P}^1_A)$ is equivalent to the derived category of the extended Dynkin quiver of type A.

- $g_A = x_1^{2k} + x_2^2 - \frac{1}{4}x_1^2x_2^2$,
  $h_A := (1 - \frac{1}{4}x_1^2)(x_2 - 2\prod_{i=1}^{k-1}(x_1 - c_i))(x_2 + 2\prod_{i=1}^{k-1}(x_1 - c_i))$,
  $-2 < c_1 < \cdots < c_{k-1} < 2$,

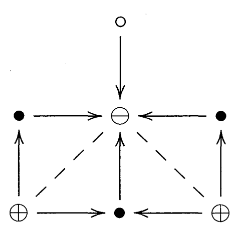
$(\widetilde{D}_{2k+2})$:



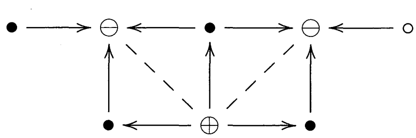
- $g_A = x_1^{2k+1} + x_2^2 - \frac{1}{4}x_1^2x_2^2$,
  $h_A := (1 - \frac{1}{4}x_1^2)(x_2^2 - 4(x_1 - c_0)\prod_{i=1}^{k-1}(x_1 - c_i)^2)$,
  $c_0 < -2 < c_1 < \cdots < c_{k-1} < 2$,
  $(\widetilde{D}_{2k+3})$:



- $g_A = x_1^3 + x_2^3 - \frac{1}{4}x_1^2x_2^2$, $h_A := (x_2 - \frac{1}{4}x_1^2 + 8)(x_2^2 - 4(x_1 + 8))$,
  $(\widetilde{E}_6)$:



- $g_A = x_1^4 + x_2^3 - \frac{1}{4}x_1^2x_2^2$, $h_A := (x_2 - \frac{1}{4}x_1^2 + 4)(x_2^2 - 4x_1^2)$,
  $(\widetilde{E}_7)$:



- $g_A = x_1^5 + x_2^3 - \frac{1}{4}x_1^2 x_2^2$, $h_A := (x_2 - \frac{1}{4}x_1^2 + 2)(x_2^2 - 4x_1^2(x_1 + 2))$, $(\widetilde{E}_8)$:

$$\begin{tikzcd}
\bullet \arrow[r] & \ominus & \bullet \arrow[l] \arrow[r] & \ominus & \circ \arrow[l] \\
\oplus \arrow[u] \arrow[r] \arrow[ur, dash, dashed] & \bullet \arrow[u] & \oplus \arrow[l] \arrow[u] \arrow[r] \arrow[ul, dash, dashed] \arrow[ur, dash, dashed] & \bullet \arrow[u] &
\end{tikzcd}$$

Finally, we give an example of a quiver with relations (Q'_A, I') for g_A with $\chi_A \leq 0$. Note that the number of vertices, the Milnor number, is given by $\mu_A = a_1 + a_2 + a_3 - 1$.

- $f_A = x_1^{a_1} + x_2^3 + x_3^2 - x_1 x_2 x_3$ $(a_1 \geq 6)$, $\mu_A \geq 10$:

$$\begin{tikzcd}
\bullet \arrow[r] & \ominus & \bullet \arrow[l] \arrow[r] & \ominus & \bullet \arrow[l] & & & \\
\oplus \arrow[u] \arrow[r] \arrow[ur, dash, dashed] & \bullet \arrow[u] & \oplus \arrow[l] \arrow[u] \arrow[r] \arrow[ul, dash, dashed] \arrow[ur, dash, dashed] & \bullet \arrow[u] & \oplus \arrow[l] \arrow[u] \arrow[r] \arrow[ul, dash, dashed] & \bullet & \oplus \arrow[l] \arrow[r] & \cdots
\end{tikzcd}$$

- $f_A = x_1^{a_1} + x_2^{a_2} + x_3^2 - x_1 x_2 x_3$ $(a_1, a_2 \geq 4)$, $\mu_A \geq 9$:

$$\begin{tikzcd}
 & & \oplus \arrow[r] \arrow[d] \arrow[dr, dash, dashed] & \bullet \arrow[d] & \oplus \arrow[l] \arrow[d] \arrow[dl, dash, dashed] & & \\
 & & \bullet \arrow[r] & \ominus & \bullet \arrow[l] & & \\
\cdots \arrow[r] & \bullet & \oplus \arrow[l] \arrow[r] \arrow[u] \arrow[ur, dash, dashed] & \bullet \arrow[u] & \oplus \arrow[l] \arrow[r] \arrow[u] \arrow[ul, dash, dashed] & \bullet & \cdots \arrow[l]
\end{tikzcd}$$

By applying suitable mutations, one can show that the derived category $D^b(\text{mod-}\mathbb{C}Q'_A/I')$ is equivalent to $D^b(\text{mod-}\mathbb{C}Q_A/I)$. This finishes the proof of Theorem 2. Q.E.D.

§4. Classical mirror symmetry

In the previous section, we compared triangulated categories. Here, we study the mirror symmetry as an isomorphism between the Frobenius manifold from the Gromov–Witten invariants of a variety and that from the deformation theory of another variety (in our situation, a singularity). It is interesting to study this isomorphism since it provides quite important information such as the number of curves in the variety in terms of the periods for the another.

4.1. Frobenius manifolds

Recall the notion of Frobenius manifolds. The definition below is taken from [24].

Definition 4.1. *Let $M = (M, \mathcal{O}_M)$ be a connected complex manifold or a formal manifold over $\mathbb{C}$ of dimension μ whose holomorphic tangent sheaf and cotangent sheaf are denoted by $\mathcal{T}_M, \Omega^1_M$ respectively and let d be a complex number. A Frobenius structure of rank μ and dimension d is a tuple $(\eta, \circ, e, E)$, where η is a non-degenerate $\mathcal{O}_M$-symmetric bilinear form on $\mathcal{T}_M$, $\circ$ is an $\mathcal{O}_M$-bilinear product on $\mathcal{T}_M$, defining an associative and commutative $\mathcal{O}_M$-algebra structure with the unit e, and E is a holomorphic vector field on M, called the Euler vector field, which are subject to the following axioms:*

(i) *The product $\circ$ is self-ajoint with respect to η : that is,*

$$\eta(\delta \circ \delta', \delta'') = \eta(\delta, \delta' \circ \delta''), \quad \delta, \delta', \delta'' \in \mathcal{T}_M. \tag{10}$$

(ii) *The* Levi–Civita *connection $\nabla : \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \to \mathcal{T}_M$ with respect to η is flat: that is,*

$$[\nabla_\delta, \nabla_{\delta'}] = \nabla_{[\delta, \delta']}, \quad \delta, \delta' \in \mathcal{T}_M. \tag{11}$$

(iii) *The tensor $C : \mathcal{T}_M \otimes_{\mathcal{O}_M} \mathcal{T}_M \to \mathcal{T}_M$ defined by $C_\delta \delta' := \delta \circ \delta'$, $(\delta, \delta' \in \mathcal{T}_M)$ is flat: that is,*

$$\nabla C = 0. \tag{12}$$

(iv) *The unit element e of the $\circ$-algebra is a ∇-flat homolophic vector field: that is,*

$$\nabla e = 0. \tag{13}$$

(v) *The metric η and the product $\circ$ are homogeneous of degree $2 - d$ $(d \in \mathbb{C})$, 1 respectively with respect to Lie derivative Lie_E of* Euler *vector field E : that is,*

$$Lie_E(\eta) = (2 - d)\eta, \quad Lie_E(\circ) = \circ. \tag{14}$$

Consider the space of horizontal sections of the connection ∇:

$$\mathcal{T}^f_M := \{\delta \in \mathcal{T}_M \mid \nabla_{\delta'} \delta = 0 \text{ for all } \delta' \in \mathcal{T}_M\} \tag{15}$$

which is, due to (11), a local system of rank μ on M such that the metric η takes constant value on $\mathcal{T}^f_M$. Namely, we have

$$\eta(\delta, \delta') \in \mathbb{C}, \quad \delta, \delta' \in \mathcal{T}^f_M. \tag{16}$$

Proposition 6. *At each point of M, there exist a local coordinate $(t_1, \ldots, t_\mu)$, called flat coordinates, such that $e = \partial_1$, $\mathcal{T}_M^f$ is spanned by $\partial_1, \ldots, \partial_\mu$ and $\eta(\partial_i, \partial_j) \in \mathbb{C}$ for all $i, j = 1, \ldots, \mu$, where we denote $\partial/\partial t_i$ by ∂_i.*

The axiom $\nabla C = 0$, implies the following:

Proposition 7. *At each point of M, there exist the local holomorphic function $\mathcal{F}$, called Frobenius potential, satisfying*

$$\eta(\partial_i \circ \partial_j, \partial_k) = \eta(\partial_i, \partial_j \circ \partial_k) = \partial_i \partial_j \partial_k \mathcal{F}, \quad i, j, k = 1, \ldots, \mu, \tag{17}$$

for any system of flat coordinates. In particular, one has

$$\eta_{ij} := \eta(\partial_i, \partial_j) = \partial_1 \partial_i \partial_j \mathcal{F}. \tag{18}$$

The associativity of the product $\circ$ implies the following *Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations*:

Proposition 8. *One has*

$$\sum_{a,b=1}^{\mu} \partial_i \partial_j \partial_a \mathcal{F} \cdot \eta^{ab} \cdot \partial_b \partial_k \partial_l \mathcal{F} = \sum_{a,b=1}^{\mu} \partial_i \partial_k \partial_a \mathcal{F} \cdot \eta^{ab} \cdot \partial_b \partial_j \partial_l \mathcal{F}, \quad i, j, k, l = 1, \ldots, \mu, \tag{19}$$

where $(\eta^{ab}) := (\eta_{ab})^{-1}$.

Recall the intersection form of Frobenius manifold (cf. Lecture 3 of [8]).

Definition 4.2. *The intersection form of the Frobenius manifold is a symmetric $\mathcal{O}_M$-bilinear form I on the cotangent sheaf Ω_M^1 defined by the formula*

$$I(dt^i, dt^j) := \sum_{k,l=1}^{\mu} \eta^{ik} \eta^{jl} E(\partial_k \partial_l \mathcal{F}), \quad i, j = 1, \ldots, \mu. \tag{20}$$

4.2. Three constructions of Frobenius manifolds

There are essentially three constructions of Frobenius manifolds from completely different origins; the theory of Gromov–Witten invariants, the invariant theory of Weyl groups and the theory of primitive forms.

4.2.1. *Gromov–Witten Theory for $\mathbb{P}^1_A$ to Frobenius Manifolds* For $g \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 0}$ and $\beta \in H_2(\mathbb{P}^1_A, \mathbb{Z})$, the moduli space (stack) $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_A, \beta)$ of orbifold (twisted) stable maps of genus g with n-marked points of degree β is defined by Chen–Ruan [6] in symplectic geometry and later by Abramovich–Graber–Vistoli [2] in algebraic geometry. It is shown that there exists a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_A, \beta)]^{vir}$ and Gromov–Witten invariants of genus g with n-marked points of degree β are defined as

$$(21)\quad \langle \gamma_1, \ldots, \gamma_n \rangle^{\mathbb{P}^1_A}_{g,n,\beta} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_A,\beta)]^{vir}} ev_1^*\gamma_1 \wedge \ldots ev_n^*\gamma_n, \quad \gamma_1, \ldots, \gamma_n \in H^*_{orb}(\mathbb{P}^1_A, \mathbb{Q}),$$

where $ev_i^* : H^*_{orb}(\mathbb{P}^1_A, \mathbb{Q}) \longrightarrow H^*(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1_A, \beta), \mathbb{Q})$ denotes the induced homomorphism by the evaluation map at i-th marked point. Then, we consider the generating function

$$(22)\qquad \mathcal{F}_g^{\mathbb{P}^1_A} := \sum_{n,\beta} \frac{1}{n!} \langle \mathbf{t}, \ldots, \mathbf{t} \rangle^{\mathbb{P}^1_A}_{g,n,\beta}, \quad \mathbf{t} = \sum_{i=1}^{\mu_A} t_i \gamma_i$$

and call it the genus g potential where $\{\gamma_1, \ldots, \gamma_{\mu_A}\}$ are $\mathbb{Q}$-basis of $H^*_{orb}(\mathbb{P}^1_A, \mathbb{Q})$ and $\{t_1, \ldots, t_{\mu_A}\}$ are the dual coordinates of the $\mathbb{C}$-basis $\{\gamma_1, \ldots, \gamma_{\mu_A}\}$ of $H^*_{orb}(\mathbb{P}^1_A, \mathbb{C}) = H^*_{orb}(\mathbb{P}^1_A, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$.

The main result in [2], [6] is the associativity of the quantum product, which yields a Frobenius manifold.

Theorem 9 (Theorem 6.2.1 of [2], Theorem 3.4.3 of [6])**.** *One has the WDVV equation for $\mathcal{F}_0^{\mathbb{P}^1_A}$. In particular, there is a structure of a formal Frobenius manifold $M_{\mathbb{P}^1_A}$ of rank μ_A and dimension one where $M_{\mathbb{P}^1_A}$ is a formal manifold whose structure sheaf $\mathcal{O}_{M_{\mathbb{P}^1_A}}$ and tangent sheaf $\mathcal{T}_{M_{\mathbb{P}^1_A}}$ are defined as the algebra $\mathbb{C}((e^{t_{\mu_A}}))[[t_1, \ldots, t_{\mu_A - 1}]]$ and $\mathcal{T}_{M_{\mathbb{P}^1_A}} := H^*_{orb}(\mathbb{P}^1_A, \mathbb{C}) \otimes_{\mathbb{C}} \mathcal{O}_{M_{\mathbb{P}^1_A}}$ where t_{μ_A} is a dual coordinate corresponding to the one-dimensional subspace $H^2_{orb}(\mathbb{P}^1_A, \mathbb{Q})$.*

Remark 10. *By the divisor axiom of the Gromov–Witten invariants, the "quantum part" of the potential $\mathcal{F}_0^{\mathbb{P}^1_A}$*

$$\sum_{n,\beta\neq 0} \frac{1}{n!} \langle \mathbf{t}, \ldots, \mathbf{t} \rangle^{\mathbb{P}^1_A}_{0,n,\beta}$$

is an element of $\mathbb{C}[[t_1, \ldots, t_{\mu_A - 1}, e^{t_{\mu_A}}]]$. Note that this formal series may not converge and hence, in order to get a Frobenius manifold, we have to assume the convergence of it on some domain in $H^*_{orb}(\mathbb{P}^1_A, \mathbb{C})$.

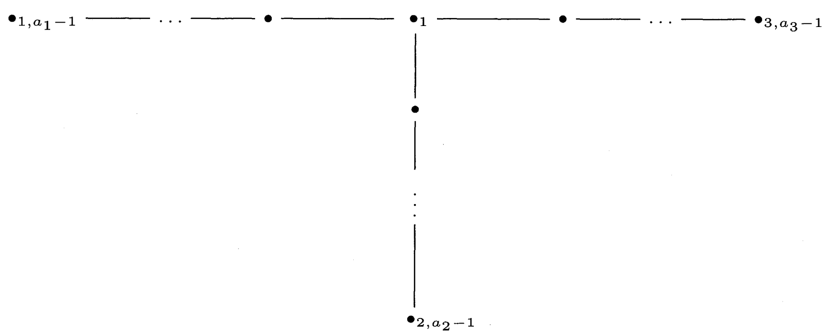


Fig. 2. The Coxeter–Dynkin diagram T_A

In this paper, for simplicity, instead of assuming the convergence we consider the formal Frobenius structure for the one constructed from Gromov–Witten invariants.

However, in our case, it turns out that the quantum product do converge (on some domain) due to the mirror isomorphism Corollary 19 since the Frobenius potential from the universal unfolding is holomorphic.

4.2.2. *Extended Weyl Groups $\widehat{W}_A$ to Frobenius Manifolds* Let T_A be the Coxeter–Dynkin diagram given in Fig. 2. Note that it can be embedded into a sub-diagram of the Coxeter–Dynkin diagram for the quiver with relations (Q_A, I) in Fig. 1.

Let $\mathfrak{h}_A$ be the complexified Cartan subalgebra associated to T_A. Denote by $\alpha_1, \dots, \alpha_{\mu_A-1} \in \mathfrak{h}_A^*$, $\mathfrak{h}_A^* := \mathrm{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$ the simple roots and by $\alpha_1^\vee, \dots, \alpha_{\mu_A-1}^\vee \in \mathfrak{h}_A$ the simple coroots. The Weyl group W_A is a group generated by the reflections

$$r_i(h) := h - \langle \alpha_i, h \rangle \alpha_i^\vee, \quad h \in \mathfrak{h}_A,\ i = 1, \dots, \mu_A - 1, \tag{23}$$

where $\langle , \rangle$ denotes the natural pairing $\langle , \rangle : \mathfrak{h}_A^* \otimes_{\mathbb{C}} \mathfrak{h}_A \longrightarrow \mathbb{C}$.

From now on, we assume that $\chi_A \neq 0$. Note that under the assumption $\chi_A \neq 0$ the Cartan matrix for T_A is non-degenerate. Set $\widehat{\mathfrak{h}}_A := \mathfrak{h}_A \times \mathbb{C}$. The affinization $\widetilde{W}_A$ of W_A acts on $\widehat{\mathfrak{h}}_A$ by

$$(h, x_{\mu_A}) \mapsto (w(h) + \sum_{i=1}^{\mu_A-1} m_i \alpha_i^\vee, x_{\mu_A}), \quad m_i \in \mathbb{Z}, \tag{24}$$

and $\mathbb{Z}$ acts on $\widehat{\mathfrak{h}}_A$ by

$$(h, x_{\mu_A}) \mapsto (h + m\omega_1^\vee, x_{\mu_A} - m), \quad m \in \mathbb{Z}, \tag{25}$$

where $\omega_1^\vee, \dots, \omega_{\mu_A-1}^\vee$ denotes the fundamental coweights, the elements of $\mathfrak{h}_A$ sarisfying $\langle \alpha_i, \omega_j^\vee \rangle = \delta_{ij}$ (where δ_{ij} is the Kronecker's delta).

It is important that $\widetilde{W}_A$ is isomorphic to the Weyl group $W_{(Q_A,I)}$ associated to the Coxeter–Dynkin diagram for the quiver with relations (Q_A, I), where $\langle \alpha_1, \alpha_{\mu_A}^\vee \rangle$, the entry of the Cartan matrix corresponding to the double dotted line, is $+2$.

Then, $\widehat{W}_A$ is defined as a group acting on $\widehat{\mathfrak{h}}_A$ generated by $\widetilde{W}_A$ and $\mathbb{Z}$ with the above actions on $\widehat{\mathfrak{h}}_A$. In particular, one has the following exact sequence

$$\{1\} \longrightarrow \widetilde{W}_A \longrightarrow \widehat{W}_A \longrightarrow \mathbb{Z} \longrightarrow \{1\}. \tag{26}$$

By the invariant theory of $\widehat{W}_A$, Dubrovin–Zhang [9] give the following:

Theorem 11 (Theorem 2.1 of [9]). *Assume that $\chi_A > 0$. There exists a unique structure of Frobenius manifold on $M_{\widehat{W}_A} := \widehat{\mathfrak{h}}_A/\widehat{W}_A$ of rank μ_A and dimension one with flat coordinates t_1, $t_{1,1}, \dots, t_{1,a_1-1}$, $t_{2,1}, \dots, t_{2,a_2-1}$, $t_{3,1}, \dots, t_{3,a_3-1}$, $t_{\mu_A} := (2\pi\sqrt{-1})x_{\mu_A}$ such that*

$$e = \frac{\partial}{\partial t_1}, \quad E = t_1 \frac{\partial}{\partial t_1} + \sum_{i=1}^{3} \sum_{j=1}^{a_i-1} \frac{a_i - j}{a_i} t_{i,j} \frac{\partial}{\partial t_{i,j}} + \chi_A \frac{\partial}{\partial t_{\mu_A}},$$

and the intersection form $I_{\widehat{W}_A}$ is given by

$$I_{\widehat{W}_A}(\alpha_i, \alpha_j) = \frac{-1}{(2\pi\sqrt{-1})^2} \langle \alpha_i, \alpha_j^\vee \rangle, \quad i, j = 1, \dots, \mu_A - 1, \tag{27}$$

$$I_{\widehat{W}_A}(\alpha_i, dx_{\mu_A}) = I_{\widehat{W}_A}(dx_{\mu_A}, \alpha_i) = 0, \quad i = 1, \dots, \mu_A - 1, \tag{28}$$

$$I_{\widehat{W}_A}(dx_{\mu_A}, dx_{\mu_A}) = \frac{1}{(2\pi\sqrt{-1})^2} \chi_A, \tag{29}$$

where we identify the cotangent space of $M_{\widehat{W}_A}$ with $\mathfrak{h}_A^* \oplus \mathbb{C}dx_{\mu_A}$.

If $\chi_A = 0$, then $\widehat{W}_A$ is defined as a certain extension of the elliptic Weyl group, which is $\widetilde{W}_A \simeq W_{(Q_A,I)}$, and $\widehat{\mathfrak{h}}_A := \mathbb{C}^{\mu_A - 1} \times \mathbb{H}$ where $\mathbb{H} := \{\tau \in \mathbb{C} \mid \mathrm{Im}(\tau) > 0\}$. Kyoji Saito and Satake construct a structure of Frobenius manifolds on the orbit space. We omit the details here and refer the reader to [23], [26] for the elliptic root systems, Weyl groups and the construction of Frobenius manifolds.

Theorem 12 ([23], [26]). *Assume that $\chi_A = 0$. There exists a structure of Frobenius manifold on $M_{\widehat{W}_A} := \widehat{\mathfrak{h}}_A/\widehat{W}_A$ of rank μ_A and dimension one.*

For $\chi_A < 0$, the construction has not yet been succeeded and it is one of most important problems.

4.2.3. *Universal Unfolding of f_A to Frobenius Manifolds* We refer the reader to [24] for the introduction to the theory of primitive forms. We follow the definitions and the terminologies described in [24].

Consider a universal unfolding F_A of f_A given by

$$F_A(x_1, x_2, x_3; s_1, s_{1,1}, \ldots, s_{3,a_3-1}, s_{\mu_A}) := x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - s_{\mu_A}^{-1} x_1 x_2 x_3 + s_1 + \sum_{i=1}^{3} \sum_{j=1}^{a_i-1} s_{i,j} x_i^j, \tag{30}$$

where we identify the parameter q in f_A with the deformation parameter $s_{\mu_A} \in \mathbb{C}\backslash\{0\}$.

Remark 13. *Let $t_1, t_{1,1}, \ldots, t_{3,a_3-1}, t_{\mu_A}$ be flat coordinates associated to the primitive form ζ_A given below, where t_{μ_A} is a flat coordinate of degree 0. The deformation parameter s_{μ_A} will turn out to be expressed as*

$$s_{\mu_A} = \begin{cases} e^{t_{\mu_A}} & \text{if} \quad \chi_A > 0, \\ e^{t_{\mu_A}} \cdot s(\mathbf{t}) & \text{if} \quad \chi_A \leq 0, \end{cases} \tag{31}$$

where $s(\mathbf{t})$ is a convergent power series in $t_{1,1}, \ldots, t_{3,a_3-1}, e^{t_{\mu_A}}$ which is invertible, is of degree zero (namely, one has $Es(\mathbf{t}) = 0$ for the Euler vector field E) and, in particular, takes value 1 if $t_{1,1} = \cdots = t_{3,a_3-1} = 0$.

Therefore, if $\chi_A > 0$, we shall always identify the parameter q (and s_{μ_A}) with $e^{t_{\mu_A}}$ from the first.

In order to define a notion of a primitive form, one needs to construct the filtered de Rham cohomology group $\mathcal{H}_{F_A}$, the Gauss–Manin connection ∇ on $\mathcal{H}_{F_A}$ and the higher residue pairings K_{F_A} on $\mathcal{H}_{F_A}$.

If $\chi_A > 0$, this is given by Sabbah [7], [21] since F_A is tame at any point on the parameter space, namely, there are no critical points coming from infinity.

If $\chi_A \leq 0$, this is a classical result developed by Kyoji Saito [22]. Therefore, it is possible to ask the existence of a primitive form. We have the following:

Theorem 14 (Ishibashi–Shiraishi–Takahashi [18]). *For the universal unfolding F_A, there exists a unique primitive form $\zeta_A \in \mathcal{H}_{F_A}$ with the minimal exponent one such that*

$$\frac{1}{(2\pi\sqrt{-1})^2}\int_{\gamma_0} \check{\zeta}_A = 1, \tag{32}$$

where $\check{\zeta}_A$ is the residue of the formal Fourier–Laplace dual of ζ_A along $\{F_A = 0\}$ and γ_0 is the 2-cycle in the Milnor fiber of F_A corresponding to the 3-cycle $\Gamma_0 := \{|x_1| = |x_2| = |x_3| = 1\} \subset \mathbb{C}^3$.

In particular, if $\chi_A > 0$, one has

$$\zeta_A = [e^{-t_{\mu_A}} dx_1 \wedge dx_2 \wedge dx_3], \tag{33}$$

$$\check{\zeta}_A = \mathrm{Res}_{F_A=0}\left[\frac{e^{-t_{\mu_A}} dx_1 \wedge dx_2 \wedge dx_3}{-F_A}\right]. \tag{34}$$

Proof. If $\chi_A > 0$, we can show the statement by the similar method, counting degrees of higher residue pairings with respect to the Euler vector field, with some calculations as the one used for the ADE singularities and simple elliptic singularities (cf. [22]). The details will be given in [18].

If $\chi_A \leq 0$, the statement is an easy consequence of Morihiko Saito's existence theorem of primitive forms for germs of isolated hypersurface singularities [25]. In particular, ζ_A is uniquely determined by the equation (32). Q.E.D.

Once we have a primitive form, we obtain a Frobenius manifold.

Corollary 15. *For a universal unfolding F_A of f_A, there exists a structure of a Frobenius manifold of rank μ_A and dimension one on the base space of the universal unfolding.*

Proof. This is obvious from Theorem 7.5 of [24]. Q.E.D.

4.3. Classical mirror conjecture

It is natural to expect the following Conjecture from the homological mirror symmetry (Conjecture 1) since the spaces of stability conditions for the triangulated categories $D^b\mathrm{coh}(\mathbb{P}^1_A)$, $D^b(\text{mod-}\mathbb{C}Q_A/I)$ and $D^b\mathrm{Fuk}^{\to}(f_A)$ should be naturally isomorphic and they should also carry natural Frobenius structures.

Conjecture 16 (cf. [27]). *There should exist isomorphisms of Frobenius manifolds between*

- $M_{\mathbb{P}^1_A}$, *the one constructed from the theory of Gromov–Witten invariants for* $\mathbb{P}^1_A$,
- $M_{\widehat{W}_A}$, *the one (that should be) constructed from the invariant theory of the extended Weyl group* $\widehat{W}_A$ *associated to the quiver with relations* (Q_A, I),
- M_{f_A,ζ_A}, *the one constructed from the universal unfolding of* f_A *by the primitive form* ζ_A.

Conjecture 16 is known to hold if $a_i = 1$ for some $i = 1, 2, 3$ by Milanov–Tseng [19] and if $\chi_A > 0$ by Rossi [20]. The next case to consider is when $\chi_A = 0$, in other words, the case when the polynomial f_A defines a simple elliptic singularity.

4.4. Our results

First, we state our result for the cases when $\chi_A = 0$.

Theorem 17 (Satake–Takahashi [27]). *Assume that* $\chi_A = 0$. *We have the following isomorphisms of Frobenius manifolds*

$$M_{\mathbb{P}^1_A} \simeq M_{\widehat{W}_A} \simeq M_{f_A,\zeta_A}, \tag{35}$$

where $M_{\widehat{W}_A}$ *denotes the Frobenius manifold constructed from the invariant theory of certain extension of the elliptic Weyl group of the corresponding type in* [23], [26].

Moreover, the genus zero Gromov–Witten potential $\mathcal{F}_0^{\mathbb{P}^1_A}$ *and the genus one Gromov–Witten potential* $\mathcal{F}_1^{\mathbb{P}^1_A}$ *are expressed by quasi-modular forms.*

Motivated by the proof of Theorem 17, we obtain the following uniqueness theorem for Frobenius manifolds. Note that we do not need any assumptions on χ_A.

Theorem 18 (Ishibashi–Shiraishi–Takahashi [17]). *There exists a unique formal Frobenius manifold* M *of rank* μ_A *and dimension one with flat coordinates* $(t_1, t_{1,1}, \dots, t_{3,a_3-1}, t_{\mu_A})$ *satisfying the following conditions*:

(i) *The unit vector field* e *and the Euler vector field* E *are given by*

$$e = \frac{\partial}{\partial t_1}, \quad E = t_1 \frac{\partial}{\partial t_1} + \sum_{i=1}^{3} \sum_{j=1}^{a_i-1} \frac{a_i - j}{a_i} t_{i,j} \frac{\partial}{\partial t_{i,j}} + \chi_A \frac{\partial}{\partial t_{\mu_A}}.$$

(ii) *The non-degenerate symmetric bilinear form η on $\mathcal{T}_M$ satisfies*

$$\eta\left(\frac{\partial}{\partial t_1},\frac{\partial}{\partial t_{\mu_A}}\right)=\eta\left(\frac{\partial}{\partial t_{\mu_A}},\frac{\partial}{\partial t_1}\right)=1,$$

$$\eta\left(\frac{\partial}{\partial t_{i_1,j_1}},\frac{\partial}{\partial t_{i_2,j_2}}\right)=\begin{cases}\frac{1}{a_{i_1}} & i_1=i_2 \text{ and } j_2=a_{i_1}-j_1,\\ 0 & \text{otherwise.}\end{cases}$$

(iii) *The Frobenius potential $\mathcal{F}$ satisfies $E\mathcal{F}|_{t_1=0}=2\mathcal{F}|_{t_1=0}$,*

$$\mathcal{F}|_{t_1=0}\in\mathbb{C}\left[[t_{1,1},\dots,t_{1,a_1-1},t_{2,1},\dots,t_{2,a_2-1},t_{3,1},\dots,t_{3,a_3-1},e^{t_{\mu_A}}]\right].$$

(iv) *Assume the condition* (iii). *The restriction of the Frobenius potential $\mathcal{F}$ to the submanifold $\{t_1=e^{t_{\mu_A}}=0\}$ is given as*

$$\mathcal{F}|_{t_1=e^{t_{\mu_A}}=0}=\mathcal{G}^{(1)}+\mathcal{G}^{(2)}+\mathcal{G}^{(3)},$$

where $\mathcal{G}^{(i)}\in\mathbb{C}[[t_{i,1},\dots,t_{i,a_i-1}]]$, $i=1,2,3$.

(v) *Assume again the condition* (iii). *In the frame $\frac{\partial}{\partial t_1}$, $\frac{\partial}{\partial t_{1,1}}$, $\dots$, $\frac{\partial}{\partial t_{3,a_3-1}}$, $\frac{\partial}{\partial t_{\mu_A}}$ of $\mathcal{T}_M$, the product $\circ$ can be extended to the limit $t_1=t_{1,1}=\dots=t_{3,a_3-1}=e^{t_{\mu_A}}=0$. The $\mathbb{C}$-algebra obtained in this limit is isomorphic to*

$$\mathbb{C}[x_1,x_2,x_3]\,/(x_1x_2,x_2x_3,x_3x_1,a_1x_1^{a_1}-a_2x_2^{a_2},a_2x_2^{a_2}-a_3x_3^{a_3}),$$

where $\partial/\partial t_{1,1},\partial/\partial t_{2,1},\partial/\partial t_{3,1}$ are mapped to x_1,x_2,x_3, respectively.

(vi) *The term*

$$\begin{cases}e^{t_{\mu_A}} & \text{if } a_1=a_2=a_3=1,\\ t_{3,1}e^{t_{\mu_A}} & \text{if } 1=a_1=a_2<a_3,\\ t_{2,1}t_{3,1}e^{t_{\mu_A}} & \text{if } 1=a_1<a_2,\\ t_{1,1}t_{2,1}t_{3,1}e^{t_{\mu_A}} & \text{if } a_1\geq 2,\end{cases}$$

occurs with the coefficient 1 *in $\mathcal{F}$.*

Actually, the condition (iv) follows from others if $A\neq(1,2,2)$ or $A\neq(2,2,r)$, $r\geq 2$. If $A=(1,2,2)$ or $A=(2,2,r)$, $r\geq 3$, then we only have to assume instead of the condition (iv) the weaker and more natural one

(iv') If $a_{i_1}=a_{i_2}$ for some $i_1,i_2\in\{1,2,3\}$, then the Frobenius potential $\mathcal{F}$ is invariant under the permutation of parameters $t_{i_1,j}$ and $t_{i_2,j}$ $(j=1,\dots,a_{i_1}-1)$.

The proof of Theorem 18 is done by solving the WDVV equations together with the above conditions. Write the Frobenius potential restricted to $\{t_1 = 0\}$ as

$$\begin{aligned}\mathcal{F}|_{t_1=0} =& \sum_{\alpha=(\alpha_{1,1},\dots,\alpha_{a_3,a_3-1})} c(\alpha, m) t^\alpha e^{mt_{\mu_A}},\\ t^\alpha =& t_{a_1,1}^{\alpha_{1,1}} \cdots t_{3,a_3-1}^{\alpha_{3,a_3-1}}.\end{aligned}$$

Define the length $|\alpha|$ of a multi-index $\alpha = (\alpha_{1,1}, \dots, \alpha_{3,a_3-1})$ by $|\alpha| := \alpha_{1,1} + \cdots + \alpha_{3,a_3-1}$. We show the uniqueness of the solution of the WDVV equations by induction on the pair $(|\alpha|, m)$.

Theorem 18 enables us to simplify the proofs given by Milanov–Tseng [19] and Rossi [20] and to generalize them to the case $\chi_A \leq 0$.

Corollary 19 ([18], [30]). *We have an isomorphism of Frobenius manifolds*

$$M_{\mathbb{P}^1_A} \simeq M_{f_A, \zeta_A}. \tag{36}$$

Proof. We check that both constructions satisfy the conditions in Theorem 18.

For $M_{\mathbb{P}^1_A}$, we can easily show the conditions (See Section 4 of [17] for details). We can choose a $\mathbb{Q}$-basis of $H^*_{orb}(\mathbb{P}^1_A, \mathbb{Q})$ whose dual coordinates satisfy the conditions (i) and (ii). The condition (iii) is clearly satisfied by the definition of the genus zero potential $\mathcal{F}_0^{\mathbb{P}^1_A}$. The condition (iv) is satisfied since the image of degree zero orbifold map with marked points on orbifold points on the source must be one of orbifold points on the target $\mathbb{P}^1_A$. The condition (v) follows from the description of the orbifold cohomology ring. The condition (vi) follows from the fact that the corresponding Gromov–Witten invariant is one, that is, the fact that a degree one map fixing three orbifold points is necessarily the identity.

On the other hand, for M_{f_A,ζ_A}, we can show the conditions by the careful study of the behavior of the Jacobian ring and the residue pairing at the limit $e^{t_{\mu_A}} \to 0$. Here, we shall explain the outline of the proof. The detail is given in [18] for the case $\chi_A > 0$ and will be given in [30] for the case $\chi_A \leq 0$.

Since the Jacobian ideal of F_A is given by

$$\left\langle G_A^{(1)} - s_{\mu_A}^{-1} x_2 x_3, G_A^{(2)} - s_{\mu_A}^{-1} x_1 x_3, G_A^{(3)} - s_{\mu_A}^{-1} x_1 x_2 \right\rangle, \tag{37}$$

where

$$G_A^{(i)} = a_i x_i^{a_1-1} + \sum_{j=1}^{a_i-1} j s_{i,j} x_i^{j-1},\ i = 1, 2, 3, \tag{38}$$

the Jacobian algebra of F_A, a free $\mathcal{O}_{M_{f_A,\zeta_A}}$-module of rank μ_A with an $\mathcal{O}_{M_{f_A,\zeta_A}}$-bilinear multiplication, can be extended to the limit $s_{\mu_A} \to 0$ by choosing the frame $\{1, x_i^j (i = 1,2,3, j = 1,\dots,a_i - 1), s_{\mu_A}^{-1} x_1x_2x_3\}$. This is the key fact for the Frobenius structure on M_{f_A,ζ_A}.

The condition (i) is satisfied since we can choose flat coordinates $t_1, t_{1,1}, \dots, t_{3,a_3-1}, t_{\mu_A}$ satisfying the following properties:

- $e = \partial/\partial s_1 = \partial/\partial t_1$ and $\partial/\partial t_{i,j}$ $(i = 1,2,3, j = 1,\dots a_i - 1)$ are eigenvectors of the Euler vector field E given by

$$E = s_1 \frac{\partial}{\partial s_1} + \sum_{i=1}^{3} \sum_{j=1}^{a_i-1} \frac{a_i - j}{a_i} s_{i,j} \frac{\partial}{\partial s_{i,j}} + \chi_A s_{\mu_A} \frac{\partial}{\partial s_{\mu_A}},$$

- $t_{i,j}$ $(i = 1,2,3, j = 1,\dots a_i - 1)$ are normalized as

$$\frac{\partial s_{i,j}}{\partial t_{i',j'}} \to \delta_{ii'}\delta_{jj'} \ (t_1 = t_{1,1} = \dots = t_{3,a_3-1} = 0, e^{t_{\mu_A}} \to 0), \tag{39}$$

- The equation (31) hold.

The condition (iii) follows since $\mathcal{F}|_{t_1=0}$ can be extended holomorphically to $s_{\mu_A} = 0$, which implies $\mathcal{F}|_{t_1=0}$ also can be extended holomorphically to $e^{t_{\mu_A}} = 0$ by the equation (31). The condition (iv) is satisfied since by putting $e^{t_{\mu_A}} = 0$ in the extended Jacobian algebra we have the quotient algebra of $\mathbb{C}[[s_1, s_{1,1}, \dots, s_{3,a_3-1}]][x_1,x_2,x_3]$ by the ideal

$$\left(x_1x_2, x_2x_3, x_3x_1, x_1G_A^{(1)} - x_2G_A^{(2)}, x_2G_A^{(2)} - x_3G_A^{(3)}\right) \tag{40}$$

again by the equation (31). By the property of flat coordinates (39), we have exactly the $\mathbb{C}$-algebra in the condition (v) after putting $t_1 = t_{1,1} = \dots = t_{3,a_3-1} = 0$.

We can show that the restriction of $[\zeta_A] \in \Omega_F$ (see [24] for the definition of Ω_F), to the subvariety of M_{f_A,ζ_A} defined by an ideal generated by $s_1 = s_{1,1} = \dots = s_{3,a_3-1} = 0$ is given by $[e^{-t_{\mu_A}} u(e^{t_{\mu_A}}) dx_1 \wedge dx_2 \wedge dx_3]$ due to the homogeneity condition and the normalization (32) for the primitive form, where $u(e^{t_{\mu_A}})$ is a convergent power series in $e^{t_{\mu_A}}$ of degree zero such that $u(0) = 1$. Note here that $s_{\mu_A} = e^{t_{\mu_A}} s(e^{t_{\mu_A}})$ on the subvariety, where $s(e^{t_{\mu_A}})$ is again a convergent power series in $e^{t_{\mu_A}}$ of degree zero such that $s(0) = 1$. Therefore, a verification of the condition (ii) can be reduced to the calculation of the following

$$\lim_{e^{t_{\mu_A}} \to 0} \left(e^{-2t_{\mu_A}} \mathrm{Res}_{\mathbb{C}^3 \times \mathbb{C}^*/\mathbb{C}^*} \begin{bmatrix} \phi(x_1,x_2,x_3) dx_1 \wedge dx_2 \wedge dx_3 \\ \dfrac{\partial f_A}{\partial x_1} \dfrac{\partial f_A}{\partial x_2} \dfrac{\partial f_A}{\partial x_3} \end{bmatrix} \right), \tag{41}$$

for $\phi(x_1,x_2,x_3) \in \{1, x_i^j (i=1,2,3, j=1,\dots,a_i-1), e^{-t_{\mu_A}}x_1x_2x_3\}$, which turns out to be non-zero only if $\phi(x_1,x_2,x_3) = e^{-t_{\mu_A}}x_1x_2x_3$. By the property of flat coordinates (39) together with the relations (37) in which we put $s_1 = s_{1,1} = \cdots = s_{3,a_3-1} = 0$, we have the condition (ii).

Note that the coefficient of the term $e^{t_{\mu_A}}$ (if $a_1 = a_2 = a_3 = 1$), $t_{3,1}e^{t_{\mu_A}}$ (if $1 = a_1 = a_2 < a_3$), $t_{2,1}t_{3,1}e^{t_{\mu_A}}$ (if $1 = a_1 < a_2$), $t_{1,1}t_{2,1}t_{3,1}e^{t_{\mu_A}}$ (if $a_1 \geq 2$) is given by the limit

$$\lim_{e^{t_{\mu_A}}\to 0}\left(e^{-t_{\mu_A}}\left.\frac{\partial^3 \mathcal{F}}{\partial t_{\mu_A}\partial t_{\mu_A}\partial t_{\mu_A}}\right|_{t_1=0}\right) \text{ if } a_1 = a_2 = a_3 = 1,$$

$$\lim_{e^{t_{\mu_A}}\to 0}\left(e^{-t_{\mu_A}}\left.\frac{\partial^3 \mathcal{F}}{\partial t_{\mu_A}\partial t_{\mu_A}\partial t_{3,1}}\right|_{t_1=t_{3,1}=\cdots=t_{3,a_3-1}=0}\right) \text{ if } 1 = a_1 = a_2 < a_3,$$

$$\lim_{e^{t_{\mu_A}}\to 0}\left(e^{-t_{\mu_A}}\left.\frac{\partial^3 \mathcal{F}}{\partial t_{\mu_A}\partial t_{2,1}\partial t_{3,1}}\right|_{t_1=t_{2,1}=\cdots=t_{3,a_3-1}=0}\right) \text{ if } 1 = a_1 < a_2,$$

$$\lim_{e^{t_{\mu_A}}\to 0}\left(e^{-t_{\mu_A}}\left.\frac{\partial^3 \mathcal{F}}{\partial t_{1,1}\partial t_{2,1}\partial t_{3,1}}\right|_{t_1=t_{1,1}=\cdots=t_{3,a_3-1}=0}\right) \text{ if } a_1 \geq 2,$$

which is always reduced to the calculation of the limit of the reidue

$$(42)\quad \lim_{e^{t_{\mu_A}}\to 0}\left(e^{-t_{\mu_A}}\cdot e^{-2t_{\mu_A}}\mathrm{Res}_{\mathbb{C}^3\times\mathbb{C}^*/\mathbb{C}^*}\left[\begin{array}{c} x_1x_2x_3dx_1\wedge dx_2\wedge dx_3 \\ \dfrac{\partial f_A}{\partial x_1}\ \dfrac{\partial f_A}{\partial x_2}\ \dfrac{\partial f_A}{\partial x_3}\end{array}\right]\right),$$

where we use that $\frac{\partial F_A}{\partial s_{i,1}} = x_i, (i=1,2,3)$ and the equations (31), (37) and (39). We obtain the condition (vi). Q.E.D.

We also have the following uniqueness theorem, which may be known for experts. We refer to Theorem 2.1 of [9] and Proposition 5.2 of [26] for the relevant statements.

Theorem 20 ([30]). *A* Frobenius manifold *of rank μ and of dimension one with the following e and E is uniquely determined by the intersection form I* :

$$(43)\qquad e = \frac{\partial}{\partial t_1},\ E = t_1\frac{\partial}{\partial t_1} + \sum_{i=1}^{3}\sum_{j=1}^{a_i-1}\frac{a_i-j}{a_i}t_{i,j}\frac{\partial}{\partial t_{i,j}} + \chi_A\frac{\partial}{\partial t_\mu}.$$

Proof. We use the following relation between the product $\circ$ and the intersection form I :

Lemma 21. *Denote by* Γ^{ij}_k *the contravariant component of the Levi–Civita connection for the intersection form* I. *Then, one has*

$$\Gamma^{ij}_k = d_j \cdot C^{ij}_k, \tag{44}$$

where d_j *is a rational number defined by* $E(t_j) = d_j \cdot t_j$ *and* $C^{ij}_k := \sum_{a,b=1}^{\mu} \eta^{ia}\eta^{jb}C_{kab}$.

Proof. See Lemma 3.4 of [8] and apply $d = 1$. Q.E.D.

One sees that C^{ij}_k can be reconstructed from the intersection form if $d_j \neq 0$. Since $d = 1$, $d_j = 0$ if and only if $j = \mu$. However, we have

$$C^{i\mu}_k = \sum_{a,b=1}^{\mu} \eta^{ia}\eta^{\mu b}C_{kab} = \delta^i_k \quad \text{(Kronecker's delta)}. \tag{45}$$

Therefore, C_{ijk} and hence the Frobenius potential $\mathcal{F}$ can be uniquely reconstructed from the intersection form. Q.E.D.

By Theorem 20, it is now possible to give an isomorphism of Frobenius manifolds between $M_{\widehat{W}_A}$ and M_{f_A,ζ_A} :

Corollary 22 ([30]). *Assume that* $\chi_A > 0$. *We have an isomorphism of Frobenius manifolds*

$$M_{\widehat{W}_A} \simeq M_{f_A,\zeta_A}. \tag{46}$$

Proof. We only have to calculate the intersection form for M_{f_A,ζ_A} and have to identify it with the one for $M_{\widehat{W}_A}$.

Let $\widetilde{\mathfrak{h}}_A$ be the complexified Cartan subalgebra of the loop Lie algebra of type A. Note that, by the results in Section 3, the Milnor lattice of f_A is isomorphic to the affine root lattice in $\widetilde{\mathfrak{h}}^*_A := \mathrm{Hom}_{\mathbb{C}}(\widetilde{\mathfrak{h}}_A, \mathbb{C})$.

Lemma 23. *Denote by* $\delta \in \widetilde{\mathfrak{h}}^*_A$ *the generator of the imaginary root. Then, one has the isomorphism of affine spaces*

$$\mathfrak{h}_A \simeq \left\{\widetilde{h} \in \widetilde{\mathfrak{h}}_A \mid \left\langle \delta, \widetilde{h} \right\rangle = 1\right\} \tag{47}$$

which is compatible with the action of the affine Weyl group $\widetilde{W}_A$ *on both sides, where the action on the LHS is defined by (see (24))*

$$h \mapsto w(h) + \sum_{i=1}^{\mu_A - 1} m_i \alpha_i^\vee, \quad m_i \in \mathbb{Z}, \tag{48}$$

and the one on the RHS is the natural one.

Proof. Some elementary calculations yield the statement. Q.E.D.

Therefore, by identifying δ and the 2-cycle γ_0 in the Milnor fiber, we see that $\mathfrak{h}_A$ can be identified as the period domain of the period mapping

$$\frac{1}{(2\pi\sqrt{-1})^2}\int \check{\zeta}_A. \tag{49}$$

Moreover, there is a formula of intersection numbers for cycles in the Milnor fiber:

Lemma 24. *One has*

$$\frac{-1}{(2\pi\sqrt{-1})^2}\sum_{a,b=1}^{\mu_A}\partial_a\left(\int_{\alpha_i}\check{\zeta}_A\right)\cdot\eta^{ab}\cdot(E\circ\partial_b)\left(\int_{\alpha_j}\check{\zeta}_A\right) = \left\langle \alpha_i,\alpha_j^\vee\right\rangle, \quad i,j=1,\ldots,\mu_A-1, \tag{50}$$

where we identified simple roots $\alpha_i,\alpha_j\in\mathfrak{h}_A^*$ with the corresponding homology classes.

Proof. See Theorem 3.4 in [22] ($\zeta^{(\frac{n}{2}-k-1)}$ must be $\zeta^{(n-k-1)}$ in the reference). Q.E.D.

By this Lemma and the identification (49), a part of the intersection form I_{f_A,ζ_A} for M_{f_A,ζ_A} can be calculated as

$$I_{f_A,\zeta_A}\left(\frac{1}{(2\pi\sqrt{-1})^2}\left(d\int_{\alpha_i}\check{\zeta}_A\right),\frac{1}{(2\pi\sqrt{-1})^2}\left(d\int_{\alpha_j}\check{\zeta}_A\right)\right) = \frac{-1}{(2\pi\sqrt{-1})^2}\left\langle\alpha_i,\alpha_j^\vee\right\rangle = I_{\widehat{W}_A}(\alpha_i,\alpha_j), \quad i,j=1,\ldots,\mu_A-1. \tag{51}$$

Recall that the coordinates x_{μ_A} and t_{μ_A} on $M_{\widehat{W}_A}$ are related by $t_{\mu_A}=(2\pi\sqrt{-1})x_{\mu_A}$. By Proposition 2 in Section 5.4 of [22], it is easy to check that

$$I_{f_A,\zeta_A}\left(\frac{1}{(2\pi\sqrt{-1})^2}\left(d\int_{\alpha_i}\check{\zeta}_A\right),\frac{dt_{\mu_A}}{(2\pi\sqrt{-1})}\right)=0, \quad i=1,\ldots,\mu_A-1, \tag{52}$$

where t_{μ_A} denotes the μ_A-th flat coordinate on M_{f_A,ζ_A}.

It is also easy to see from the definition of the intersection form that

$$(53)\quad I_{f_A,\zeta_A}\left(\frac{dt_{\mu_A}}{(2\pi\sqrt{-1})},\frac{dt_{\mu_A}}{(2\pi\sqrt{-1})}\right) = \frac{1}{(2\pi\sqrt{-1})^2}\chi_A = I_{\widehat{W}_A}(dx_{\mu_A},dx_{\mu_A}).$$

This completes the proof of Corollary 22 . Q.E.D.

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