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Singularities in Generic Geometry
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Singularities of frontals

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§1. Introduction

In this survey article we introduce the notion of frontals, which provides a class of generalised submanifolds with singularities but with well-defined tangent spaces. We present a review of basic theory and known studies on frontals in several geometric problems from singularity theory viewpoints. In particular, in this paper, we try to give some of detailed proofs and related ideas, which were omitted in the original papers, to the basic and important results related to frontals.

We start with one of theoretical motivations for our notion "frontal". Let M be a C^∞ manifold of dimension m, which is regarded as an ambient space. Suppose $n \leq m$ and let $f : N \to M$ be an *immersion* of an n-dimensional C^∞ manifold N, which is regarded as a parameter space, to M. Then for each point $t \in N$, we have the n-plane $f_*(T_tN)$, the image of the differential map $f_* : T_tN \to T_{f(t)}M$ at t in the tangent space $T_{f(t)}M$. Thus we have a field of tangential n-planes $\{f_*(T_tN)\}_{t\in N}$ along the immersion f. Moreover if M is endowed with a Riemannian metric, then we have also a field of tangential $(m-n)$-planes $f_*(T_tN)^\perp$ along f. Taking those vector bundles we can develop differential topology, theory of characteristic classes and so on of immersed submanifolds. Besides, taking local adapted frames for immersions, we can develop differential geometry of immersed submanifolds in terms of frames. Then a natural and challenging problem arises to us on the possibility to find a natural class of singular mappings enjoying the same properties as immersed submanifolds and to develop generalised topological and geometric theories on them.

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In this paper we introduce such a class of generalised submanifolds in terms of Grassmannians: Let $\mathrm{Gr}(n, TM)$ denote the Grassmannian of tangential n-planes in the tangent bundle TM over an m-dimensional C^∞ manifold M with the canonical projection $\pi : \mathrm{Gr}(n, TM) \to M$ (see §3). Let N be a C^∞ manifold of dimension n with $0 \leq n \leq m$ and take a point $a \in N$. Then a C^∞ map-germ $f : (N, a) \to M$ is called a *frontal map-germ* or a *frontal* in short if there exists a "Legendre" lifting of f, that is, there exist an open neighbourhood U of a and a C^∞ lift $\widetilde{f} : U \to \mathrm{Gr}(n, TM)$ of f, $\pi \circ \widetilde{f} = f|_U$, such that the image of differential $f_*(T_tN)$ is contained in $\widetilde{f}(t)$, for any $t \in U$. Note that $\widetilde{f}(t)$ is an n-plane in $T_{f(t)}M$. Moreover a C^∞ mapping $f : N \to M$ is called a *frontal mapping* or a *frontal* in short if, the germ $f : (N, a) \to M$ at any point $a \in N$ is a frontal. See §4 for details. The formulation using Grassmannians is very natural and satisfactory from the viewpoint of differential systems and their geometric solutions as well. See for instance [101][52][53].

Note that, if $\dim(N) = 1$, then any frontal $f : N \to M$ has a global Legendre lift $\widetilde{f} : N \to \mathrm{Gr}(1, TM)$ (Lemma 12.3). However, if $\dim(N) = 2$, then a frontal $f : N \to M$ not necessarily has a global Legendre lift (Example 2.5). This fact seems to be found first in the present paper. Also note that any mapping $f : N \to M$ is a frontal if $\dim(N) = \dim(M)$ (Remark 4.2). Any constant mapping $f : N \to M$ is a frontal.

The notion of "frontals" was introduced already in many papers, e.g. [38][105][95][85][10][11], in the case of hypersurfaces as a natural generalisation of wave-fronts. See §2.

We are going to give a survey on local classification of singularities appearing in frontals in various geometric contexts. Basically we mean by the "singularities" of frontals, as usual, the equivalence classes of germs of frontals under the following equivalence relation:

Definition 1.1. Two map-germs $f : (N, a) \to (M, f(a))$ and $g : (N', a') \to (M', f'(a'))$ are *right-left equivalent* or $\mathcal{A}$-*equivalent* or *diffeomorphic*, if there exist diffeomorphism-germs $\varphi : (N, a) \to (N', a')$ and $\Phi : (M, f(a)) \to (M', f'(a'))$ such that the following diagram commutes:

$$\begin{array}{ccc} (N, a) & \xrightarrow{f} & (M, f(a)) \\ \varphi \downarrow & & \downarrow \Phi \\ (N', a') & \xrightarrow{g} & (M', f'(a')). \end{array}$$

As the typical singularities of frontals, we introduce cuspidal edges, swallowtails, folded umbrellas, open swallowtails, open folded umbrellas and so on.

The *cuspidal edge* is defined as the equivalence class of the map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^m, 0)$, $m \geq 3$,

$$(t, s) \mapsto (t + s,\ t^2 + 2st,\ t^3 + 3st^2,\ 0,\ \ldots,\ 0),$$

which is diffeomorphic to $(u, w) \mapsto (u, w^2, w^3, 0, \ldots, 0)$. The cuspidal edge singularities are originally defined only in the three dimensional space. Here we are generalising the notion of the cuspidal edge in higher dimensional ambient space. It will be often emphasised it by writing "embedded" cuspidal edge.

The *folded umbrella* (or the *cuspidal cross cap*) is defined as the equivalence class of the map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$,

$$(t, s) \mapsto (t + s,\ t^2 + 2st,\ t^4 + 4st^3),$$

which is diffeomorphic to $(u, t) \mapsto (u,\ t^2 + ut,\ t^4 + \frac{2}{3}ut^3)$.

The *open folded umbrella* is defined as the equivalence class of the map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^m, 0)$, $m \geq 4$,

$$(t, s) \mapsto (t + s,\ t^2 + 2st,\ t^4 + 4st^3,\ t^5 + 5st^4,\ 0,\ \ldots,\ 0),$$

which is diffeomorphic to $(u, t) \mapsto (u,\ t^2{+}ut,\ t^4{+}\frac{2}{3}ut^3,\ t^5{+}\frac{5}{8}ut^4,\ 0,\ \ldots,\ 0)$. The open folded umbrella appeared for instance as a frontal-symplectic singularity in the paper [48].

The *swallowtail* is defined as the equivalence class of the map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^3, 0)$,

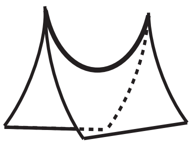
$$(t, s) \mapsto (t^2 + s,\ t^3 + \tfrac{3}{2}st,\ t^4 + 2st^2),$$

which is diffeomorphic to $(u, t) \mapsto (u,\ t^3 + ut,\ t^4 + \frac{2}{3}ut^2)$.

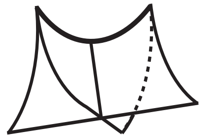
The *open swallowtail* is defined as the equivalence class of the map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^m, 0)$, $m \geq 4$,

$$(t, s) \mapsto (t^2 + s,\ t^3 + \tfrac{3}{2}st,\ t^4 + 2st^2,\ t^5 + \tfrac{5}{2}st^3,\ 0,\ \ldots,\ 0),$$

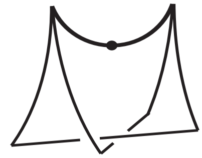
which is diffeomorphic to $(u, t) \mapsto (u,\ t^3{+}ut, t^4{+}\frac{2}{3}ut^2,\ t^5{+}\frac{5}{9}ut^3,\ 0,\ \ldots,\ 0)$. The open swallowtail singularity was introduced by Arnol'd (see [6]) as a singularity of Lagrangian varieties in symplectic geometry. Here we abstract its diffeomorphism class as the singularity of parametrised surfaces (see [26][43]).



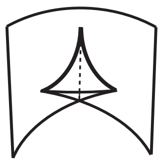
cuspidal edge



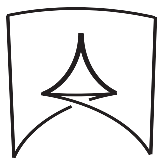
folded umbrella



open folded umbrella



swallowtail



open swallowtail

In Part I, we provide basic studies for an intrinsic understanding of frontals as parametrised singular submanifolds with well-defined tangent spaces.

We give the exact definition of frontals in §2 in the case of hypersurfaces and, after the description of Grassmannian bundles and canonical (or generalised contact) distributions in §3, we give the general definition in §4. In §5, we have introduced the density function as a main notion for the theory of frontals.

A frontal $f : N \to M$ is called a *proper frontal* in the present paper if the singular (non-immersive) locus $S(f)$ is nowhere dense in N (§6). In [44][45][46][47], "frontal" maps were defined as proper frontals, namely, the density of regular locus was assumed. Note that proper frontals are not generic in the space of all frontals for C^∞-topology in general (Remark 6.4). In §7, we introduce the tangent bundles to frontals.

Viewed from our generalisation, the notion of frontals turns to be closely related to the notion of *openings*. Though the notion of openings of mappings seems to be noticed naively in many previous contributions, it is introduced in the author's recent papers [44][45]. An opening separates the self-intersections of the original map-germ, preserving its singularities. For example, the swallowtail is an opening of the *Whitney's cusp* map-germ $(\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ defined by $(t, s) \mapsto (t^2 + s,\ t^3 + \frac{3}{2}st)$ which is diffeomorphic to $(u, t) \mapsto (u,\ t^3 + ut)$ and the open swallowtail is a versal opening of them. Openings of map-germs appear as typical singularities in several problems of geometry and its applications. Note that the process of unfoldings of map-germs $(\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ preserves the "relative dimension" $m - n$. On the other hand, the process of openings preserves n but changes m, and it gives bridges between map-germs of different relative dimensions. We recall also the related notions, "Jacobi modules" $\mathcal{J}_f$ and "ramification modules" $\mathcal{R}_f$. They play important role to analysis and classification of singularities of mappings f, in particular, the study on symplectic singularities, contact singularities and singularities of tangent surfaces ([29][30][31][34][35][37][39][40][49]). Moreover those notions seem to be related to recognition problem of singularities (see Definition 8.4). Note that we used the notation, for the ramification module of f, '$\mathcal{D}_f$' instead of $\mathcal{R}_f$ in [29], relating Mather's $\mathcal{C}$-equivalence, and we denoted it by 'H_f' in [30][31][33][34], because it can be regarded as a cohomological invariant. Note that the notion of openings, Jacobi modules and ramification modules for *multi-germs* is naturally introduced in the paper [44]. We give a review on the theory of opening related to frontals in §8 and §9. Moreover in §10 we give ideas of "subfrontals" and "superfrontals" related to frontals.

In [97], it is introduced the related notion of "coherent tangent bundles" as generalised Riemannian manifolds. Moreover Saji, Umehara, Yamada are developing the intrinsic studies of frontals in terms of singular metrics introduced by Kossowski [72]. We intend to give an abstract differential-topological feature of frontals, which is invariant under diffeomorphisms, by proving another way to study intrinsically frontals in terms of the theory of C^∞-rings (§11).

In part II, we give a survey of several results on frontals as an application of the basic theory presented in part I.

In §12, we treat frontal curves and give basic results on them. Let $\gamma : N \to \mathbf{R}^2$ be a planar frontal curve with $\dim(N) = 1$. By Lemma 12.3, there exists a global Legendre lifting $\widetilde{\gamma} : I \to P(T^*\mathbf{R}^2)$. Thus it is possible to perform differential geometry of planar frontals, as a generalised differential geometry of planar immersions, in terms of Legendre curves covering frontals. In fact geometric studies on planar frontals, evolutes and involutes, are given in a series of papers [21][22][23][24]. As a related topics to planar frontals, we gave a review the "Goursat Monster tower" found by Zhitomirskii, Montgomery, Mormul and others (cf. [80][81][16]) and the "Legendre-Goursat duality" related to it in [46].

The study on singularities and bifurcations of wavefronts based on Legendre singularity theory are established by Arnold-Zakalyukin's theory ([5][7][103][104]). The application of singularity theory to differential geometry has been developed by many authors (see for instance [90][91][14][63]). The geometric study of submanifolds in hyperbolic space H^{n+1} based on singularity theory was initiated by Izumiya et al.([66][67][68]). The Legendre duality developed in [62][18] enables us to unify the theory of framed curves in any space form as describes in [42].

We recall Legendre duality (see [13][82][58][52][18]) in the framework of moving frames and flags and discuss its generalisation and relation with the theory of frontals in §13, §14 and §15.

Let $\gamma : I \to \mathbf{R}^3$ be a space frontal curve. Then the tangent surface (tangent developable) $\mathrm{Tan}(\gamma)$ is defined as the surface ruled by tangent lines to γ. Then the tangent surface has zero Gaussian curvature, therefore it is flat with respect to Euclidean metric of $\mathbf{R}^3$ at least off the singular locus. Thus the tangent surfaces serve main parts of "flat frontals" (§16). Flat fronts or flat frontals are studied also in [83][84].

The notion of tangent surfaces ruled by "tangent lines" to directed curves is naturally generalised in various ways: For a curve in a projective space, we regard tangent projective lines as "tangent lines". The classification is generalised to A_n-geometry (§17). For a curve in a Riemannian manifold, we regard tangent geodesics as "tangent lines. In fact, tangent surfaces are defined for proper frontal curves (directed curves) in a manifold with an affine connection (§18). After discussing useful criteria of singularities in §19, we define null tangent surface to a null curve of a semi (pseudo)-Riemannian manifold, regarding null geodesics as "tangent lines" (see §20). In particular we pick up several results related to D_n-geometry ([57]). For a horizontal curve of a sub-Riemannian manifold, we regard "tangent lines" by abnormal geodesics (see §21). In particular the classification result of singularities of tangent surfaces to generic integral curves to Cartan distribution with G_2-symmetry is introduced.

Speaking of G_2, we note that the work on frontals may be related to the rolling ball problem [1][9][8][79]. We will treat "rolling frontals" as a generalisation of rolling bodies [4] in a forthcoming paper.

In the last section (§22), as an appendix, we show the Malgrange's preparation theorem on differentiable algebras ([74]) from the ordinary Malgrange-Mather's preparation theorem (see for example [15]), relating to the theory of C^∞-rings which we have utilised in this paper.

The author hopes very much that this survey paper helps to raise wider reader's interest to the mathematics on frontals.

The author would like to thank the anonymous referee for the valuable comment to improve the paper.

In this paper a manifold or a mapping is supposed to be of class C^∞ unless otherwise stated. The symbol $\subseteq$ of inclusion is often used, which has the same meaning as $\subset$ just to stress that the equality may occur.

Part I. Basic Theory

§2. The case of hypersurfaces

Let M be a manifold of dimension m. Let $P(T^*M)$ denote the projective cotangent bundle of M, which consists of non-zero cotangent vectors somewhere on M considered up to a non-zero scalar multiplication. Note that $P(T^*M)$ is naturally identified with the Grassmannian bundle $\mathrm{Gr}(m-1, TM)$ (see §3) by sending each class $(x, [\alpha]) \in P(T^*M)$ of a non-zero covector $\alpha \in T^*M$ to its kernel $\mathrm{Ker}(\alpha) \in \mathrm{Gr}(m-1, T_xM)$. Note that $\alpha \in T^*_xM$, that $\alpha : T_xM \to \mathbf{R}$ is a non-zero linear map, and that $\mathrm{Ker}(\alpha) \subset T_xM$ is an $(m-1)$-plane. Then the $(2m-1)$-dimensional manifold $P(T^*M)$ has a canonical contact structure $D \subset TP(T^*M)$. In fact it is defined by $D = \bigcup_{(x,[\alpha])} D_{(x,[\alpha])}$, and $D_{(x,[\alpha])} = \pi_*^{-1}(\mathrm{Ker}(\alpha))$, where $\pi : P(T^*M) \to M$ is the canonical projection.

We recall the coordinate description of the contact structure, which will be needed for the detailed computation on singularities.

Let $(x^1, x^2, \ldots, x^m)$ be a local coordinate system on an open subset U of M. Let

$$(x^1, x^2, \ldots, x^m, p_1, p_2, \ldots, p_m)$$

be the associated system of coordinates on T^*U such that any element $\alpha \in T^*U$ is expressed as

$$\alpha = p_1 dx^1 + p_2 dx^2 + \cdots + p_m dx^m,$$

by its coordinates. Set $V_i = \{p_i \neq 0\} \subset T^*U, 1 \leq i \leq m$. Then we have a local system of coordinates of $P(T^*M)$ associated to V_i,

$$x^1, x^2, \ldots, x^m, -p_1/p_i, \ldots, -p_{i-1}/p_i, -p_{i+1}/p_i, \ldots, -p_m/p_i.$$

To avoid non-essential complexity, we will discuss just for $i = m$ in what follows. Then set $a_i = -p_i/p_m, 1 \leq i \leq m-1$. Then

$$x^1, x^2, \ldots, x^m, a_1, a_2, \ldots, a_{m-1}$$

give a local system of coordinates of $P(T^*M)$ and the contact structure $D \subset TP(T^*M)$ is given locally by

$$dx^m - (a_1 dx^1 + a_2 dx^2 + \cdots + a_{m-1} dx^{m-1}) = 0.$$

Let N be a submanifold of dimension n with $n < m$. Then the submanifold N induces the *projective conormal bundle*

$$\widetilde{N} = P(T^*_N M) = \{(x, [\alpha]) \in P(T^*M) \mid \alpha|_{T_xN} = 0\},$$

which satisfies that $T\widetilde{N} \subset D$ and $\dim(\widetilde{N}) = m-1$, in other words, a Legendre submanifold in the contact manifold $P(T^*M)$.

In particular, suppose $n = m-1$, that is, N is a hypersurface of M. Then $\pi|_{\widetilde{N}} : \widetilde{N} \to N$ is a diffeomorphism. Its inverse $N \to \widetilde{N}$ is given by $x \mapsto (x, T_xN)$.

Let $f : N \to M$ be an immersion of an $(m-1)$-dimensional manifold N to an m-dimensional manifold M. Then we have an immersion $\widetilde{f} : N \to P(T^*M)$ defined by $\widetilde{f}(t) = (f(t), f_*(T_tN))$. Then $\widetilde{f}$ is a lift of f and $\widetilde{f}$ is D-integral, i.e. $\widetilde{f}_*(T_tN) \subset D_{\widetilde{f}(t)}$ for any $t \in N$. In other words, $\widetilde{f}$ is a Legendre immersion.

Remark 2.1. Set $\widetilde{f}(t) = (f(t), [\alpha(t)]) \in P(T^*_{f(t)}M)$. Then the condition $\widetilde{f}_*(T_tN) \subset D_{\widetilde{f}(t)}$ is equivalent to that $\alpha(t)|_{f_*(T_tN)} = 0$.

Definition 2.2. Let N be a manifold of dimension $m-1$ and M a manifold of dimension m. A map-germ $f : (N, a) \to M$ is called a *wave-front* or a *front* in short if there exists a germ of Legendre immersion $\widetilde{f} : (N, a) \to P(T^*M)$ with $\pi \circ \widetilde{f} = f$.

A mapping $f : N \to M$ is called a *wave-front* or a *front* in short if, for any point $a \in N$, the germ of f at a is a front.

A map-germ $f : (N, a) \to M$ is a front if and only if there exists a representative of f, which is a front.

Remark 2.3. In the original and naive context, the image $f(N)$ was called a wave-front rather than the parametrisation f itself.

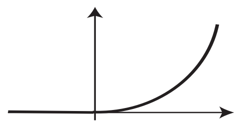
Definition 2.4. Let N be a manifold of dimension $m-1$ and M a manifold of dimension m. Let $a \in N$. A map-germ $f : (N, a) \to M$ is called a *frontal map-germ* or a *frontal* in short if there exist a germ of Legendre lifting $\widetilde{f} : (N, a) \to P(T^*M)$ of f, that is, there exist an open neighbourhood U of a, a representative $f : U \to M$ of f and a Legendre lifting $\widetilde{f} : U \to P(T^*M)$ of $f|_U$, i.e. $\widetilde{f}_*(T_tN) \subset D_{\widetilde{f}(t)}$ for any $t \in U$ and $\pi \circ \widetilde{f} = f|_U$. Here we do not assume that $\widetilde{f}$ is an immersion.

A mapping $f : N \to M$ is called a *frontal mapping* or a *frontal* in short if, for any $a \in N$, the germ of f at a is a frontal.

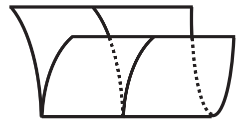
A map-germ $f : (N, a) \to M$ is a frontal if and only if there exists a representative of f, which is a frontal.

In Definition 2.4 we have defined the notion of frontals by the *local* existence of its Legendre liftings. A frontal $f : N \to M$ not necessarily has its *global* Legendre lifting $\widetilde{f} : N \to P(T^*M)$.

Example 2.5. Define a C^∞ function $\varphi : \mathbf{R} \to \mathbf{R}$ by $\varphi(t) = e^{-1/t^2}(t > 0), \varphi(t) = 0(t \leq 0)$. Then define $h : \mathbf{R}^2 \to \mathbf{R}^3$ by $h(t_1, t_2) = (t_1, t_2^2, t_2^3 + \varphi(t_1)t_2)$, which we will call a *half cuspidal edge*.



The graph of $\varphi(t)$.



The image of the half cuspidal edge.

The mapping h is not frontal. In fact the local existence of Legendre lift for h does not hold at the origin $(t_1, t_2) = (0, 0)$. Moreover h is a front on $\mathbf{R}^2 \setminus \{(0,0)\}$ with cuspidal edge along $\{(t_1, 0) \mid t_1 < 0\}$ and the Legendre lifting $\widetilde{h} : \mathbf{R}^2 \setminus \{(0,0)\} \to P(T^*\mathbf{R}^3) \cong \mathbf{R}^3 \times \mathbf{R}P^2$ is not homotopically trivial. In fact $\widetilde{h}$ restricted to a loop around the origin of $\mathbf{R}^2$ generates the fundamental group $\pi_1(P(T^*\mathbf{R}^3)) \cong \pi_1(\mathbf{R}P^2) \cong \mathbf{Z}/2\mathbf{Z}$.

Define $k : \mathbf{R}^2 \to \mathbf{R}^2$ by $k(t_1, t_2) = \varphi(t_1^2 + t_2^2 - 1)(t_1, t_2)$. Then k is a C^∞ mapping which collapses the unit disc to the origin and maps the outside of the unit disc to $\mathbf{R}^2 \setminus \{(0,0)\}$ diffeomorphically. Set $f = h \circ k : \mathbf{R}^2 \to \mathbf{R}^3$. Then

(1) f is a frontal.

(2) There does not exist a global Legendre lifting $\widetilde{f} : \mathbf{R}^2 \to P(T^*\mathbf{R}^3)$ of f.

To see (1), let $t = (t_1, t_2)$ satisfy $t_1^2 + t_2^2 < 1$. Then the germ of f at t is constant and therefore it is a frontal. Let t satisfy $t_1^2 + t_2^2 > 1$. Then the germ of f at t is right equivalent to h at $k(t) \in \mathbf{R}^2 \setminus \{(0,0)\}$, which is a frontal. Let t satisfy $t_1^2 + t_2^2 = 1$. Then any local extension of $\widetilde{h} \circ k$ to $(\mathbf{R}^2, t)$ turns to be a Legendre lift of the germ of f at t. Therefore f is a frontal. Thus we have (1). To see (2), it is sufficient to observe that $\widetilde{h} \circ k : \mathbf{R}^2 \setminus D^2 \to P(T^*\mathbf{R}^3)$, which is the unique Legendre lift of f restricted to $\mathbf{R}^2 \setminus D^2$, is never extended continuously to $\mathbf{R}^2$.

§3. Grassmannian bundle and generalised contact distribution

Let N be a manifold of dimension n and M a manifold of dimension m with $n \leq m$. Note that in the previous section we assumed $m = n+1$. However in the next section we treat the general case under the weaker condition $n \leq m$.

To treat the general cases, we recall the Grassmannian bundle associated to the tangent bundle TM of M. For each $x \in M$, $\mathrm{Gr}(n, T_xM)$ denotes the Grassmannian manifold consisting of n-dimensional subspaces of T_xM. Then let $\mathrm{Gr}(n, TM) = \bigcup_{x \in M} \mathrm{Gr}(n, T_xM)$. Note that

$\mathrm{Gr}(n, TM)$ is a bundle over M with fibres $\mathrm{Gr}(n, T_xM)$ and that the dimension $\dim(\mathrm{Gr}(n, T_xM)) = n(m-n)$. Note also that $\mathrm{Gr}(n, T_xM)$ is identified with $\mathrm{Gr}(m-n, T_x^*M)$ and therefore that, when $m = n+1$, $\mathrm{Gr}(n, TM)$ is identified with $P(T^*M)$. Let $\pi : \mathrm{Gr}(n, TM) \to M$ be the canonical projection, $\pi(x, V) = x$ for any $(x, V) \in \mathrm{Gr}(n, TM)$ with $V \in \mathrm{Gr}(n, T_xM), x \in M$. If $n = m$, then $\pi : \mathrm{Gr}(m, TM) \to M$ is a diffeomorphism.

Lemma 3.1. *Let $\Phi : M \to M'$ be a diffeomorphism. Let n be an integer with $0 \leq n \leq m = \dim(M)$. Let $\Phi_\sharp : \mathrm{Gr}(n, TM) \to \mathrm{Gr}(n, TM')$ denote the diffeomorphism induced by the differential map Φ_* which is regarded as the bundle isomorphism $\Phi_* : TM \to TM'$ covering Φ. Then we have $\pi \circ \Phi_\sharp = \Phi \circ \pi : \mathrm{Gr}(n, TM) \to M'$. Here π means the canonical projection $\mathrm{Gr}(n, TM') \to M'$ as well as $\mathrm{Gr}(n, TM) \to M$.*

Proof: Let $(x, V) \in \mathrm{Gr}(n, TM)$. Then $\Phi_\sharp(x, V) = (\Phi(x), \Phi_*(V))$. Therefore $(\pi \circ \Phi_\sharp)(x, V) = \pi(\Phi(x), \Phi_*(V)) = \Phi(x) = (\Phi \circ \pi)(x, V)$. $\square$

We recall the coordinate description of Grassmannians. Let $(x_0, V_0) \in \mathrm{Gr}(n, TM)$. Here $x_0 \in M$ and $V_0 \in \mathrm{Gr}(n, T_{x_0}M)$ so that $V_0 \subset T_{x_0}M$ is a fixed n-plane. The suffix 0 is used to indicate that (x_0, V_0) becomes the centre of the local coordinate system we are going to provide. Let us take a local coordinate system $(x^1, \dots, x^n, x^{n+1}, \dots, x^m)$ on a coordinate neighbourhood $U \subset M$ with the centre at x_0 such that $\partial/\partial x^1, \dots, \partial/\partial x^n$ generate V_0 at x_0. Let $\pi' : U \to \mathbf{R}^n$ denote the coordinate projection defined by $(x^1, \dots, x^n, \dots, x^m) \mapsto (x^1, \dots, x^n)$. Let $\Omega \subset \pi^{-1}(U)$ be the set of (x, V) with $x \in U, V \in \mathrm{Gr}(n, T_xM)$ such that V is mapped isomorphically by $\pi'_* : TU \to T\mathbf{R}^n$ to $\pi'_*(V)$. Then, for any (x, V), there exist unique real numbers $a_j^k, (1 \leq j \leq n, n+1 \leq k \leq m)$ such that the n-plane V has the basis $h_1, h_2, \dots, h_n$ of the form

$$\begin{cases} h_1 = \dfrac{\partial}{\partial x^1}(x) + a_1^{n+1}\dfrac{\partial}{\partial x^{n+1}}(x) + \cdots + a_1^m \dfrac{\partial}{\partial x^m}(x), \\ h_2 = \dfrac{\partial}{\partial x^2}(x) + a_2^{n+1}\dfrac{\partial}{\partial x^{n+1}}(x) + \cdots + a_2^m \dfrac{\partial}{\partial x^m}(x), \\ \vdots \qquad \ddots \\ h_n = \dfrac{\partial}{\partial x^n}(x) + a_n^{n+1}\dfrac{\partial}{\partial x^{n+1}}(x) + \cdots + a_n^m \dfrac{\partial}{\partial x^m}(x). \end{cases}$$

Thus we have a system of coordinates $(x^1, \dots, x^m, a_j^k, (1 \leq j \leq n, n+1 \leq k \leq m))$ on Ω of $\mathrm{Gr}(n, TM)$ with the centre at (x_0, V_0).

We call the coordinate systems constructed as above *Grassmannian coordinates*.

The *canonical distribution* $D \subset T(\mathrm{Gr}(n, TM))$ on the Grassmann bundle $\mathrm{Gr}(n, TM)$ is defined by $D = \bigcup_{(x,V)} D_{(x,V)}$ where (x, V) runs over $\mathrm{Gr}(n, TM)$, V being an n-plane of T_xM, $x \in M$, and, for $v \in T_{(x,V)}(\mathrm{Gr}(n, TM))$,

$$v \in D_{(x,V)} \Longleftrightarrow \pi_*(v) \in V (\subset T_xM).$$

We call the canonical distribution D on $\mathrm{Gr}(n, TM)$ also the *canonical differential system* and also the *contact distribution*, in a generalised and wider sense. If $n = m-1$, then D is the contact distribution in the strict sense. Note that, if $n = m$, then $D = T(\mathrm{Gr}(n, TM)) \cong TM$.

Definition 3.2. A mapping $F : N \to \mathrm{Gr}(n, TM)$ is called an *integral mapping* of the contact distribution $D \subset T\mathrm{Gr}(n, TM)$ or a *D-integral mapping* if $F_*(TN) \subset D$. If $\dim(N) = n$, then we call an integral mapping $f : N^n \to \mathrm{Gr}(n, TM)$ of the contact structure $D \subset T\mathrm{Gr}(n, TM)$ a *Legendre mapping* in a generalised and wider sense.

Lemma 3.3. *A mapping $F : N^n \to \mathrm{Gr}(n, TM)$ is a Legendre mapping if and only if, $(\pi \circ F)_*(T_tN) \subseteq F(t), (t \in N)$. If F is Legendre and $\pi \circ F$ is an immersion at $t \in N$, then $F(t) = (\pi \circ F)_*(T_tN)$.*

Proof: By definition, F is Legendre if and only if, for any $t \in N$, $F_*(T_tN) \subset D_{F(t)}$. Since $D_{F(t)} = \pi_*^{-1}(F(t))$, regarding $F(t)$ as an n-plane in $T_{(\pi\circ F)(t)}M$, the condition is equivalent to that $\pi_*(F_*(T_tN)) \subseteq F(t)$, that is, $(\pi \circ F)_*(T_tN) \subseteq F(t)$, for any $t \in N$. Moreover if $\pi \circ F$ is an immersion at $t \in N$, then $\dim((\pi \circ F)_*(T_tN)) = n$. Therefore we have $(\pi \circ F)_*(T_tN) = F(t)$. $\square$

The following result shows one of fundamental properties of the canonical differential systems (the generalised contact distributions).

Proposition 3.4. *Let $\Phi : M \to M'$ be a diffeomorphism. Let $0 \leq n \leq m = \dim(M)$. Let D denote the contact distribution of $\mathrm{Gr}(n, TM')$ as well as that of $\mathrm{Gr}(n, TM)$. Then, for any $(x, V) \in \mathrm{Gr}(n, TM)$, we have*

$$(\Phi_\sharp)_*(D_{(x,V)}) = D_{(\Phi(x),\Phi_*(V))}.$$

In particular we have $(\Phi_\sharp)_*(D) = D \subset T(\mathrm{Gr}(n, TM'))$ (see Lemma 3.1).

Proof: Let $v \in D_{(x,V)}$. Then $\pi_*v \in V$. Then we have, by Lemma 3.1,

$$\pi_*((\Phi_\sharp)_*(v)) = (\pi \circ \Phi_\sharp)_*(v) = (\Phi \circ \pi)_*(v) = \Phi_*(\pi_*v) \in \Phi_*(V).$$

Therefore we have $(\Phi_\sharp)_*(D_{(x,V)}) \subseteq D_{(\Phi(x),\Phi_*(V))}$. The converse inclusion is obtained by considering Φ^{-1}, or, by counting the dimension of the vector spaces. $\square$

We conclude this section by the coordinate description of the contact distribution: Take the Grassmannian coordinates $(x^1,\dots,x^m, a^k_j, (1 \leq j \leq n, n+1 \leq k \leq m))$ of $\mathrm{Gr}(n,TM)$ on an open set $\Omega \subset \mathrm{Gr}(n,TM)$. Set

$$\theta^k := dx^k - \sum_{j=1}^{n} a^k_j dx^j, \ (n+1 \leq k \leq m).$$

Lemma 3.5. *Let* $0 \leq n \leq \dim(M)$. *The local description of the contact distribution* D *of* $\mathrm{Gr}(n,TM)$ *is given by*

$$D|_{T\Omega} = \{v \in T\Omega \mid \theta^{n+1}(v) = 0, \dots, \theta^m(v) = 0\}.$$

Proof: Let $(x,V) \in \Omega$ and $v = \sum_{i=1}^m b^i \partial/\partial x^i + \sum_{j,k} c^k_j \partial/\partial a^k_j \in T_{(x,V)}\Omega$. Then $v \in D_{(x,V)}$ if and only if $\pi_*(v) \in V$. Now $V = \langle h_1,\dots,h_n\rangle_{\mathbf{R}}$ in terms of the above basis (described after Lemma 3.1). Then the condition is equivalent to that $\sum_{i=1}^m b^i\partial/\partial x^i = \sum_{j=1}^n \lambda^j h_j$ for some $\lambda^1,\dots,\lambda^n \in \mathbf{R}$, which is equivalent to that $b^j = \lambda^j, 1 \leq j \leq n$ and $b^k = \sum_{j=1}^n b^j a^k_j, n+1 \leq k \leq m$, and thus equivalent to that $\theta^k(v) = 0$, $n+1 \leq k \leq m$. $\square$

§4. Generalised frontals

We give the exact definition of our main notion in this paper:

Definition 4.1. Let N be an n-dimensional manifold and M an m-dimensional manifold with $n \leq m$. A map-germ $f : (N,a) \to M$ is called a *frontal map-germ* or a *frontal* in a generalised sense, if there exists a germ of Legendre lift $\widetilde{f} : (N,a) \to \mathrm{Gr}(n,TM)$ of f, that is, if there exists an open neighbourhood U of a and a D-integral lift $\widetilde{f} : U \to \mathrm{Gr}(n,TM)$ of f for the canonical distribution $D \subset T\mathrm{Gr}(n,TM)$ and for the canonical projection $\pi : \mathrm{Gr}(n,TM) \to M$, which satisfies that $f_*(T_tN) \subseteq \widetilde{f}(t)$ for any $t \in U$ and $\pi \circ \widetilde{f} = f$.

We call a mapping $f : N^n \to M^m$ a *frontal mapping* or a *frontal* in a generalised sense, if, for any point $a \in N$, the germ of f at a is a frontal.

Remark 4.2. Note that, in the equi-dimensional case $n = m$, any mapping $f : N \to M$ is a frontal. In fact the mapping $\widetilde{f} : N \to \mathrm{Gr}(m,TM)$ defined by $\widetilde{f}(t) := T_{f(t)}M$ is a Legendre lift of f.

Proposition 4.3. *Let $f : (N, a) \to (M, f(a))$ and $g : (N', a') \to (M', f(a'))$ be map-germs. If f is a frontal and g is right-left equivalent to f, then g is a frontal.*

Proof: Suppose $g \circ \varphi = \Phi \circ f$ for some diffeomorphism-germs $\varphi : (N, a) \to (N', a')$ and $\Phi : (M, f(a)) \to (M', f(a'))$. Let $\widetilde{f} : (N, a) \to \mathrm{Gr}(n, TM)$ be a Legendre lift of f. Set $\widetilde{g} := \Phi_\sharp \circ \widetilde{f} \circ \varphi^{-1} : (N', a') \to \mathrm{Gr}(n, TM')$. For $t' \in (N, a')$, we have, by Proposition 3.4,

$$\widetilde{g}_*(T_{t'}N') = (\Phi_\sharp)_*(\widetilde{f}_*(\varphi_*^{-1}(T_{t'}N))) = (\Phi_\sharp)_*(\widetilde{f}_*(T_{\varphi^{-1}(t)}N)) \subset (\Phi_\sharp)_* D = D.$$

Therefore $\widetilde{g}$ is Legendre. Moreover, by Lemma 3.1, we have

$$\pi \circ \widetilde{g} = \pi \circ \Phi_\sharp \circ \widetilde{f} \circ \varphi^{-1} = \Phi \circ \pi \circ \widetilde{f} \circ \varphi^{-1} = \Phi \circ f \circ \varphi^{-1} = g.$$

Therefore $\widetilde{g}$ is a Legendre lifting of g, and hence g is a frontal. □

Definition 4.4. A map-germ $f : (N, a) \to M$ is called a *front* in the generalised sense if there exists a Legendre lift $\widetilde{f} : (N, a) \to \mathrm{Gr}(n, TM)$ of f such that $\widetilde{f}$ is an immersion-germ. A mapping $f : N^n \to M^m$ is called a *front* in the generalised sense if, for any $a \in N$, the germ of f at a is a front.

A map-germ $f : (N, a) \to M$ is a front in the generalised sense if and only if there exists a representative of f which is a front. The condition that $f : N \to M$ is a front in the generalised sense is equivalent to the local existence, at each point of N, of an immersive lift $\widetilde{f} : U \to \mathrm{Gr}(n, TM)$ of f satisfying $f_*(T_tN) \subset \widetilde{f}(t), (t \in U)$.

§5. Density function

The notion of density functions is a key to understand the geometry of frontals, which was introduced in [71][25][95] first. We introduce its generalisation (see also [59][60]):

Proposition 5.1. *Let $f : (N, a) \to M$ be a map-germ with $\dim(N) = n \leq m = \dim(M)$. Then the following conditions are equivalent*:
(1) *f is a frontal map-germ.*
(2) *There exists a frame $h_1, h_2, \dots, h_n : (N, a) \to TM$ along f and a function-germ $\sigma : (N, a) \to \mathbf{R}$ such that*

$$\left(\frac{\partial f}{\partial t_1} \wedge \frac{\partial f}{\partial t_2} \wedge \cdots \wedge \frac{\partial f}{\partial t_n}\right)(t) = \sigma(t)(h_1 \wedge h_2 \wedge \cdots \wedge h_n)(t),$$

as germs of n-vector fields $(N, a) \to \wedge^n TM$ over f. Here $t_1, t_2, \dots, t_n$ are coordinates on (N, a).

The function $\sigma : (N, a) \to \mathbf{R}$ in Proposition 5.1 is called a *signed area density function* or briefly an *s-function* of the frontal f associated with the frame. Note that the function σ is essentially the same thing with the function λ introduced in [71][25] in the case $\dim(M) = 3$.

Two function-germs $\sigma, \widetilde{\sigma} : (N, a) \to \mathbf{R}$ are called $\mathcal{K}$-*equivalent* if there exists a diffeomorphism-germ $T : (N, a) \to (N, a)$ and a non-vanishing function-germ $c : (N, a) \to \mathbf{R}$, $c(a) \neq 0$, such that $\widetilde{\sigma}(T(t)) = c(t)\sigma(t), (t \in (N, a))$ (see [75]).

Lemma 5.2. *The $\mathcal{K}$-equivalence class of a signed area density function σ is independent of the choice of the frame $h_1, h_2, \dots, h_n$ and of the coordinates $t_1, t_2, \dots, t_n$ on $(\mathbf{R}^n, a)$ and depend only on the frontal f.*

Proof: Let us take another frame $k_1, \dots, k_n$. Then there exists $A = (a_{ij}) : (\mathbf{R}^n, a) \to \mathrm{GL}(n, \mathbf{R})$ such that $(h_1, \dots, h_n) = (k_1, \dots, k_n)A$. Then $h_1 \wedge h_2 \wedge \cdots \wedge h_n = (\det A)(k_1 \wedge k_2 \wedge \cdots \wedge k_n)$. Therefore σ is transformed to $(\det A)\sigma$. Let us take another coordinates $T_1, T_2, \dots, T_n$ on $(\mathbf{R}^n, a)$. Then

$$\left(\frac{\partial f}{\partial T_1} \wedge \frac{\partial f}{\partial T_2} \wedge \cdots \wedge \frac{\partial f}{\partial T_n}\right)(T(t)) = J(t)\left(\frac{\partial f}{\partial t_1} \wedge \frac{\partial f}{\partial t_2} \wedge \cdots \wedge \frac{\partial f}{\partial t_n}\right)(t),$$

where $J(t)$ is the Jacobian function $\partial(T_1, \dots, T_n)/\partial(t_1, \dots, t_n)$ at t. Therefore $\sigma(t)$ is transformed to the function $J(t)\sigma(T(t))$. Thus we have the required result. □

We call the signed density function of a frontal, considered up to $\mathcal{K}$-equivalence, a *density function* of the frontal. The singular locus (non-immersive locus) $S(f)$ of f coincides with the zero locus $\{\sigma = 0\}$ of the density function σ.

§6. Proper frontals

Frontals can be collapsing in general. For example, any constant mapping $f : N \to M$ is a frontal. In fact any lifting $F : N \to \mathrm{Gr}(n, TM)$ of f is Legendre in that case. See also Example 2.5.

Definition 6.1. A frontal $f : N \to M$ is called a *proper frontal* if the regular locus

$$R(f) := \{t \in N \mid f_* : T_tN \to T_{f(t)}M \text{ is injective.}\}$$

of f is dense in N. A germ of frontal $f : (N, a) \to M$ is called a *germ of proper frontal* if there exists a representative of f which is a proper frontal.

Note that $R(f)$ is an open subset of N in general. Then the condition that f is a proper frontal requires that $R(f)$ is open and dense.

The fundamental property of proper frontals is the following:

Proposition 6.2. *Let $f : N \to M$ be a proper frontal. Then there exists the unique global Legendre (i.e. D-integral) lift $\widetilde{f} : N \to \mathrm{Gr}(n, TM)$ of f, for the canonical projection $\pi : \mathrm{Gr}(n, TM) \to M$, $\pi \circ \widetilde{f} = f$. Here D is the contact distribution on $\mathrm{Gr}(n, TM)$, $n = \dim(N)$, introduced in §4.*

Proof: Consider the mapping $F : R(f) \to \mathrm{Gr}(n, TM)$ defined by $F(t) = f_*(T_tN) \in \mathrm{Gr}(n, T_{f(t)}M) \subset \mathrm{Gr}(n, TM)$. Then F is a D-integral mapping and $\pi \circ F = f|_{R(f)}$. By Lemma 3.3, F is a unique Legendre lifting of $f|_{R(f)}$. Since f is a frontal, for any $a \in N$, there exists an open neighbourhood U of a and a D-integral lift $\widetilde{f} : U \to \mathrm{Gr}(n, TM)$ of f. Then by the uniqueness of F, we have $\widetilde{f} = F$ on $U \cap R(f)$. Since f is a proper frontal, $R(f)$ is dense in N, and therefore $U \cap R(f)$ is dense in U. Thus the Legendre lift F of f is uniquely extended to $U \cup R(f)$. Since a is arbitrary, we have the unique Legendre lift $\widetilde{f} : N \to \mathrm{Gr}(n, TM)$ of f. □

Proposition 6.3. *If $f : N \to M$ is a frontal and it has a unique Legendre lift $\widetilde{f} : N \to \mathrm{Gr}(n, TM)$, then f is a proper frontal.*

Proof: Suppose the regular locus $R(f)$ of f is not dense in N. Then there exists a non-void open subset $U \subset N$ such that the maximal rank of $f|_U$ is $\ell < n$. Then there exists a non-void open subset $V \subset U$ such that $f|_V$ is of constant rank ℓ. Then there exists a non-void open subset $W \subset V$ and an open subset $\Omega \subset M$ such that $f|_W : W \to \Omega$ is right-left equivalent to $h : \mathbf{R}^n \to \mathbf{R}^m$ which is defined by $h(s_1, \dots, s_\ell, s_{\ell+1}, \dots, s_n) = (s_1, \dots, s_\ell, 0, \dots, 0)$ ("Rank theorem", see [15]). Let $\widetilde{f} : N \to \mathrm{Gr}(n, TM)$ be a Legendre lift and $\widetilde{h} : \mathbf{R}^n \to \mathrm{Gr}(n, T\mathbf{R}^m)$ be the induced lift of h by $\widetilde{f}|_W$ (cf. Proposition 4.3). Then $T_{h(s)}(\mathbf{R}^\ell \times \{0\}) \subset \widetilde{h}(s), (s \in \mathbf{R}^n)$. Then there exists a non-trivial perturbation of $\widetilde{h}$ therefore of $\widetilde{f}$ with compact support. □

Remark 6.4. *Proper frontals are not generic* in C^∞-topology in general. In fact the frontal mapping $f : \mathbf{R}^2 \to \mathbf{R}^3$ constructed in Example 2.5 can not be approximated by any proper frontal.

Now we introduce the notion of non-degenerate frontals which was originated in [71].

Definition 6.5. We say that a frontal $f : (N, a) \to M$ has a *non-degenerate* singular point at a if the density function σ of f satisfies that

$\sigma(a) = 0$ and $d\sigma(a) \neq 0$. Note that the condition is invariant under the $\mathcal{K}$-equivalence of σ (see Proposition 5.2).

To study the property of non-degenerate singular points of frontals, we recall the following result.

Lemma 6.6. *Let N be a manifold of dimension n. Let $g : (N, a) \to (N, g(a))$ be a map-germ. Let J_g denote the Jacobi matrix of g and $\det(J_g) : (N, a) \to \mathbf{R}$ the Jacobian determinant of g. Suppose $(\det J_g)(a) = 0$. Then $(d\det(J_g))(a) = 0$ if $\operatorname{rank}(J_g)(a) \leq n - 2$, that is, if g is of corank ≥ 2 at a.*

Proof: It is easy to see, as a fundamental fact in the linear algebra, for the determinant function $\det : M(n, n; \mathbf{R})$ on the space of $n \times n$-matrices, and for any $A \in M(n, n; \mathbf{R})$ with $\det(A) = 0$, $(d\det)(A) = 0$ if and only if $\operatorname{rank}(A) \leq n - 2$. Then we have, if $\operatorname{rank}(J_g) \leq n - 2$, then $(d\det(J_g))(p) = (J_g)^*(d\det)(p) = 0$. □

Lemma 6.7. *If a frontal $f : (N, a) \to M$ has a non-degenerate singular point at a, then f is of corank 1 such that the singular locus $S(f) \subset (N, a)$ is a regular hypersurface.*

Proof: Let us take a representative $f : U \to M$ of f, using the same symbol, satisfying that $d\sigma(t) \neq 0$ for any $t \in U$. Then $S(f) = \{t \in U \mid \sigma(t) = 0\}$ is a regular hypersurface of U. In particular $S(f)$ is nowhere dense in U. Therefore f is a proper frontal. Let $\widetilde{f} : U \to \mathrm{Gr}(n, TM)$ be the unique Legendre lifting of f. Set $V = \widetilde{f}(a) \subset T_{f(p)}M$. Take a local coordinate system $(x^1, \dots, x^n, x^{n+1}, \dots, x^m)$ around $f(a)$ of M such that $V = \langle (\partial/\partial x^1)(f(a)), \dots, (\partial/\partial x^n)(f(a)) \rangle_{\mathbf{R}}$ Define $g : U \to \mathbf{R}^n$ by $g = (x^1 \circ f, \dots, x^n \circ f)$, deleting U if necessary. Then the rank of g_* at a is equal to the rank of f_* at a. Moreover the signed area density function of g is $\mathcal{K}$-equivalent to that of f. Note that the signed area density function of g is $\mathcal{K}$-equivalent to the Jacobian determinant of g. Suppose the rank of g_* at p is less than $n - 1$. Then by Lemma 6.6 we see that $(d\sigma)(a) = d(\det(J_g))(a) = 0$. This leads a contradiction to the assumption of non-degeneracy. Therefore we have that the rank of g_* is equal to $n - 1$. Thus we have the required result. □

§7. Tangent bundles and complementary bundles

Let $f : N \to M$ be a proper frontal. Let $\dim(N) = n$ and $\widetilde{f} : N \to \mathrm{Gr}(n, TM)$ be the unique Legendre lifting of f (Proposition 6.2). Then we have a subbundle T_f of the pull-back bundle f^*TM over N defined by

$$T_f := \{(t, v) \in N \times TM \mid v \in \widetilde{f}(t)\} \subset f^*TM.$$

We call T_f the *tangent bundle* to the proper frontal f. Moreover we call the quotient bundle $Q_f := f^*TM/T_f$ the *complementary bundle*.

Definition 7.1. A proper frontal $f : N \to M$ is called *oriented* (resp. *co-oriented*) if the bundle T_f (resp. Q_f) is oriented. f is called *orientable* (resp. *co-orientable*) if T_f (resp. Q_f) is orientable.

Example 7.2. The proper front $f : S^1(\subset \mathbf{C}) \to \mathbf{R}^2(= \mathbf{C})$ defined by $z \mapsto 2z - \overline{z}^2$, for $z \in \mathbf{C}, |z| = 1$ ("cardioid") is not orientable nor co-orientable. The half cuspidal edge $h : \mathbf{R}^2 \setminus \{(0,0)\} \to \mathbf{R}^3$ (see Example 2.5) restricted to $\mathbf{R}^2 \setminus \{(0,0)\}$ is a proper front which is not orientable nor co-orientable. The mapping $\mathbf{R}^2 \to \mathbf{R}^3$ defined by the normal form of the cuspidal edge (resp. folded umbrella) is a proper front (resp. frontal) which is orientable and co-orientable.

Let $f : N \to M$ be a proper frontal. Then the bundle homomorphism $\varphi_f : TN \to T_f, \varphi(t, v) = (t, f_*(v))$ is induced. Then we have

$$\begin{aligned} R(f) &= \{t \in N \mid f_* : T_tN \to T_{f(t)}M \text{ is injective}\} \\ &= \{t \in N \mid \varphi_f \text{ is injective at } t\}. \end{aligned}$$

The notion of frontals will play important role in differential geometry. Therefore the following observations are important. First we treat the case of hypersurfaces ($m = n + 1$).

Lemma 7.3. *If M is endowed with a Riemannian metric, then $f : N^n \to M^{n+1}$ is a frontal if and only if, for any $a \in N$, there exists an open neighbourhood U of a and a unit vector field ν along f such that $\nu(t)$ is normal to the subspace $f_*(T_tN)$ for any $t \in U$.*

Proof: Let f be frontal. Let $\widetilde{f}(t) = (f(t), [\alpha(t)])$ be a Legendre lifting of f. It defines the local integral tangential hyperplane field $\mathrm{Ker}(\alpha(t))$ along f. Then we associate the normal line field $\mathrm{Ker}(\alpha(t))^\perp$ with $\widetilde{f}(t)$ and take a local unit frame $\nu(t)$ of $\mathrm{Ker}(\alpha(t))^\perp$. Conversely let $\nu(t)$ be a local unit normal field along f with $f_*(T_tN)$. Regarding the metric, we associate a non-zero cotangent vector field $\alpha(t)$ with $\nu(t)$ so that $\mathrm{Ker}(\alpha(t)) = \nu(t)^\perp$. Then the tangential hyperplane field $\nu(t)^\perp$ satisfies the condition $f_*(T_tN) \subseteq \nu(t)^\perp$. The condition is equivalent to that $\widetilde{f}(t) = (f(t), [\alpha(t)])$ is a Legendre map. $\square$

Lemma 7.4. *If M is endowed with a Riemannian metric, then $f : N^n \to M^{n+1}$ is a front if and only if locally there exists a normal unit vector field ν along f such that (f, ν) is an immersion to the unit tangent bundle T_1M.*

Proof: Regarding each unit vector $\nu \in T_xM$ as an element of T_x^*M by $v \mapsto \nu \cdot v$, we have the natural double covering $T_1M \to P(T^*M)$. Therefore we have required result by Lemma 7.3. $\square$

In generalised cases, we have:

Lemma 7.5. *If M is endowed with a Riemannian metric, then $f : N^n \to M^m$ is a frontal if and only if, for any $a \in N$, there exists an open neighbourhood U of a and a system of orthonormal vector fields $\nu_1, \dots, \nu_{m-n}$ over U along f such that $\nu_i(t)$ is normal to the subspace $f_*(T_tN)$ for any $t \in U$, $i = 1, \dots, m-n$.*

Proof: Suppose f is a frontal. For any a, let $\widetilde{f} : U \to \mathrm{Gr}(n, TM)$ be a Legendre local lifting of $f|_U$. Deleting U if necessary, take an orthonormal frame $h_1, \dots, h_n, \nu_1, \dots, \nu_{m-n}$ on U such that $h_1(t), \dots, h_n(t)$ form a basis of $\widetilde{f}(t) \subset T_{f(t)}M$ for any $t \in U$. Then $\nu_1, \dots, \nu_{m-n}$ satisfy the required condition. Conversely we may set $\widetilde{f}(t) = \langle \nu_1(t), \dots, \nu_{m-n}(t)\rangle^{\perp}$. Then $\pi \circ \widetilde{f} = f$ and $(\pi \circ \widetilde{f})_*(T_tN) = f_*(T_tN) \subset \widetilde{f}(t)$, hence $\widetilde{f}$ is Legendre by Lemma 3.3. $\square$

The following is clear:

Lemma 7.6. *If M is a Riemannian manifold, then the condition that $f : N^n \to M^m$ is a front is equivalent to the local existence of an orthonormal unit frame $\nu_1, \dots, \nu_n$ along f such that $t \mapsto (f(t), \langle \nu_1(t), \dots, \nu_n(t)\rangle^{\perp})$ is an immersion to $\mathrm{Gr}(n, TM)$.*

Let $f : N \to M$ be a proper frontal. If M is endowed with a Riemannian metric, then we define the *normal bundle* to f by

$$N_f := \{(t, w) \in N \times TM \mid w \in \widetilde{f}(t)^{\perp}\} \subset f^*TM,$$

which is isomorphic to the complementary bundle Q_f (see §7). Note that both bundles T_f and N_f have induced Riemannian bundle structures from TM.

§8. Openings and frontals

In this section, we review the known results on "geometric" openings.

We denote by $\mathcal{E}_{N,a}$ the $\mathbf{R}$-algebra of C^∞ function-germs on (N, a) with the maximal ideal $\mathfrak{m}_{N,a}$. If $(N, a) = (\mathbf{R}^n, 0)$ is the origin, then we use $\mathcal{E}_n, \mathfrak{m}_n$ instead of $\mathcal{E}_{N,a}, \mathfrak{m}_{N,a}$ respectively.

Definition 8.1. ([30][35]) Let $f : (N, a) \to (M, b)$ be a C^∞ map-germ with $\dim(N) = n \le m = \dim(M)$. We define the *Jacobi module* of f:

$$\mathcal{J}_f := \mathcal{E}_{N,a}\, d(f^*\Omega_{M,b}) = \left\{ \sum_{j=1}^{m} a_j\ df^j \mid\ a_j \in \mathcal{E}_{N,a}, 1 \le j \le m \right\}$$

in the space $\Omega^1_{N,a}$ of 1-form germs on (N, a). Here $f^j = x^j \circ f$, for a system of coordinates $(x^1, \dots, x^m)$ of (M, b). Further we define the *ramification module* $\mathcal{R}_f$ by

$$\mathcal{R}_f := \{h \in \mathcal{E}_{N,a} \mid dh \in \mathcal{J}_f\}.$$

Example 8.2. Let μ be a positive integer and $g : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ a map-germ defined by $g(t) = t^\mu$. Then $\mathcal{J}_g = \mathfrak{m}_1^{\mu-1} dt$ and $\mathcal{R}_g = \mathbf{R} + \mathfrak{m}_1^\mu$. Here $\mathfrak{m}_1^\mu = t^\mu \mathcal{E}_1 = \{h \in \mathcal{E}_1 \mid \frac{d^k h}{dt^k}(0) = 0, (0 \le k \le \mu)\}$. In fact, since $dg = \mu t^{\mu-1} dt$, we gave $\mathcal{J}_g = \mathfrak{m}_1^{\mu-1} dt$. Moreover, for a $k \in \mathcal{E}_1$, we have that $k \in \mathcal{R}_f$ if and only if $\frac{dk}{dt} \in \mathfrak{m}_1^{\mu-1}$ if and only if $k \in \mathbf{R} + \mathfrak{m}_1^\mu$.

Note that $\mathcal{J}_f$ is just the first order component of the graded differential ideal $\mathcal{J}_f^\bullet$ in $\Omega^\bullet_{N,a}$ generated by $df^1, \dots, df^m$. Then the singular locus is given by $S(f) = \{x \in (N, a) \mid \operatorname{rank} \mathcal{J}_f(x) < n\}$. Also we consider the *kernel field* $\operatorname{Ker}(f_* : TN \to TM)$, of f near a. Then we see that, for another map-germ $f' : (N, a) \to (M', b')$ with $\mathcal{J}_{f'} = \mathcal{J}_f$, $n \le m'$, we have $\mathcal{S}_{f'} = \mathcal{S}_f$ and $\operatorname{Ker}(f'_*) = \operatorname{Ker}(f_*)$. Note that related notion was introduced in [78].

Lemma 8.3. *Let $f : (N, a) \to (M, b)$ be a map-germ. Then we have*:
(1) *$f^*\mathcal{E}_{M,b} \subset \mathcal{R}_f \subset \mathcal{E}_{N,a}$ and $\mathcal{R}_f$ is an $\mathcal{E}_{M,b}$-module via f^*.*
(2) *For another map-germ $f' : (N, a) \to (M', b')$, $\mathcal{J}_{f'} = \mathcal{J}_f$ if and only if $\mathcal{R}_{f'} = \mathcal{R}_f$.*
(3) *If $\tau : (M, b) \to (M', b')$ is a diffeomorphism-germ, then $\mathcal{R}_{\tau\circ f} = \mathcal{R}_f$. If $\sigma : (N', a') \to (N, a)$ is a diffeomorphism-germ, then $\mathcal{R}_{f\circ\sigma} = \sigma^*(\mathcal{R}_f)$.*

Proof: (1) follows from that, if $h \in \mathcal{R}_f$ and $dh = \sum_{j=1}^m p_j df_j$, then we have

$$d\{(k \circ f)h\} = \sum_{j=1}^{m} \{(k \circ f)p_j + h\,(\partial k/\partial y_j)\}\, df_j.$$

(2) It is clear that $\mathcal{J}_{f'} = \mathcal{J}_f$ implies $\mathcal{R}_{f'} = \mathcal{R}_f$. Conversely suppose $\mathcal{R}_{f'} = \mathcal{R}_f$. Then any component f'_j of f' belongs to $\mathcal{R}_{f'} = \mathcal{R}_f$, hence $df_j \in \mathcal{J}_f$. Therefore $\mathcal{J}_{f'} \subset \mathcal{J}_f$. By the symmetry we have $\mathcal{J}_{f'} = \mathcal{J}_f$.

(3) follows from that $\mathcal{J}_{\tau\circ f} = \mathcal{J}_f$ and $\mathcal{J}_{f\circ\sigma} = \sigma^*(\mathcal{J}_f)$. $\square$

Definition 8.4. Let $f : (N, a) \to M$ and $g : (N', a') \to M'$ be map-germs. Then f and g are called *$\mathcal{J}$-equivalent* if there exists a diffeomorphism-germ $\sigma : (N, a) \to (N', a')$ such that $\mathcal{J}_{g\circ\sigma} = \mathcal{J}_f$. Here $\mathcal{J}_f = \mathcal{E}_{N,t_0} f^*\Omega^1_{M,f(t_0)}$ (see Definition 8.1). Note that $\dim(M)$ and $\dim(M')$ can be different.

Definition 8.5. Let $f : (\mathbf{R}^n, a) \to (\mathbf{R}^m, b)$ be a map-germ. Let $h_1, \dots, h_r \in \mathcal{R}_f$. Then the map-germ $F : (\mathbf{R}^n, a) \to \mathbf{R}^m \times \mathbf{R}^r = \mathbf{R}^{m+r}$ defined by

$$F = (f_1, \dots, f_m, h_1, \dots, h_r)$$

is called an *opening* of f, while f is called a *closing* of F.

Proposition 8.6. *Let M, N be a manifold of dimension m, n respectively. A map-germ $f : (N, a) \to M$ is a frontal if and only if f is right-left equivalent to an opening of a map-germ $g : (\mathbf{R}^n, 0) \to (\mathbf{R}^n, 0)$.*

Proof: Suppose $f : (N, a) \to M$ be a frontal map-germ. Then, since $\widetilde{f}(a)$ is an n-dimensional vector subspace of $T_{f(a)}M$, there exists of a system of local coordinates of $(M, f(a))$

$$y_1, \dots, y_n, z_1, \dots, z_k, (k = m - n),$$

such that, for $g = (y_1 \circ f, \dots, y_n \circ f)$, the component $z_j \circ f, (1 \leq j \leq k)$ belongs to the $\mathcal{R}_g$ and therefore f is right-left equivalent to an opening of g. Conversely, suppose f is right-left equivalent to an opening $G : (\mathbf{R}^n, a) \to \mathbf{R}^{n+k} = \mathbf{R}^m$ of a germ $g = (g_1, \dots, g_n) : (\mathbf{R}^n, a) \to (\mathbf{R}^n, g(a))$. Set $G = (g_1, \dots, g_\ell, h_1, \dots, h_k)$. Then, since $h_j \in \mathcal{R}_g, 1 \leq j \leq n$ $dh_j = \sum_{i=1}^n a^i_j dg_i$, for some function-germs $a^i_j : (\mathbf{R}^n, a) \to \mathbf{R}$. Define $\widetilde{G} : (\mathbf{R}^n, a) \to \mathrm{Gr}(n, T\mathbf{R}^m)$, in terms of Grassmannian coordinates,

$$\widetilde{G}(t) = \left(g_1(t), \dots, g_\ell(t), h_1(t), \dots, h_k(t), a^i_j(t)\right), \quad (t \in (\mathbf{R}^n, a)).$$

Then, by Lemma 3.5, $\widetilde{G}$ is a D-integral lift of g. Therefore G is an ℓ-frontal, and so is f. □

§9. Versal openings

Definition 9.1. An opening $F = (f, h_1, \dots, h_r)$ of f is called a *versal opening* (resp. a *mini-versal opening*) of $f : (\mathbf{R}^n, a) \to (\mathbf{R}^m, b)$, if $1, h_1, \dots, h_r$ form a (minimal) system of generators of $\mathcal{R}_f$ as an $\mathcal{E}_{\mathbf{R}^m,b}$-module via $f^* : \mathcal{E}_{\mathbf{R}^m,b} \to \mathcal{E}_{\mathbf{R}^n,a}$.

A C^∞ map-germ $f : (\mathbf{R}^n, a) \to (\mathbf{R}^m, b)$ is called *analytic* if f is right-left equivalent to a real analytic map-germ ([35]). Moreover f is called a *finite map-germ* if $\mathcal{E}_{\mathbf{R}^n,a}$ is a finite $f^*(\mathcal{E}_{\mathbf{R}^m,b})$-module. Then f is finite if and only if $\dim_{\mathbf{R}}(\mathcal{E}_{\mathbf{R}^n,a}/\langle f_1, \dots, f_m\rangle_{\mathcal{E}_{\mathbf{R}^n,a}} < \infty$. If f is analytic, then f is finite if and only if its complexification has isolated zero set ([100]). By $\mathfrak{m}_{\mathbf{R}^m,a}$, we denote the maximal ideal of $\mathcal{E}_{\mathbf{R}^m,a}$ which consists of function-germs vanishing at 0. By the projection $\pi_m : \mathbf{R}^{m+r} = \mathbf{R}^m \times \mathbf{R}^r \to \mathbf{R}^m$ we regard $\mathbf{R}^{m+r}$ as an affine bundle over $\mathbf{R}^m$. If f is finite and analytic, then, in the analytic category, $\mathcal{R}^\omega f$ is a finite $\mathcal{O}_{\mathbf{R}^m,b}$-module.

We summarise the known results on the existence of versal openings:

Theorem 9.2. *Let $f : (\mathbf{R}^n, a) \to (\mathbf{R}^m, b)$ be a map-germ. Suppose that* (I) *f is finite and of corank at most one*, *or* (II) *f is finite analytic. Then we have*
(1) *The ramification module $\mathcal{R}_f$ of f is a finitely generated $f^*(\mathcal{E}_{\mathbf{R}^m,b})$-module. In particular f has a versal opening.*
(2) $1, h_1, \dots, h_r \in \mathcal{R}_f$ *form a system of generators of $\mathcal{R}_f$ as a $f^*(\mathcal{E}_{\mathbf{R}^m,b})$-module if and only if the residue classes $1, \overline{h}_1, \dots, \overline{h}_r$ form an $\mathbf{R}$-basis of the vector space $V = \mathcal{R}_f/f^*(\mathfrak{m}_{\mathbf{R}^m,b})\mathcal{R}_f$. In particular there exists a versal opening $F = (f, h_1, \dots, h_r)$ of f. If $r = \dim_{\mathbf{R}} V - 1$, then F is called a mini-versal opening of f.*
(3) *For any versal opening $F : (\mathbf{R}^n, a) \to \mathbf{R}^{m+r}$ of f and for any opening $G : (\mathbf{R}^n, a) \to \mathbf{R}^{m+s}$ of f, there exists an affine bundle map $\Psi : (\mathbf{R}^{m+r}, F(a)) \to (\mathbf{R}^{m+s}, G(a))$ such that $G = \Psi \circ F$.*
(4) *For any two mini-versal openings $F, F' : (\mathbf{R}^n, a) \to \mathbf{R}^{m+r}$ of f, there exists an affine bundle isomorphism $\Phi : (\mathbf{R}^{m+r}, F(0)) \to (\mathbf{R}^{m+r}, F'(0))$ such that $F' = \Phi \circ F$.*

Proof of Theorem 9.2: Case (I): (1) is proved as Lemma 2.1 of [35]. Then $f^* : \mathcal{E}_{\mathbf{R}^m,b} \to \mathcal{R}_f$ is a homomorphism of differentiable algebras in the sense of Malgrange [74]. Therefore, by Malgrange's preparation theorem ([74] Corollary 4.4) we have (2). Case (II): Let $1, h_1, \dots, h_r$ generate $\mathcal{R}^\omega f$ over $\mathcal{O}_{\mathbf{R}^m,b}$ via f^*. Then $1, h_1, \dots, h_r$ generate $\mathcal{R}_f$ over $\mathcal{E}_{\mathbf{R}^m,b}$ via f^* by Proposition 5.2 of [45]. Therefore we have (1) and (2). The assertion (3) is clear from the definitions. (4) follows from (2). $\square$

We do not repeat the proofs of Lemma 2.1 in [35] nor Proposition 5.2 in [45]. However we give an exposition, relating the theory of C^∞-rings, on Malgrange's preparation theorem in §22 of this paper.

§10. Subfrontals and superfrontals

Relating the theory of frontals with that of openings, we are led to the following generalisations of frontals naturally.

Definition 10.1. Let M be a manifold and ℓ an integer with $0 \leq \ell \leq \dim(M)$. A map-germ $f : (N, a) \to M$ is called an *ℓ-frontal* if there exists a D-integral lift $\widetilde{f} : (N, a) \to \mathrm{Gr}(\ell, TM)$ of f. Here we do not assume that $\ell = \dim(N)$ and D is the contact distribution on $\mathrm{Gr}(\ell, TM)$. The condition on the C^∞ mapping $\widetilde{f}$ is that $f_*(T_tN) \subset \widetilde{f}(t) \in \mathrm{Gr}(\ell, T_{f(t)}M)$ for any $t \in (N, a)$. If $0 < \ell < \dim(N)$, then an ℓ-frontal is called a *subfrontal*. If $\dim(N) < \ell < \dim(M)$, then an an ℓ-frontal is called a *superfrontal*.

Proposition 10.2. *Let $f : (N, a) \to M$ be an ℓ-frontal and $f' : (N', a') \to M'$ be right-left equivalent to f. Then also f' is an ℓ-frontal.*

Proof: The proof is performed similarly to Proposition 4.3 using Proposition 3.4. □

Proposition 10.3. *Let M, N be a manifold of dimension m, n respectively and ℓ an integer with $0 \leq \ell \leq m$. A map-germ $f : (N, a) \to M$ is an ℓ-frontal if and only if f is right-left equivalent to an opening of a map-germ $g : (\mathbf{R}^n, 0) \to (\mathbf{R}^\ell, 0)$.*

Proof: Suppose $f : (N, a) \to M$ be an ℓ-frontal map-germ. Then, since $\widetilde{f}(a)$ is an ℓ-dimensional vector subspace of $T_{f(a)}M$, there exists of a system of local coordinates of $(M, f(a))$

$$y_1, \dots, y_\ell, z_1, \dots, z_k, (k = m - \ell),$$

such that, for $g = (y_1 \circ f, \dots, y_\ell \circ f)$, the component $z_j \circ f, (1 \leq j \leq k)$ belongs to the $\mathcal{R}_g$ and therefore f is right-left equivalent to an opening of g. Conversely, suppose f is right-left equivalent to an opening $G : (\mathbf{R}^n, a) \to \mathbf{R}^{\ell+k} = \mathbf{R}^m$ of a germ $g = (g_1, \dots, g_\ell) : (\mathbf{R}^n, a) \to (\mathbf{R}^\ell, g(a))$. Set $G = (g_1, \dots, g_\ell, h_1, \dots, h_k)$. Then, since $h_j \in \mathcal{R}_g, 1 \leq j \leq \ell$, $dh_j = \sum_{i=1}^\ell a_j^i dg_i$, for some function-germs $a_j^i : (\mathbf{R}^n, a) \to \mathbf{R}$. Define $\widetilde{G} : (\mathbf{R}^n, a) \to \mathrm{Gr}(\ell, T\mathbf{R}^m)$, in terms of Grassmannian coordinates,

$$\widetilde{G}(t) = \left(g_1(t), \dots, g_\ell(t), h_1(t), \dots, h_k(t), a_j^i(t)\right), \quad (t \in (\mathbf{R}^n, a)).$$

Then, by Lemma 3.5, $\widetilde{G}$ is a D-integral lift of g. Therefore G is an ℓ-frontal, and so is f. □

§11. Algebraic openings

In this section we will utilise the notion of sheaves which describes locally-defined objects ([12]) to introduce an algebraic notion which is related to frontals.

Let N be a manifold. Let $\mathcal{E}_N$ denote the sheaf of C^∞ function-germs on N. For any open subset $U \subset N$, $\mathcal{E}_N(U) = C^\infty(U)$, the C^∞-ring of all real-valued C^∞ functions on U. Note that $\mathcal{E}_N$ has the natural structure of C^∞-ring sheaf. See §22 for the notion of C^∞-rings.

Definition 11.1. Let $\mathcal{F}$ be a sub C^∞-ring sheaves of $\mathcal{E}_N$. The *versal opening* $\breve{\mathcal{F}}$ of $\mathcal{F}$ is defined as follows: For any open subset $U \subset N$, $\breve{\mathcal{F}}(U)$ is the set of $h \in \mathcal{E}_N(U)$ satisfying that, for any $p \in U$, there exists $g_1, \dots, g_r \in \mathcal{F}_p$ and $a_1, \dots, a_r \in \mathcal{E}_{N,p}$ such that

$$dh = \sum_{i=1}^{r} a_i dg_i$$

in $\Omega^1_{N,p}$. Here Ω^1_N means the sheaf of C^∞ 1-form-germs on N, $\mathcal{E}_{N,p}$ (resp. $\Omega^1_{N,p}$) the stalk of $\mathcal{E}_N$ (resp. of Ω^1_N) at p, i.e. the set of germs at p, and $d : \mathcal{E}_{N,p} \to \Omega^1_{N,p}$ the exterior differential.

Let $\mathcal{F}, \mathcal{G}$ be a sub C^∞-ring sheaves of $\mathcal{E}_N$. Then $\mathcal{G}$ is called an *opening* of $\mathcal{F}$ if $\mathcal{F} \subseteq \mathcal{G} \subseteq \breve{\mathcal{F}}$.

We have that following basic properties of algebraic openings.

Proposition 11.2. *Let N be a C^∞ manifold and let $\mathcal{E}_N$ denote the sheaf of C^∞ function-germs on N. Let $\mathcal{F}$ be a sub C^∞-ring sheaf of $\mathcal{E}_N$. Then we have*
(1) *$\breve{\mathcal{F}}$ is a sub C^∞-ring sheaf of $\mathcal{E}_N$.* (2) *$\mathcal{F} \subset \breve{\mathcal{F}}$.* (3) *$\breve{\breve{\mathcal{F}}} = \breve{\mathcal{F}}$.*

Proof: (1) Let $p \in N$. Let $h_1, \dots, h_r \in \breve{\mathcal{F}}_p$ and $f \in C^\infty(\mathbf{R}^r)$. Let $dh_i = \sum_{j=1}^{s_i} a_{ij} dg_{ij}$ for some $a_{ij} \in \mathcal{E}_{N,p}$, $g_{ij} \in \mathcal{F}_p$. Then

$$d(f(h_1, \dots, h_r)) = \sum_{i=1}^{r} \frac{\partial f}{\partial x_i}(h_1, \dots, h_r)\, dh_i = \sum_{i=1}^{r} \sum_{j=1}^{s_i} \left(\frac{\partial f}{\partial x_i}(h_1, \dots, h_r) a_{ij} \right) dg_{ij}.$$

Therefore $f(h_1, \dots, h_r) \in \breve{\mathcal{F}}_p$. (2) Let $p \in N$ and $g \in \mathcal{F}_p$. Then we have $dg = 1 \cdot dg$, and therefore $g \in \breve{\mathcal{F}}_p$. (3) Let $p \in N$ and $h \in \breve{\breve{\mathcal{F}}}_p$. Then $dh = \sum_{i=1}^{r} a_i dh_i$ for some $a_i \in \mathcal{E}_{N,p}, h_i \in \breve{\mathcal{F}}_p$. Since $h_i \in \breve{\mathcal{F}}_p$ for each i, $dh_i = \sum_{j=1}^{s_i} b_{ij} dg_{ij}$ for some $b_{ij} \in \mathcal{E}_{N,p}$ and $g_{ij} \in \mathcal{F}_p$. Then we have $dh = \sum_{i=1}^{r} \sum_{j=1}^{s_i} (a_i b_{ij}) dg_{ij}$, therefore $h \in \breve{\mathcal{F}}_p$. $\square$

We call $\mathcal{F}$ *full* if $\widetilde{\mathcal{F}} = \mathcal{F}$. Then Proposition 11.2 shows that $\widetilde{\mathcal{F}}$ is the minimal full sheaf containing $\mathcal{F}$.

Let $\varphi : N' \to N$ be a C^∞ mapping and $\mathcal{F}$ a subsheaf of $\mathcal{E}_N$ on N. We define a subsheaf $\varphi^*\mathcal{F}$ of $\mathcal{E}_{N'}$ on N' by $(\varphi^*\mathcal{F})_q = \varphi^*(\mathcal{F}_{\varphi(q)})$, where $\varphi^* : \mathcal{E}_{N,\varphi(q)} \to \mathcal{E}_{N',q}$ is defined by $\varphi^*(h) = h \circ \varphi, (h \in \mathcal{E}_{N,\varphi(q)})$. If $\varphi = \Phi$ is a diffeomorphism, then $(\Phi^*\mathcal{F})(U') = \Phi^*(\mathcal{F}(\Phi(U')))$ for any open $U' \subset N'$.

Then we have the naturality of versal openings:

Proposition 11.3. *Let $\mathcal{F}$ be a sub C^∞-ring sheaves of $\mathcal{E}_N$. For any diffeomorphism $\Phi : N' \to N$ from another manifold N', we have $\widetilde{\Phi^*\mathcal{F}} = \Phi^*\widetilde{\mathcal{F}}$.*

Proof: Let $q \in N'$ and $h \in \widetilde{\Phi^*\mathcal{F}}_q$. Then $dh = \sum_{i=1}^r a_i d(\Phi^* g_i)$ for some $a_i \in \mathcal{E}_{N',q}$ and $g_i \in \mathcal{F}_{\Phi(q)}$. Then $d(\Phi^{-1*}h) = \Phi^{-1*}(\sum_{i=1}^r a_i d(\Phi^* g_i)) = \sum_{i=1}^r (\Phi^{-1*}a_i) dg_i$. Since $\Phi^{-1*}a_i \in \mathcal{E}_{N,\Phi(q)}$, we see $\Phi^{-1*}h \in \widetilde{\mathcal{F}}_{\Phi(q)}$, therefore $h \in \Phi^*((\widetilde{\mathcal{F}})_{\Phi(q)}) = (\Phi^*\widetilde{\mathcal{F}})_q$. Thus we have $\widetilde{\Phi^*\mathcal{F}}_q \subseteq (\Phi^*\widetilde{\mathcal{F}})_q$. Applying the same argument to Φ^{-1} and $\Phi^*\mathcal{F}$, then we have $\widetilde{\Phi^*\mathcal{F}}_q \supseteq (\Phi^*\widetilde{\mathcal{F}})_q$. Therefore we have the required equality. $\square$

Definition 11.4. Let N be a manifold. Let $\mathcal{F}$ be a sub C^∞-ring sheaf of $\mathcal{E}_N$. A mapping $f : N \to M$ is called a *realisation* of $\mathcal{F}$ if $\mathcal{F} = f^*\mathcal{E}_M$.

The following is clear:

Proposition 11.5. *Let $f : (\mathbf{R}^n, a) \to (\mathbf{R}^m, b)$ be a map-germ. Let $\mathcal{F} = f^*\mathcal{E}_{\mathbf{R}^m,b}$ be the germ of subsheaf of $\mathcal{E}_{\mathbf{R}^n,a}$. Let $F : (\mathbf{R}^n, a) \to \mathbf{R}^{m+r}$ be an opening of f. Then F is a versal opening of f if and only if F is a realisation of the algebraic opening $\widetilde{\mathcal{F}}$ of $\mathcal{F}$.*

Definition 11.6. A mapping $f : N \to M$ is called *locally injective* if for any $a \in N$, there exists an open neighbourhood U of a in N such that $f|_U : U \to M$ is injective.

Proposition 11.7. *Let $f : N \to M$ be a finite mapping and $F : N \to M'$ a realisation of the versal opening $\widetilde{f^*\mathcal{E}_M}$ of $f^*\mathcal{E}_M$. Then F is locally injective.*

Proof: Let $a \in N$. Then $F^*\mathcal{E}_{M',F(a)} = \widetilde{f^*\mathcal{E}_M}_a = \mathcal{R}_{f,a}$. Then the germ of F at a is a versal opening of f. Therefore by Proposition 2.16 of [45] we have the result. $\square$

Definition 11.8. Let $\mathcal{F}$ be a sub C^∞-ring of $\mathcal{E}_N$. We call $\mathcal{F}$ *locally injective* if for any $a \in N$, there exist $h_1, \dots, h_r \in \mathcal{F}_a$ such that $(h_1, \dots, h_r) : (N, a) \to \mathbf{R}^r$ has an injective representative.

Proposition 11.9. *If $f : N \to M$ is a realisation of a locally injective sub C^∞-ring $\mathcal{F}$ of $\mathcal{E}_N$, then f is locally injective.*

Proof: Let $a \in N$. There exist $h_1, \ldots, h_r \in \mathcal{F}_a$ such that $(h_1, \ldots, h_r) : (N, a) \to \mathbf{R}^r$ has an injective representative. There exists a $g_i \in \mathcal{E}_{M,f(a)}$ such that $h_i = g_i \circ f$ for each $i, 1 \leq i \leq r$. After taking representatives of germs we have $h_i = g_i \circ f : U \to \mathbf{R}, (1 \leq i \leq r)$ on an open neighbourhood of a. Deleting U if necessary, $(h_1, \ldots, h_r) = (g_1, \ldots, g_r) \circ f : U \to \mathbf{R}^r$ is injective. Therefore $f|_U$ is injective. □

Part II. Advanced studies and applications

§12. Frontal curves

Let us give several observations on frontal map-germs and frontal maps $N \to M$ with $\dim(N) = 1$.

Let $f : (N, a) \to M$ be a map-germ with $\dim(N) = 1$. We consider the classification problem of germs up to the right-left equivalence. To simplify this, let $(N, a) = (\mathbf{R}, 0)$ and $(M, f(a)) = (\mathbf{R}^m, 0)$. Let t be the coordinate of $(\mathbf{R}, 0)$ and $x^1, \ldots, x^m$ of $(\mathbf{R}^m, 0)$. We define the *order* of f at 0 by

$$\mathrm{ord}(f) := \inf \left\{ k \in \mathbf{N} \;\middle|\; \frac{d^k f}{dt^k}(0) \neq 0 \right\}$$

If the Taylor infinite series of f is 0, then we set $\mathrm{ord}(f) = \infty$. It is easy to see that $\mathrm{ord}(f)$ is invariant under right-left equivalence.

Lemma 12.1. *If* $\mathrm{ord}(f) < \infty$, *then f is a frontal. Moreover f is right-left equivalent to an opening of the map-germ $g : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ defined by $t \mapsto t^\mu$, where $\mu = \mathrm{ord}(f)$.*

Proof: For a diffeomorphism-germ $\sigma : (\mathbf{R}, 0) \to (\mathbf{R}, 0)$ and a linear transformation $\Phi : (\mathbf{R}^m, 0) \to (\mathbf{R}^m, 0)$, $\Phi \circ f \circ \sigma$ is of form:

$$\Phi \circ f \circ \sigma = (t^\mu,\ h_2(t),\ \ldots\ ,\ h_m(t)),$$

with $h_i \in \mathfrak{m}_1^{\mu+1}, 2 \leq i \leq m$. Set $g(t) = t^\mu$. Then $\mathcal{R}_g = \mathbf{R} + \mathfrak{m}_1^\mu$ and $h_i \in \mathcal{R}_g, 2 \leq i \leq m$ (see Example 8.2). Therefore $\Phi \circ f \circ \sigma$ is an opening of g. □

Since a constant map-germ is a frontal and a non-constant analytic curve-germ has a finite order, by Lemma 12.1 we have

Corollary 12.2. *If $f : (\mathbf{R}, a) \to \mathbf{R}^m$ is an analytic map-germ, then f is a frontal.*

As for a global result, we have:

Lemma 12.3. *Let* $\dim(N) = 1$ *and* $f : N \to M$ *a frontal. Then there exists a global Legendre lift* $\widetilde{f} : N \to M$.

Proof: Let $R(f)$ denote the immersion locus of f and set $S := N \setminus \overline{R(f)}$. We have the Legendre lift $F : R(f) \to \mathrm{Gr}(1, TM)$ of $f|_{R(f)}$ which is defined by $F(t) = f_*(T_tN)$. The mapping F is extended to $\overline{R(f)}$ continuously. Since f is a frontal. F is extended to a C^∞ Legendre lift of f on an open neighbourhood of $\overline{R(f)}$. Now take any connected component J of the open set S. Then J is diffeomorphic to S^1 or an open interval. In the case that J is diffeomorphic to S^1, then $f|_J$ is of constant rank 0 and it is a constant mapping. Let us consider the case that $J \subset N$ is diffeomorphic to an open interval. Take the closure $I = \overline{J}$ in N, which is diffeomorphic to an interval, $[0, 1], (0, 1]$ or $(0, 1)$. In the case I is diffeomorphic to $[0, 1]$, consider the boundary points of I, which belong to $R(f)$ necessarily. Since the fibre of $\pi : \mathrm{Gr}(1, TM) \to M$ is diffeomorphic to the projective space $\mathrm{Gr}(1, \mathbf{R}^m) = \mathbf{R}P^{m-1}$ which is connected, we can extend the given Legendre lift $F : R(f) \to \mathrm{Gr}(1, TM)$ to a Legendre lift on an open set containing $\overline{R(f)} \cup I$. The extension is performed independently for each connected component of S. Thus we have a global Legendre lift $\widetilde{f} : N \to M$. $\square$

Remark 12.4. If $\dim(N) = 2$, then a frontal $f : N \to M$ does not necessarily have a global Legendre lifting. See Example 2.5.

Next we study the genericity problem of frontal curves. To simplify the story we treat frontals $f : \mathbf{R} \to \mathbf{R}^m$. Let $\widetilde{f} : \mathbf{R} \to \mathrm{Gr}(1, T\mathbf{R}^m) = P(T\mathbf{R}^m) = \mathbf{R}^m \times \mathbf{R}P^{m-1}$ be an integral lifting of f (see Lemma 12.3). Then, turning upside-down the view point, we start from an integral map $F : \mathbf{R} \to \mathrm{Gr}(1, T\mathbf{R}^m)$. Let F be, in terms of Grassmannian coordinates $x^1, \dots, x^m, a^2, \dots, a^m$,

$$F(t) = (x^1(t), \dots, x^m(t), a^2(t), \dots, a^m(t)),$$

which satisfies $F^*\theta^2 = 0, \dots, F^*\theta^m = 0$, namely that

$$dx^2 - a^2 dx^1 = 0, \dots, dx^m - a^m dx^1 = 0.$$

The condition is equivalent to that

$$\frac{dx^2}{dt}(t) = a^2(t)\frac{dx^1}{dt}(t), \ \dots, \ \frac{dx^m}{dt}(t) = a^m(t)\frac{dx^1}{dt}(t).$$

Therefore, if functions $x^1(t), a^2(t), \dots, a^m(t)$ and values $x^2(0), \dots, x^m(0)$ are arbitrarily given, then the integral mapping F is uniquely determined. Thus we can apply ordinary transversality theorem to discuss the genericity of frontal curves through Legendre curves.

Remark 12.5. In general we can apply a transversality argument to Legendre mappings of corank ≤ 1 and obtain the classification of generic singularities (see [33][40]). However the similar argument does not work for Legendre mappings having singularities of corank ≥ 2 (see Example 2.5, Remark 6.4).

§13. Frames and flags

As refinements of the notion of frontal curves, we consider framed curves or "flagged" curves. Flagged curves and framed curves in a space-form play important roles in topology, geometry and singularity theory. For example, as it is well-known, the self-linking number in 3-space is defined via framing ([88]). The fundamental theory of curves is formulated via osculation framing. Surface boundaries have adapted framings, etc. Two kinds of frames, adapted frames and osculating frames, are considered in [43] from the viewpoint of duality. We classify the singularities of envelopes associated to framed curves. The singularities of envelopes in E^3 were studied in [41] to apply to the flat extension problem of a surface with boundary. The problem on extensions by tangentially degenerate surfaces motivates to study the envelopes associated to framings on curves in a space form.

In this article already we have used Grassmannians to introduce the frontals. Then we are naturally led to the following definitions.

Let M be a manifold of dimension m and $\ell_1, \dots, \ell_r$ integers with $0 \leq \ell_1 < \cdots < \ell_r \leq m$. Define the *flag bundle* $\mathrm{Fl}(\ell_1, \dots, \ell_r; TM)$ over M of type $(\ell_1, \dots, \ell_r)$ as the totality of flags $V_{\ell_1} \subset \cdots \subset V_{\ell_r} \subset T_xM$ with $\dim(V_{\ell_i}) = \ell_i, (1 \leq i \leq r)$, x running over M. Then $\pi : \mathrm{Fl}(\ell_1, \dots, \ell_r; TM) \to M$ is a fibration with fibres of dimension

$$\ell_1(m - \ell_1) + (\ell_2 - \ell_1)(m - \ell_2) + \cdots + (\ell_r - \ell_{r-1})(m - \ell_r)$$

Moreover $\mathrm{Fl}(n; TM) = \mathrm{Gr}(n, TM)$.

Set $\mathcal{F} = \mathrm{Fl}(\ell_1, \dots, \ell_r; TM)$. Suppose M is endowed with an affine connection ∇. Let $\gamma(t) = (x(t), V_{\ell_1}(t), \dots, V_{\ell_r}(t))$ be a curve on $\mathcal{F}$. Let vectors $v_1(t), \dots v_{\ell_r}(t) \in T_{x(t)}M$ satisfy that $V_{\ell_j}(t) = \langle v_1(t), \dots, v_{\ell_j}(t)\rangle_{\mathbf{R}}$ for each t and $1 \leq j \leq r$. Then consider the condition

$$x'(t) \in V_{\ell_1}(t),\ \nabla v_1(t), \dots, \nabla v_{\ell_j}(t) \in V_{\ell_{j+1}}(t), (1 \leq j < r),$$

at t. Here, for a vector field $v(t)$ along a curve $x(t)$ in M, we define $\nabla v(t) := \nabla_{x'(t)} v(t)$, the covariant derivative of $v(t)$ by the velocity vector $x'(t)$. By this condition we define the distribution $D \subset T\mathcal{F}$, which depends on the given affine connection.

If M is a projective space, then the above construction is more clarified ([43]). Let V be a real vector space of dimension $m+1$ and $n_1, \dots, n_s$ integers satisfying $0 < n_1 < \cdots < n_s \leq m+1$. Define the *flag manifold* $\mathrm{Fl}(n_1, \dots, n_s, V)$ of type $(n_1, \dots, n_s)$ by the totality of flags $V_{n_1} \subset \cdots \subset V_{n_s} \subset V$ of linear subspaces with $\dim(V_{n_i}) = n_i, (1 \leq i \leq s)$. Set $\mathcal{F} = \mathrm{Fl}(n_1, \dots, n_s, V)$. Then the *canonical distribution* $D \subset T\mathcal{F}$ is defined as follows: Denote by $\pi_i : \mathcal{F} \to \mathrm{Gr}(n_i, V)$ the canonical projection to the i-th member of the flag. Then, for $v \in T_{\mathbf{V}}\mathcal{F}, \mathbf{V} \in \mathcal{F}$,

$$v \in D_{\mathbf{V}} \Longleftrightarrow \pi_{i*}(v) \in T\mathrm{Gr}(n_i, V_{n_{i+1}}) (\subset T\mathrm{Gr}(n_i, V)), (1 \leq i \leq s-1).$$

Then D is a subbundle of $T\mathcal{F}$ with

$$\mathrm{rank}(D) = n_1(n_2-n_1)+(n_2-n_1)(n_3-n_2)+\cdots+(n_s-n_{s-1})(m+1-n_s).$$

Note that the flag bundle $\mathrm{Fl}(\ell_1, \dots, \ell_r; TP(V))$ is naturally identified with the flag manifold $\mathrm{Fl}(1, \ell_1+1, \dots, \ell_r+1, V)$. Therefore the canonical differential system on $\mathrm{Fl}(\ell_1, \dots, \ell_s; TP(V))$ is induced. Then the canonical distribution D on the Grassmannian bundle $\mathrm{Gr}(n, T(P(V)) = \mathrm{Fl}(1, n+1, V)$ introduced in §3 coincides with that introduced here.

Definition 13.1. Let V be a real vector space of dimension $m+1$. A curve-germ $f : (N, a) \to P(V), \dim(N) = 1$ is called a *flagged curve* if there exists a D-integral lift $\widetilde{f} : (N, a) \to \mathrm{Fl}(1, 2, \dots, m, V)$ of f with respect to the projection $\pi_1 : \mathrm{Fl}(1, 2, \dots, m, V) = \mathrm{Gr}(1, V) = P(V)$.

Let $\gamma : N \to \mathbf{R}P^m$ be a curve and $t_0 \in N$. Take a system of projective local coordinates $(x_1, x_2, \dots, x_m)$ of $\mathbf{R}P^m$ with the centre at $\gamma(t_0)$ and the local affine representation $(\mathbf{R}, t_0) \to (\mathbf{R}^m, 0)$,

$$\gamma(t) = {}^T(x_1(t), x_2(t), \dots, x_m(t))$$

of γ. Consider the $(m \times k)$-matrix

$$W_k(t_0) := \left(\gamma'(t_0), \gamma''(t_0), \cdots, \gamma^{(k)}(t_0)\right)$$

for any integer $k \geq 1$ and $k = \infty$. Note that the rank of $W_k(t_0)$ is independent of the choice on representations for γ.

Definition 13.2. We call γ *of finite type* at $t = t_0 \in N$ if the $(m \times \infty)$-matrix

$$W_\infty(t_0) = \Big(\gamma'(t_0),\ \gamma''(t_0),\ \cdots,\ \gamma^{(k)}(t_0),\ \cdots\cdots\Big)$$

is of rank m. Define, for $1 \le i \le m$, $a_i := \min\{k \mid \operatorname{rank} W_k(t_0) = i\}$. Then we have a sequence of natural numbers $1 \le a_1 < a_2 < \cdots < a_m$, and we call γ *of type* $(a_1, a_2, \dots, a_m)$ at $t = t_0 \in N$.

If $(a_1, a_2, \dots, a_m) = (1, 2, \dots, m)$, then $t = t_0$ is called an *ordinary point* of γ.

Let $f : N \to \mathbf{R}P^m$ be of finite type at $t_0 \in N$. Then the *osculating flag* to f at t_0 is defined by

$$O_1(t_0) \subset O_2(t_0) \subset \cdots \subset O_k(t_0) \subset \cdots \subset O_m(t_0) = T_{f(t_0)}\mathbf{R}P^m,$$

where O_r is the linear subspace of $T_{f(t_0)}\mathbf{R}P^m$ generated by

$$\gamma'(t_0),\ \gamma''(t_0),\ \cdots,\ \gamma^{(k)}(t_0).$$

The corresponding projective subspace through $f(t_0)$ to $O_k(t_0)$ is also regarded ([34]). Then there exists unique integral lift

$$\widetilde{f} : N \to \mathrm{Fl}(1, 2, \dots, k, \dots, m, V)$$

of f.

The classification results on singularities which are related to flagged curves are given in [41][42][43].

§14. Legendre duality

The Legendre duality is a natural geometric framework where the frontals play fundamental roles. In this section we review several studies of frontals in specified (semi-)Riemannian manifolds from [42][53].

Let $\mathbf{R}^{n,m}$ denote the metric vector space of signature (n, m), n minus and m plus ([27][87]).

We write $\mathbf{R}^{0,n}$ as $\mathbf{R}^n$ simply. Recall the space-models, the *sphere* and the *hyperbolic space*,

$$S^{n+1} = \{x \in \mathbf{R}^{n+2} \mid x\cdot x = 1\},\ H^{n+1} = \{x \in \mathbf{R}^{1,n+1} \mid x\cdot x = -1,\ x^0 > 0\},$$

where $\mathbf{R}^{1,n+1} = \mathbf{R}^{n+2}_1 = \{(x^0, x^1, \dots, x^{n+1})\}$ is the Minkowski space of index $(1, n+1)$ (See for instance [66][27]). The inner product in $\mathbf{R}^{1,n+1}$ is defined by $x \cdot y = -x^0y^0 + \sum_{i=1}^{n+1} x^iy^i$. Moreover we identify Euclidean space E^{n+1} with $\{x \in \mathbf{R}^{n+2} \mid x^0 = 1\} \subset \mathbf{R}^{n+2}$ if necessary.

Let X denote S^{n+1}, H^{n+1} or E^{n+1}. Set $Z = \widetilde{\mathrm{Gr}}(n, TX)$, the oriented Grassmannian bundle over X. Then Z is a double covering of $\mathrm{Gr}(n, TX)$. The space Z is identified with T_1X, the unit tangent bundle to X. In fact,

$$\begin{array}{rcl} T_1S^{n+1} & = & \{(x, y) \in S^{n+1} \times S^{n+1} \mid x \cdot y = 0\}, \\ T_1H^{n+1} & = & \{(x, y) \in H^{n+1} \times S^{1,n} \mid x \cdot y = 0\}, \end{array}$$

where $S^{1,n} = \{x \in \mathbf{R}^{1,n+1} \mid x \cdot x = 1\}$ is the *de Sitter space*. Note that $Z = T_1H^{n+1}$ is identified with $T_{-1}S^{1,n} = \{(y, v) \mid y \in S^{1,n}, v \in T_yS^{1,n}, v \cdot v = -1\}$. Moreover $T_1E^{n+1} = E^{n+1} \times S^n$. We set $Y = S^{n+1}, S^{1,n}, \mathbf{R} \times S^n$ corresponding to $S^{n+1}, H^{n+1}, E^{n+1}$ respectively. Define $\pi_1 : Z \to X$ by the projection to the first component in three cases. Define $\pi_2 : Z \to Y$ by the projection to the second component in the cases $(X, Y) = (S^{n+1}, S^{n+1}), (X, Y) = (H^{n+1}, S^{1,n})$. In the case $(X, Y) = (E^{n+1}, \mathbf{R} \times S^n)$, we define $\pi_2 : Z = E^{n+1} \times S^n \to \mathbf{R} \times S^n$ by $\pi_2(x, y) = (-x \cdot y,\, y)$. The space has the canonical contact structure and all fibres of π_1 and π_2 are Legendre submanifolds. Therefore π_1 and π_2 are Legendre fibrations. Then we have the double Legendre fibration in each case:

$$X \xleftarrow{\pi_1} Z \xrightarrow{\pi_2} Y.$$

As the model of duality, we do have the projective duality ([98][58]): We set

$$Z \;=\; \mathcal{I}_{n+2} \;:=\; \{([x], [y]) \in P^{n+1} \times P^{n+1*} \mid x \cdot y = 0\}.$$

Here P^{n+1*} is the dual projective space and $\cdot$ means the natural paring. The contact structure on $\mathcal{I}_{n+2}$ is defined by $dx \cdot y = x \cdot dy = 0$ ([58]). The projections $\pi_1 : \mathcal{I}_{n+2} \to X = P^{n+1}, \pi_2 : \mathcal{I}_{n+2} \to Y = P^{n+1*}$ are both Legendre fibrations.

The following fact is basic to unify our treatment:

Proposition 14.1. ([52][53]) *All Legendre double fibrations $X \longleftarrow Z \longrightarrow Y$ constructed above are locally isomorphic to each other. In particular each of them is locally isomorphic to the double fibration of the projective duality $P^{n+1} \longleftarrow \mathcal{I}_{n+2} \longrightarrow P^{n+1*}$.*

Let $f : N^n \to X$ be a *co-oriented* proper frontal (see Definition 7.1). Then there arises naturally the Legendre lift $\widetilde{f} : N \to T_1X = Z$ for $\pi_1 : Z \to X$ by attaching the unit normal vector field along f. The *Legendre dual* of f is defined by $f^\vee := \pi_2 \circ \widetilde{f} : N \to Y$. Then $f^\vee$ is a frontal. If $f^\vee$ is a proper frontal. Then we have the equality $f^{\vee\vee} = f$.

Let $\gamma : I \to X$ be a C^∞ immersion from an interval or a circle I. In general, we mean by a *framing* of the immersed curve γ, an oriented orthonormal frame $(e_1, e_2, \dots, e_{n+1})$ along γ. An immersion γ is called *framed* if a framing is given.

Remark 14.2. Note that in [62][18], more general framings are considered to treat also light cone in Minkowski space.

If $X = S^{n+1}$, then we set $e_0(t) = \gamma(t) \in S^{n+1}$, and we have the moving frame $\widetilde{\gamma} = (e_0, e_1, \dots, e_{n+1}) : I \to G = SO(n+2) \subset \mathrm{GL}_+(n+2, \mathbf{R})$.

If $X = H^{n+1}$, then we set $e_0(t) = \gamma(t) \in H^{n+1}$, and we have the moving frame $\widetilde{\gamma} = (e_0, e_1, \dots, e_{n+1}) : I \to G = SO(1, n+1) \subset \mathrm{GL}_+(n+2, \mathbf{R})$.

In any of three cases, the frame manifold G is identified with an open subset of the oriented flag manifold $\widetilde{\mathcal{F}}_{n+2}$ consisting of oriented complete flags

$$V_1 \subset V_2 \subset \cdots \subset V_{n+1} \subset \mathbf{R}^{n+2}$$

in $\mathbf{R}^{n+2}$. For each $g = (e_0, e_1, \dots, e_{n+1}) \in \mathrm{GL}_+(n+2, \mathbf{R})$, we set the oriented subspace

$$V_i = \langle e_0, e_1, \dots, e_{i-1} \rangle_{\mathbf{R}} \subset \mathbf{R}^{n+2}, \ (1 \le i \le n+1).$$

This induces an open embedding $G \to \widetilde{\mathcal{F}}_{n+2} = \mathrm{Fl}(1, 2, \dots, n+1)$. Thus, for a framed curve $\gamma : I \to X$ in $X = E^{n+1}, S^{n+1}, H^{n+1}$, with the frame $(e_1, \dots, e_{n+1})$, we have the flagged curve $\widetilde{\gamma}$ by setting

$$V_i(t) = \langle e_0(t), e_1(t), \dots, e_{i-1}(t) \rangle_{\mathbf{R}} \subset \mathbf{R}^{n+2}, \ (1 \le i \le n+1).$$

Then $\widetilde{\gamma}$ is a lifting of γ for the projection $\pi_1 : \widetilde{\mathrm{Fl}}(1, 2, \dots, n+1, \mathbf{R}^{n+2}) \to \widetilde{\mathrm{Gr}}(1, \mathbf{R}^{n+2})$ to Grassmannian of oriented lines in $\mathbf{R}^{n+2}$. Note that there is the natural open embedding $X \subset \widetilde{\mathrm{Gr}}(1, \mathbf{R}^{n+2})$ in each of three cases.

The projective duality plays an essential role, for instance, to formulate the famous Plücker-Klein's formula, to analyse generic projective hypersurface (Bruce, Platonova, Landis [7]), tangent surfaces and Monge-Ampère equations ([52]).

Let $f : N^n \longrightarrow \mathbf{R}P^{n+1}$ be a frontal. Then we have the Legendre lifting $\widetilde{f} : N \longrightarrow Gr(n, T\mathbf{R}P^{n+1}) = PT^*\mathbf{R}P^{n+1}$. Then we get the *projective dual* $f^\vee : N \longrightarrow \mathbf{R}P^{n+1*}$ of f by the composition of $\widetilde{f}$ with the projection $\pi^* : PT^*\mathbf{R}P^{n+1*} \longrightarrow \mathbf{R}P^{n+1*}$. If f is sufficiently generic, then $f^\vee$ is also frontal, and we get the presumable equality $f^{\vee\vee} = f$.

Viewed from Legendre duality, we consider the class of tangentially degenerate frontals.

Definition 14.3. Let $f : N \to \mathbf{R}P^{n+1}(S^{n+1}, H^{n+1}, E^{n+1})$ be a proper frontal. Then f is called *tangentially degenerate* if the regular locus $R(f^\vee) = \{t \in N \mid f^\vee$ is an immersion at $t\}$ of the dual $f^\vee$ of f is not dense in N.

See the basic text [3] for the tangentially degenerate submanifolds.

§15. Grassmannian frontals

With the notion of frontals, we are naturally led to the following generalization of the projective duality.

Let $f : N^n \longrightarrow \mathbf{R}P^m$ be a frontal of codimension $r = m - n$. Then, consider the Legendre lifting of f :

$$\begin{array}{rcl} \widetilde{f} : N \longrightarrow \mathrm{Gr}(n, T\mathbf{R}P^m) & \hookrightarrow & \mathrm{Gr}(1, \mathbf{R}^{m+1}) \times \mathrm{Gr}(n+1, \mathbf{R}^{m+1}) \\ & \cong & \mathrm{Gr}(1, \mathbf{R}^{m+1}) \times \mathrm{Gr}(r, \mathbf{R}^{m+1*}). \end{array}$$

The Grassmannian bundle $\mathrm{Gr}(n, T\mathbf{R}P^m)$ is identified with

$$\mathcal{I} = \{(p, q) \in \mathrm{Gr}(1, \mathbf{R}^{m+1}) \times \mathrm{Gr}(r, \mathbf{R}^{m+1*}) \mid p \subseteq q^\vee\}.$$

Here, for $q \in \mathrm{Gr}(r, \mathbf{R}^{m+1*})$, we set $q^\vee := \{v \in \mathbf{R}^{m+1} \mid \alpha(v) = 0 (\alpha \in q)\}$.

Therefore we are naturally led to define the *Grassmannian dual* $f^\vee : N \longrightarrow \mathrm{Gr}(r, \mathbf{R}^{m+1*})$ of $f : N \longrightarrow \mathbf{R}P^m$ by $\widetilde{f}$ composed with the projection to the second component, $(p, q) \mapsto q$.

Definition 15.1. A proper (co-oriented) frontal $f : N^n \to \mathbf{R}P^m$ $(S^{n+1}, H^{n+1}, E^{n+1})$ is called *tangentially degenerate* if the regular locus $R(f^\vee) = \{t \in N \mid f^\vee$ is an immersion at $t\}$ of the Grassmannian dual $f^\vee$ of f is not dense in N.

Returning to the general case, we remark that the equality "$f^{\vee\vee} = f$" does not have any meaning, even if $f^\vee$ is a proper frontal in the sense of Definition 4.1. Therefore, for a mapping into a Grassmannian, it is natural to specialise the definition of frontals as follows:

Let $f : N \longrightarrow \mathrm{Gr}(r, \mathbf{R}^{m+1})$ be a C^∞ mapping with $n + r \leq m + 1$. Set $s = m + 1 - n - r$. Then f is called *Grassmannian frontal* if there exists the *unique* integral lift $\widetilde{f} : M \longrightarrow (\mathcal{I}, \mathcal{D})$ of f with respect to a fibration $\pi_1 : \mathcal{I} \longrightarrow \mathrm{Gr}(r, \mathbf{R}^{m+1})$ and a distribution $\mathcal{D}$ on $\mathcal{I}$ defined as follows: First set

$$\mathcal{I} := \{(p, q) \in \mathrm{Gr}(r, \mathbf{R}^{m+1}) \times \mathrm{Gr}(s, \mathbf{R}^{m+1*}) \mid p \subseteq q^\vee\},$$

and consider the projection $\pi_1 : I \to \mathrm{Gr}(r, \mathbf{R}^{n+2})$ (resp. $\pi_2 : I \to \mathrm{Gr}(s, \mathbf{R}^{m+1*})$). Moreover set

$$\mathcal{P} := \{(p, q, p') \in \mathrm{Gr}(r, \mathbf{R}^{m+1}) \times \mathrm{Gr}(s, \mathbf{R}^{m+1*}) \times \mathrm{Gr}(r, \mathbf{R}^{m+1}) \mid p \subseteq q^\vee, p' \subseteq q^\vee\},$$

and consider the projection $\rho : \mathcal{P} \to \mathcal{I}$ to the first and second factors (resp. $\varphi : \mathcal{P} \to \mathrm{Gr}(r, \mathbf{R}^{m+1})$ to the third factor). Then we get the double fibration $(\rho,\ \varphi)$:

$$\mathcal{I} \xleftarrow{\rho} \mathcal{P} \xrightarrow{\varphi} \mathrm{Gr}(r, \mathbf{R}^{m+1}).$$

For each $c = (p, q) \in \mathcal{I}$, we consider $\rho^{-1}(c)$. Then we consider its projection

$$\varphi(\rho^{-1}(c)) = \{p' \in \mathrm{Gr}(r, \mathbf{R}^{m+1}) \mid p' \subseteq q^{\vee}\}$$

by φ, which is regarded as $\mathrm{Gr}(r, \mathbf{R}^{r+n})$. Note that $\dim q^{\vee} = r + n$, $p \in \varphi(\rho^{-1}(c))$ and that $\varphi(\rho^{-1}(c)) \subset \mathrm{Gr}(r, \mathbf{R}^{m+1})$ is a submanifold of codimension $r(m + 1 - r) - rn = rs$.

Define the *tautological subbundle* $\mathcal{D} \subset T\mathcal{I}$ of codimension rs, for each $c = (p, q) \in \mathcal{I}$, by

$$\mathcal{D}_c = \pi_*^{-1}(T_p(\varphi(\rho^{-1}(c)))) \subset T_c\mathcal{I}.$$

Note that, if $r \neq 1$, or, $r \neq n + 1$, then the "system of tangential linear subspaces" $\{\varphi(\rho^{-1}(c)) \mid c \in I\}$ in the Grassmannian $\mathrm{Gr}(r, \mathbf{R}^{m+1})$ defined by $\mathcal{D}$ does not represent general tangential linear subspaces of the Grassmannian.

If we take local Grassmannian coordinates $(a_{ij})_{1\leq i\leq r, 1\leq j\leq n+s}$ of $\mathrm{Gr}(r, \mathbf{R}^{m+1})$ and $(b_{k\ell})_{1\leq k\leq n+r, 1\leq \ell\leq s}$ of $\mathrm{Gr}(s, \mathbf{R}^{m+1*})$, then $\mathcal{I}$ is defined by the system of equations

$$b_{ij} + a_{i1}b_{r+1\,j} + \cdots + a_{in}b_{r+n\,j} + a_{i\,n+j} = 0,\ 1 \leq i \leq r, 1 \leq j \leq s,$$

and $\mathcal{D}$ is defined by the system of 1-forms

$$b_{r+1\,j}da_{i1} + \cdots + b_{r+n\,j}da_{in} + da_{i\,n+j} = 0,\ 1 \leq i \leq r, 1 \leq j \leq s.$$

The integral lifting $\widetilde{f}$ is called the *Legendre lifting* of f in the generalised sense. The relation to the original definition of frontals is as follows:

Lemma 15.2. *Let* $F : (\mathbf{R}^n, 0) \longrightarrow (\mathcal{I}, (p_0, q_0))$ *be an integral map-germ to the distribution* $\mathcal{D} \subset T\mathcal{I}$. *Then* $f = \pi_1 \circ F : (\mathbf{R}^n, 0) \longrightarrow (\mathrm{Gr}(r, \mathbf{R}^{m+1}), p_0)$ *is Grassmannian frontal if and only if* $\kappa \circ f$ *is proper, i.e.* $S(\kappa \circ f) \subset (\mathbf{R}^n, 0)$ *is nowhere dense, for some projection*

$$\kappa : (\mathrm{Gr}(r, \mathbf{R}^{m+1}), p_0) \hookrightarrow (Hom(\mathbf{R}^r, \mathbf{R}^{n+s}), 0) \xrightarrow{i^*} (Hom(\mathbf{R}, \mathbf{R}^{n+s}), 0) \hookrightarrow \mathbf{R}P^{n+s-1},$$

induced from a linear inclusion $i : \mathbf{R} \hookrightarrow \mathbf{R}^r$.

Now, from the duality, we have another distribution $\mathcal{D}' \subset T\mathcal{I}$ from the projection $\pi' : \mathcal{I} \longrightarrow \mathrm{Gr}(s, \mathbf{R}^{m+1*})$ to the second factor, setting

$$\mathcal{P}' = \{(q', p, q) \in \mathrm{Gr}(s, \mathbf{R}^{m+1*}) \times \mathrm{Gr}(r, \mathbf{R}^{n+2}) \times \mathrm{Gr}(s, \mathbf{R}^{m+1*}) \mid q \subseteq p^\vee,\ q' \subseteq p^\vee\}.$$

Then the fundamental result is the following:

Proposition 15.3. *Two distributions $\mathcal{D}$ and $\mathcal{D}'$ on the incidental manifold $\mathcal{I}$ coincide.*

We conclude this section by the following observation:

Proposition 15.4. *Let $F : N^n \to I \subset Gr(r, \mathbf{R}^{m+1}) \times Gr(s, \mathbf{R}^{m+1*})$ be an integral mapping to the distribution $\mathcal{D}$ with $n + r + s = m + 1$. Suppose $\pi \circ F =: f$ and $\pi' \circ F =: f^\vee$ are Grassmannian frontals respectively. Then we have $f^{\vee\vee} = f$.*

§16. Tangent varieties

Given a curve in Euclidean 3-space $\mathbf{E}^3 = \mathbf{R}^3$, the embedded tangent lines to the curve draw a surface in $\mathbf{R}^3$, which is called the *tangent surface* (or *tangent developable*) to the curve ([17][35]). It is known that the tangent surfaces (tangent developables) are developable surfaces. Developable surfaces which are locally isometric to the plane keep on interesting many mathematicians, for instance, Monge (1764), Euler (1772), Cayley (1845), Lebesgue (1899). See [73] for details. Therefore the tangent surfaces are regarded as generalised solutions (with singularities) of the Monge-Ampère equation

$$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2 = 0$$

on spacial surfaces $z = z(x, y)$. Tangent surfaces are flat in E^3. However they are not flat but "extrinsically flat" or tangentially degenerate in S^3 and H^3 (cf. [3][71]). See also §14. The notion of *types* (a_1, a_2, a_3) for a curve-germ is introduced (Definition 13.2). Then the cuspidal edge, (resp. the swallowtail, the cuspidal beaks (Mond surface), the cuspidal butterfly) is obtained as the tangent developable of a curve of type $(1, 2, 3)$ (resp. $(2, 3, 4)$, $(1, 3, 4)$, $(3, 4, 5)$).

This property is related to "projective duality": *The projective dual of a tangent surface collapse to a curve* (*the dual curve*). See [36].

Let $\gamma : \mathbf{R} \to \mathbf{R}^3$ be an immersed curve. Then the tangent surface has the natural parametrization

$$\mathrm{Tan}(\gamma) : \mathbf{R}^2 \to \mathbf{R}^3, \quad \mathrm{Tan}(\gamma)(t, s) := \gamma(t) + s\gamma'(t).$$

The tangent surface necessarily has singularities at least along γ, "the edge of regression".

It is known that the tangent surface to a generic curve $\gamma : \mathbf{R} \to \mathbf{R}^3$ in $\mathbf{R}^3$ has singularities only along γ and is locally diffeomorphic to the cuspidal edge or to the folded umbrella (also called, the cuspidal cross cap), as is found by Cayley and Cleave (1980). Cuspidal edge singularities appear along ordinary points where $\gamma', \gamma'', \gamma'''$ are linearly independent, while the folded umbrellas appear at isolated points of zero torsion where $\gamma', \gamma'', \gamma'''$ are linearly dependent but $\gamma', \gamma'', \gamma''''$ are linearly independent.

In a higher dimensional space $\mathbf{R}^m, m \geq 4$, for an immersed curve $\gamma : \mathbf{R} \to \mathbf{R}^m$, we define the tangent surface $\mathrm{Tan}(\gamma) : \mathbf{R}^2 \to \mathbf{R}^m$ by $\mathrm{Tan}(\gamma)(t, s) := \gamma(t) + s\gamma'(t)$. Then we have generically that $\gamma', \gamma'', \gamma'''$ are linearly independent and $\mathrm{Tan}(\gamma)$ is locally diffeomorphic to the (embedded) cuspidal edge in $\mathbf{R}^m$. Now we give the general definition:

Definition 16.1. Let N be an n-dimensional manifold. Let $f : N^n \to \mathbf{R}^m$ be a proper frontal. Let $\widetilde{f} : N \to \mathrm{Gr}(n, T\mathbf{R}^m)$ be the Legendre lift of f. Then the *tangent mapping* $\mathrm{Tan}(f) : T_f \to \mathbf{R}^m$ of f is defined by, for $t \in N$ and $v \in \widetilde{f}(t) \subset T_{f(t)}\mathbf{R}^m$,

$$\mathrm{Tan}(f)(t, v) := f(t) + v, \quad (t, v) \in T_f,$$

using the affine structure of $\mathbf{R}^m$. Then we define the *tangent variety* of f as the parametrised variety which is defined by the right equivalence class of $\mathrm{Tan}(f)$. If $(t_1, \dots, t_n)$ is a system of local coordinates of N, and $(t_1, \dots, t_n, s_1, \dots, s_n)$ the induced system of local coordinates of T_f induced by a system of local frame $v_1(t), \dots, v_n(t)$ of $\widetilde{f}$, then $\mathrm{Tan}(f)$ is given by

$$\mathrm{Tan}(f)(t, s) = f(t) + \sum_{j=1}^{n} s_j v_j(t).$$

Also note that we can define similarly the tangent varieties of mappings to a projective space. Tangent varieties appear in various geometric problems and applications naturally ([3][14][54][55][56][64][69][65][90][73][102]). See [36][43], for the geometric exposition on the local classification problem of tangent varieties. In particular it is proved in [43][56] the following:

Proposition 16.2. *Let* $\gamma : (N, t_0) \to \mathbf{R}P^m$ *be a curve-germ of finite type* (*Definition 13.2*). *Then* $\mathrm{Tan}(\gamma) : (N \times \mathbf{R}, (t_0, 0)) \to \mathbf{R}P^m$ *is a proper frontal.*

A proper frontal $f : N \to M$ is called a *directed curve* if $\dim(N) = 1$ ([59][60][61]). A directed curve γ is called *orientable* if there exists a frame $u : N \to TM$, $u(t) \neq 0$, along γ such that $\gamma'(t) \in \langle\, u(t)\, \rangle_{\mathbf{R}}$, $t \in \mathbf{R}$, which projects to the unique Legendre lift $\widetilde{\gamma} : N \to P(TM) = \mathrm{Gr}(1, TM)$ of γ satisfying $\gamma'(t) \in \widetilde{\gamma}(t)$, $(t \in \mathbf{R})$.

Let $\gamma : N \to M$ be a directed curve and $\widetilde{\gamma} : N \to P(TM)$ the unique D-integral lift of f. Recall that the *tangent bundle* to f is defined by $T_\gamma := \{(t, v) \in N \times TM \mid v \in \widetilde{\gamma}(t)\}$, which is a line bundle over N (see §7). Let M be a manifold with an affine connection. We define the *tangent mapping* $\mathrm{Tan}(\gamma) : T_\gamma \to M$ by $(t, v) \to \exp(v)$, using the exponential map (see §18).

Remark 16.3. By Lemma 12.3, there exists a global Legendre lift $\widetilde{\gamma} : N \to P(TM)$ of f. Then the orientability condition means that the line bundle $T_{\widetilde{f}}$ over N is orientable.

Let $M = \mathbf{R}^m$ and $\gamma : N \to \mathbf{R}^m$ be a directed and orientable curve. Then the tangent surface $\mathrm{Tan}(\gamma) : N \times \mathbf{R} \to \mathbf{R}^m$ of a directed curve γ is defined by

$$\mathrm{Tan}(\gamma)(t, s) := \gamma(t) + s\, u(t)$$

The right equivalence class of $\mathrm{Tan}(\gamma)$ is independent of the choice of frame u.

The singularities of the tangent surface $\mathrm{Tan}(\gamma)$ for a generic directed curve $\gamma : \mathbf{R} \to \mathbf{R}^m$ on a neighbourhood of the curve are only the cuspidal edge, the folded umbrella, and swallowtail if $m = 3$, and the embedded cuspidal edge and the open swallowtail if $m \geq 4$. See [20][59]. Several degenerate cases are studied in [76][77][32][34][35][36].

§17. Grassmannian geometry

We will give a series of classification results of singularities of tangent surfaces in A_n-geometry, i.e. the geometry associated to the group $\mathrm{PGL}(n + 1, \mathbf{R})$ (see [57]).

Let $V = \mathbf{R}^{m+1}$ be the vector space of dimension $m + 1$ and consider a flag in V of the following type (a complete flag):

$$V_1 \subset V_2 \subset V_3 \subset \cdots \subset V_m \subset V, \quad \dim(V_i) = i.$$

The set of such flags form a manifold $\mathrm{Fl}(1, 2, 3, \ldots, m)$ of dimension $\frac{n(n+1)}{2}$.

A curve $\gamma : \mathbf{R} \to P(V) = P(V^{m+1})$ induces a D-integral curve $\Gamma : \mathbf{R} \to \mathrm{Fl}(1, 2, 3, \ldots, m)$ for the canonical distribution D on the flag manifold $T\mathrm{Fl}(1, 2, 3, \ldots, m)$, if we regard its osculating planes: the curve itself is given by $V_1(t)$, the tangent line is given by $V_2(t)$, the osculating plane is given by $V_3(t)$ and so on.

Let $m = 2$. Let $V_1(t) \subset V_2(t) \subset V = \mathbf{R}^3$ be an admissible curve. For each a, planes V_2 satisfying $V_1(a) \subset V_2 \subset V$ form the tangent line to the curve $\{V_1(t)\}$ at $t = a$ in $P(V) = P^2$. Similarly lines V_1 satisfying $V_1 \subset V_2(a)$ form the tangent line to the dual curve $\{V_2(t)\}$ at $t = a$ in $\mathrm{Gr}(2, V) = P(V^*) = P^{2*}$, the dual projective plane. For a generic admissible curve, we have the duality on "tangent maps":

Let $m = 3$. Let $\Gamma : \mathbf{R} \to \mathrm{Fl}(1, 2, 3)$ be a D-integral curve. Set $\Gamma(t) = (V_1(t), V_2(t), V_3(t))$, $V_1(t) \subset V_2(t) \subset V_3(t) \subset V = \mathbf{R}^4$. Then Γ induces the curve $\pi_1 \circ \Gamma$ in $P^3 = P(\mathbf{R}^4)$, the curve $\pi_2 \circ \Gamma$ in $\mathrm{Gr}(2, \mathbf{R}^4)$ and the curve $\pi_3 \circ \Gamma$ in $P^{3*} = \mathrm{Gr}(3, \mathbf{R}^4)$. Then we have the following duality on their tangent surfaces in A_3-geometry:

3	4	3
Cuspidal Edge	Cuspidal Edge	Cuspidal Edge
Swallow Tail	Cuspidal Edge	Folded Umbrella
Mond Surface	Open Swallowtail	Mond Surface
Folded Umbrella	Cuspidal Edge	Swallow Tail

In general for a generic D-integral curve $\Gamma : \mathbf{R} \to \mathrm{Fl}(1, 2, 3, \dots, m)$,

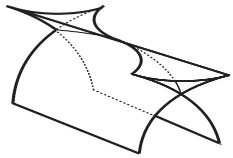
$$V_1(t) \subset V_2(t) \subset V_3(t) \subset \cdots \subset V_m(t) \subset V = \mathbf{R}^{m+1},$$

we have the classification of singularities of tangent surfaces:

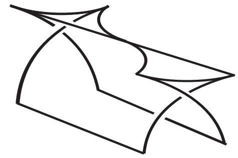
Theorem 17.1. ($A_n, n \geq 4$) *The classification list consists of* $n+1$ *cases for curves in Grassmannians*:

P^n	$\mathrm{Gr}(2,V)$	$\mathrm{Gr}(3,V)$	$\mathrm{Gr}(4,V)$	$\cdots$	$\mathrm{Gr}(n,V)$
CE	CE	CE	CE	$\cdots$	CE
OSW	CE	CE	CE	$\cdots$	CE
OM	OSW	CE	CE	$\cdots$	OFU
OFU	CE	OSW	CE	$\cdots$	OM
CE	CE	CE	OSW	$\cdots$	CE
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
CE	CE	CE	CE	$\cdots$	OSW

The *cuspidal edge* (resp. *open swallowtail*, *open Mond surface*, *open folded umbrella*) is defined as a diffeomorphism equivalence class of the tangent surface-germ to a curve of type $(1, 2, 3, \cdots)$ (resp. $(2, 3, 4, 5, \cdots)$, $(1, 3, 4, 5, \cdots)$, $(1, 2, 4, 5, \cdots)$) in an affine space.



Mond surface (cuspidal-beaks)



open Mond surface

§18. Affine connection and tangent surface

Now let us consider the case of directed curves in a Riemannian manifold, or more generally, the case of directed curves in a manifold with any affine connection, which is not necessarily projectively flat. For any directed curve, we have the well-defined tangent geodesic to each point of the curve. If we regard it as the "tangent line", then we have the well-defined tangent surface for the directed curve.

It is proved in [59], for any affine connection on a manifold of dimension $m \geq 3$, the singularities of the tangent surface to a generic directed curve on a neighbourhood of the curve are only the *cuspidal edge*, the *folded umbrella*, and *swallowtail* if $m = 3$, and the *embedded cuspidal edge* and the *open swallowtail* if $m \geq 4$. Moreover we have:

Theorem 18.1. ([59]) *Let ∇ be any torsion-free affine connection on a manifold M. Let $\gamma : \mathbf{R} \to M$ be a C^∞ curve.*

(1) *Let* $\dim(M) = 3$. *If $(\nabla\gamma)(a)$, $(\nabla^2\gamma)(a)$, $(\nabla^3\gamma)(a)$ are linearly independent at $t = a \in \mathbf{R}$, then the tangent surface* $\mathrm{Tan}(\gamma)$ *is locally diffeomorphic to the cuspidal edge at $(a, 0) \in \mathbf{R}^2$. If $(\nabla\gamma)(a)$, $(\nabla^2\gamma)(a)$, $(\nabla^3\gamma)(a)$ are linearly dependent, and $(\nabla\gamma)(a)$, $(\nabla^2\gamma)(a)$, $(\nabla^4\gamma)(a)$ are linearly independent, then the tangent surface* $\mathrm{Tan}(\gamma)$ *is locally diffeomorphic to the folded umbrella at $(a, 0) \in \mathbf{R}^2$. If $(\nabla\gamma)(a) = 0$ and $(\nabla^2\gamma)(a)$, $(\nabla^3\gamma)(a)$, $(\nabla^4\gamma)(a)$ are linearly independent, then the tangent surface* $\mathrm{Tan}(\gamma)$ *is locally diffeomorphic to the swallowtail at $(a, 0) \in \mathbf{R}^2$.*

(2) *Let* $\dim(M) \geq 4$. *If $(\nabla\gamma)(a), (\nabla^2\gamma)(a), (\nabla^3\gamma)(a)$ are linearly independent at $t = a \in \mathbf{R}$, then the tangent surface* $\mathrm{Tan}(\gamma)$ *is locally diffeomorphic to the embedded cuspidal edge at $(a, 0) \in \mathbf{R}^2$. If $(\nabla\gamma)(a) = 0$ and $(\nabla^2\gamma)(a), (\nabla^3\gamma)(a), (\nabla^4\gamma)(a), (\nabla^5\gamma)(a)$ are linearly independent at $t = a \in \mathbf{R}$, then the tangent surface* $\mathrm{Tan}(\gamma)$ *is locally diffeomorphic to the open swallowtail at $(a, 0) \in \mathbf{R}^2$.*

For the proof of Theorem 18.1, we apply the characterisation theorems found in [71][25][43].

In [59][60], singularities of tangent surfaces of torsionless curves are studied. In that case, so called fold singularities and $(2, 5)$-cuspidal edges appear. See also [28].

§19. Characterisation of frontal singularities

When we treat singularities in a general ambient space as in the previous section, we need the intrinsic characterisations of singularities. Note that the characterization of swallowtails was applied to hyperbolic geometry in [71] and to Euclidean and affine geometries in [52]. The characterization of folded umbrellas is applied to Lorenz-Minkowski geometry in [25]. In Theorem 18.1, we apply to non-flat projective geometry the characterisations and their some generalization via the notion of openings introduced in §8.

Let $f : (\mathbf{R}^2, p) \to M^3$ be a frontal with a non-degenerate singular point at p (see Lemma 6.7) and $\widetilde{f} : (\mathbf{R}^2, p) \to \mathrm{Gr}(2, TM)$ the integral lifting of f. Let $V_1, V_2 : (\mathbf{R}^2, p) \to TM$ be an associated frame with $\widetilde{f}$. Let $L : (\mathbf{R}^2, p) \to T^*M \setminus \zeta$ be an annihilator of $\widetilde{f}$. The condition is that $\langle L, V_1\rangle = 0, \langle L, V_2\rangle = 0$. Here ζ means the zero section. Let $c : (\mathbf{R}, t_0) \to (\mathbf{R}^2, p)$ be a parametrization of the singular locus $S(f)$, $p = c(t_0)$, and $\eta : (\mathbf{R}^2, p) \to T\mathbf{R}^2$ be a vector field which restricts to the kernel field of f_* on $S(f)$. Suppose that $V_2(p) \notin f_*(T_p\mathbf{R}^2)$. Then, for any affine connection ∇ on M, we define

$$\psi(t) := \langle L(c(t)), (\nabla^f_\eta V_2)(c(t))\rangle.$$

Note that the vector field $(\nabla^f_\eta V_2)(c(t))$ is independent of the extension η and the choice of affine connection ∇, since $\eta|_{S(f)}$ is a kernel field of f_*. We call the function $\psi(t)$ the *characteristic function* of f.

Then the following characterisations of cuspidal edges and folded umbrellas are given in [71][25]:

Theorem 19.1. (Theorem 1.4 of [25]). *Let $f : (\mathbf{R}^2, p) \to M^3$ be a germ of frontal with a non-degenerate singular point at p. Let $c : (\mathbf{R}, t_0) \to (\mathbf{R}^2, p)$ be a parametrization of the singular locus of f. Suppose $f_*c'(t_0) \neq 0$. Then, for the characteristic function ψ,*
(1) f is diffeomorphic to the cuspidal edge if and only if $\psi(t_0) \neq 0$.
(2) f is diffeomorphic to the folded umbrella if and only if $\psi(t_0) = 0$, $\psi'(t_0) \neq 0$.

We can summarise several known results as those on openings of the fold:

Theorem 19.2. ([59][60]) *Let $f : (\mathbf{R}^2, p) \to M^m, m \geq 2$ be a germ of frontal with a non-degenerate singular point at p, $\widetilde{f} : (\mathbf{R}^2, p) \to$ $\mathrm{Gr}(2, TM)$ the integral lifting of f and $V_1, V_2 : (\mathbf{R}^2, p) \to TM$ an associated frame with $\widetilde{f}$. Let $c : (\mathbf{R}, t_0) \to (\mathbf{R}^2, p)$ be a parametrization of the singular locus of f. Suppose $f_*c'(t_0) \neq 0$. Then f is diffeomorphic to*

an opening of the fold, namely to the germ $(u, w) \mapsto (u, \frac{1}{2}w^2)$. *Moreover we have*:
(0) *Let* $m = 2$. *Then* f *is diffeomorphic to the fold.*
(1) *Let* $m \geq 3$. *Then* f *is diffeomorphic to the cuspidal edge if and only if* $\psi(t_0) \neq 0$.
(2) *Let* $m = 3$. *Then* f *is diffeomorphic to the folded umbrella if and only if* $\psi(t_0) = 0, \psi'(t_0) \neq 0$.

Based on results in [71] and [43], we summarise the characterization results on openings of the Whitney's cusp map-germ:

Theorem 19.3. ([59][61]) *Let* $f : (\mathbf{R}^2, p) \to M^m, m \geq 2$ *be a germ of frontal with a non-degenerate singular point at* p, $V_1, V_2 : (\mathbf{R}^2, p) \to TM$ *an associated frame with* $\widetilde{f}$ *with* $V_2(p) \notin f_*(T_p\mathbf{R}^2)$, *and* $\eta : (\mathbf{R}^2, p) \to T\mathbf{R}^2$ *an extension of a kernel field along of* f_*. *Let* $c : (\mathbf{R}, t_0) \to (\mathbf{R}^2, p)$ *be a parametrization of the singular locus of* f. *Set* $\gamma = f \circ c : (\mathbf{R}, t_0) \to M$. *Suppose* $(\nabla\gamma)(t_0) = 0$ *and* $(\nabla^2\gamma)(t_0) \neq 0$. *Then* f *is diffeomorphic to an opening of Whitney's cusp, namely to the germ* $(u, t) \mapsto (u, t^3 + ut)$. *Moreover we have*

(0) *Let* $m = 2$. *Then* f *is diffeomorphic to Whitney's cusp.*

(1) *Let* $m = 3$. *Then* f *is diffeomorphic to the swallowtail if and only if*

$$V_1(c(t_0)),\ V_2(c(t_0)),\ (\nabla^f_\eta V_2)(c(t_0))$$

are linearly independent in $T_{f(p)}M$.

(2) *Let* $m \geq 4$. *Then* f *is diffeomorphic to the open swallowtail if and only if*

$$(V_1 \circ c)(t_0),\ (V_2 \circ c)(t_0),\ ((\nabla^f_\eta V_2) \circ c)(t_0),\ (\nabla^\gamma_{\partial/\partial t}((\nabla^f_\eta V_2) \circ c))(t_0)$$

are linearly independent in $T_{f(p)}M$.

Note that the conditions appeared in Theorem 19.1 are invariant under diffeomorphism equivalence introduced in Introduction. In fact the conditions are invariant under a weaker equivalence relation. In Definition 8.4, we have introduce the notion of $\mathcal{J}$-equivalence of map-germs.

Corollary 19.4. *Let* $f : (\mathbf{R}^2, 0) \to (\mathbf{R}^m, 0)$ *be a frontal. Then* f *is* $\mathcal{J}$*-equivalent to Whitney's cusp if and only if* f *is diffeomorphic to an opening of Whitney's cusp. Moreover. if* $m = 2$, *then* f *is diffeomorphic to Whitney's cusp. If* $m = 3$ *and* f *is a front, then* f *is diffeomorphic to swallowtail.*

The known criteria of singularities (see for instance [93][94][70]) seem to be closely related with frontals, openings and $\mathcal{J}$-equivalence. The detailed relations are still open to be studied.

§20. Null frontals

Let (M, g) be a semi-Riemannian manifold with an indefinite metric g. Denote by $\mathcal{C} \subset TM$ the null cone field associated with the indefinite metric g, i.e. $\mathcal{C}$ is the set of null vectors:

$$\mathcal{C} = \bigcup_{x \in M} \mathcal{C}_x, \quad \mathcal{C}_x = \{u \in T_x M \mid \ g_x(u, u) = 0\}.$$

Let $\pi : \mathcal{C} \to M$ be the canonical projection.

Definition 20.1. A mapping $f : N \to M$ is called *totally null* (resp. *null*) if the induced metric f^*g is identically zero (resp. f^*g is degenerate everywhere). The condition that f is totally null is equivalent to that $f_*(T_t N) \subset \mathcal{C}_{f(t)}$ (resp. $f_*(T_t N)$ is tangent to $\mathcal{C}_{f(t)}$), for any $t \in N$.

Definition 20.2. A curve-germ $\gamma : (\mathbf{R}, a) \to M$ is called *null* if $\gamma'(t) \in \mathcal{C} \quad (t \in (\mathbf{R}, a))$. Moreover $\gamma : (\mathbf{R}, a) \to M$ is called *null-directed* if there exists a lift $u : (\mathbf{R}, a) \to \mathcal{C}$ such that $\pi \circ u = \gamma, u(t) \neq 0, \gamma'(t) \in \langle u(t) \rangle_{\mathbf{R}},\ t \in (\mathbf{R}, a)$.

A map-germ is null (resp. null-directed) if and only if it is totally null (resp. totally null frontal).

Definition 20.3. Let $\gamma : (\mathbf{R}, a) \to M$ be null-directed. Define the *null tangent surface* $\mathrm{Tan}(\gamma) : (\mathbf{R}^2, a \times \mathbf{R}) \to M$ of γ as the ruled surface by null geodesics through points $\gamma(t)$ with the directions $u(t)$.

The right equivalence class of $\mathrm{Tan}(\gamma)$ is independent of the choice of the lift u.

We have the following classification results. For the details see [56][57]: The singularities of tangent surface $\mathrm{Tan}(\gamma)$ for a generic null directed curve $\gamma : \mathbf{R} \to \mathbf{R}^{2,2}$ are cuspidal edges and open swallowtails. The singularities of tangent surface $\mathrm{Tan}(\gamma)$ for a generic null directed curve $\gamma : \mathbf{R} \to \mathbf{R}^{2,3}$ are cuspidal edges, open swallowtails, open Mond surfaces and unfurled folded umbrellas. The singularities of tangent surface $\mathrm{Tan}(\gamma)$ for a generic null directed curve $\gamma : \mathbf{R} \to \mathbf{R}^{3,3}$ (the projection of a generic "Engel integral" curve) are embedded cuspidal edges, open swallowtails and open Mond surfaces. See [43] for the normal forms and pictures of the singularities.

In general the tangent surface to a null curve is a ruled surface by null lines, which is not necessarily a totally null surface, but a null surface, which we call the *null tangent surface*.

Let X be a 3-dimensional Lorentzian manifold (with signature $(1, 2)$). A smooth map-germ $\varphi : (\mathbf{R}^2, 0) \to X$ is called a *null frontal surface* or a *null frontal* in short if there exists a smooth lift $\widetilde{\varphi} : (\mathbf{R}^2, 0) \to PT^*X = \mathrm{Gr}(2, TX)$ of φ such that $\widetilde{\varphi}(t)$ is a lightlike plane in $T_{\varphi(t)}X$ and $\varphi_*(T_t\mathbf{R}^2) \subset \widetilde{\varphi}(t)$, for any $t \in (\mathbf{R}^2, 0)$. The notion of null frontals is a natural generalization of null immersions to singular surfaces. We have presented several classification results of singularities which arise in null frontals up to local diffeomorphisms and up to $O(2,3)$-conformal transformations in the conformally flat case (cf. [54]). The classification is achieved by using the fact that null frontals are obtained as tangent surfaces to null curves in X, as well as "associated varieties" to Legendre curves in the space Y of null geodesics on X (cf. [55][56][57]). A related result is obtained in [19].

§21. Abnormal frontals

Let M be a 5-dimensional manifold and $\mathcal{D} \subset TM$ a distribution of rank 2. Then $\mathcal{D}$ is called a *Cartan distribution* if it has growth $(2, 3, 5)$, namely, if $\mathrm{rank}(\mathcal{D}^{(2)}) = 3$ and $\mathrm{rank}(\mathcal{D}^{(3)}) = 5$, where, we define in terms of Lie bracket, $\mathcal{D}^{(2)} = \mathcal{D} + [\mathcal{D}, \mathcal{D}]$ and $\mathcal{D}^{(3)} = \mathcal{D}^2 + [\mathcal{D}, \mathcal{D}^2]$. It is known that, for any point x of M and for any direction $\ell \subset \mathcal{D}_x$, there exists an abnormal geodesic, which is unique up to parametrisations, through x with the given direction ℓ (see [50][51]).

Then, for a given $\mathcal{D}$-directed curve γ, we define *abnormal tangent surface* of γ, which is ruled by abnormal geodesics through points $\gamma(t)$ with the directions $u(t)$.

On $\mathbf{R}^5$ with coordinates $(\lambda, \nu, \mu, \tau, \sigma)$, define the distribution $\mathcal{D} \subset T\mathbf{R}^5$ generated by the pair of vector fields

$$\begin{aligned} \eta_1 &= \frac{\partial}{\partial \lambda} + \nu \frac{\partial}{\partial \mu} - (\lambda\nu - \mu)\frac{\partial}{\partial \tau} + \nu^2 \frac{\partial}{\partial \sigma}, \\ \eta_2 &= \frac{\partial}{\partial \nu} - \lambda \frac{\partial}{\partial \mu} + \lambda^2 \frac{\partial}{\partial \tau} - (\lambda\nu + \mu)\frac{\partial}{\partial \sigma}. \end{aligned}$$

Then $\mathcal{D} \subset T\mathbf{R}^5$ is a Cartan distribution and it has maximal symmetry of dimension 14, maximal among all Cartan distributions, which is of type G_2, one of simple Lie algebras.

For a generic G_2-Cartan directed curve $\gamma : \mathbf{R} \to \mathbf{R}^5$, the tangent surfaces at any point $a \in \mathbf{R}$ is classified, up to local diffeomorphisms, into *embedded cuspidal edge*, *open Mond surface*, and *generic open folded pleat* (see [55] for details). The classification of singularities in abnormal tangent surfaces to generic Cartan directed curves for general Cartan distributions seems to be un-known yet.

§22. Appendix: Malgrange preparation theorem on differentiable algebras

We show the Malgrange's preparation theorem on differentiable algebras [74] from the ordinary Malgrange-Mather's preparation theorem (see for example [15]), relating to the theory of C^∞-rings which we have utilised in this paper.

An $\mathbf{R}$-algebra A is called *local* if it has a unique maximal ideal $\mathfrak{m}_A$.

Example 22.1. Let $\mathcal{E}_n$ denote the $\mathbf{R}$-algebra of C^∞-functions-germs $(\mathbf{R}^n, 0) \to \mathbf{R}$. Then $\mathcal{E}_n$ is a local $\mathbf{R}$-algebra with the unique maximal ideal $\mathfrak{m}_n = \{h \in \mathcal{E}_n \mid h(0) = 0\}$.

Definition 22.2. ([74]) A local $\mathbf{R}$-algebra A is called a *differentiable algebra* if a surjective $\mathbf{R}$-algebra homomorphism, mapping 1 to 1, $\pi : \mathcal{E}_m \to A$, for some $m \in \mathbf{N}$ is endowed.

A differentiable algebra A has the unique maximal ideal $\mathfrak{m}_A = \pi(\mathfrak{m}_m)$.

Let A and B be differentiable algebras with the surjective homomorphisms $\pi : \mathcal{E}_m \to A$ and $\psi : \mathcal{E}_n \to B$ respectively. An $\mathbf{R}$-algebra homomorphism $u : A \to B$ is called a *morphism* of differentiable algebras if there exists a C^∞ map-germ $g : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ such that the diagram

$$\begin{array}{ccc} \mathcal{E}_m & \xrightarrow{g^*} & \mathcal{E}_n \\ \pi \downarrow & & \downarrow \psi \\ A & \xrightarrow{u} & B \end{array}$$

commutes.

A morphism $u : A \to B$ of differentiable algebras is called *finite* (resp. *quasi-finite*) if B is a finite A-module via u (resp. $B/\mathfrak{m}_A B$ is a finite dimensional $\mathbf{R}$-vector space).

If u is finite, then it is quasi-finite. Then we have:

Theorem 22.3. (*Malgrange preparation theorem on differentiable algebras.* Theorem 4.1 in [74] p.73) *Let $u : A \to B$ be a morphism of differentiable algebras. Then u is finite if and only if it is quasi-finite. Moreover $b_1, \dots, b_r \in B$ generate B over A via u if and only if $\overline{b}_1, \dots, \overline{b}_r \in B/\mathfrak{m}_A B$ generate $B/\mathfrak{m}_A B$ over $\mathbf{R}$.*

Theorem 22.4. (*Malgrange-Mather's preparation theorem*: Theorem 6.5, Corollary 6.6 in [15]) *Let $f : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ be a C^∞ map-germ with the induced homomorphism $f^* : \mathcal{E}_m \to \mathcal{E}_n$. Let C be a finite $\mathcal{E}_n$-module. Then C is a finite $\mathcal{E}_m$-module via f^* if and only if*

$C/\mathfrak{m}_m C$ is a finite dimensional $\mathbf{R}$-vector space. Moreover $c_1, \dots, c_r \in C$ generate C over $\mathcal{E}_m$ via f^* if and only if $\overline{c}_1, \dots, \overline{c}_r \in C/\mathfrak{m}_m C$ generate $C/\mathfrak{m}_m C$ over $\mathbf{R}$.

Proof of the statement that Theorem 22.4 implies Theorem 22.3:
Let u be quasi-finite. Suppose $b_1, \dots, b_r \in B$ and $\overline{b}_1, \dots, \overline{b}_r \in B/\mathfrak{m}_A B$ generate $B/\mathfrak{m}_A B$ over $\mathbf{R}$. Let $g : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ and $g^* : \mathcal{E}_m \to \mathcal{E}_n$ cover $u : A \to B$. Note B is a finite $\mathcal{E}_n$-module via ψ. In fact $1 \in B$ generates B over $\mathcal{E}_n$ via the surjection ψ. Also note that $\mathfrak{m}_m B = \pi(\mathfrak{m}_m)B \subseteq \mathfrak{m}_A B$. Then $\overline{b}_1, \dots, \overline{b}_r \in B/\mathfrak{m}_m B$ generate $B/\mathfrak{m}_m B$ over $\mathbf{R}$ via $u \circ \pi = \psi \circ g^*$. Therefore, by Theorem 22.4, $b_1, \dots, b_r \in B$ generate B over A. Thus u is finite. This implies also the remaining statement naturally. □

Definition 22.5. A commutative ring A is called a C^∞*-ring* if the following conditions are satisfied:
(1) A contains the field $\mathbf{R}$ of real numbers.
(2) For any positive integer r, for any $a_1, \dots, a_r \in A$, and for any C^∞ function $f \in C^\infty(\mathbf{R}^r)$, an element $f(a_1, \dots, a_r) \in A$ is assigned, such that the equality

$$(g(f_1, \dots, f_s))(a_1, \dots, a_r) = g(f_1(a_1, \dots, a_r), \dots, f_s(a_1, \dots, a_r))$$

holds for any $g \in C^\infty(\mathbf{R}^s), f_1, \dots, f_s \in C^\infty(\mathbf{R}^n)$.
(3) The operations on A by C^∞ functions are compatible with the structure of $\mathbf{R}$-algebra on A, i.e. if f is a polynomial, $f = P(x_1, \dots, x_r) \in \mathbf{R}[x_1, \dots, x_r] \subset C^\infty(\mathbf{R}^r)$, then $f(a_1, \dots, a_r)$ is equal to the element $P(a_1, \dots, a_r)$ obtained just by substitutions (see [29]).

Note that by the condition (1), a C^∞-ring is naturally an $\mathbf{R}$-algebra. A C^∞-ring A is called a local C^∞-ring if A is a local $\mathbf{R}$-algebra. Let $\mathfrak{m}_A$ denote the unique maximal ideal of a local C^∞-ring. Let A be a C^∞-ring. We say that $a_1, \dots, a_n \in A$ generate A as the C^∞-ring if for any $a \in A$, there exists $f \in C^\infty(\mathbf{R}^n)$ such that $a = f(a_1, \dots, a_n)$. A is called a *finitely generated* C^∞-ring if there exists a finite number of elements generating A as the C^∞-ring. Let $\pi : A \to A/\mathfrak{m}_A$ denote the natural projection and $i : \mathbf{R} \to A$ the inclusion.

Lemma 22.6. *Let A be a differentiable algebra with a surjective $\mathbf{R}$-algebra homomorphism $\pi : \mathcal{E}_m \to A$. Then we have*:
(1) *A has the induced structure of a local C^∞-ring.*
(2) *A is generated by $\pi(x_1), \dots, \pi(x_m)$ as the C^∞-ring. Here $(x_1, \dots, x_m)$ is a system of coordinates of $(\mathbf{R}^m, 0)$ with the centre at 0.*
(3) *$\pi \circ i : \mathbf{R} \to A/\mathfrak{m}_A$ is a bijection.*

Proof: (1) For any positive integer r, for any $a_1, \dots, a_r \in A$, and for any C^∞ function $f \in C^\infty(\mathbf{R}^r)$, we take a system of lifts $\widetilde{a}_1, \dots, \widetilde{a}_r \in \mathcal{E}_m$ for π and define $f(a_1, \dots, a_r) := \pi(f(\widetilde{a}_1, \dots, \widetilde{a}_r))$. If we take another system of lifts $\widehat{a}_1, \dots, \widehat{a}_r \in \mathcal{E}_m$ for π, we have

$$f(\widetilde{a}_1, \dots, \widetilde{a}_r) - f(\widehat{a}_1, \dots, \widehat{a}_r) = \sum_{i=1}^{r} g_i(\widetilde{a}, \widehat{a})(\widetilde{a}_i - \widehat{a}_i) \in \mathrm{Ker}(\pi),$$

for some C^∞ functions $g_i(\widetilde{x}_1, \dots, \widetilde{x}_r; \widehat{x}_1, \dots, \widehat{x}_r) \in C^\infty(\mathbf{R}^{2r}), 1 \leq i \leq r$. Thus $\pi(f(\widehat{a}_1, \dots, \widehat{a}_r)) = \pi(f(\widetilde{a}_1, \dots, \widetilde{a}_r))$. Moreover, take any $a \in A$. Then there exists $h \in \mathcal{E}_m$ such that $a = \pi(h)$. (2) Take an $H \in C^\infty(\mathbf{R}^m)$ having h as the germ at 0. Then $H(\pi(x_1), \dots, \pi(x_m)) = \pi(H(x_1, \dots, x_m)) = \pi(h) = a$. (3) $\mathbf{R} \cong \mathcal{E}_m/\mathfrak{m}_m \cong A/\mathfrak{m}_A$. □

Proposition 22.7. *Let $(A, \mathfrak{m}_A)$ be a local C^∞-ring. Then the following conditions are equivalent*:
(1) *A is finitely generated as C^∞-ring and the natural map $\pi \circ i : \mathbf{R} \to A/\mathfrak{m}_A$ is bijective.*
(2) *A is a differentiable algebra in the sense of Malgrange.*

Proof: (1) $\Rightarrow$ (2): Let $a_1, \dots, a_m$ be a system of generators of A as C^∞-ring. Define $\Pi : C^\infty(\mathbf{R}^m) \to A$ by $\Pi(f) = f(a_1, \dots, a_m)$. Then Π is surjective. Set $I = \Pi^{-1}(\mathfrak{m}_A)$ which is a maximal ideal of $C^\infty(\mathbf{R}^m)$ with $C^\infty(\mathbf{R}^m)/I \cong \mathbf{R}$. Then there exists a point $p \in \mathbf{R}^m$ such that $I = \{f \in C^\infty(\mathbf{R}^m) \mid f(p) = 0\}$ (see Proposition 2.1 [2] for instance). Moreover $\mathrm{Ker}(\Pi) \subset I$. Set $J = \{f \in C^\infty(\mathbf{R}^m) \mid \text{the germ of } f \text{ at } p \text{ is zero}\}$. We show that $J \subseteq \mathrm{Ker}(\Pi)$. Let $h \in J$. Then there exists $k \in C^\infty(\mathbf{R}^m)$ such that $k(p) \neq 0$ and $hk = 0$. Then

$$0 = (hk)(a_1, \dots, a_m) = h(a_1, \dots, a_m)k(a_1, \dots, a_m).$$

On the other hand $k(a_1, \dots, a_m) \notin \mathfrak{m}_A$. Hence $k(a_1, \dots, a_m)$ is invertible. Then $\Pi(h) = h(a_1, \dots, a_m) = 0$ and thus $h \in \mathrm{Ker}(\Pi)$. Now $\Pi : C^\infty(\mathbf{R}^n) \to A$ induces a surjective homomorphism $\pi' : \mathcal{E}_{\mathbf{R}^m,p} \to A$. Define $\pi : \mathcal{E}_m = \mathcal{E}_{\mathbf{R}^m,0} \to A$ by $\pi(f) = \pi'(\widetilde{f})$, where $\widetilde{f}(x) = f(x - p)$.
The implication (2) $\Rightarrow$ (1) follows by Lemma 22.6. □

Example 22.8. Let $I = \{h \in C^\infty(\mathbf{R}) \mid \exists n_0 \in \mathbf{N}, h(n) = 0 (n \in \mathbf{N}, n \geq n_0)\}$. Then I is a maximal ideal of $C^\infty(\mathbf{R})$. Let $A = C^\infty(\mathbf{R})_I$ be the localisation (a localisation at infinity). Then A is an $\mathbf{R}$-algebra with the unique maximal ideal $\mathfrak{m}_A$. However A is not a differentiable algebra in the sense of Malgrange. In fact, the quotient field $A/\mathfrak{m}_A \cong C^\infty(\mathbf{R})/I$ is a Robinson's *hyper-real number field* [92].

We call an $\mathbf{R}$-algebra homomorphism u a *C^∞-ring homomorphism* if

$$u(f(a_1, \dots, a_r)) = f(u(a_1), \dots, u(a_r)),$$

for any $r \geq 1$, for any $a_1, \dots, a_r \in A$ and for any $f \in C^\infty(\mathbf{R}^r)$.

Lemma 22.9. *Let $\varphi : \mathcal{E}_m \to \mathcal{E}_n$ be an $\mathbf{R}$-algebra homomorphism. Then the following conditions are equivalent*:
(1) *There exists a C^∞ map-germ $g : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ such that $\varphi = g^*$.*
(2) *φ is a C^∞-ring homomorphism.*

Proof: (1) $\Rightarrow$ (2): Let $a_1, \dots, a_r \in \mathcal{E}_m$ and $h \in C^\infty(\mathbf{R}^r)$. Then

$$\begin{aligned} h(\varphi(a_1), \dots, \varphi(a_r)) &= h(g^*a_1, \dots, g^*a_r) = h \circ (a_1, \dots, a_r) \circ g \\ &= g^*(h(a_1, \dots, a_r)) = \varphi(h(a_1, \dots, a_r)). \end{aligned}$$

(2) $\Rightarrow$ (1): Let $x_1, \dots, x_m$ be coordinates of $(\mathbf{R}^m, 0)$. Then we have $\varphi(x_1), \dots, \varphi(x_m) \in \mathfrak{m}_n$. Take representatives $\widetilde{g}_i : U \to \mathbf{R}$ of $\varphi(y_i)$ over a common open neighbourhood of 0 in $\mathbf{R}^n$, $(1 \leq i \leq m)$. We set $\widetilde{g} = (\widetilde{g}_1, \dots, \widetilde{g}_m) : U \to \mathbf{R}^m$. Then $\widetilde{g}(0) = 0$. Take the germ $g : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$ of $\widetilde{g}$ at 0. Let $h \in \mathcal{E}_m$. Take a representative $\widetilde{h} \in C^\infty(\mathbf{R}^m)$. Then we have $\varphi(h) = \varphi(\widetilde{h}(x_1, \dots, x_m)) = \widetilde{h}(\varphi(x_1), \dots, \varphi(x_m)) = h \circ g = g^*(h)$. Therefore $\varphi = g^*$. $\square$

Lemma 22.10. *Let $u : A \to B$ be an $\mathbf{R}$-algebra homomorphism of differentiable algebras. Then the following conditions are equivalent*:
(1) *u is a morphism of differentiable algebras.*
(2) *u is a C^∞-ring homomorphism.*

Proof: (1) $\Rightarrow$ (2): Let $a_1, \dots, a_r \in A$ and $f \in C^\infty(\mathbf{R}^r)$. Take $\widetilde{a}_i \in \mathcal{E}_m$ with $\pi(\widetilde{a}_i) = a_i$. Then $\psi(g^*\widetilde{a}_i) = u(a_i)$. Then $u(f(a_1, \dots, a_r) = u(\pi(f(\widetilde{a}_1, \dots, \widetilde{a}_r))) = \psi(f \circ (\widetilde{a}_1, \dots, \widetilde{a}_r) \circ g) = \psi(f(g^*\widetilde{a}_1, \dots, g^*\widetilde{a}_r)) = f(u(a_1), \dots, u(a_r))$.
(2) $\Rightarrow$ (1): Take $g_i \in \mathcal{E}_n$ with $u(\pi(x_i)) = \psi(g_i)$. Since $\psi(g_i) \in \mathfrak{m}_B$, we have $g_i \in \mathfrak{m}_m$. Set $g = (g_1, \dots, g_m) : (\mathbf{R}^n, 0) \to (\mathbf{R}^m, 0)$. Let $h \in \mathcal{E}_m$ and take a representative $H \in C^\infty(\mathbf{R}^m)$ of the germ h. Then we have

$$\begin{aligned} u(\pi(h)) &= u(H(\pi(x_1), \dots, \pi(x_m))) = H(u(\pi(x_1)), \dots, u(\pi(x_m))) \\ &= H(\psi(g_1), \dots, \psi(g_m)) = \psi(H(g_1, \dots, g_m) = \psi(g^*(h)). \end{aligned}$$

$\square$

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