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Characteristic classes of coherent sheaves on singular varieties

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Dedicated to Professor Takuo Fukuda on his sixtieth birthday

For a compact singular variety V, there are several definitions of Chern classes, the Mather class, the Schwartz-MacPherson class, the Fulton-Johnson class and so forth ([BrSc], [F], [FJ], [M], [Sc1], see also [Al], [BLSS], [PP] and [Y] for recent developements). They are in the homology of V and, if V is non-singular, they all reduce to the Poincaré dual of the Chern class $c^*(TV)$ of the tangent bundle TV of V. On the other hand, for a coherent sheaf $\mathcal{F}$ on V, the (cohomology) Chern character $\mathrm{ch}^*(\mathcal{F})$ or the Chern class $c^*(\mathcal{F})$ makes sense if either V is non-singular or $\mathcal{F}$ is locally free. In this article, we propose a definition of the homology Chern character $\mathrm{ch}_*(\mathcal{F})$ or the Chern class $c_*(\mathcal{F})$ for a coherent sheaf $\mathcal{F}$ on a possibly singular variety V. In this direction, the homology Chern character or the Chern class is defined in [Sc2] (see also [K]) using the Nash type modification of V relative to the linear space associated to the coherent sheaf $\mathcal{F}$. Also, the homology Todd class $\tau(\mathcal{F})$ is introduced in [BFM] to describe their Riemann-Roch theorem. Our class is closely related to the latter.

The variety V we consider in this article is a local complete intersection defined by a section of a holomorphic vector bundle over the ambient complex manifold M. If $\mathcal{F}$ is a locally free sheaf on V, then the class $\mathrm{ch}_*(\mathcal{F})$ coincides with the image of $\mathrm{ch}^*(\mathcal{F})$ by the Poincaré homomorphism $H_*(V) \to H^*(V)$. This fact follows from the Riemann-Roch theorem for the embedding of V into M, which we prove at the level of Čech-de Rham cocycles. We also compute the Chern character and the Chern class of the tangent sheaf of V when V has only isolated singularities.

In section 1, we discuss characteristic cocycles in the Čech-de Rham complex and define local Chern classes and characters in the Čech-de

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Rham cohomology. We prove a lemma which gives an explicit relation between the cocycle for the product of two symmetric series and the product of cocycles for these series (Lemma 1.5, also Proposition 1.6), which is fundamental in the proof of the Riemann-Roch theorem at the cocycle level. In section 2, we describe the Thom class of the variety V in M as above and, in section 3, we prove the Riemann-Roch theorem mentioned above (Theorem 3.1, Corollaries 3.4 and 3.5). In section 4, we introduce the homology Chern character for a coherent sheaf on V (Definition 4.1). For this definition, we only need that V be a local complete intersection. Finally in section 5, we compute the Chern character and the Chern class of the tangent sheaf of V (Theorem 5.1).

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§1. Local Chern classes and characters in the Čech-de Rham cohomology

As to the theory of characteristic classes, we use the Chern-Weil theory modified to fit in the framework of Čech-de Rham cohomology. For the Chern-Weil theory of characteristic classes of vector bundles, we refer to [BB], [Bo] and [MS]. For the background on the Čech-de Rham cohomology, we refer to [BT]. The integration and characteristic classes in this cohomology theory are first studied in [Le1-4]. See also [Su2] for these material. They are also briefly summarized in the section 1 of [Su3] and we freely use the notation and facts there, except we indicate cohomology Chern classes by superscripts in this article.

(A) Characteristic forms

Let M be a C^∞ manifold of dimension m and let $(T^\vee_{\mathbb{R}}M)^c$ be the complexified cotangent bundle of M. For a C^∞ complex vector bundle E over M, we denote by $A^p(E)$ the vector space of sections of the bundle $\Lambda^p(T^\vee_{\mathbb{R}}M)^c \otimes E$ on M. Recall that a connection ∇ for E is a linear map $A^0(E) \to A^1(E)$ satisfying the Leibniz rule. Let K be the curvature of ∇, which is an element in $A^2(\mathrm{End}(E))$. We set $A = (\sqrt{-1}/2\pi)K$ and define

$$
\begin{aligned}
c^*(\nabla) &= \det(I + A), \\
\mathrm{ch}^*(\nabla) &= \mathrm{tr}(e^A), \\
\mathrm{td}(\nabla) &= \det\left(\frac{A}{I - e^{-A}}\right).
\end{aligned}
\tag{1.1}
$$

Note that $I - e^{-A}$ is divisible by A and the result is invertible so that

$$
\mathrm{td}^{-1}(\nabla) = \det\left(\frac{I - e^{-A}}{A}\right)
$$

also makes sense. If we denote by $c^i(\nabla)$ the homogeneous piece in $c^*(\nabla)$ of degree i in the entries of A, it is a closed $2i$-form on M and its class $[c^i(\nabla)]$ in the de Rham cohomology $H^{2i}(M;\mathbb{C})$ is the i-th Chern class $c^i(E)$ of E. The class of $c^*(\nabla)$ in $H^*(M;\mathbb{C})$ is the total (cohomology) Chern class $c^*(E)$ of E. If we set $s^i(\nabla) = \operatorname{tr}(A^i)$, then it is a closed $2i$-form on M. Denoting by r the rank of E, we have

$$c^*(\nabla) = 1 + \sum_{i=1}^{r} c^i(\nabla) \qquad \text{and} \qquad \operatorname{ch}^*(\nabla) = r + \sum_{i\geq 1} \frac{s^i(\nabla)}{i!}.$$

The forms $c^i = c^i(\nabla)$ and $s^i = s^i(\nabla)$ are related by Newton's formula :

$$(1.2) \qquad s^i - c^1 s^{i-1} + c^2 s^{i-2} - \cdots + (-1)^i i\, c^i = 0, \qquad i \geq 1.$$

The class of $\operatorname{ch}^*(\nabla)$ in $H^*(M;\mathbb{C})$ is the (cohomology) Chern character $\operatorname{ch}^*(E)$ of E. Each homogeneous piece of $\operatorname{td}(\nabla)$ is also closed and the class of $\operatorname{td}(\nabla)$ in $H^*(M;\mathbb{C})$ is the Todd class $\operatorname{td}(E)$ of E. Note that the constant term in $\operatorname{td}(\nabla)$ is 1 and that $\operatorname{td}(\nabla)$ can be expressed as a series (in fact a polynomial) in $c^i(\nabla)$. We have the following fundamental formula [HL, III, Corollary 5.4] :

$$(1.3) \qquad \sum_{i=0}^{r} (-1)^i \operatorname{ch}^*(\Lambda^i \nabla^\vee) = \operatorname{td}^{-1}(\nabla) \cdot c^r(\nabla),$$

where $\nabla^\vee$ denotes the connection for $E^\vee$ dual to ∇ and $\Lambda^i\nabla^\vee$ the connection for $\Lambda^i E^\vee$ induced by $\nabla^\vee$. Here we set $\Lambda^0 E^\vee = M \times \mathbb{C}$ (the trivial line bundle) and $\Lambda^0 \nabla^\vee = d$. See, e.g., [H, Theorem 10.1.1] for the above formula in cohomology.

Let $\xi = \sum_{i=0}^{q} (-1)^i E_i$ be a virtual bundle and $\nabla^\bullet = (\nabla^{(q)}, \dots, \nabla^{(0)})$ a family of connections, each $\nabla^{(i)}$ being a connection for E_i. We set

$$c^*(\nabla^\bullet) = \prod_{i=0}^{q} c^*(\nabla^{(i)})^{\epsilon(i)} \qquad \text{and} \qquad \operatorname{ch}^*(\nabla^\bullet) = \sum_{i=0}^{q} (-1)^i \operatorname{ch}^*(\nabla^{(i)}),$$

where $\epsilon(i) = (-1)^i$. If we denote by $c^i = c^i(\nabla^\bullet)$ and $s^i/i! = s^i(\nabla^\bullet)/i!$ the homogeneous pieces of degree $2i$ in $c^*(\nabla^\bullet)$ and $\operatorname{ch}^*(\nabla^\bullet)$, respectively, they are again related by (1.2). More generally, if $\varphi = \varphi(c^1, c^2, \dots)$ is a series in c^i (we call such a series a symmetric series), we set $\varphi(\nabla^\bullet) = \varphi(c^1(\nabla^\bullet), c^2(\nabla^\bullet), \dots)$. Then it is a closed form and its class $\varphi(\xi)$ in the cohomology ring $H^*(M;\mathbb{C})$ is the characteristic class of ξ with respect to φ. Suppose further that we have two families of connections

$\nabla^\bullet_\nu = (\nabla^{(q)}_\nu, \dots, \nabla^{(0)}_\nu)$, $\nu = 0, 1$, for ξ. Then, we have a form $\varphi(\nabla^\bullet_0, \nabla^\bullet_1)$ alternating in $(\nabla^\bullet_0, \nabla^\bullet_1)$ such that

$$(1.4) \qquad d\,\varphi(\nabla^\bullet_0, \nabla^\bullet_1) = \varphi(\nabla^\bullet_1) - \varphi(\nabla^\bullet_0),$$

which shows that the class $\varphi(\xi)$ does not depend on the choice of family of connections. We recall the construction of $\varphi(\nabla^\bullet_0, \nabla^\bullet_1)$ for later use ([Bo, p.65], [Su2, Ch.II, (8.2)]). Thus, for each $i = 0, \dots, q$, we consider the vector bundle $E_i \times \mathbb{R} \to M \times \mathbb{R}$ and let $\tilde{\nabla}^{(i)}$ be the connection for it given by $\tilde{\nabla}^{(i)} = (1-t)\nabla^{(i)}_0 + t\nabla^{(i)}_1$. We set $\tilde{\nabla}^\bullet = (\tilde{\nabla}^{(q)}, \dots, \tilde{\nabla}^{(0)})$. Denoting by π_* the integration along the fibers of the projection $\pi : M \times [0,1] \to M$, we define $\varphi(\nabla^\bullet_0, \nabla^\bullet_1) = \pi_*\varphi(\tilde{\nabla}^\bullet)$. Note that the "higher difference forms" for more than two families of connections are constructed similarly.

Now we prove a lemma which will be used in the next paragraph to describe explicitly the difference between the cocycle for the product of two symmetric series and the product of cocycles for these series. Note that $\varphi\psi(\nabla^\bullet) = \varphi(\nabla^\bullet) \cdot \psi(\nabla^\bullet)$, for symmetric series φ and ψ and a family of connections $\nabla^\bullet$.

Lemma 1.5. *In the above situation, for two symmetric series φ and ψ, we have*

$$\varphi\psi(\nabla^\bullet_0, \nabla^\bullet_1) = \varphi(\nabla^\bullet_0) \cdot \psi(\nabla^\bullet_0, \nabla^\bullet_1) + \varphi(\nabla^\bullet_0, \nabla^\bullet_1) \cdot \psi(\nabla^\bullet_1) - d\,\tau_{01},$$

where

$$\tau_{01} = \pi_*(\varphi(\pi^*\nabla^\bullet_0, \tilde{\nabla}^\bullet) \cdot d\psi(\pi^*\nabla^\bullet_1, \tilde{\nabla}^\bullet)).$$

Proof. By definition, the left hand side is equal to $\pi_*(\varphi(\tilde{\nabla}^\bullet) \cdot \psi(\tilde{\nabla}^\bullet))$ and the sum of the first two terms in the right hand side is equal to

$$\pi_*(\varphi(\pi^*\nabla^\bullet_0) \cdot \psi(\tilde{\nabla}^\bullet) + \varphi(\tilde{\nabla}^\bullet) \cdot \psi(\pi^*\nabla^\bullet_1)).$$

We have

$$\begin{aligned}\varphi(\tilde{\nabla}^\bullet) \cdot \psi(\tilde{\nabla}^\bullet) &- (\varphi(\pi^*\nabla^\bullet_0) \cdot \psi(\tilde{\nabla}^\bullet) + \varphi(\tilde{\nabla}^\bullet) \cdot \psi(\pi^*\nabla^\bullet_1)) \\ &= d\varphi(\pi^*\nabla^\bullet_0, \tilde{\nabla}^\bullet) \cdot d\psi(\pi^*\nabla^\bullet_1, \tilde{\nabla}^\bullet) - \pi^*(\varphi(\nabla^\bullet_0) \cdot \psi(\nabla^\bullet_1)).\end{aligned}$$

If we denote by i the embedding of the boundary $\{0,1\}$ of $[0,1]$ into $[0,1]$ and by $\partial\pi$ the restriction of π to $\{0,1\}$, the lemma follows from the identities $\pi_* \circ \pi^* = 0$,

$$\pi_* \circ d + d \circ \pi_* = (\partial\pi)_* \circ i^*$$

[Bo, (3.10) Theorem] and

$$(\partial\pi)_* \circ i^*(\varphi(\pi^*\nabla_0^\bullet, \tilde{\nabla}^\bullet) \cdot d\psi(\pi^*\nabla_1^\bullet, \tilde{\nabla}^\bullet))$$
$$= \varphi(\nabla_0^\bullet, \nabla_1^\bullet) \cdot d\psi(\nabla_1^\bullet, \nabla_1^\bullet) - \varphi(\nabla_0^\bullet, \nabla_0^\bullet) \cdot d\psi(\nabla_1^\bullet, \nabla_0^\bullet) = 0.$$

Q.E.D.

(B) Characteristic cocycles in the Čech-de Rham complex

Let M be as above. For an open covering $\mathcal{U}$ of M, we denote by $(A^*(\mathcal{U}), D)$ the Čech-de Rham complex associated to $\mathcal{U}$ [Su2, Ch.II,3]. The complex defines the Čech-de Rham cohomology $H^*(A^*(\mathcal{U}))$, which is canonically isomorphic with the de Rham cohomology $H^*(M; \mathbb{C})$. We recall this cohomology when $\mathcal{U}$ consists of two open sets U_0 and U_1 (the "Mayer-Vietoris situation"). In this case, a cochain σ in $A^p(\mathcal{U})$ is written as

$$\sigma = (\sigma_0, \sigma_1, \sigma_{01}),$$

where σ_0 and σ_1 are r-forms on U_0 and U_1, respectively, and σ_{01} is an $(r-1)$-form on $U_{01} = U_0 \cap U_1$, and the differential $D : A^p(\mathcal{U}) \to A^{p+1}(\mathcal{U})$ is given by

$$D\sigma = (d\sigma_0, d\sigma_1, \sigma_1 - \sigma_0 - d\sigma_{01}).$$

The Čech-de Rham cohomology is also equipped with the cup product, which is defined on the cochain level by assigning to σ in $A^p(\mathcal{U})$ and τ in $A^q(\mathcal{U})$ the cochain $\sigma \smile \tau$ in $A^{p+q}(\mathcal{U})$ given by

$$\sigma \smile \tau = (\sigma_0 \cdot \tau_0, \sigma_1 \cdot \tau_1, (-1)^p \sigma_0 \cdot \tau_{01} + \sigma_{01} \cdot \tau_1),$$

where the product is the exterior product. The cup product is compatible with the usual one in $H^*(M; \mathbb{C})$.

If $\xi = \sum_{i=0}^{q}(-1)^i E_i$ is a virtual bundle, we take a family of connections $\nabla_\nu^\bullet = (\nabla_\nu^{(q)}, \ldots, \nabla_\nu^{(0)})$ for ξ on each U_ν, $\nu = 0, 1$, and for the collection $\nabla_\star^\bullet = (\nabla_0^\bullet, \nabla_1^\bullet)$ and a symmetric series φ, we define the cochain $\varphi(\nabla_\star^\bullet)$ in $A^*(\mathcal{U})$ by

$$\varphi(\nabla_\star^\bullet) = (\varphi(\nabla_0^\bullet), \varphi(\nabla_1^\bullet), \varphi(\nabla_0^\bullet, \nabla_1^\bullet)).$$

Then by (1.4), $\varphi(\nabla_\star^\bullet)$ is a cocycle and defines a class $[\varphi(\nabla_\star^\bullet)]$ in $H^*(A^*(\mathcal{U}))$. It does not depend on the choice of the collection of families of connections $\nabla_\star^\bullet$ and corresponds to the class $\varphi(\xi)$ under the isomorphism $H^*(A^*(\mathcal{U})) \simeq H^*(M; \mathbb{C})$.

From Lemma 1.5, we have the following :

Proposition 1.6. *For two symmetric series φ and ψ, we have, in $A^*(\mathcal{U})$,*

$$\varphi\psi(\nabla_\star^\bullet) = \varphi(\nabla_\star^\bullet) \smile \psi(\nabla_\star^\bullet) + D\tau,$$

where $\tau = (0, 0, \tau_{01})$ *with* τ_{01} *a form on* U_{01} *as given in* Lemma 1.5.

In the sequel, we use the above formula only for a collection $\nabla_\star = (\nabla_0, \nabla_1)$ of connections for a single vector bundle.

(C) Localization

In this paper, we consider the following two types of localizations :
(I) localization of the top Chern class of a vector bundle by a non-vanishing section,
and
(II) localization of the Chern classes of a virtual bundle by exactness.

To describe these, let M be as above and let V be a closed set in M. Letting $U_0 = M \setminus V$ and U_1 a neighborhood of V in M, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. We set

$$A^p(\mathcal{U}, U_0) = \{\, \sigma \in A^p(\mathcal{U}) \mid \sigma_0 = 0 \,\}.$$

Then $A^*(\mathcal{U}, U_0)$ is a subcomplex of $A^*(\mathcal{U})$ and the cohomology it defines is canonically isomorphic with the relative cohomology $H^*(M, M \setminus V; \mathbb{C})$. Note that the cup product of a cochain in $A^*(\mathcal{U})$ and a cochain in $A^*(\mathcal{U}, U_0)$ is in $A^*(\mathcal{U}, U_0)$ and this induces a natural $H^*(M;\mathbb{C})$-module structure on $H^*(M, M \setminus V; \mathbb{C})$.

Remark 1.7. In the situation of Proposition 1.6, if $\psi(\nabla_\star^\bullet)$ is in $A^*(\mathcal{U}, U_0)$, i.e., if $\psi(\nabla_0^\bullet) = 0$, then so is $\varphi\psi(\nabla_\star^\bullet)$, since $\varphi\psi(\nabla_0^\bullet) = \varphi(\nabla_0^\bullet) \cdot \psi(\nabla_0^\bullet)$. The proposition shows that the class $\varphi\psi(\xi)$ coincides with $\varphi(\xi) \smile \psi(\xi)$ in $H^*(M, M \setminus V; \mathbb{C})$, since τ is also in $A^*(\mathcal{U}, U_0)$.

We start with the type (I). Thus let E be a vector bundle of rank r over M and s a non-vanishing section of E on U_0. We say that a connection ∇ for E is s-trivial if $\nabla s = 0$. Recall that, for an s-trivial connection ∇, we have $c^r(\nabla) = 0$ [Su2, Ch.II, Proposition 9.1]. Let ∇_0 be an s-trivial connection for E on U_0 and ∇_1 an arbitrary connection for E on U_1. The top Chern class $c^r(E)$ of E is represented by the cocycle

$$c^r(\nabla_\star) = (c^r(\nabla_0),\, c^r(\nabla_1),\, c^r(\nabla_0, \nabla_1))$$

in $A^{2r}(\mathcal{U})$. Since ∇_0 is s-trivial, we have $c^r(\nabla_0) = 0$ and $c^r(\nabla_\star)$ is in fact in $A^{2r}(\mathcal{U}, U_0)$. Thus it defines a class in $H^{2r}(M, M \setminus V; \mathbb{C})$, which we denote by $c^r(E, s)$. It is sent to the class $c^r(E)$ by the canonical homomorphism

$$j^* : H^{2r}(M, M \setminus V; \mathbb{C}) \to H^{2r}(M; \mathbb{C}).$$

It does not depend on the choice of the s-trivial connection ∇_0 or on the choice of the connection ∇_1. We call $c^r(E, s)$ the localization of $c^r(E)$ with respect to the section s.

For the type (II), let

$$(1.8) \qquad 0 \longrightarrow E_q \xrightarrow{h_q} \cdots \xrightarrow{h_1} E_0 \longrightarrow 0$$

be a complex of C^∞ complex vector bundles over M which is exact on U_0. Then we will see below that, for each $i > 0$, there is a canonical localization $c^i_V(\xi)$ in $H^{2i}(M, M \setminus V; \mathbb{C})$ of the Chern class $c^i(\xi)$ in $H^{2i}(M; \mathbb{C})$ of the virtual bundle $\xi = \sum_{i=0}^q (-1)^i E_i$.

Following [BB], we say that a family of connections $\nabla^\bullet = (\nabla^{(q)}, \dots, \nabla^{(0)})$ for ξ is compatible with the sequence (1.8) if, for each $i = 1, \dots, q$, the following diagram is commutative :

$$\begin{array}{ccc} A^0(E_i) & \xrightarrow{\nabla^{(i)}} & A^1(E_i) \\ {\scriptstyle h_i}\downarrow & & \downarrow{\scriptstyle 1\otimes h_i} \\ A^0(E_{i-1}) & \xrightarrow{\nabla^{(i-1)}} & A^1(E_{i-1}). \end{array}$$

Note that for a given exact sequence, there is always a family $\nabla^\bullet$ of connections compatible with the sequence. We have the following "vanishing theorem" [BB, Lemma (4.22)] :

Lemma 1.9. *If $\nabla^\bullet_0$ is a family of connections on U_0 compatible with* (1.8), *then, for each $i > 0$,*

$$c^i(\nabla^\bullet_0) = 0.$$

In fact, the above holds for a finite number of families of connections $\nabla^\bullet_{0,0}, \dots, \nabla^\bullet_{p,0}$ on U_0 compatible with (1.8), i.e., $c^i(\nabla^\bullet_{0,0}, \dots, \nabla^\bullet_{p,0}) = 0$. Thus, for a symmetric series φ without constant term, we also have $\varphi(\nabla^\bullet_{0,0}, \dots, \nabla^\bullet_{p,0}) = 0$.

Let $\nabla^\bullet_0$ be a family of connections compatible with (1.8) on U_0 and $\nabla^\bullet_1$ an arbitrary family of connections for $\xi = \sum_{i=0}^q (-1)^i E_i$ on U_1. Then the class $c^i(\xi)$ is represented by the cocycle

$$c^i(\nabla^\bullet_\star) = (c^i(\nabla^\bullet_0), c^i(\nabla^\bullet_1), c^i(\nabla^\bullet_0, \nabla^\bullet_1))$$

in $A^{2i}(\mathcal{U})$. By Lemma 1.9, we have $c^i(\nabla^\bullet_0) = 0$ and thus the cocycle is in $A^{2i}(\mathcal{U}, U_0)$ and it defines a class $c^i_V(\xi)$ in $H^{2i}(M, M \setminus V; \mathbb{C})$. It is sent to $c^i(\xi)$ by the canonical homomorphism j^*. It is not difficult to see that the class $c^i_V(\xi)$ does not depend on the choice of the family of connections $\nabla^\bullet_0$ compatible with (1.8) or on the choice of the family of connections $\nabla^\bullet_1$.

If φ is a symmetric series without constant term, we may also define the localized class $\varphi_V(\xi)$ of $\varphi(\xi)$. In particular, noting that the alternating sum of the ranks of E_i is zero, if $M \setminus V \neq \emptyset$, we have the localized Chern character $\mathrm{ch}_V^*(\xi)$ in the relative cohomology $H^*(M, M \setminus V; \mathbb{C})$, which is sent to $\mathrm{ch}^*(\xi)$ by the homomorphism j^*. It is the class of the cocycle

$$\mathrm{ch}^*(\nabla_\star^\bullet) = (0,\, \mathrm{ch}^*(\nabla_1^\bullet),\, \mathrm{ch}^*(\nabla_0^\bullet, \nabla_1^\bullet))$$

in $A^*(\mathcal{U}, U_0)$.

Let E be another vector bundle over M and ∇ a connection for E on M. Then its Chern character $\mathrm{ch}^*(E)$ is the class of the cocycle

$$\mathrm{ch}^*(\nabla) = (\mathrm{ch}^*(\nabla),\, \mathrm{ch}^*(\nabla),\, 0)$$

in $A^*(\mathcal{U})$. The complex

$$0 \longrightarrow E \otimes E_q \longrightarrow \cdots \longrightarrow E \otimes E_0 \longrightarrow 0$$

is exact on U_0 and the family $\nabla \otimes \nabla_0^\bullet = (\nabla \otimes \nabla_0^{(q)}, \ldots, \nabla \otimes \nabla_0^{(0)})$ of connections is compatible with the above sequence on U_0. We set $E \otimes \xi = \sum_{i=0}^q (-1)^i E \otimes E_i$ and let $\nabla \otimes \nabla_1^\bullet$ denote the family $(\nabla \otimes \nabla_1^{(q)}, \ldots, \nabla \otimes \nabla_1^{(0)})$. Then $\mathrm{ch}^*(E \otimes \xi)$ is the class of the cocycle

$$\mathrm{ch}^*(\nabla \otimes \nabla_\star^\bullet) = (0,\, \mathrm{ch}^*(\nabla \otimes \nabla_1^\bullet),\, \mathrm{ch}^*(\nabla \otimes \nabla_0^\bullet, \nabla \otimes \nabla_1^\bullet)).$$

We have

$$\begin{aligned} \mathrm{ch}^*(\nabla \otimes \nabla_1^\bullet) &= \mathrm{ch}^*(\nabla) \cdot \mathrm{ch}^*(\nabla_1^\bullet), \\ \mathrm{ch}^*(\nabla \otimes \nabla_0^\bullet, \nabla \otimes \nabla_1^\bullet) &= \mathrm{ch}^*(\nabla) \cdot \mathrm{ch}^*(\nabla_0^\bullet, \nabla_1^\bullet). \end{aligned}$$

Hence, recalling the definition of the cup product, we have

$$\mathrm{ch}^*(\nabla \otimes \nabla_\star^\bullet) = \mathrm{ch}^*(\nabla) \smile \mathrm{ch}^*(\nabla_\star^\bullet). \tag{1.10}$$

in $A^*(\mathcal{U}, U_0)$. In particular, we have

$$\mathrm{ch}_V^*(E \otimes \xi) = \mathrm{ch}^*(E) \smile \mathrm{ch}_V^*(\xi).$$

Remark 1.11. The local Chern characters defined as above have all the necessary properties and should coincide with the ones in [I]. Hence they are in the cohomology $H^*(M, M \setminus V; \mathbb{Q})$ with $\mathbb{Q}$ coefficients. Also, the local Chern classes above are in the image of $H^*(M, M \setminus V; \mathbb{Z}) \to H^*(M, M \setminus V; \mathbb{C})$. See also [BFM] for local Chern characters.

Now let M be a complex manifold and denote by $\mathcal{O}_M$ and $\mathcal{A}_M$, respectively, the sheaves of germs of holomorphic functions and of real analytic functions on M. If U is a relatively compact open set in M and if $\mathcal{S}$ is a coherent $\mathcal{O}_U$-module, there is a complex of real analytic vector bundles on U as (1.8) such that at the sheaf level

$$(1.12) \qquad 0 \longrightarrow \mathcal{A}_U(E_q) \longrightarrow \cdots \longrightarrow \mathcal{A}_U(E_0) \longrightarrow \mathcal{A}_U \otimes_{\mathcal{O}_U} \mathcal{S} \longrightarrow 0$$

is exact [AH1]. We call such a sequence a resolution of $\mathcal{S}$ by vector bundles. We define the Chern character $\mathrm{ch}^*(\mathcal{S})$ of $\mathcal{S}$ by $\mathrm{ch}^*(\mathcal{S}) = \mathrm{ch}^*(\xi)$, $\xi = \sum_{i=0}^{q}(-1)^i E_i$. Then it does not depend on the choice of the resolution. If we denote by V the support of $\mathcal{S}$, then it is an analytic set in U and on $U \setminus V$, the sequence (1.8) is exact. Thus we have the localized Chern character $\mathrm{ch}^*_V(\mathcal{S})$ in $H^*(U, U \setminus V; \mathbb{C})$. If E is a vector bundle over U, the characteristic classes of $E \otimes \mathcal{S}$ are those of $E \otimes \xi$. Hence, from (1.10), we have

$$(1.13) \qquad \mathrm{ch}^*_V(E \otimes \mathcal{S}) = \mathrm{ch}^*(E) \smile \mathrm{ch}^*_V(\mathcal{S}).$$

Note that the above equality also holds if we replace E by a virtual bundle over U.

§2. Thom class

Let M be a complex manifold of dimension $n + k$ and V a compact analytic subvariety (reduced analytic subspace) of pure dimension n in M. We denote by i the embedding $V \hookrightarrow M$. If $V = \bigcup_{\alpha=1}^{\ell} V_\alpha$ is the irreducible decomposition of V, we set $[V] = \sum_{\alpha=1}^{\ell}[V_\alpha]$ in $H_n(V; \mathbb{C})$.

We define the *Thom homomorphism* $T : H^p(V; \mathbb{C}) \to H^{p+2k}(M, M \setminus V; \mathbb{C})$ by $T = A^{-1} \circ P$ so that we have the commutative diagram

$$\begin{array}{ccc} H^p(V;\mathbb{C}) & \xrightarrow{T} & H^{p+2k}(M, M\setminus V;\mathbb{C}) \\ \Big\downarrow P & & \wr\Big\downarrow A \\ H_{2n-p}(V;\mathbb{C}) & \xrightarrow{=} & H_{2n-p}(V;\mathbb{C}), \end{array}$$

where A and P denote, respectively, the Alexander isomorphism and the Pioncaré homomorphism [Su2, Ch.VI, 4]. Recall that P is given by the cap product with the class $[V]$. For the class $[1]$ in $H^0(V; \mathbb{C})$, we denote $T([1])$ in $H^{2k}(M, M \setminus V; \mathbb{C})$ by Ψ_V, and call it the *Thom class* of V in M.

Remark 2.1. In [Br], these homomorphisms are defined in cohomology with $\mathbb{Z}$ coefficients by a combinatorial method. See [Ab] for a related work.

Let U be a regular neighborhood of V in M with continuous retraction $\rho : U \to V$. We have, by excision, $H^*(M, M \setminus V; \mathbb{C}) \simeq H^*(U, U \setminus V; \mathbb{C})$. Note that for σ in $H^*(U; \mathbb{C})$ and τ in $H^*(U, U \setminus V; \mathbb{C})$, we have

$$A(\sigma \smile \tau) = i^*\sigma \frown A(\tau).$$

Hence the Thom homomorphism T is given, for a class α in $H^p(V; \mathbb{C})$, by

$$T(\alpha) = \rho^*(\alpha) \smile \Psi_V. \tag{2.2}$$

We define the Gysin homomorphism $i_* : H^p(V; \mathbb{C}) \to H^{p+2k}(M; \mathbb{C})$ by $i_* = j^* \circ T$. Note that, if M is compact, we have the commutative diagram

$$\begin{array}{ccccc}
H^p(V;\mathbb{C}) & \xrightarrow{T} & H^{p+2k}(M, M \setminus V;\mathbb{C}) & \xrightarrow{j^*} & H^{p+2k}(M;\mathbb{C}) \\
\downarrow P_V & & \wr\downarrow A & & \wr\downarrow P_M \\
H_{2n-p}(V;\mathbb{C}) & \xrightarrow{=} & H_{2n-p}(V;\mathbb{C}) & \xrightarrow{i_*} & H_{2n-p}(M;\mathbb{C}).
\end{array}$$

In this and the subsequent sections, we consider the following two cases :
(i) V is non-singular,
(ii) V is a local complete intersection defined by a section (see Definition 2.3 below).

First, suppose V is non-singular and let $p : N_V \to V$ be the normal bundle of V in M. In this case, P and T are isomorphisms. We may take as U above a tubular neighborhood so that ρ is C^∞. Then $\rho : U \to V$ is isomorphic with $p : W \to V$ for a neighborhood W of the zero section in N_V, which we identify with V. The bundle $\rho^* N_V$ is also isomorphic with $p^* N_V$. Thus we have an isomorphism

$$H^*(M, M \setminus V; \mathbb{C}) \simeq H^*(N_V, N_V \setminus V; \mathbb{C}).$$

The Thom class Ψ_V of V corresponds to the Thom class Ψ_{N_V} of the bundle N_V under this isomorphism and the Thom homomorphism corresponds to the Thom isomorphism

$$T_{N_V} : H^p(V; \mathbb{C}) \xrightarrow{\sim} H^{p+2k}(N_V, N_V \setminus V; \mathbb{C}).$$

Note that, if we denote by s_Δ the diagonal section of the bundle $p^* N_V$ over N_V, its zero set is V and we have [Su2, Ch.III, Theorem 4.4]

$$\Psi_{N_V} = c^k(p^* N_V, s_\Delta).$$

Second, recall that a subvariety V of codimension k in M is a local complete intersection (abbreviated as LCI) in M if the ideal sheaf $\mathcal{I}_V$ in $\mathcal{O}_M$ of functions vanishing on V is locally generated by k functions. In this case, the normal sheaf $\mathcal{N}_V = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V)$ is a locally free $\mathcal{O}_V$-module, $\mathcal{O}_V = \mathcal{O}_M/\mathcal{I}_V$. We denote by N_V the associated vector bundle.

Definition 2.3. We say that a subvariety V of codimension k in M is an *LCI defined by a section* if there exist a holomorphic vector bundle N of rank k over M and a holomorphic section s of N such that the local components of s generate $\mathcal{I}_V$.

Thus a subvariety V in M is an LCI defined by a section if and only if there exist a holomorphic vector bundle N over M and a holomorphic section s of N such that (*) s is regular [F, B.3] and the analytic subspace defined by s is reduced and is equal to V. Furthermore, the condition (*) is equivalent to saying that s is generically transverse to the zero section and V is the zero set of s ([T], [Lo, VI.1.6], see also [Su3, Remark 4.10.3]). In this case, we have $N_V = N|_V$. Note that an LCI defined by a section is a "strong" local complete intersection in the sense of [LS]. Note also that for any hypersurface ($k = 1$) V in M, there is a natural line bundle N such that V is an LCI defined by a section of N.

We recall the following theorem, which is proved in [Su2]. See [F, §14.1] for the algebraic case.

Theorem 2.4. *Let V be a compact LCI defined by a section s of a bundle N over M. Then the localization $c^k(N,s)$ in $H^{2k}(M, M \setminus V; \mathbb{C})$ of $c^k(N)$ with respect to s corresponds to $[V]$ under the Alexander duality $H^{2k}(M, M \setminus V; \mathbb{C}) \xrightarrow{\sim} H_{2n}(V; \mathbb{C})$.*

Thus, if V is an LCI defined by a section, Theorem 2.4 shows that

$$\Psi_V = c^k(N, s). \tag{2.5}$$

§3. Riemann-Roch theorem for embeddings

Let V be a compact subvariety in a complex manifold M, which is either of type (i) or (ii) in the previous section. Let U be a regular neighborhood of V in M with a continuous retraction $\rho : U \to V$. In the case (ii), suppose V is defined by a section s of a vector bundle N over M. In the case (i), (M, V) is C^∞ diffeomorphic with (N_V, V) and, in the latter, V is defined by the diagonal section s_Δ of the bundle p^*N_V over N_V. In what follows we write N_V by M anew and set $N = p^*N_V$ and $s = s_\Delta$. Thus in either case we may express the Thom class Ψ_V as

(2.5). In the case (i), we may take as U a tubular neighborhood and we may assume that ρ is the restriction of p to U.

Let $U_0 = M \setminus V$ and U_1 a neighborhood of V as before. Also, let ∇_0 be an s-trivial connection for N on U_0 and ∇_1 an arbitrary connection for N on U_1. We consider the vector bundle $N \times \mathbb{R}$ over $U_{01} \times \mathbb{R}$ and let $\tilde{\nabla}$ be the connection for it given by $\tilde{\nabla} = (1-t)\nabla_0 + t\nabla_1$. Let $\Lambda^\bullet \nabla_\nu^\vee$ denote the family of connections $(\Lambda^k \nabla_\nu^\vee, \dots, \Lambda^0 \nabla_\nu^\vee)$ on U_ν, for $\nu = 0, 1$. Also denote by $\Lambda^\bullet \tilde{\nabla}^\vee$ the family $(\Lambda^k \tilde{\nabla}^\vee, \dots, \Lambda^0 \tilde{\nabla}^\vee)$. Let $\pi : U_{01} \times [0,1] \to U_{01}$ be the projection. Recall that, in $A^*(\mathcal{U})$,

$$\mathrm{ch}^*(\Lambda^\bullet \nabla_\star^\vee) = (\mathrm{ch}^*(\Lambda^\bullet \nabla_0^\vee), \mathrm{ch}^*(\Lambda^\bullet \nabla_1^\vee), \mathrm{ch}^*(\Lambda^\bullet \nabla_0^\vee, \Lambda^\bullet \nabla_1^\vee))$$

whose class in $H^*(M; \mathbb{C})$ is $\mathrm{ch}^*(\lambda_{N^\vee})$, $\lambda_{N^\vee} = \sum_{i=0}^k (-1)^i \Lambda^i N^\vee$.

Theorem 3.1. *The cocycle* $\mathrm{ch}^*(\Lambda^\bullet \nabla_\star^\vee)$ *is in* $A^*(\mathcal{U}, U_0)$ *and is given by*

$$\mathrm{ch}^*(\Lambda^\bullet \nabla_\star^\vee) = \mathrm{td}^{-1}(\nabla_\star) \smile c^k(\nabla_\star) + D\tau,$$

where $\tau = (0, 0, \tau_{01})$, $\tau_{01} = \pi_*(\mathrm{td}^{-1}(\pi^*\nabla_0, \tilde{\nabla}) \cdot d\, c^k(\pi^*\nabla_1, \tilde{\nabla}))$.

Proof. By (1.3), we have

$$\begin{aligned}
\mathrm{ch}^*(\Lambda^\bullet \nabla_0^\vee) &= \mathrm{td}^{-1}(\nabla_0) \cdot c^k(\nabla_0) = 0, \\
\mathrm{ch}^*(\Lambda^\bullet \nabla_1^\vee) &= \mathrm{td}^{-1}(\nabla_1) \cdot c^k(\nabla_1), \\
\mathrm{ch}^*(\Lambda^\bullet \nabla_0^\vee, \Lambda^\bullet \nabla_1^\vee) &= \pi_* \,\mathrm{ch}^*(\Lambda^\bullet \tilde{\nabla}^\vee) \\
&= \pi_*(\mathrm{td}^{-1}(\tilde{\nabla}) \cdot c^k(\tilde{\nabla})) = (\mathrm{td}^{-1} \cdot c^k)(\nabla_0, \nabla_1).
\end{aligned}$$

Hence we see that

$$\mathrm{ch}^*(\Lambda^\bullet \nabla_\star^\vee) = (\mathrm{td}^{-1} \cdot c^k)(\nabla_\star)$$

and the theorem follows from Proposition 1.6 (see also Remark 1.7). Q.E.D.

Note that $\tau = 0$ when $k = 1$.

Remark 3.2. Consider the Koszul complex associated to s [F, B.3]:

$$0 \longrightarrow \Lambda^k N^\vee \longrightarrow \cdots \longrightarrow \Lambda^1 N^\vee \longrightarrow \Lambda^0 N^\vee \longrightarrow 0, \tag{3.3}$$

which is exact on $U_0 = M \setminus V$. It is not difficult to see that the family $\Lambda^\bullet \nabla_0^\vee$ is compatible with the sequence (3.3) on U_0. The fact that $\mathrm{ch}^*(\Lambda^\bullet \nabla_0^\vee) = 0$ also follows from this (cf. Lemma 1.9).

Let $\mathcal{F}$ be a coherent $\mathcal{O}_V$-module. The direct image $i_!\mathcal{F}$ is a coherent $\mathcal{O}_M$-module, which is simply $\mathcal{F}$ extended by zero on $M \setminus V$, and thus we have the localized Chern character $\mathrm{ch}^*_V(i_!\mathcal{F})$ in $H^*(M, M \setminus V; \mathbb{C})$.

In the case (i), we take a resolution of $\mathcal{F}$ of the form (1.12) on V. Then we have $\mathrm{ch}^*(\mathcal{F}) = \mathrm{ch}^*(\xi)$, $\xi = \sum_{i=0}^q (-1)^i E_i$. Let $\nabla^{(i)}$ be a connection for E_i, $i = 0, \dots, q$, and denote by $\nabla^{\mathcal{F}}$ the family of connections $(\rho^*\nabla^{(0)}, \dots, \rho^*\nabla^{(q)})$, for the virtual bundle $\rho^*\xi$ over U.

In the case (ii), we assume that $\mathcal{F}$ is locally free and thus $\mathcal{F} = \mathcal{O}_V(F)$ for some vector bundle F over V. Since the classification of continuous vector bundles and that of C^∞ vector bundles coincide over paracompact manifolds, we may assume that ρ^*F is a C^∞ vector bundle and let $\nabla^{\mathcal{F}}$ be a connection for ρ^*F on U.

In either case, let $\mathrm{ch}^*(\nabla^{\mathcal{F}}_\star)$ denote the cocycle

$$\mathrm{ch}^*(\nabla^{\mathcal{F}}_\star) = (\mathrm{ch}^*(\nabla^{\mathcal{F}}), \mathrm{ch}^*(\nabla^{\mathcal{F}}), 0)$$

in $A^*(\mathcal{U})|_U$, whose class in $H^*(U; \mathbb{C})$ is $\rho^* \mathrm{ch}^*(\mathcal{F})$.

Corollary 3.4. *In the above situation, we have*

$$\mathrm{ch}^*(\nabla^{\mathcal{F}}_\star) \smile \mathrm{ch}^*(\Lambda^\bullet \nabla^\vee_\star) = \mathrm{ch}^*(\nabla^{\mathcal{F}}_\star) \smile \mathrm{td}^{-1}(\nabla_\star) \smile c^k(\nabla_\star) + D(\mathrm{ch}^*(\nabla^{\mathcal{F}}_\star) \smile \tau)$$

in $A^*(\mathcal{U}, U_0)|_U$.

Corollary 3.5. *Let V be a compact subvariety in M and $\mathcal{F}$ a coherent $\mathcal{O}_V$-module. We have the following formulas in either one of the cases*:
(i) *V is non-singular,*
(ii) *V is an LCI defined by a section and $\mathcal{F}$ is locally free.*

$$\begin{aligned} \mathrm{ch}^*_V(i_!\mathcal{F}) &= T(\mathrm{ch}^*(\mathcal{F}) \smile \mathrm{td}^{-1}(N_V)) && \text{in } H^*(M, M \setminus V; \mathbb{C}), \\ \mathrm{ch}^*(i_!\mathcal{F}) &= i_*(\mathrm{ch}^*(\mathcal{F}) \smile \mathrm{td}^{-1}(N_V)) && \text{in } H^*(M; \mathbb{C}). \end{aligned}$$

Proof. The Koszul complex (3.3) gives a locally free resolution of $i_!\mathcal{O}_V$:

$$0 \longrightarrow \mathcal{O}_M(\Lambda^k N^\vee) \longrightarrow \cdots \longrightarrow \mathcal{O}_M(\Lambda^0 N^\vee) \longrightarrow i_!\mathcal{O}_V \longrightarrow 0.$$

If we compute the local class $\mathrm{ch}^*_V(i_!\mathcal{O}_V)$ using this resolution, we see that it is represented by $\mathrm{ch}^*(\Lambda^\bullet \nabla^\vee_\star)$. We have, by (1.13),

$$\mathrm{ch}^*_V(i_!\mathcal{F}) = \begin{cases} \mathrm{ch}^*(\rho^*\xi \otimes i_!\mathcal{O}_V) = \mathrm{ch}^*(\rho^*\xi) \smile \mathrm{ch}^*_V(i_!\mathcal{O}_V), & \text{in the case (i)} \\ \mathrm{ch}^*(\rho^*F \otimes i_!\mathcal{O}_V) = \mathrm{ch}^*(\rho^*F) \smile \mathrm{ch}^*_V(i_!\mathcal{O}_V), & \text{in the case (ii).} \end{cases}$$

Recall that either $\mathrm{ch}^*(\rho^*\xi)$ or $\mathrm{ch}^*(\rho^*F)$ is represented by $\mathrm{ch}^*(\nabla_*^{\mathcal{F}})$. Recalling also that $N|_U \simeq \rho^* N_V$ and $c^k(N,s) = \Psi_V$ (the Thom class), by Corollary 3.4, we get

$$\mathrm{ch}_V^*(i_!\mathcal{F}) = \rho^*(\mathrm{ch}^*(\mathcal{F}) \smile \mathrm{td}^{-1}(N_V)) \smile \Psi_V.$$

By (2.2), we get the first formula. The second follows from the first. Q.E.D.

Remarks 3.6. 1. The equalities in Corollary 3.5 hold in cohomology with $\mathbb{Q}$ coefficients (cf. Remarks 1.11 and 2.1).
2. In the case V is non-singular, the formulas are proved in [AH2]. If, furthermore, V is algebraic, the second formula in Corollary 3.5 is a special case of the Grothendieck-Riemann-Roch theorem [BoSe].
3. In [I], a similar formula is proved for the Thom class of a vector bundle. Namely, let $p : E \to X$ be a complex vector bundle of rank r over a topological space X. Then, in our natation,

$$\mathrm{ch}_X^*(\lambda_{E^\vee}) = p^*\,\mathrm{td}^{-1}(E) \smile \Psi_E,$$

where $\lambda_{E^\vee} = \sum_{i=0}^r (-1)^i \Lambda^i p^* E^\vee$ and Ψ_E denotes the Thom class of E. When X is a C^∞ manifold, this formula can be proved at the level of Čech-de Rham cocycles as above; in the situation of Theorem 3.1, simply let $M = E$, $V = X$ (identified with the zero section of E), $N = p^*E$ and $s = s_\Delta$ and note that $\Psi_E = c^r(p^*E, s_\Delta)$.
4. In the algebraic category, the formulas are proved for a locally free $\mathcal{O}_V$-module on an LCI by analyzing the graph construction in [BFM, 3. Proposition]. Note that their general Riemann-Roch theorem does not directly imply the formulas.
5. These formulas are also proved at the level of differential forms and currents in [HL]. See also [Bi].

§4. Homology Chern characters and classes

Let V be a subvariety of pure codimension k in a complex manifold M. Suppose that V is an LCI. Thus the ideal sheaf $\mathcal{I}_V$ of functions vanishing on V is locally generated by k functions and the normal sheaf $\mathcal{N}_V = \mathcal{H}om_{\mathcal{O}_V}(\mathcal{I}_V/\mathcal{I}_V^2, \mathcal{O}_V)$ is locally free. We denote by N_V the associated vector bundle and let $\tau_V = TM|_V - N_V$ be the virtual tangent bundle of V. Note that it does not depend on the embedding $i : V \hookrightarrow M$.

Definition 4.1. For a coherent $\mathcal{O}_V$-module $\mathcal{F}$, we define the homology Chern character $\mathrm{ch}_*(\mathcal{F})$ by

$$\mathrm{ch}_*(\mathcal{F}) = \mathrm{td}\, N_V \frown A(\mathrm{ch}_V^*(i_!\mathcal{F})).$$

Remarks 4.2. 1. If V is an LCI defined by a section of a vector bundle N over M, we may write

$$\mathrm{ch}_*(\mathcal{F}) = A(\mathrm{td}\, N \smile \mathrm{ch}_V^*(i_!\mathcal{F})).$$

2. The above definition is related to the (homology) Todd class $\tau(\mathcal{F})$ of $\mathcal{F}$ in [BFM] by

$$\mathrm{ch}_*(\mathcal{F}) = (\mathrm{td}^{-1}\,\tau_V) \frown \tau(\mathcal{F}).$$

In [BFM], $\tau(\mathcal{F})$ is defined using an embedding of V, but it is shown that $\tau(\mathcal{F})$ is independent of the embedding for a projective variety V. Thus $\mathrm{ch}_*(\mathcal{F})$ is also independent of the embedding in this case.

The following directly follows from the definition.

Proposition 4.3. (1) *For an exact sequence of coherent $\mathcal{O}_V$-modules*

$$0 \longrightarrow \mathcal{F}_q \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow 0,$$

we have

$$\sum_{i=0}^{q}(-1)^i\,\mathrm{ch}_*(\mathcal{F}_i) = 0.$$

(2) *For a vector bundle E over V and a coherent $\mathcal{O}_V$-module $\mathcal{F}$,*

$$\mathrm{ch}_*(E \otimes \mathcal{F}) = \mathrm{ch}^*(E) \frown \mathrm{ch}_*(\mathcal{F}).$$

The following is a direct consequence of Corollary 3.5.

Proposition 4.4. *Suppose either V is non-singular or V is defined by a section and $\mathcal{F}$ is locally free. Then we have*

$$\mathrm{ch}_*(\mathcal{F}) = \mathrm{ch}^*(\mathcal{F}) \frown [V].$$

In particular, for the structure sheaf $\mathcal{O}_V$,

$$\mathrm{ch}_*(\mathcal{O}_V) = [V].$$

If $\mathrm{ch}_*(\mathcal{F})$ is in the image of the Poincaré homomorphism $H^*(V) \to H_*(V)$, we may define the homology Chern class $c_*(\mathcal{F})$ via Newton's formula. Namely, suppose

$$\mathrm{ch}_*(\mathcal{F}) = \sigma^* \frown [V],$$

for some σ^* in $H^*(V)$ and write $\sigma^* = \sum_{i\geq 0}\sigma^i/i!$ with σ^i in $H^{2i}(V)$. Then we define $\gamma^* = 1 + \sum_{i\geq 1}\gamma^i$ with γ^i in $H^{2i}(V)$ by

$$\sigma^i - \gamma^1\sigma^{i-1} + \gamma^2\sigma^{i-2} - \cdots + (-1)^i i\gamma^i = 0, \qquad i \geq 1.$$

If we define the homology Chern class $c_*(\mathcal{F})$ of $\mathcal{F}$ by

$$c_*(\mathcal{F}) = \gamma^* \frown [V],$$

then it is not difficult to check that the definition does not depend on the choice of σ^*.

Example 4.5. Suppose either V is non-singular or V is defined by a section and $\mathcal{F}$ is locally free. Then, from Proposition 4.4,

$$c_*(\mathcal{F}) = c^*(\mathcal{F}) \frown [V].$$

In particular,

$$c_*(\mathcal{O}_V) = [V].$$

§5. Characteristic classes of the tangent sheaf

Let V be an LCI defined by a section of a vector bundle N over a complex manifold M. Denoting by Ω_M and Ω_V the sheaves of holomorphic 1-forms on M and V, respectively, we have the exact sequence

$$0 \longrightarrow \mathcal{I}_V/\mathcal{I}_V^2 \longrightarrow \Omega_M \otimes_{\mathcal{O}_M} \mathcal{O}_V \longrightarrow \Omega_V \longrightarrow 0.$$

Let $\Theta_M = \mathcal{O}_M(TM)$ be the tangent sheaf of M. We define the tangent sheaf Θ_V of V by $\Theta_V = \mathcal{H}om_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$, which is independent of the embedding $V \hookrightarrow M$. From the above sequence, we have the exact sequence

$$0 \longrightarrow \Theta_V \longrightarrow \Theta_M \otimes_{\mathcal{O}_M} \mathcal{O}_V \longrightarrow \mathcal{N}_V \longrightarrow \mathcal{E}xt^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V) \longrightarrow 0.$$

Setting $\mathcal{E} = \mathcal{E}xt^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)$, we get, from Propositions 4.3 and 4.4,

$$\mathrm{ch}_*(\Theta_V) = \mathrm{ch}^*(\tau_V) \frown [V] + \mathrm{ch}_*(\mathcal{E}).$$

If p is an isolated singular point of V, by the Riemann-Roch theorem for the embedding $p \hookrightarrow M$, we have $\mathrm{ch}_*(\mathcal{E}) = \tau(V,p)[p]$, where $\tau(V,p) = \dim \mathcal{E}xt^1_{\mathcal{O}_V}(\Omega_V, \mathcal{O}_V)_p$ is the Tjurina number of V at p. Thus we have the following :

Theorem 5.1. *Let V be an LCI of dimension n (≥ 1) defined by a section with isolated singularities $p_1, \dots, p_s$. For the tangent sheaf Θ_V of V, we have*

$$\mathrm{ch}_*(\Theta_V) = \mathrm{ch}^*(\tau_V) \frown [V] + \sum_{i=1}^{s} \tau(V, p_i)\,[p_i],$$

$$c_*(\Theta_V) = c^*(\tau_V) \frown [V] + (-1)^{n+1}(n-1)! \sum_{i=1}^{s} \tau(V, p_i)\,[p_i].$$

Note that the class $c^*(\tau_V) \frown [V]$ coincides with the canonical class of [F, Example 4.2.6], [FJ] in this case.

Let (V, p) be an isolated complete intersection singularity. If it admits a good $\mathbb{C}^*$-action in the sense of [Loo, 9.B], $\tau(V,p) = \mu(V,p)$, the Milnor number of V at p ([G, 3. Satz], [Loo, (9.10) Proposition]). On the other hand, for a variety as in Theorem 5.1, the Schwartz-MacPherson class $c_*(V)$ of V is given by [Su1]

$$c_*(V) = c^*(\tau_V) \frown [V] + (-1)^{n+1} \sum_{i=1}^{s} \mu(V, p_i)\,[p_i].$$

Hence we have

Corollary 5.2. *Let V be as in* Theorem 5.1 *with $n = 1$ or 2. If V admits a good $\mathbb{C}^*$-action near each singular point p_i, then*

$$c_*(\Theta_V) = c_*(V).$$

Remark 5.3. It would be an interesting problem to compare the class $\mathrm{ch}_*(\mathcal{F})$ with the homology Chern character of $\mathcal{F}$ as defined in [Sc2].

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