On elliptic Lax pairs and isomonodromic deformation systems for elliptic lattice equations

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In honour of Professor Noumi for the occasion of his 60th birthday

Abstract.

In a previous article [11] a novel class of elliptic Lax pairs for integrable lattice equations was introduced. The present article proposes a de-autonomisation of those Lax pairs leading to a class of elliptic discrete isomonodromic deformation problems. We analyse the systems of compatibility conditions using some (possibly novel) higher order elliptic identities.

§1. Introduction

Ever since Richard Fuchs in 1905, [18], investigated the problem posed by his father, Lazarus Fuchs, namely to describe the iso-monodromic deformation of a linear second order ordinary differential equation (ODE) with 3 fixed and one moving regular singularity, and thus discovered for the first time the nonlinear differential equation for an apparent singularity that has to be included in the coefficients, which is now known by the name Painlevé VI equation (P_{VI}), the study of linear systems of equations giving rise to nonlinear equations as compatibility conditions has formed a main theme in the subject called *integrable systems*. Such systems of linear equations are nowadays usually loosely called *Lax pairs*, referring to the work by Peter Lax, [31], on the general structure of the linear system found earlier by Gardner, Greene, Kruskal and Miura giving rise to the Korteweg-de Vries (KdV) equation.

Richard Fuchs' problem constitutes probably the first ever example of a non-trivial Lax pair, although as an *isomonodromic deformation problem* it is of a different type than the 'autonomous' Lax pairs that have become prominent in the theory of soliton systems and integrable PDEs and their discrete counterparts, cf. e.g. [1].

The history of isomonodromic deformation theory, in connection with Painlevé type equations, has undergone some spectacular developments especially in recent years. R.Fuchs' 1905 paper[1] was followed up by his more often cited paper of 1907, [19], as well as by the work Ludwig Schlesinger, [48], on the isomonodromic deformations of more general matrix systems. Furthermore, René Garnier in [20] extended Fuchs' scheme to higher order, allowing multiple moving singularities, leading to coupled systems of ODEs (one for each moving singularity), which are compatible through some additional partial differential relations. It is worth mentioning that the Garnier system amounts to what in modern terminology would be called a *Painlevé VI hierarchy*. Fast forward almost seventy years, when the isomonodromic deformation theory in the sense of Painlevé equations, was rediscovered on the one hand in the seminal work of the Kyoto school, [23, 24, 25] and by Flaschka and Newell [12], and where the context of reductions (from integrable partial differential equations) allowed systematic constructions of isomonodromic deformation problems using the techniques of integrable systems. An overview of how isomonodromic deformation problems in combination with the Riemann-Hilbert techniques was used to access the transcendental solutions of Painelvé equations can be found in the monographs [22, 15].

In the early 1990s, when *discrete Painlevé equations* made their appearance, discrete isomonodromic deformation problems were provided and studied in [33, 14, 43, 27]. The first isomonodromic deformation problem of q-difference type was provided in [43] for a q-version of the P_{III} equation. The latter, in fact, in its most general alternate form is equivalent to the q-version of P_{VI} of Jimbo and Sakai, [26], who provided a q-isomonodromic deformation problem closer in spirit to the Fuchs' system for the continuous PVI equation, (cf. also [21] for the connection between these two different isomonodromic deformation problems of q-difference type). The first proposals for the higher-order analogues, i.e., discrete Garnier type systems, were given in [36, 47], while [46] delivered a full q-analogue of the original Garnier system. The general theory of discrete isomonodromic deformation theory was developed in various

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[1]The discovery of P_{VI} in this paper was commemorated in the special issue [10] where also some of the early history was summarised.

recent papers, [6, 9, 29], where the Riemann-Hilbert aspects go back to work by Birkhoff and collaborators in the early 20th century, [7, 8].

In this paper we present a general class of isomonodromic deformation problems which form (in some sense) the non-autonomous counterparts of the elliptic Lax systems studied in the article [11]. In the spirit of the early paper [43], cf. also [21], where we introduced the *de-autonomization* procedure for differential and q-difference Lax pairs which were obtained from periodic reductions of integrable lattice equations, leading to Lax pairs for associated discrete Painlevé equations, we will introduce isomonodromic deformation problems of elliptic type. A general theory of elliptic type isomonodromic deformation problems, and the associated Riemann-Hilbert problem, was presented in [29], but the compatibility conditions were not worked out in that paper. On the other hand, the problem of finding a Lax pair for the famous elliptic discrete Painlevé equation of Sakai, [45], was addressed by several authors in recent years, [50, 44], cf. also [37], based on the birational geometry behind the Painlevé equations (see [28] for a recent review). We mention, in passing, that in the continuous case, various elliptic isomonodromic deformation problems were studied in the literature [30, 49], essentially going back to the pioneering work by Okamoto in the 1970s, [38, 39, 40]. We point out that in contrast to the above works on elliptic isomonodromic systems, our approach is very natural from the perspective of the lattice systems studied in [11] and is essentially based on the exploration of elliptic addition formulae. Along the way we present some seemingly novel elliptic identities of arbitrary order (i.e, in terms of arbitrary number of free arguments) which facilitate the analysis of the compatibility conditions. We will present the general scheme of equations, for arbitrary rank and order, and highlight some explicit cases for the sake of illustration.

§2. Autonomous elliptic Lax pairs for Integrable lattice systems

In a previous article [11] a general elliptic Lax scheme of rank N, generalizing a novel Lax representation of Adler's lattice equation (Q4), [3], was presented. In this section, mostly for the sake of introducing the notations, we outline the main construction of the corresponding autonomous lattice systems. For the basic definitions and standard formulae regarding the Weierstrass family of elliptic functions $\sigma(x)$, $\zeta(x)$ and $\wp(x)$ we refer to Appendix A.

We consider a compatible linear system for a N-component vector function $\boldsymbol{\chi}_\kappa = \boldsymbol{\chi}_\kappa(n,m)$, depending on a complex-valued spectral parameter κ (we prefer to indicate the dependence on the spectral parameter by means of an index rather than as a function argument to make the formulae more transparent), as well as on discrete variables n, m. In fact, the dependent variables of the systems under consideration, such as the functions $\xi_i(n,m)$ below, are functions of discrete independent variables n and m, and for convenience we denote the unit shifts in these variables (i.e. elementary shifts along the lattice) by the notation:

$$\widetilde{\xi}_i(n,m) := \xi_i(n+1,m) \ , \quad \widehat{\xi}_i(n,m) := \xi_i(n,m+1) \ ,$$

while multiple shifts are denoted, e.g. by $\widehat{\widetilde{\xi}}_i(n,m) := \xi(n+1,m+1)$, etc. The linear system (Lax pair) is given by the simultaneous pair of shift relations:

$$(2.1) \qquad\qquad \widetilde{\boldsymbol{\chi}}_\kappa = \boldsymbol{L}_\kappa\,\boldsymbol{\chi}_\kappa \ , \quad \widehat{\boldsymbol{\chi}}_\kappa = \boldsymbol{M}_\kappa\,\boldsymbol{\chi}_\kappa \ ,$$

defining horizontal and vertical shits of the vector function $\boldsymbol{\chi}_\kappa$, according to the diagram:
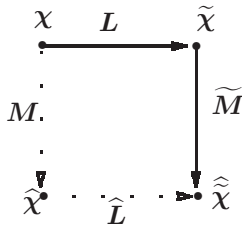

Fig. 1. Lattice compatibility configuration.

The compatibility condition (discrete zero-curvature condition) leads to the matrix relation

$$(2.2) \qquad\qquad \widehat{\boldsymbol{L}}_\kappa \boldsymbol{M}_\kappa = \widetilde{\boldsymbol{M}}_\kappa \boldsymbol{L}_\kappa \ .$$

The Lax matrices[2] $\boldsymbol{L}_\kappa$ and $\boldsymbol{M}_\kappa$ attached to the edges are assumed to be of the form:

(2.3a) $\qquad (\boldsymbol{L}_\kappa)_{i,j} = \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha)h_j \ ,$

(2.3b) $\qquad (\boldsymbol{M}_\kappa)_{i,j} = \Phi_\kappa(\widehat{\xi}_i - \xi_j - \beta)k_j \ , \quad (j = 1, \ldots, N)$

where the coefficients h_j and k_j are assumed to be independent of the spectral parameter κ, as are the functions $\xi_i = \xi_i(n, m)$ appearing in the argument of the function Φ_κ which is defined as

(2.4) $$\Phi_\kappa(x) := \frac{\sigma(\kappa + x)}{\sigma(\kappa)\,\sigma(x)} \ .$$

The parameters α and β in (2.3) are constants, but as they are associated with the lattice directions we refer to them as lattice parameters.

Working out the compatibility condition (2.2) by means of the basic addition formulae given in Appendix A, in particular (B.1), the consistency requirement gives rise to

$$\sum_{l=1}^N \widehat{h}_l k_j \left[\zeta(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha) + \zeta(\widehat{\xi}_l - \xi_j - \beta) + \zeta(\kappa) \right.$$
$$\left. - \zeta(\kappa + \widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta) \right]$$
$$= \sum_{l=1}^N \widetilde{k}_l h_j \left[\zeta(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha) + \zeta(\kappa) \right.$$
$$\left. - \zeta(\kappa + \widehat{\widetilde{\xi}}_i - \xi_j - \alpha - \beta) \right]$$
$$(i, j = 1, \ldots, N) \ .$$

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[2]We prefer to indicate the dependence of these matrices on the spectral parameter κ by an index notation, even though κ is generally a complex valued quantity (being a uniformising variable for the Weierstrass elliptic curve), in order to make the κ dependence clearly visible.

Due to the (arbitrary) dependence on the spectral parameter κ these equations separate into two parts:

(2.5a)
$$\left(\sum_{l=1}^{N} \widehat{h}_l\right) k_j = \left(\sum_{l=1}^{N} \widetilde{k}_l\right) h_j, \quad (j = 1, \dots, N),$$

(2.5b)
$$\sum_{l=1}^{N} \widehat{h}_l \left[\zeta(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha) + \zeta(\widehat{\xi}_l - \xi_j - \beta)\right] k_j$$
$$= \sum_{l=1}^{N} \widetilde{k}_l \left[\zeta(\widehat{\widetilde{\xi}}_i - \widetilde{\xi}_l - \beta) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha)\right] h_j$$
$$(i, j = 1, \dots, N).$$

From this system of equations we want to extract a closed form system of lattice equations for the main dependent variables $\xi_i(n, m)$, eliminating the coefficient variables h_i, k_i. In order to do so we have to distinguish now between two cases which we referred to in [11] as *Landau-Lifschitz type* (spin non-zero) and *Krichever-Novikov type* (spin zero) respectively, depending on whether $\sum_l h_l$ is non-vanishing or zero. The latter case (for $N = 2$) corresponds to the case of Q4, in which case we obtain readily the 3-leg form of that equation. The main part of the paper [11] was dedicated to analysing the higher-rank situation of that case, deriving a coupled set of implicit lattice equations, analogous to the 3-leg form, for $N = 3$, we will not summarize those results here as they require a lot of additional notation.

In the case where $\sum_l h_l \neq 0$ we have that the variables h_j, k_j are proportional to each other, $k_j = \rho h_j$, and after summation we obtain the (multiplicative) conservation law:

(2.6)
$$\frac{\sum_{l=1}^{N} \widehat{h}_l}{\sum_{l=1}^{N} h_l} = \frac{\sum_{l=1}^{N} \widetilde{k}_l}{\sum_{l=1}^{N} k_l},$$

so that the Lax equations reduce to

$$\sum_{l=1}^{N} \left[\zeta(\widehat{\widetilde{\xi}}_i - \widehat{\xi}_l - \alpha)\rho\widehat{h}_l - \zeta(\widehat{\widetilde{\xi}}_l - \widetilde{\xi}_j - \beta)\widetilde{k}_l\right]$$
$$= \sum_{l=1}^{N} \left[\zeta(\xi_j - \widehat{\xi}_l + \beta)\rho\widehat{h}_l - \zeta(\xi_j - \widetilde{\xi}_l + \alpha)\widetilde{k}_l\right],$$
$$(i, j = 1, \dots, N).$$

Under the further assumption that the Centre of Mass (CoM) motion obeys the equation

$$(2.7) \qquad \widetilde{\Xi} + \widehat{\Xi} = \widehat{\widetilde{\Xi}} + \Xi \ , \quad \text{where} \quad \Xi := \sum_{j=1}^{N} \xi_j$$

we can analyse the Lax equations most conveniently by considering the following elliptic function and its expansion in terms of Weierstrass ζ-functions, using the elliptic Lagrange interpolation formula (B.7) in Appendix B:

$$
\begin{aligned}
F(\xi) &:= \prod_{l=1}^{N} \frac{\sigma(\xi - \widehat{\widetilde{\xi}}_l)\sigma(\xi - \xi_l - \alpha - \beta)}{\sigma(\xi - \widehat{\xi}_l - \alpha)\sigma(\xi - \widetilde{\xi}_l - \beta)} \\
&= \sum_{l=1}^{N} \left[\zeta(\xi - \widehat{\xi}_l - \alpha) - \zeta(\eta - \widehat{\xi}_l - \alpha) \right] H_l \\
&\quad + \sum_{l=1}^{N} \left[\zeta(\xi - \widetilde{\xi}_l - \beta) - \zeta(\eta - \widetilde{\xi}_l - \beta) \right] K_l \ .
\end{aligned}
$$

The latter holds as an identity for any four sets of variables ξ_l, $\widehat{\xi}_l$, $\widetilde{\xi}_l$, $\widehat{\widetilde{\xi}}_l$ such that the relation (2.7) for their sums holds. Here η denotes any one of the zeroes (i.e., $\widehat{\widetilde{\xi}}_i$ or $\xi_i + \alpha + \beta$), and we have the explicit expressions for the coefficients:

$$(2.8a) \quad H_l \;=\; \frac{\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widehat{\widetilde{\xi}}_k + \alpha)\sigma(\widehat{\xi}_l - \xi_k - \beta)}{\left[\prod_{k=1}^{N} \sigma(\widehat{\xi}_l - \widetilde{\xi}_k + \alpha - \beta) \right] \prod_{k \neq l} \sigma(\widehat{\xi}_l - \widehat{\xi}_k)} \ ,$$

$$(2.8b) \quad K_l \;=\; \frac{\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\widetilde{\xi}}_k + \beta)\sigma(\widetilde{\xi}_l - \xi_k - \alpha)}{\left[\prod_{k=1}^{N} \sigma(\widetilde{\xi}_l - \widehat{\xi}_k + \beta - \alpha) \right] \prod_{k \neq l} \sigma(\widetilde{\xi}_l - \widetilde{\xi}_k)} \ .$$

We note that by construction the coefficients obey the identity $\sum_{l=1}^{N}(H_l + K_l) = 0$.

Using the identities above, taking $\xi = \widehat{\widetilde{\xi}}_i$, $\eta = \xi_j + \alpha + \beta$ in $F(\xi)$, and comparing the result with the Lax equations, we can identify:

$$tH_l = \rho \widehat{h}_l \ , \quad -tK_l = \widetilde{k}_l = \widetilde{\rho} \widetilde{h}_l \ , \quad l = 1, \ldots, N \ ,$$

with t an arbitrary proportionality factor. Thus, inserting the explicit expressions for H_l and K_l we obtain a system of $N+2$ equations for the

$N + 2$ unknowns: $\xi_1, \ldots, \xi_N$, ρ, t. This comprises the set of equations

$$\frac{\widetilde{t}}{\underset{\sim}{\rho}}\widetilde{H}_l + \frac{\widehat{t}}{\underset{\approx}{\rho}}\widehat{K}_l = 0 , \quad (l = 1, \ldots, N) , \quad \widetilde{\widetilde{\Xi}} + \widehat{\widehat{\Xi}} = \widehat{\widehat{\widetilde{\Xi}}} + \Xi ,$$

which yields the system of N 7-point equations:

$$(2.9) \quad \prod_{k=1}^{N} \frac{\sigma(\xi_l - \widetilde{\xi}_k + \alpha)\,\sigma(\xi_l - \underset{\sim}{\xi}_k - \beta)\,\sigma(\xi_l - \widehat{\xi}_k + \gamma)}{\sigma(\xi_l - \widehat{\xi}_k + \beta)\,\sigma(\xi_l - \underset{\sim}{\xi}_k - \alpha)\,\sigma(\xi_l - \widetilde{\xi}_k - \gamma)} = p$$

for $N + 1$ variables ξ_i ($i = 1, \ldots, N$) and $p = -\underset{\sim}{t}\,\rho\,/(\underset{\sim}{t}\,\rho)$, supplemented with the relation (2.7), which fixes the CoM dynamics. The under-accents $\underset{\sim}{\cdot}$ and $\underset{\frown}{\cdot}$ in (2.9) denote reverse lattice shifts: $\underset{\sim}{\xi}_i(n, m) = \xi_i(n - 1, m)$, $\underset{\frown}{\xi}_i(n, m) = \xi_i(n, m - 1)$. We note that the implicit system of PΔEs arises from the following Lagrangian:

$$\mathscr{L} = \sum_{i,j}\left[f(\xi_i - \widetilde{\xi}_j + \alpha) - f(\xi_i - \widehat{\xi}_j + \beta) - f(\widehat{\xi}_i - \widetilde{\xi}_j + \alpha - \beta)\right] - \ln|p|\,\Xi$$

in which the function f is the elliptic dilogarithm $f(x) = \int^x \ln \sigma(\xi)\,d\xi$. For reasons that we will not go into here we expect the system of equations for $N = 2$ to constitute an implicit form of a lattice version of the Landau-Lifschitz (LL) equations, in a similar way as the 3-leg equations are an implicit form of Q4. Lattice versions of the LL equations were proposed in [32, 2, 4], but so far no relation between these different proposals has been established, nor is anything known yet about the solution structure of those models.

Remark: The one-step periodic reduction, $\widetilde{\chi}_\kappa = \lambda\chi_\kappa$, reduces the Lax system (2.1) to the Lax pair of the discrete-time Ruijsenaars system that was constructed in [34]. In that case the Lax equation $\widehat{L}_\kappa M_\kappa = M_\kappa L_\kappa$ yields an implicit system of second order OΔEs which constitutes the discrete-time equations of motion for the discrete-time Ruijsenaars system. In the simplest case, these will re-emerge as the autonomous limit of the systems we will consider next in the simplest case. In a sense the results described in the remainder of the paper could be thought of as constituting a similarity reduction of the system of 7-point equations (2.9). We will come back to this in the Conclusions.

§3. Elliptic isomonodromic deformation scheme

In this section, we introduce a general class of isomonodromic deformation problems on the torus, inspired by the form of the lattice systems of the previous section.

Using a similar notation as before we now consider the compatibility of the system of linear equations

$$(3.1a) \qquad \chi_{\kappa+\tau} = T_\kappa \chi_\kappa ,$$
$$(3.1b) \qquad \widetilde{\chi}_\kappa = L_\kappa \chi_\kappa ,$$

where the first relation (3.1a) is a linear first order difference equation on the torus, i.e. defining a shift over a fixed increment τ in the uniformizing spectral variable κ, while the second relation (3.1b) defines a shift $\widetilde{\ }$ of the vector function χ_κ in some additional discrete variable n as according to (2.1). Eventually we will equip the system (3.1) with additional lattice directions, in additional variables such as m as in section 2, each of which corresponding to a linear equation such as (3.1b).

3.1. General scheme

We take the matrices L_κ and T_κ of the form

$$(3.2a) \qquad (L_\kappa)_{i,j} = H_{i,j}\ \sigma(\kappa)\ \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha) ,$$

$$(3.2b) \qquad (T_\kappa)_{i,j}$$
$$= \sum_{l_1,\dots,l_{m-1}} S_{i,j}^{(l_1,\dots,l_{m-1})} \prod_{\nu=1}^{m} \sigma(\kappa - \kappa_\nu)\ \Phi_{\kappa-\kappa_\nu}(\xi_{l_{\nu-1}}^{(\nu-1)} - \xi_{l_\nu}^{(\nu)} - \gamma_\nu) ,$$
$$(i,j = 1,\dots,N)$$

where the matrices $H = (H_{i,j})$ and the quantities $S_{i,j}^{l,l_1,\dots,l_{m-1}}$ remain to be specified, and in which we identify:

$$\xi_{l_0}^{(0)} = \xi_i , \quad \xi_{l_m}^{(m)} = \xi_j .$$

The parameters κ_ν, $(\nu = 1,\dots,m)$, are fixed, while the quantities γ_ν, $(\nu = 1,\dots,m)$, are assumed to be functions of the discrete variable n, the precise dependence of which follows from the analysis below. All quantities are assumed to be independent of κ unless explicitly indicated.

The compatibility condition

$$(3.3) \qquad \widetilde{T}_\kappa L_\kappa = L_{\kappa+\tau} T_\kappa ,$$

gives rise to

$$\sum_{l_1,\dots,l_m} \widetilde{S}_{i,l_m}^{(l_1,\dots,l_{m-1})} H_{l_m,j}\, \sigma(\kappa)\Phi_\kappa(\widetilde{\xi}_{l_m} - \xi_j - \alpha)$$

$$\times \left[\prod_{\nu=1}^{m} \sigma(\kappa - \kappa_\nu)\, \Phi_{\kappa-\kappa_\nu}(\widetilde{\xi}_{l_{\nu-1}}^{(\nu-1)} - \widetilde{\xi}_{l_\nu}^{(\nu)} - \widetilde{\gamma}_\nu) \right]$$

$$= \sum_{l,l'_1,\dots,l'_{m-1}} H_{i,l}\, S_{l,j}^{(l'_1,\dots,l'_{m-1})} \sigma(\kappa+\tau)\Phi_{\kappa+\tau}(\widetilde{\xi}_i - \xi_l - \alpha)$$

$$\times \prod_{\nu=1}^{m} \sigma(\kappa - \kappa_\nu)\, \Phi_{\kappa-\kappa_\nu}(\xi_{l'_{\nu-1}}^{(\nu-1)} - \xi_{l'_\nu}^{(\nu)} - \gamma_\nu)$$

in which we set in addition to the above identifications:

$$\xi_{l'_0}^{(0)} = \xi_l \ , \quad \xi_{l'_m}^{(m)} = \xi_j \ .$$

Using the relation $\Phi_\kappa(\tau)\Phi_{\kappa+\tau}(x) = \Phi_\kappa(\tau + x)\Phi_\tau(x)$ on the right hand side, as well as the identity (C.1) we can rewrite both sides of the latter equality to yield:

$$\sum_{l_1,\dots,l_m} \widetilde{S}_{i,l_m}^{(l_1,\dots,l_{m-1})} H_{l_m,j}$$

$$\times \sum_{\nu'=1}^{m+1} \Phi_{\kappa-\kappa_{\nu'}}(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}) \prod_{\substack{\nu=1\\\nu\neq\nu'}}^{m+1} \Phi_{\kappa_{\nu'}-\kappa_\nu}(\widetilde{\xi}_{l_{\nu-1}}^{(\nu-1)} - \widetilde{\xi}_{l_\nu}^{(\nu)} - \widetilde{\gamma}_\nu)$$

$$= \sum_{l,l'_1,\dots,l'_{m-1}} H_{i,l}\, S_{l,j}^{(l'_1,\dots,l'_{m-1})} \sigma(\tau)\Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)$$

$$\times \sum_{\nu'=0}^{m} \Phi_{\kappa-\kappa_{\nu'}}(\widetilde{\xi}_i - \xi_j - \alpha + \tau - \gamma) \prod_{\substack{\nu=0\\\nu\neq\nu'}}^{m} \Phi_{\kappa_{\nu'}-\kappa_\nu}(\xi_{l'_{\nu-1}}^{(\nu-1)} - \xi_{l'_\nu}^{(\nu)} - \gamma_\nu) \ ,$$

where (in order to avoid having to separate the sums and products) we have introduced the notations:

$$\kappa_0 := 0 \ , \quad \kappa_{m+1} := 0 \ , \quad \gamma_{m+1} := 0 \ ,$$

$$\xi_{l'_{-1}}^{(-1)} = \widetilde{\xi}_i \ , \quad \widetilde{\xi}_{l_{m+1}}^{(m+1)} = \xi_j + \alpha \ , \quad \gamma := \sum_{\nu=1}^{m} \gamma_\nu \ .$$

Setting now $\widetilde{\gamma} = \gamma - \tau$ (implying that at this point we take the γ_ν to depend on n such that their sum is a linear function of the discrete

variable) the terms depending on κ can be identified leading to the system of relations:

$$
(3.4a) \qquad \sum_{l_1,\ldots,l_m} \widetilde{S}_{i,l_m}^{(l_1,\ldots,l_{m-1})} H_{l_m,j} \prod_{\substack{\nu=1 \\ \nu \neq \nu'}}^{m+1} \Phi_{\kappa_{\nu'}-\kappa_\nu}(\widetilde{\xi}_{l_{\nu-1}}^{(\nu-1)} - \widetilde{\xi}_{l_\nu}^{(\nu)} - \widetilde{\gamma}_\nu)
$$

$$
= \sum_{l,l_1',\ldots,l_{m-1}'} H_{i,l} S_{l,j}^{(l_1',\ldots,l_{m-1}')} \sigma(\tau) \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)
$$

$$
\times \prod_{\substack{\nu=0 \\ \nu \neq \nu'}}^{m} \Phi_{\kappa_{\nu'}-\kappa_\nu}(\xi_{l_{\nu-1}'}^{(\nu-1)} - \xi_{l_\nu'}^{(\nu)} - \gamma_\nu) ,
$$

$$
\nu' = 1, \ldots, m , \quad i,j = 1, \ldots, N,
$$

together with

$$
(3.4b) \qquad \sum_{l,l_1,\ldots,l_{m-1}} \widetilde{S}_{i,l}^{(l_1,\ldots,l_{m-1})} H_{l,j} \prod_{\nu=1}^{m} \Phi_{-\kappa_\nu}(\widetilde{\xi}_{l_{\nu-1}}^{(\nu-1)} - \widetilde{\xi}_{l_\nu}^{(\nu)} - \widetilde{\gamma}_\nu)
$$

$$
= \sum_{l,l_1',\ldots,l_{m-1}'} H_{i,l} S_{l,j}^{(l_1',\ldots,l_{m-1}')} \sigma(\tau) \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)
$$

$$
\times \prod_{\nu=1}^{m} \Phi_{-\kappa_\nu}(\xi_{l_{\nu-1}'}^{(\nu-1)} - \xi_{l_\nu'}^{(\nu)} - \gamma_\nu) ,
$$

$$
i,j = 1, \ldots, N.
$$

The fundamental system of relations (3.4) is the basis of further analysis. Like in the autonomous system described in the section 2, the coefficient matrices $H_{i,j}$ and, in this case, $S_{i,j}^{(l_1',\ldots,l_{m-1}')}$ have to be eliminated. To remain in the spirit of the previous case we will make some simplifying assumptions, for instance that the matrix of coefficient (H_{ij}) is of rank 1. Eliminating the those coefficients would yield a system of of equations for the main quantities ξ_i including the quantities $\xi_i^{(\nu)}$. For consistency also some 'global' conditions may be needed on the latter quantities, such as certain restrictions on the sums $\Xi^{(\nu)} = \sum_l \xi_l^{(\nu)}$.

3.2. Example: First order scheme

In order to make the structure of the Lax system (3.4) more transparant, we will first illustrate them by means of some simpler examples, namely the cases where $m = 1$ and $m = 2$ in (3.2b).

First order scheme $(m = 1)$ In this case we have the elliptic discrete isomonodromic system (3.1) with Lax matrices of the form:

(3.5a) $$(\boldsymbol{L}_\kappa)_{i,j} = H_{i,j}\, \sigma(\kappa)\, \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha)\ ,$$

(3.5b) $$(\boldsymbol{T}_\kappa)_{i,j} = S_{i,j}\, \sigma(\kappa - \kappa_1)\, \Phi_{\kappa-\kappa_1}(\xi_i - \xi_j - \gamma)\ ,$$
$$(i, j = 1, \ldots, N)\ .$$

The coefficients $H_{i,j}$, $S_{i,j}$ do not depend on the spectral parameter κ and remain to be determined, while the $\xi_i = \xi_i(n)$ are the main independent variables. We observe in this case that the forms of the matrices (3.5) are reminiscent of those of the discrete zero-curvature Lax pair (2.3), except that we include here extra factors $\sigma(\kappa)$ which turn out to be necessary in order to separate out the κ-dependence in the consistency conditions[3]. Furthermore, the coefficients h_j, k_j in (2.3) correspond to a rank 1 restriction on the coefficient matrices $\boldsymbol{H} = (H_{ij})$ and $\boldsymbol{S} = (S_{ij})$, which here we don't want to impose from the start in order to allow for more freedom in the analysis. We observe, furthermore, that in this case we have a single variable γ, which depends linearly on n via the relation

$$\widetilde{\gamma} = \gamma - \tau \quad \Rightarrow \quad \gamma = \gamma(n) = \gamma(0) - n\tau\ ,$$

on the discrete variables n. In this first order case the system (3.4) adopts the form:

(3.6a) $$\sum_{l=1}^{N} \widetilde{S}_{il} H_{lj}\, \Phi_{\kappa_1}(\widetilde{\xi}_l - \xi_j - \alpha)$$
$$= \sum_{l=1}^{N} H_{il} S_{lj}\, \sigma(\tau)\Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\, \Phi_{\kappa_1}(\tau + \widetilde{\xi}_i - \xi_l - \alpha)\ ,$$

(3.6b) $$\sum_{l=1}^{N} \widetilde{S}_{il} H_{lj}\, \Phi_{-\kappa_1}(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma})$$
$$= \sum_{l=1}^{N} H_{il} S_{lj}\, \sigma(\tau)\Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\, \Phi_{-\kappa_1}(\xi_l - \xi_j - \gamma)\ ,$$
$$i, j = 1, \ldots, N\ .$$

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[3]In the autonomous case those extra factors can be readily removed by means of a simple gauge transformation.

Let us first note that for $N = 1$ (scalar case) this system of equations becomes quite simple and reduces to the equality

$$\frac{\widetilde{S}}{S} = \frac{\sigma(\tau)\,\Phi_\tau(\widetilde{\xi} - \xi - \alpha)\,\Phi_{\kappa_1}(\tau + \widetilde{\xi} - \xi - \alpha)}{\Phi_{\kappa_1}(\widetilde{\xi} - \xi - \alpha)}$$
$$= \frac{\sigma(\tau)\,\Phi_\tau(\widetilde{\xi} - \xi - \alpha)\,\Phi_{-\kappa_1}(-\gamma)}{\Phi_{-\kappa_1}(-\widetilde{\gamma})} \ ,$$

where $S = S_{11}$ and $\xi = \xi_1$. Multiplying out the denominators from the second equality, the latter reduces further by using the addition formula (B.1) and yields the simple relation

$$\zeta(\tau + \widetilde{\xi} - \xi - \alpha) + \zeta(\widetilde{\gamma}) - \zeta(\widetilde{\xi} - \xi - \alpha) - \zeta(\gamma) = 0 \ ,$$

from which the parameter κ_1 has disappeared. This last equation can be resolved by using (A.4) and yields the following two branches of solutions:

$$\widetilde{\xi} - \xi - \alpha + \gamma \doteq 0 \qquad \text{and} \qquad \widetilde{\xi} - \xi - \alpha - \widetilde{\gamma} \doteq 0 \ ,$$

(in which "$\doteq 0$" indicates that the equality holds modulo the period lattice of the Weierstrass elliptic functions). Thus, in this simple case we find that the dependent variable ξ depends quadratically on the discrete variable n:

$$(3.7) \qquad \xi(n) = \xi(0) + (\alpha \mp \gamma(0) + \Omega)n \pm \tfrac{1}{2}n(n \mp 1)\tau \ ,$$

(in which Ω denotes any integer combination of the periods of the Weierstrass functions[4]). This "scalar Lax" case is not quite trivial, even though it can be integrated explicitly, recalling that the representation involves functions $\xi(n)$ which appear in the arguments of elliptic functions, and that the quadratic dependence on the independent variable is reminiscent of certain cases of Painlevé type equations which can be linearized.

To analyse the cases $N \geq 2$ it is convenient to introduce a somewhat unconventional matrix notation. For any two $N \times N$ matrices $\boldsymbol{A} = (A_{ij})$

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[4]In principle the choice of period Ω does not need to be fixed, but could alter under application of the map. We will not consider that possibility in this paper, but only note that the presence of the freedom of choosing periods in (3.7) will not alter the dependence on n in quantities as $u := \wp(\xi(n))$ which would obey rational counterparts of the equations considered.

and $\boldsymbol{B} = (B_{ij})$, let us introduce the operation of "gluing" two matrices, i.e., the entry-by-entry multiplication creating the "glued" matrix denoted by $[\boldsymbol{AB}]$ having entries:

$$([\boldsymbol{AB}])_{ij} := A_{ij}B_{ij} .$$

This allows us to rewrite (3.6) in the following short-hand way:

(3.8a) $$\widetilde{\boldsymbol{S}} \cdot [\boldsymbol{A}_{\kappa_1}\boldsymbol{H}] = [\boldsymbol{A}_{\kappa_1+\tau}\boldsymbol{H}] \cdot \boldsymbol{S} ,$$
(3.8b) $$[\widetilde{\boldsymbol{G}}_{-\kappa_1}\widetilde{\boldsymbol{S}}] \cdot \boldsymbol{H} = [\boldsymbol{A}_{\tau}\boldsymbol{H}] \cdot [\boldsymbol{G}_{-\kappa_1}\boldsymbol{S}] ,$$

where we have introduced the matrices

$$\begin{aligned}
(\boldsymbol{A}_{\kappa})_{ij} &:= \sigma(\kappa)\Phi_{\kappa}(\widetilde{\xi}_i - \xi_j - \alpha) , \\
(\boldsymbol{G}_{\kappa})_{ij} &:= \sigma(\kappa)\Phi_{\kappa}(\xi_i - \xi_j - \gamma) .
\end{aligned}$$

As in the autonomous case of section 2, we are interested primarily in the case that the matrix $\boldsymbol{H}$ is of rank 1, in which case from (3.8b) we have that either the glued matrix $[\boldsymbol{A}_{\tau}\boldsymbol{H}]$ must be singular, or the matrix $[\boldsymbol{G}_{-\kappa_1}\boldsymbol{S}]$ is singular. In the former case, as a consequence of the Frobenius determinant formula (B.4) of Appendix B, we must have that

$$\det(\boldsymbol{A}_{\tau}) = 0 \quad \Rightarrow \quad \tau + \widetilde{\Xi} - \Xi - N\alpha = 0 , \quad \text{with} \quad \Xi = \sum_{j=1}^{N} \xi_j .$$

For $N = 2$ this condition on the sums of the ξ-variables is a sufficient condition for $[\boldsymbol{A}_{\tau}\boldsymbol{H}]$ to be of rank 1, as the glued matrix is, up to multiplication by diagonal matrices, a Frobenius (i.e., elliptic Cauchy) matrix. Since, as a consequence, in that case in general position the matrices $\boldsymbol{A}_{\kappa_1}$ and $\boldsymbol{A}_{\kappa_1+\tau}$ are invertible (provided we avoid values $\kappa_1 \doteq 0$), and consequently also the glued matrices $[\boldsymbol{A}_{\kappa_1}\boldsymbol{H}]$ and $[\boldsymbol{A}_{\kappa_1+\tau}\boldsymbol{H}]$ are generically invertible, we can solve $\widetilde{\boldsymbol{S}}$ from (3.8a) and get the equation

(3.9) $$[\widetilde{\boldsymbol{G}}_{-\kappa_1}([\boldsymbol{A}_{\kappa_1+\tau}\boldsymbol{H}] \cdot \boldsymbol{S} \cdot [\boldsymbol{A}_{\kappa_1}\boldsymbol{H}]^{-1})] \cdot \boldsymbol{H} = [\boldsymbol{A}_{\tau}\boldsymbol{H}] \cdot [\boldsymbol{G}_{-\kappa_1}\boldsymbol{S}] .$$

Taking the rank 1 matrix $\boldsymbol{H} = \boldsymbol{h}^+(\boldsymbol{h}^-)^T = (h_i^+ h_j^-)$ in the form of a bi-vector, and using the formula for the inverse of a Frobenius matrix,

(B.9), this equation can be written explicitly in the following way

$$\sum_{l,l',l''} \Phi_{\kappa_1+\tau}(\widetilde{\xi}_i - \xi_l - \alpha)\, \Phi_{\kappa_1}(\tau)\, \Phi_{\kappa_1-\tau}(\xi_{l'} - \widetilde{\xi}_{l''} + \alpha)\, \Phi_{-\kappa_1}(\widetilde{\xi}_i - \widetilde{\xi}_{l''} - \widetilde{\gamma})$$

$$\times \left[\frac{\prod_k \sigma(\xi_{l'} - \widetilde{\xi}_k + \alpha)}{\prod_{k\neq l'} \sigma(\xi_{l'} - \xi_k)}\right] \left[\frac{\prod_k \sigma(\widetilde{\xi}_{l''} - \xi_k - \alpha)}{\prod_{k\neq l''} \sigma(\widetilde{\xi}_{l''} - \widetilde{\xi}_k)}\right] h_l^- S_{l,l'}(h_{l'}^-)^{-1}\,,$$

$$= \sum_l \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\, \Phi_{-\kappa_1}(\xi_l - \xi_j - \gamma)\, h_l^- S_{l,j}(h_j^-)^{-1}$$

(3.10)

which constitutes a system of N^2 homogeneous linear equations for the N^2 quantities $h_i^- S_{ij}(h_j^-)^{-1}$ involving the entries of the matrix $\boldsymbol{S}$. In order for the equation to lead to nontrivial solutions for the latter quantities the coefficient determinant must vanish, and this leads to a first-order difference equation in terms of the main dependent variables $\xi_j(n)$, which is subject to the additional determinant condition $\det(\boldsymbol{A}_\tau) = 0$. In the case $N = 2$ we are thus led to a first order coupled equation for $\xi_1(n)$ and $\xi_2(n)$, but also subject to the condition $\tau + \xi_1 + \xi_2 - 2\alpha = 0$. Thus, we get effectively a first order equation in terms of a single dependent variable, which we expect would again be linearizable. However due to the complexity of the determinant condition following from (3.10) it is hard to do the integration explicitly. Thus, unlike the autonomous case, in order to get nontrivial equations for the ξ_j we must proceed to a higher order scheme, which we will do in the next section.

3.3. Revised scheme

We noted from the analysis of the previous subsection that not only the κ_1 parameter becomes irrelevant in the final equations, but also that from the rank 1 matrix $\boldsymbol{H} = \boldsymbol{h}^+(\boldsymbol{h}^-)^T$ the dependence on the first factor $\boldsymbol{h}^+$ effectively disappears, while from (3.10) it is apparent that the glued matrix $[\boldsymbol{G}_{-\kappa_1}\boldsymbol{S}]$ can be taken to be of rank 1. Calling the latter matrix $\overline{\boldsymbol{S}} = (\overline{S}_{ij})$ we see that we can rewrite the Lax matrix (3.5b) as follows

$$(\boldsymbol{T}_\kappa)_{ij} = S_{ij}\sigma(\kappa)\, \Phi_\kappa(\xi_i - \xi_j - \gamma - \kappa_1)\, \sigma(-\kappa_1)\Phi_{-\kappa_1}(\xi_i - \xi_j - \gamma)$$

$$=: \overline{S}_{ij}\sigma(\kappa)\, \Phi_\kappa(\xi_i - \xi_j - \overline{\gamma})\,,$$

where we have written $\overline{\gamma} := \gamma + \kappa_1$. Thus, the scheme given by (3.5) is equivalent to one in which for the matrix $\boldsymbol{T}_\kappa$ we take the form

(3.11) $$(\boldsymbol{T}_\kappa)_{i,j} = \overline{S}_{i,j}\,\sigma(\kappa)\, \Phi_\kappa(\xi_i - \xi_j - \overline{\gamma})\,,$$

instead of the original form (3.5b). Working out the compatibility of the system (3.3) we have now the following alternative computation to the

original one. The left-hand side corresponds to

$$\sum_{l=1}^{N} \widetilde{\overline{S}}_{il}\, H_{lj}\, \Phi_{\kappa}(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\overline{\gamma}})\Phi_{\kappa}(\widetilde{\xi}_l - \xi_j - \alpha)\ ,$$

$$= \sum_{l=1}^{N} \widetilde{\overline{S}}_{il}\, H_{lj}\, \Phi_{\kappa}(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\overline{\gamma}})$$

$$\times \left[\zeta(\kappa) - \zeta(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\overline{\gamma}}) + \zeta(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\overline{\gamma}}) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha) \right]$$

while the right-hand side yields:

$$\sum_{l=1}^{N} H_{il}\, \overline{S}_{lj}\, \sigma(\tau)\Phi_{\kappa}(\tau)\, \Phi_{\kappa+\tau}(\widetilde{\xi}_i - \xi_l - \alpha)\, \Phi_{\kappa}(\xi_l - \xi_j - \overline{\gamma})$$

$$= \sum_{l=1}^{N} H_{il}\, \overline{S}_{lj}\, \sigma(\tau)\Phi_{\tau}(\widetilde{\xi}_i - \xi_l - \alpha)\, \Phi_{\kappa}(\tau + \widetilde{\xi}_i - \xi_l - \alpha)\, \Phi_{\kappa}(\xi_l - \xi_j - \overline{\gamma})$$

$$= \sum_{l=1}^{N} H_{il}\, \overline{S}_{lj}\, \sigma(\tau)\Phi_{\tau}(\widetilde{\xi}_i - \xi_l - \alpha)\, \Phi_{\kappa}(\tau + \widetilde{\xi}_i - \xi_j - \alpha - \overline{\gamma})$$

$$\times \left[\zeta(\kappa) - \zeta(\kappa + \tau + \widetilde{\xi}_i - \xi_j - \alpha - \overline{\gamma}) + \zeta(\tau + \widetilde{\xi}_i - \xi_l - \alpha) + \zeta(\xi_l - \xi_j - \overline{\gamma}) \right]$$

Setting as before $\widetilde{\overline{\gamma}} = \overline{\gamma} - \tau$, and identifying on both sides the terms that contain κ and those that don't, we arrive at the system of equations:

(3.12a)
$$\sum_{l=1}^{N} \widetilde{\overline{S}}_{il} H_{lj} = \sum_{l=1}^{N} H_{il} \overline{S}_{lj}\, \sigma(\tau)\, \Phi_{\tau}(\widetilde{\xi}_i - \xi_l - \alpha)\ ,$$

(3.12b)
$$\sum_{l=1}^{N} \widetilde{\overline{S}}_{il} H_{lj}\, \sigma(-\tau)\, \Phi_{-\tau}(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\overline{\gamma}})\, \Phi_{-\tau}(\widetilde{\xi}_l - \xi_j - \alpha)$$

$$= \sum_{l=1}^{N} H_{il} \overline{S}_{lj}\, \Phi_{-\tau}(\xi_l - \xi_j - \overline{\gamma}),$$

$$i, j = 1, \ldots, N\ ,$$

which is essentially equivalent to (3.6), except that the system is written in terms of variables $\overline{S}_{ij}$ rather than S_{ij}. Since the latter quantities were to be determined from the Lax compatibility in the first place, where now the parameter κ_1 is absorbed in the yet to be determined variables, it is clear that the final equations for ξ will not involve the latter parameter. We note also that the way in which the second relation (3.12b)

was obtained, was achieved by extracting the the terms containing the ζ functions in the compatibility, and then recombining them with the appropriate terms from the first relation (3.12a) in such a way that we essentially get the full Lax compatibility relation back from which we started but for any fixed value of the spectral parameter, $\kappa = \kappa_0$ say. Since we only need one such relation together with (3.12a) to have the full set of compatibility relations, it suffices to set $\kappa_0 = -\tau$ in order to obtain (3.12b). In matrix form, using the gluing convention introduced earlier, the resulting system can be written conveniently as follows:

$$(3.13a) \qquad \widetilde{\overline{S}} \cdot H = [A_\tau H] \cdot \overline{S},$$

$$(3.13b) \qquad [\widetilde{\overline{G}}_{-\tau}\widetilde{\overline{S}}] \cdot [A_{-\tau}H] = H \cdot [\overline{G}_{-\tau}\overline{S}] \ ,$$

where $\overline{G}_\kappa$ denotes the matrix G_κ in which γ is replaced by $\overline{\gamma}$.

We will now analyse the system (3.13), which for generic κ_1 is equivalent[5] to (3.8). First, it is easily derived from the explicit form (3.12), and by using the relation (B.3) of Appendix B, that for $N = 1$ we obtain once again a linearizable system system for $\xi(n) := \xi_1(n)$, namely

$$\wp(\widetilde{\xi} - \xi - \alpha) = \wp(\overline{\gamma}) \quad \Rightarrow \quad \widetilde{\xi} - \xi - \alpha \doteq \pm\overline{\gamma} \ ,$$

leading to the solution

$$\xi(n) = \xi(0) + (\alpha \pm \overline{\gamma}(0) + \Omega)n \pm \tfrac{1}{2}n(n-1)\tau \ .$$

When $N = 2$, taking as before the matrix H to be of rank 1, there are two possibilities: from (3.13a) either, *i)* $\overline{S}$ is of rank 1, or, *ii)* $[A_\tau H]$ is of rank 1, implying that $\det(A_\tau) = 0$. In case *i)* we would conclude that $\det(A_{-\tau}) = 0$ (since otherwise $\det(\overline{G}_{-\tau}) = 0$ and this would lead to special conditions on γ), while in case *ii)* we would conclude that $[\overline{G}_{-\tau}\overline{S}]$ is of rank 1. Both options lead to similar results, so for convenience let is pursue the case *i)*. In that case we have the condition:

$$\det(A_{-\tau}) = 0 \quad \Rightarrow \quad -\tau + \widetilde{\Xi} - \Xi - 2\alpha \doteq 0 \ ,$$

for $\Xi = \xi_1 + \xi_2$, as follows from the Frobenius determinant formula (B.4).

To resolve this case *i)*, let us write once again $H = h^+(h^-)^T$ and $\overline{S} = s^+(s^-)^T$, with entries $\overline{S}_{ij} = s_i^+ s_j^-$, then the the first relation

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[5]In fact, from the Lax equation (3.3) for (3.5), we have that $[\widetilde{G}_{\kappa-\kappa_1}\widetilde{S}] \cdot [A_\kappa H] = [A_{\kappa+\tau}H] \cdot [G_{\kappa-\kappa_1}S]$ for arbitrary fixed κ, and this will lead to either system (3.8) or (3.13) with suitable choices for κ.

(3.13a) allows us to identify $s_j^- = \rho h_j^-$ (for some scalar function ρ), and consequently:

$$(\widetilde{s}^- \cdot h^+)\frac{\widetilde{s}_i^+}{h_i^+} = \sum_{l=1}^{2} (A_\tau)_{il} s_l^- s_l^+ \ .$$

The second relation (3.13b) leads to the condition

$$\frac{\widetilde{s}_i^+}{h_i^+} \sum_{l=1}^{2} (\widetilde{\overline{G}}_{-\tau})_{il} \widetilde{s}_l^- h_l^+ (A_{-\tau})_{lj} = \sum_{l=1}^{2} s_l^- s_l^+ (\overline{G}_{-\tau})_{lj} \ ,$$

Expressing all the entries of the first and second relation in terms of $s_l^- s_l^+ =: S_l$, $\widetilde{s}_l^- h_l^+ =: H_l$ we get a system of equations comprising on the one hand

$$(3.14a) \qquad \left(1 + \frac{H_2}{H_1}\right)\widetilde{S}_1 = A_{11}^+ S_1 + A_{12}^+ S_2 \ ,$$

$$(3.14b) \qquad \left(\frac{H_1}{H_2} + 1\right)\widetilde{S}_2 = A_{21}^+ S_1 + A_{22}^+ S_2 \ ,$$

where we abbreviated $A_{ij}^{\pm} := (A_{\pm\tau})_{ij}$, and on the other hand, with $G_{ij}^{\pm} := (\overline{G}_{\pm\tau})_{ij}$,

$$(3.15a) \quad G_{11}^- S_1 + G_{21}^- S_2 = \left(\widetilde{G}_{11}^- A_{11}^- + \widetilde{G}_{12}^- A_{21}^- \frac{H_2}{H_1}\right)\widetilde{S}_1$$

$$= \left(\widetilde{G}_{21}^- A_{11}^- \frac{H_1}{H_2} + \widetilde{G}_{22}^- A_{21}^-\right)\widetilde{S}_2 \ ,$$

$$(3.15b) \quad G_{12}^- S_1 + G_{22}^- S_2 = \left(\widetilde{G}_{11}^- A_{12}^- + \widetilde{G}_{12}^- A_{22}^- \frac{H_2}{H_1}\right)\widetilde{S}_1$$

$$= \left(\widetilde{G}_{21}^- A_{12}^- \frac{H_1}{H_2} + \widetilde{G}_{22}^- A_{22}^-\right)\widetilde{S}_2 \ .$$

Rewriting these relations in terms of $X = H_2/H_1$ and $Y = S_2/S_1$, we are led to:

$$(3.16a) \quad \frac{\widetilde{Y}}{X} = \frac{A_{21}^+ + A_{22}^+ Y}{A_{11}^+ + A_{12}^+ Y} = \frac{\widetilde{G}_{11}^- A_{11}^- + \widetilde{G}_{12}^- A_{21}^- X}{\widetilde{G}_{21}^- A_{11}^- + \widetilde{G}_{22}^- A_{21}^- X} \ ,$$

$$(3.16b) \quad (1+X)(G_{11}^- + G_{21}^- Y) = (A_{11}^+ + A_{12}^+ Y)(\widetilde{G}_{11}^- A_{11}^- + \widetilde{G}_{12}^- A_{21}^- X) \ ,$$

$$(3.16c) \quad A_{12}^-/A_{11}^- = A_{22}^-/A_{21}^- = \frac{G_{12}^- + G_{22}^- Y}{G_{11}^- + G_{21}^- Y} \ .$$

These constitute, in fact, a system of four independent relations for X, Y and $\widetilde{Y}$ with coefficients in terms of ξ_1 and ξ_2, and can be solved by

direct computation. Eliminating X, Y and $\widetilde{Y}$ we get a rather compli-
cated first order difference equation in terms of ξ_1 and ξ_2, which we
refrain from writing down here, and which is subject to the relation
$-\tau + \xi_1 + \xi_2 - 2\alpha + \dot{=} 0$. Because of the latter condition we expect
the resulting equation for one of the variables, say ξ_1, and containing
three free parameters, α, $\Xi(0)$ and γ_0 (apart from the step size τ), to be
linearisable, but we have not yet done so.

Although the first order scheme described in this section and the first
order elliptic difference equations resulting from them may be interesting
in their own right, the scheme seems not yet rich enough to obtain higher
order OΔEs, e.g. elliptic difference equations of Painlevé type. This is
as expected, as the parallel with the monodromy problem for P$_{\text{VI}}$ and its
connection with lattice KdV systems, cf. [35], indicates that we need at
least two lattice directions to get interesting transcendental equations.
Thus, we will next investigate the structure of the higher order elliptic
scheme.

§4. Higher order revised scheme

We noted in the previous section that the monodromy part of the
Lax pair, i.e., (3.5b), can be simplified by redefining the coefficient ma-
trix S_{ij} and the moving parameter γ, absorbing the parameter κ_1. This
applies also to the general discrete monodromy problem (3.2b), which
accordingly can be simplified to the following form:

$$(4.1) \quad (\boldsymbol{T}_\kappa)_{i,j} = \sum_{l_1,\ldots,l_{m-1}} \overline{S}_{i,j}^{(l_1,\ldots,l_{m-1})} \prod_{\nu=1}^{m} \sigma(\kappa) \, \Phi_\kappa(\xi_{l_{\nu-1}}^{(\nu-1)} - \xi_{l_\nu}^{(\nu)} - \overline{\gamma}_\nu) \,,$$

by redefining

$$\overline{S}_{i,j}^{(l_1,\ldots,l_{m-1})} := S_{i,j}^{(l_1,\ldots,l_{m-1})} \prod_{\nu=1}^{m} \sigma(-\kappa_\nu) \, \Phi_{-\kappa_\nu}(\xi_{l_{\nu-1}}^{(\nu-1)} - \xi_{l_\nu}^{(\nu)} - \gamma_\nu) \,,$$

$$\overline{\gamma}_\nu := \gamma_\nu + \kappa_\nu \,.$$

The latter redefinition is possible since *ab initio* we don't specify the co-
efficient matrices, requiring them only to be independent of the spectral
parameter κ. All further properties of the coefficients should follow from
the consistency conditions and additional natural choices (e.g., imposing
a rank 1 condition on the coefficient matrix in (3.2a) in accordance with
the choices of section 2). We now reexamine the consistency condition
(3.3) of (3.2a) and (4.1) for the higher order case.

4.1. Second order scheme

As is clear from the first order case, treated in subsection 3.3, this requires functional identities for products of the form $\prod_\nu \Phi_\kappa(x_\nu)$, i.e. of functions Φ_κ for the same value of the label. Such identities generalizing (B.1), which is equivalent to one of the standard addition formulae for Weierstrass functions, were discussed in Appendix C, cf. (C.8) and (C.9), and the generalized form is described in the Lemma of the Appendix. The key feature of these higher order elliptic addition formulae is that they allow us to separate the spectral parameter dependence of the Lax compatibility conditions, and thus to derive a set of basic relations from which the coefficient matrices can be eliminated. A general closed-form formula is hard to give (Lemma 1 gives us a general prescription) for arbitrary orders, in contrast to the original scheme presented in section 3. However, the latter contains redundant parameters which are hard to get rid of in higher orders, when trying to capture what goes on in concrete formulae. Thus, we will restrict ourselves in this section to the case of a second order scheme (i.e. $m = 2$), which conveys adequately the ideas. The corresponding monodromy problem in revised form, in that case is given by:

$$(4.2) \qquad \chi_{\kappa+\tau} = T'_\kappa \, \chi_\kappa \ ,$$

$$(T'_\kappa)_{i,j} := \sigma^2(\kappa) \sum_{l'=1}^{N} S^{(l')}_{i,j} \Phi_\kappa(\xi_i - \eta_{l'}) \Phi_\kappa(\eta_{l'} - \xi_j - \gamma) \ ,$$

$$(i, j = 1, \ldots, N) \ ,$$

where, for notational convenience, we have omitted the notation and renamed the $\xi^{(1)}_{l_1} + \overline{\gamma}_1 =: \eta_{l'}$, denoting $l_1 = l'$. The coefficient variables $S^{(l')}_{i,j}$ remain to be determined when we consider this difference equation on the torus in conjunction with the lattice Lax system given by

$$(4.3a) \qquad \widetilde{\chi}_\kappa = L_\kappa \, \chi_\kappa \ , \quad (L_\kappa)_{i,j} = H_{i,j} \, \sigma(\kappa) \, \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha) \ ,$$

$$(4.3b) \qquad \widehat{\chi}_\kappa = M_\kappa \, \chi_\kappa \ , \quad (M_\kappa)_{i,j} = K_{i,j} \, \sigma(\kappa) \, \Phi_\kappa(\widehat{\xi}_i - \xi_j - \beta) \ ,$$

which constitutes a system of the type considered in section 2, but without prejudice for now about the form of the coefficient matrices $H = (H_{ij})$ and $K = (K_{ij})$, but which we will assume in due course to be of rank 1 in accordance with the derivation in section 2. The only further assumption on the latter is that they are independent of the spectral parameter κ.

To give a motivation of the monodromy problem associated with (4.2), we consider the variable η as an intermediate dependent variable

on a 2-step configuration in a multidimensional lattice, where the elementary shifts correspond to Lax operations of the type (4.3) but in perhaps additional lattice directions. Thus, the monodromy problem given in the form of the elliptic difference equation (4.2) would amount to a de-autonomization of a 2-step periodic reduction on that lattice. The 2-step periodic reduction, illustrated in the diagram below, leads to a spectral problem of the form: $\widehat{\widetilde{\chi}} = \lambda\chi$, constituting the spectral part of a Lax pair describing a higher-order stationary discrete flow. A subsequent de-autonomization, in the spirit of the paper [43], on the level of the Lax representation by making the replacement $\lambda\chi \rightsquigarrow \chi_{\kappa+\tau}$, leads then to a monodromy problem of the form (4.2), in which the intermediate value η of the dependent variable ξ, shifted in a relevant direction, can be left unspecified. This allows us to determine the value η from the consistency of the Lax pair.
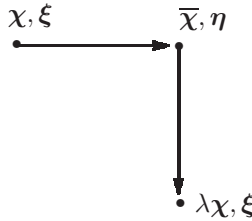

Fig. 2. 2-step periodic reduction.

The elliptic isomonodromic deformation system comprising (4.2) and (4.3) leads to the following set of compatibility conditions:

$$(4.4\text{a}) \qquad \widetilde{T}'_{\kappa}\, L_{\kappa} \;=\; L_{\kappa+\tau}\, T'_{\kappa}\,,$$

$$(4.4\text{b}) \qquad \widehat{T}'_{\kappa}\, M_{\kappa} \;=\; M_{\kappa+\tau}\, T'_{\kappa}\,,$$

$$(4.4\text{c}) \qquad \widehat{L}_{\kappa}\, M_{\kappa} \;=\; \widetilde{M}_{\kappa}\, L_{\kappa}\,.$$

We note that the combination of multiple lattice shifts appearing in the monodromy problems of Painlevé type equations was first made apparent in [35, 36] where it was made manifest in the monodromy problem of P_{VI}, and in the structure of lattice Garnier systems. Recently, in [41], this was also exploited in the construction of difference and q-difference Garnier systems. In what follows, we will focus on one the lattice shifts, namely the one generated by (4.3a), and the compatibility condition (4.4a), while (4.4c) was analysed in section 2.

To analyse the system (4.4a) most effectively we need to use the (seemingly novel) elliptic identities presented in Appendix C, the structure of which is summarised in the Lemma. In particular, where we used (B.1) in the first order case, we now need identities such as (C.8) and (C.9) to analyse the structure of the compatibility relations. The consistency condition (4.4a) can be worked out as follows. The left-hand side can be written as

$$\sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')} H_{lj} \Phi_\kappa(\widetilde{\xi}_i - \widetilde{\eta}_{l'}) \Phi_\kappa(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \widetilde{\gamma}) \Phi_\kappa(\widetilde{\xi}_l - \xi_j - \alpha)$$

$$= \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')} H_{lj} \tfrac{1}{2}\Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma})$$

$$\times \left[\Big(\zeta(\kappa) - \zeta(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}) + \zeta(\widetilde{\xi}_i - \widetilde{\eta}_{l'}) \right.$$

$$+ \zeta(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \widetilde{\gamma}) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha) \Big)^2$$

$$+ \wp(\kappa) - \Big(\wp(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}) + \wp(\widetilde{\xi}_i - \widetilde{\eta}_{l'})$$

$$\left. + \wp(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \widetilde{\gamma}) + \wp(\widetilde{\xi}_l - \xi_j - \alpha) \Big) \right],$$

where we have made use of the identity (C.8). The right-hand side, using again the identity $\Phi_\kappa(\tau)\,\Phi_{\kappa+\tau}(x) = \Phi_\tau(x)\,\Phi_\kappa(\tau+x)$ leads to

$$\sum_{l,l'=1}^{N} H_{il} S_{lj}^{(l')} \sigma(\tau)\Phi_\kappa(\tau) \Phi_{\kappa+\tau}(\widetilde{\xi}_i - \xi_l - \alpha) \Phi_\kappa(\xi_l - \eta_{l'}) \Phi_\kappa(\eta_{l'} - \xi_j - \gamma)$$

$$= \sum_{l,l'=1}^{N} H_{il} S_{lj}^{(l')} \sigma(\tau) \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha) \Phi_\kappa(\tau + \widetilde{\xi}_i - \xi_l - \alpha)$$

$$\times \Phi_\kappa(\xi_l - \eta_{l'}) \Phi_\kappa(\eta_{l'} - \xi_j - \gamma)$$

$$= \sum_{l,l'=1}^{N} H_{il} S_{lj}^{(l')} \tfrac{1}{2}\sigma(\tau)\Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)\Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau)$$

$$\times \left[\Big(\zeta(\kappa) - \zeta(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau) + \zeta(\tau + \widetilde{\xi}_i - \xi_l - \alpha) \right.$$

$$+ \zeta(\xi_l - \eta_{l'}) + \zeta(\eta_{l'} - \xi_j - \gamma) \Big)^2$$

$$+ \wp(\kappa) - \Big(\wp(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau) + \wp(\tau + \widetilde{\xi}_i - \xi_l - \alpha)$$

$$\left. + \wp(\xi_l - \eta_{l'}) + \wp(\eta_{l'} - \xi_j - \gamma) \Big) \right].$$

Setting (once again) $\widetilde{\gamma} = \gamma - \tau$ the common factor $\Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}) = \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau)$ on both sides cancel out, and after factoring them out the remaining terms can be separated in accordance with their different dependence on κ. The latter only appears in combination with the external indices i, j and do not mix with the summation indices. Thus, we only have three types of terms w.r.t. to the dependence on κ: constant terms, terms linear in $(\zeta(\kappa) - \zeta(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}))$ and terms of the form $(\zeta(\kappa) - \zeta(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau)^2 + \wp(\kappa) - \wp(\kappa + \widetilde{\xi}_i - \xi_j - \alpha - \gamma + \tau))$. These then yield the following relations (in reverse order):

$$(4.5a) \quad \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')} H_{lj} = \sum_{l,l'=1}^{N} \sigma(\tau)\Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha) H_{il} S_{lj}^{(l')},$$

$$(4.5b) \quad \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')} H_{lj} \left[\zeta(\widetilde{\xi}_i - \widetilde{\eta}_{l'}) + \zeta(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \gamma + \tau) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha) \right]$$

$$= \sum_{l,l'=1}^{N} \sigma(\tau)\Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha) H_{il} S_{lj}^{(l')}$$

$$\times \left[\zeta(\tau + \widetilde{\xi}_i - \xi_l - \alpha) + \zeta(\xi_l - \eta_{l'}) + \zeta(\eta_{l'} - \xi_j - \gamma) \right],$$

and

$$(4.5c)$$

$$\sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')} H_{lj} \left[\left(\zeta(\widetilde{\xi}_i - \widetilde{\eta}_{l'}) + \zeta(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \gamma + \tau) + \zeta(\widetilde{\xi}_l - \xi_j - \alpha) \right)^2 \right.$$

$$\left. - \wp(\widetilde{\xi}_i - \widetilde{\eta}_{l'}) - \wp(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \gamma + \tau) - \wp(\widetilde{\xi}_l - \xi_j - \alpha) \right]$$

$$= \sum_{l,l'=1}^{N} H_{il} S_{lj}^{(l')} \sigma(\tau) \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha)$$

$$\times \left[\left(\zeta(\tau + \widetilde{\xi}_i - \xi_l - \alpha) + \zeta(\xi_l - \eta_{l'}) + \zeta(\eta_{l'} - \xi_j - \gamma) \right)^2 \right.$$

$$\left. - \wp(\tau + \widetilde{\xi}_i - \xi_l - \alpha) - \wp(\xi_l - \eta_{l'}) - \wp(\eta_{l'} - \xi_j - \gamma) \right]$$

By combining (4.5b) with (4.5a), and using (A.4), the former can also be re-cast in the form

(4.6)
$$\sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')} \, H_{lj} \, \frac{\sigma(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma})\sigma(\widetilde{\eta}_{l'} - \xi_j - \widetilde{\gamma} - \alpha)\sigma(\widetilde{\xi}_i + \widetilde{\xi}_l - \xi_j - \widetilde{\eta}_{l'} - \alpha)}{\sigma(\widetilde{\xi}_i - \widetilde{\eta}_{l'})\sigma(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \widetilde{\gamma})\sigma(\widetilde{\xi}_l - \xi_j - \alpha)}$$
$$= \sum_{l,l'=1}^{N} H_{il} \, S_{lj}^{(l')} \, \frac{\sigma(\widetilde{\xi}_i - \eta_{l'} + \tau - \alpha)\sigma(\xi_l - \xi_j - \gamma)\sigma(\widetilde{\xi}_i - \xi_l - \xi_j + \eta_{l'} - \alpha - \widetilde{\gamma})}{\sigma(\widetilde{\xi}_i - \xi_l - \alpha + \tau)\sigma(\xi_l - \eta_{l'})\sigma(\eta_{l'} - \xi_j - \gamma)} \, ,$$

while combining (4.5c) with both (4.5b) and (4.5a), and using again (C.8), we can obtain a simpler form of the third relation, namely

(4.7)
$$\sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')} \, H_{lj} \, \sigma(-\tau) \, \Phi_{-\tau}(\widetilde{\xi}_i - \widetilde{\eta}_{l'})$$
$$\times \Phi_{-\tau}(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \widetilde{\gamma}) \, \Phi_{-\tau}(\widetilde{\xi}_l - \xi_j - \alpha)$$
$$= \sum_{l,l'=1}^{N} H_{il} \, S_{lj}^{(l')} \, \Phi_{-\tau}(\xi_l - \eta_{l'}) \, \Phi_{-\tau}(\eta_{l'} - \xi_j - \gamma) \, .$$

As before, we want to eliminate the quantities $S_{i,j}^{(l')}$ and $H_{i,j}$ in these relations to obtain a (possibly coupled) system of equations for the variables ξ_j and η_j alone. To do the analysis it may prove helpful to use the notation introduced in section 3, using glued matrices. In that notation eqs. (4.5a) and (4.7) can be written as

(4.8a) $$[\widetilde{S}] \cdot H = [A_\tau H] \cdot [S] \, ,$$
(4.8b) $$[\widetilde{E}_{-\tau}[\widetilde{S}]\widetilde{F}_{-\tau}] \cdot [A_{-\tau}H] = H \cdot [E_{-\tau}[S]F_{-\tau}] \, ,$$

with $A_{\pm\tau}$ as given in section 3, and where $[S]$ denotes the matrix with entries $([S])_{i,j} = \sum_{l'} S_{ij}^{(l')}$. Furthermore, we are compelled to introduce a somewhat ad-hoc notation for

$$\sum_{l'} S_{ij}^{(l')} \, \sigma^2(-\tau)\Phi_{-\tau}(\xi_i - \eta_{l'}) \, \Phi_{-\tau}(\eta_{l'} - \xi_j - \gamma) =: [E_{-\tau}[S]G_{-\tau}]_{ij}$$

a kind of 'doubly glued' matrix involving the upper index in the quantity $S_{i,j}^{(l')}$, and where E_κ and F_κ (for arbitrary κ) denote the matrices with entries

$$(E_\kappa)_{i,j} := \sigma(\kappa) \, \Phi_\kappa(\xi_i - \eta_j) \, , \quad (F_\kappa)_{ij} := \sigma(\kappa) \, \Phi_\kappa(\eta_i - \xi_j - \gamma) \, .$$

The 'middle' relation (4.6), or (modulo the first relation) equivalently (4.5b), is the more complicated one to write in matrix form. To achieve that we rewrite the original form (4.5b) as follows:

$$\sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')} H_{lj} \Big[\zeta(\widetilde{\xi}_i - \widetilde{\eta}_{l'}) + \zeta(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \gamma + \tau) + \zeta(-\tau) - \zeta(\widetilde{\xi}_i - \widetilde{\xi}_l - \gamma)$$

$$+ \zeta(\widetilde{\xi}_i - \widetilde{\xi}_l - \gamma) + \zeta(\tau) + \zeta(\widetilde{\xi}_i - \xi_j - \alpha) - \zeta(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}) \Big]$$

$$= \sum_{l,l'=1}^{N} H_{il} S_{lj}^{(l')} \sigma(\tau) \Phi_\tau(\widetilde{\xi}_i - \xi_l - \alpha) \Big[\zeta(\tau + \widetilde{\xi}_i - \xi_l - \alpha) + \zeta(\xi_l - \xi_j - \widetilde{\gamma}) + \zeta(-\tau)$$

$$- \zeta(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}) + \zeta(\xi_l - \eta_{l'}) + \zeta(\eta_{l'} - \xi_j - \gamma) + \zeta(\tau) - \zeta(\xi_l - \xi_j - \widetilde{\gamma}) \Big] \ ,$$

and apply the identity (B.1) on each quadruple of ζ-terms in the summands, thus, obtaining:

(4.9)

$$\sum_{l,l'=1}^{N} \frac{\widetilde{S}_{il}^{(l')} H_{lj}}{\Phi_{-\tau}(\widetilde{\xi}_i - \widetilde{\xi}_l - \widetilde{\gamma})} \Big[\Phi_{-\tau}(\widetilde{\xi}_i - \widetilde{\eta}_{l'}) \, \Phi_{-\tau}(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \widetilde{\gamma})$$

$$+ \Phi_{-\tau}(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}) \, \Phi_\tau(\widetilde{\xi}_l - \xi_j - \alpha) \Big]$$

$$= \sum_{l,l'=1}^{N} H_{il} \frac{S_{lj}^{(l')}}{\Phi_\tau(\xi_l - \xi_j - \gamma)} \sigma(\tau) \, \Phi_\tau(\widetilde{\xi} - \xi_l - \alpha)$$

$$\times \Big[\Phi_\tau(\widetilde{\xi} - \xi_j - \alpha - \widetilde{\gamma}) \, \Phi_{-\tau}(\tau + \widetilde{\xi}_i - \xi_l - \alpha) + \Phi_\tau(\xi_l - \eta_{l'}) \, \Phi_\tau(\eta_{l'} - \xi_j - \gamma) \Big] \ .$$

This relation can be written more concisely using the notation of glued matrices, extending the latter further by introducing the notation

$$([A/B])_{ij} := A_{ij}/B_{ij} \ ,$$

in the following form

$$\big[[\widetilde{E}_{-\tau}[\widetilde{S}]\widetilde{F}_{-\tau}]/\widetilde{G}_{-\tau} \big] \cdot H - \Big[\big([\widetilde{S}/\widetilde{G}_{-\tau}] \cdot [A_\tau H] \big) C_{-\tau} \Big]$$

(4.10) $\quad = \big[(H \cdot [S/G_\tau]) \, C_\tau \big] - [A_\tau H] \cdot \big[[E_\tau[S]F]/G_\tau \big] \ ,$

where C_κ (for arbitrary κ) is given by

$$(C_\kappa)_{ij} = \sigma(\kappa) \, \Phi_\kappa(\widetilde{\xi}_i - \xi_j - \alpha - \widetilde{\gamma}) \ .$$

In spite of the unconventional notation, we believe this way of writing the relations to which the coefficients are subject are somewhat more insightful than the expressions in terms of components.

4.2. Case $N = 1$

For $N = 1$ the original system of equations (4.5) takes a simple form in terms of the single variables $\xi := \xi_1$, $\eta := \eta_1$, and leads to the coupled system of equations:

(4.11a)
$$\zeta(\widetilde{\xi} - \widetilde{\eta}) + \zeta(\widetilde{\eta} - \widetilde{\xi} - \widetilde{\gamma}) + \zeta(\widetilde{\xi} - \xi - \alpha)$$
$$= \zeta(\widetilde{\xi} - \xi + \tau - \alpha) + \zeta(\xi - \eta) + \zeta(\eta - \xi - \gamma) \,,$$

(4.11b)
$$\wp(\widetilde{\xi} - \widetilde{\eta}) + \wp(\widetilde{\eta} - \widetilde{\xi} - \widetilde{\gamma}) + \wp(\widetilde{\xi} - \xi - \alpha)$$
$$= \wp(\widetilde{\xi} - \xi - \alpha + \tau) + \wp(\xi - \eta) + \wp(\eta - \xi - \gamma) \,,$$

for ξ and η, together with the relation

$$\frac{\widetilde{S}}{S} = \sigma(\tau)\Phi_\tau(\widetilde{\xi} - \xi - \alpha) \,,$$

for $S := S_{1,1}$. There are various solutions of the system (4.11) via the viable identifications of the terms, namely

Case i)

(4.12a)
$$\left. \begin{array}{rcl} \widetilde{\xi} - \xi - \alpha & \doteq & \eta - \xi - \gamma \\ \widetilde{\xi} - \widetilde{\eta} & \doteq & \xi - \eta \end{array} \right\} \quad \Rightarrow$$

$$\begin{cases} \xi(n) = \xi(0) + (\eta(0) - \xi(0) - \gamma(0) + \alpha + \Omega)\,n + \tfrac{1}{2}n(n-1)(\tau + \Omega') \\ \eta(n) = \eta(0) + (\eta(0) - \xi(0) - \gamma(0) + \alpha + \Omega)\,n + \tfrac{1}{2}n(n-1)(\tau + \Omega') \end{cases}$$

Case ii)

(4.12b)
$$\left. \begin{array}{rcl} \widetilde{\xi} - \xi - \alpha & \doteq & \xi - \eta \\ \widetilde{\eta} - \widetilde{\xi} - \widetilde{\gamma} & \doteq & \eta - \xi - \gamma \end{array} \right\} \quad \Rightarrow$$

$$\begin{cases} \xi(n) = \xi(0) + (\xi(0) - \eta(0) + \alpha + \Omega)\,n + \tfrac{1}{2}n(n-1)(\tau + \Omega') \\ \eta(n) = \eta(0) + (\xi(0) - \eta(0) + \alpha + \Omega)\,n + \tfrac{1}{2}n(n-3)(\tau + \Omega') \end{cases}$$

Case iii)

(4.12c)
$$\left. \begin{array}{rcl} \widetilde{\xi} - \xi - \alpha & \doteq & \eta - \xi - \gamma \\ \widetilde{\eta} - \widetilde{\xi} - \widetilde{\gamma} & \doteq & \xi - \eta \end{array} \right\} \quad \Rightarrow$$

$$\begin{cases} \xi(n) = \xi(0) + \left(\alpha - \tfrac{1}{2}\gamma_0 - \tfrac{1}{4}\tau + \Omega'\right)n + \tfrac{1}{4}n(n-1)\tau \\ \qquad + \tfrac{1}{2}\left(\xi(0) - \eta(0) - \tfrac{1}{4}\tau + \tfrac{1}{2}(\gamma_0 + \Omega)\right)((-1)^n - 1) \\ \eta(n) = \eta(0) + \left(\alpha - \tfrac{1}{2}\gamma_0 - \tfrac{1}{4}\tau + \Omega'\right)n + \tfrac{1}{4}n(n-3)\tau \\ \qquad + \tfrac{1}{2}\left(\eta(0) - \xi(0) + \tfrac{1}{4}\tau - \tfrac{1}{2}(\gamma_0 + \Omega)\right)((-1)^n - 1) \end{cases}$$

Case iv)

(4.12d)
$$\left.\begin{array}{rcl} \widetilde{\widetilde{\xi}} - \xi - \alpha & \doteq & \xi - \eta \\ \widetilde{\widetilde{\xi}} - \widetilde{\eta} & \doteq & \eta - \xi - \gamma \end{array}\right\} \Rightarrow$$

$$\begin{cases} \xi(n) = \xi(0) + \left(\alpha - \tfrac{1}{2}(\gamma_0 + \Omega) - \tfrac{1}{4}\tau + \Omega'\right) n + \tfrac{1}{4}n(n-1)\tau \\ \qquad - \tfrac{1}{2}\left(\xi(0) - \eta(0) + \tfrac{1}{2}(\gamma_0 + \Omega) + \tfrac{1}{4}\tau\right)((-1)^n - 1) \\ \eta(n) = \eta(0) + \left(\alpha - \tfrac{1}{2}(\gamma_0 + \Omega) - \tfrac{1}{4}\tau + \Omega'\right) n + \tfrac{1}{4}n(n-3)\tau \\ \qquad + \tfrac{3}{2}\left(\eta(0) - \xi(0) - \tfrac{1}{4}\tau - \tfrac{1}{2}(\gamma_0 + \Omega)\right)((-1)^n - 1) \end{cases}$$

in which Ω and Ω' denote arbitrary periods of the elliptic functions. Note that cases *iii)* and *iv)* differ from the cases *i)* and *ii)* by the appearance of alternating terms. As in the first order scheme we see that in the linearisable case there is quadratic dependence on the independent variable n, but in the higher order scheme it appears within a coupled system for $\xi(n)$ and $\eta(n)$.

4.3. Case $N = 2$

We will discuss the strategy to analyse the system of relations (4.5), or, in the shorthand notation we introduced, comprising (4.8) together with (4.10) for the case $N = 2$. As before, we assume the matrix $\boldsymbol{H}$ to be of rank 1, where we can write as before $\boldsymbol{H} = \boldsymbol{h}^+(\boldsymbol{h}^-)^T$. From (4.8a) we observe that either $\det(\boldsymbol{A}_\tau) = 0$ or that the matrix $[\boldsymbol{S}]$ has to be singular (in fact, of rank 1 when $N = 2$). We pursue for convenience the latter case; in fact we will assume the coefficient $S_{ij}^{(l')}$ to be of the form $S_{ij}^{(l')} = s_i^+ s_{l'}^0 s_j^-$, in other words: fully factorized. Going back to the original form of the constitutive relations (4.5), and inserting this *Ansatz* into (4.5a) we find that $s_j^- = \rho h_j^-$ for some factor ρ and furthermore the following relation:

(4.13a)
$$(H_1 + H_2) \frac{\widetilde{S}^0}{S^0} \frac{\widetilde{S}_i}{H_i} = \sum_{l=1}^{2} (\boldsymbol{A}_\tau)_{il} S_l ,$$

using as in section 2 the notation $S_l = s_l^- s_l^+$, $H_l = \widetilde{s}_l^- h_l^+$, and introducing $S^0 := s_1^0 + s_2^0$. Eq. (4.7) yields

(4.13b)
$$\frac{\widetilde{S}_i}{H_i} \sum_{l=1}^{2} (\widetilde{\boldsymbol{K}}_{-\tau})_{il} H_l (\boldsymbol{A}_{-\tau})_{lj} = \sum_{l=1}^{2} S_l (\boldsymbol{K}_{-\tau})_{lj} ,$$

in which

$$(\boldsymbol{K}_\kappa)_{ij} := \sum_{l'=1}^{2} s_{l'}^0 \, (\boldsymbol{E}_\kappa)_{il'} \, (\boldsymbol{F}_\kappa)_{l'j} .$$

From (4.13b) it follows that the matrix $\boldsymbol{A}_{-\tau}$ must be of rank 1, and hence we have again the condition

$$-\tau + \widetilde{\widetilde{\Xi}} - \Xi - 2\alpha \doteq 0 \ .$$

Finally, from (4.5b) we have

$$(4.13c) \qquad \frac{\widetilde{S}_i}{H_i} \sum_{l=1}^{2} H_l \left[(\widetilde{\boldsymbol{Z}})_{il} + \widetilde{S}^0 \zeta(\widetilde{\xi}_l - \xi_j - \alpha) \right]$$

$$= \sum_{l=1}^{2} (\boldsymbol{A}_\tau)_{il} S_l \left[S^0 \zeta(\tau + \widetilde{\xi}_i - \xi_l - \alpha) + (\boldsymbol{Z})_{lj} \right] \ ,$$

where the matrix $\boldsymbol{Z}$ has entries

$$(\boldsymbol{Z})_{ij} := \sum_{l'=1}^{2} s_{l'}^0 \left[\zeta(\xi_i - \eta_{l'}) + \zeta(\eta_{l'} - \xi_j - \gamma) \right] \ .$$

The resolution of the system (4.13) can in principle be done following similar lines as the parallel system in section 3. However, in this case we have more variables at our disposal, including the coefficients s_i^0 and the intermediate variables η_i, $(i = 1, 2)$, exploiting also the additional relation (4.13b). Elimination of the coefficient variables are expected to yield a coupled set of first order equations for $\xi_1(n)$ and $\eta_1(n)$, but the full analysis and assessment of the consequences, as well as generalizations to higher rank ($N \geq 3$) and higher order ($m \geq 3$), remain to be done and will be pursued in a follow-up paper.

§5.   Conclusions

In this paper we proposed a general system of elliptic discrete isomonodromic deformation problems from de-autonomisation of elliptic Lax pairs presented in our earlier paper [11]. While the first order scheme for $N = 1$ and $N = 2$ only leads to linearisable equations, we laid out the structure for the higher order (i.e., two-step) scheme and derived the constitutive relations by using some possibly novel elliptic identities. In principle the analysis in section 4 can be readily extended to the 3-step, or multi-step, case by using higher-order elliptic identities such as (C.9) and the ones described in the Lemma of Appendix C. These constitutive relations contain coefficient matrices that are to be eliminated in order to yield a system of nonlinear non-autonomous elliptic ordinary difference equations which we expect to constitute higher-order

and higher-rank versions of elliptic type Painlevé equations (i.e. elliptic Garnier and Schlesinger systems[6]), but further analysis is needed to confirm those expectations.

While we refer to the Lax systems investigated in this paper as isonomodromic deformation problems, it remains to be investigated in what sense these Lax systems are indeed isomonodromic. A general elliptic version of isomonodromic deformation theory was established some time ago by Krichever, [29], and a closer comparison with that work may establish that the systems proposed here are indeed isomonodromic in the sense of that theory. However, in that paper the compatibility conditions of the isomonodromic system were not pursued, while here we have shown how to obtain a handle on that problem by using a system of elliptic identities which is excellently suitable for that purpose. This is the main purpose of the present paper. The explicit formulae we obtained for the cases $N = 1$ and $N = 2$ confirm that the resulting equations have behaviour that one would expect from discrete Painlevé type equations, but further work is needed to make those assertions rigorous.

We, furthermore, point out that, as a byproduct of the higher order isomonodromic scheme laid out in section 4, we can consider its autonomous limit, which amounts to a 2-step higher-time flow of the elliptic discrete-time Ruijsenaars model of [34]. The Lax pair in that case is a discrete iso-spectral problem given by

$$(5.1) \qquad \boldsymbol{T}'_\kappa \boldsymbol{\chi}_\kappa = \lambda \boldsymbol{\chi}_\kappa \ , \quad \widetilde{\boldsymbol{\chi}}_\kappa = \boldsymbol{L}_\kappa \, \boldsymbol{\chi}_\kappa \ , \quad \widehat{\boldsymbol{\chi}}_\kappa = \boldsymbol{M}_\kappa \, \boldsymbol{\chi}_\kappa \ ,$$

obtained by supplementing (4.3) with $\tau = 0$ and γ constant. In the stationary case the compatibility relations become

$$(5.2) \qquad \widetilde{\boldsymbol{T}}'_\kappa \, \boldsymbol{L}_\kappa = \boldsymbol{L}_\kappa \, \boldsymbol{T}'_\kappa \ , \quad \widehat{\boldsymbol{T}}'_\kappa \, \boldsymbol{M}_\kappa = \boldsymbol{M}_\kappa \, \boldsymbol{T}'_\kappa \ ,$$

together with (2.2).

The compatibility of (5.2) follows similar analysis as the one for non-autonomous case, making use of (C.8) and the result is the following set

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[6]We note, in proof, that in a recent preprint [42], which appeared after acceptance of the present paper, an elliptic Garnier system was proposed. Since that approach uses a multiplicative representation of the elliptic functions, a comparison with our Lax approach is not immediately obvious.

of constitutive relations:

$$(5.3a) \qquad \sum_{l,l'=1}^{N} H_{il}\, S_{lj}^{(l')} = \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj} \ ,$$

$$\sum_{l,l'=1}^{N} H_{il}\, S_{lj}^{(l')}\, \frac{\sigma(\widetilde{\xi}_i - \eta_{l'} - \alpha)\sigma(\xi_l - \xi_j - \gamma)\sigma(\widetilde{\xi}_i - \xi_l - \xi_j + \eta_{l'} - \alpha - \gamma)}{\sigma(\widetilde{\xi}_i - \xi_l - \alpha)\sigma(\xi_l - \eta_{l'})\sigma(\eta_{l'} - \xi_j - \gamma)}$$

$$= \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj}\, \frac{\sigma(\widetilde{\xi}_i - \widetilde{\xi}_l - \gamma)\sigma(\widetilde{\eta}_{l'} - \xi_j - \gamma - \alpha)\sigma(\widetilde{\xi}_i + \widetilde{\xi}_l - \xi_j - \widetilde{\eta}_{l'} - \alpha)}{\sigma(\widetilde{\xi}_i - \widetilde{\eta}_{l'})\sigma(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \gamma)\sigma(\widetilde{\xi}_l - \xi_j - \alpha)} \ ,$$

$$(5.3b)$$

and

$$(5.3c) \quad \sum_{l,l'=1}^{N} H_{il}\, S_{lj}^{(l')}\, \Phi_{\kappa_0}(\widetilde{\xi}_i - \xi_l - \alpha)\, \Phi_{\kappa_0}(\xi_l - \eta_{l'})\, \Phi_{\kappa_0}(\eta_{l'} - \xi_j - \gamma)$$

$$= \sum_{l,l'=1}^{N} \widetilde{S}_{il}^{(l')}\, H_{lj}\, \Phi_{\kappa_0}(\widetilde{\xi}_i - \widetilde{\eta}_{l'})\, \Phi_{\kappa_0}(\widetilde{\eta}_{l'} - \widetilde{\xi}_l - \gamma)\, \Phi_{\kappa_0}(\widetilde{\xi}_l - \xi_j - \alpha) \ ,$$

where in the latter we can fix κ_0 to be any non-singular fixed value. The relations (5.3a) and (5.3b) can be directly obtained by setting $\tau = 0$ in the corresponding relations for the non-autonomous case, while (5.3c) is just the Lax compatibility for any fixed value κ_0 of the spectral parameter (in this case we cannot set $\kappa_0 = -\tau$ as in (4.7)). We will leave the problem of attaining an explicit resolution of this system leading to closed-form expressions for higher discrete-time flows, to a future publication.

Appendix A: Weierstrass elliptic functions

Here, we collect some useful formulae for elliptic functions, see also the standard textbooks e.g. [5, 51]. The Weierstrass sigma-function is defined by

$$(A.1) \qquad \sigma(x) = x \prod_{(k,\ell)\neq(0,0)} \left(1 - \frac{x}{\omega_{k\ell}}\right) \exp\left[\frac{x}{\omega_{k\ell}} + \frac{1}{2}\left(\frac{x}{\omega_{k\ell}}\right)^2\right] \ ,$$

with $\omega_{kl} = 2k\omega_1 + 2\ell\omega_2$ and $2\omega_{1,2}$ being a fixed pair of the primitive periods. The relations between the Weierstrass elliptic functions are given by

$$(A.2) \qquad \zeta(x) = \frac{\sigma'(x)}{\sigma(x)} \ , \quad \wp(x) = -\zeta'(x) \ ,$$

where $\sigma(x)$ and $\zeta(x)$ are odd functions and $\wp(x)$ is an even function of its argument. We recall also that the $\sigma(x)$ is an entire function, and $\zeta(x)$ is a meromorphic function having simple poles at ω_{kl}, both being quasi-periodic, obeying

$$\zeta(x + 2\omega_{1,2}) = \zeta(x) + 2\eta_{1,2} \, , \quad \sigma(x + 2\omega_{1,2}) = -\sigma(x)e^{2\eta_{1,2}(x+\omega_{1,2})} \, ,$$

in which $\eta_{1,2} = \zeta(\omega_{1,2})$ satisfy $\eta_1\omega_2 - \eta_2\omega_1 = \frac{\pi i}{2}$, whereas $\wp(x)$ is doubly periodic. The most important properties, for the sake of the computations in the main text, are the addition formulae, which are functional relations holding for arbitrary values (apart from singular points) for the variables in the arguments. The most fundamental is perhaps the three-term relation for $\sigma(x)$, which can be written as (A.4)

$$(A.3) \qquad \sigma(x + a)\,\sigma(x - a)\,\sigma(y + b)\,\sigma(y - b)$$
$$- \sigma(x + b)\,\sigma(x - b)\,\sigma(y + a)\,\sigma(y - a)$$
$$= \sigma(x + y)\,\sigma(x - y)\,\sigma(a + b)\,\sigma(a - b) \, .$$

A limiting case of the latter is he relation

$$(A.4) \quad \zeta(\alpha) + \zeta(\beta) + \zeta(\gamma) - \zeta(\alpha + \beta + \gamma) = \frac{\sigma(\alpha + \beta)\sigma(\beta + \gamma)\sigma(\gamma + \alpha)}{\sigma(\alpha)\sigma(\beta)\sigma(\gamma)\sigma(\alpha + \beta + \gamma)} \, ,$$

between the σ- and ζ-functions. Furthermore, we have as a consequence of the latter

$$(A.5) \qquad \zeta(\alpha + \beta) - \zeta(\alpha) - \zeta(\beta) = \frac{1}{2}\frac{\wp'(\alpha) - \wp'(\beta)}{\wp(\alpha) - \wp(\beta)} \, .$$

as well as the addition formula for the Weierstrass elliptic $\wp$-function:

$$(A.6) \qquad \Big(\zeta(\alpha + \beta) - \zeta(\alpha) - \zeta(\beta)\Big)^2 = \wp(\alpha) + \wp(\beta) + \wp(\alpha + \beta) \, .$$

Finally we have the fundamental relation between σ- and $\wp$-functions

$$(A.7) \qquad \frac{\sigma(x + y)\,\sigma(x - y)}{\sigma^2(x)\,\sigma^2(y)} = \wp(y) - \wp(x) \, ,$$

which in turn gives back the three-term relation (A.3) by using the identity

$$(\wp(x) - \wp(a))\,(\wp(y) - \wp(b)) - (\wp(x) - \wp(b))\,(\wp(y) - \wp(a))$$
$$= (\wp(x) - \wp(y))\,(\wp(a) - \wp(b)) \, ,$$

thus, showing that no information gets lost if we reduce one functional relation to another.

Appendix B: The function $\Phi_\kappa(x)$ and determinantal identities

The function $\Phi(x)$ was introduced in (2.4) and in terms of this function the various addition formulae of Appendix A can be conveniently expressed. Thus, eq. (A.4) can be cast into the form

$$(B.1) \quad \Phi_\kappa(x)\Phi_\kappa(y) = \Phi_\kappa(x+y)\left[\zeta(\kappa) + \zeta(x) + \zeta(y) - \zeta(\kappa+x+y)\right] ,$$

while (A.3) can be rewritten as

$$(B.2) \quad \Phi_\kappa(x)\Phi_\lambda(y) = \Phi_\kappa(x-y)\Phi_{\kappa+\lambda}(y) + \Phi_{\kappa+\lambda}(x)\Phi_\lambda(y-x) ,$$

which can be considered as an elliptic analogue of the partial fraction expansion. Furthermore, (A.7) takes the form

$$(B.3) \qquad\qquad \Phi_\kappa(x)\,\Phi_{-\kappa}(x) = \wp(x) - \wp(\kappa) .$$

Furthermore, in terms of this function we have the famous Frobenius formula, which can be considered to be an elliptic version of the well-known Cauchy determinantal identity. It reads:

$$(B.4) \qquad \det\left(\Phi_\kappa(x_i - y_j)\right) = \Phi_\kappa(\Sigma)\sigma(\Sigma)\frac{\prod_{k<\ell}\sigma(x_k - x_\ell)\sigma(y_\ell - y_k)}{\prod_{k,\ell}\sigma(x_k - y_\ell)} ,$$

$$\text{where } \Sigma \equiv \sum_i (x_i - y_i) ,$$

cf. [17]. From (B.4), by expanding along one of its rows or columns, an elliptic form of the Lagrange interpolation formula can obtained, which reads:

$$(B.5) \qquad \prod_{i=1}^N \frac{\sigma(\xi - x_i)}{\sigma(\xi - y_i)} = \sum_{i=1}^N \Phi_{-\Sigma}(\xi - y_i)\frac{\prod_{j=1}^N \sigma(y_i - x_j)}{\prod_{\substack{j=1\\j\neq i}}^N \sigma(y_i - y_j)} ,$$

for $\Sigma \neq 0$, where

$$(B.6) \qquad\qquad\qquad \Sigma \equiv \sum_{i=1}^N (x_i - y_i) .$$

When $\Sigma = 0$ we recover the following formula

$$(B.7) \quad \prod_{i=1}^N \frac{\sigma(\xi - x_i)}{\sigma(\xi - y_i)} = \sum_{i=1}^N \left[\zeta(\xi - y_i) - \zeta(x - y_i)\right]\frac{\prod_{j=1}^N \sigma(y_i - x_j)}{\prod_{\substack{j=1\\j\neq i}}^N \sigma(y_i - y_j)} ,$$

in which x denotes any one of the zeroes x_i. Note that in this case the left hand side is a meromorphic function on the elliptic curve as a consequence of Abel's theorem. Using (B.5) it can be easily verified that eq. (B.7) is independent of the choice of x. In fact, this follows from the key property that

$$(B.8) \qquad \sum_{i=1}^{N} \frac{\prod_{j=1}^{N} \sigma(y_i - x_j)}{\prod_{\substack{j=1 \\ j \neq i}}^{N} \sigma(y_i - y_j)} = 0 \;,$$

whenever $\sum_i (x_i - y_i) = 0$. This latter relation (B.8) is nothing else than a rewriting of (B.5). Finally, we give the expression for the inverse of the elliptic Cauchy matrix, namely

$$(B.9) \qquad \left[(\Phi_\kappa(x. - y.))^{-1} \right]_{ij} = \Phi_{\kappa + \Sigma}(y_i - x_j) \frac{X(y_i) Y(x_j)}{Y_1(y_i) X_1(x_j)} \;,$$

(with Σ as in (B.6)), in terms of the elliptic polynomials

$$X(\xi) = \prod_{k=1}^{N} \sigma(\xi - x_k) \;, \quad Y(\xi) = \prod_{k=1}^{N} \sigma(\xi - y_k) \;,$$

and

$$(B.10) \qquad X_1(x_j) = \prod_{k \neq j} \sigma(x_j - x_k) \;, \quad Y_1(y_i) = \prod_{k \neq i} \sigma(y_i - y_k) \;.$$

Equation (B.9) can be derived using (B.5) and (B.7).

Appendix C: Higher-order identities

The addition formulae in terms of the function $\Phi_\kappa(x)$ lend themselves fairly easily to a higher-order generalizations, which can be proven by induction from the basic ones. Thus, from (B.2) one can prove the following general product identity

$$(C.1) \qquad \prod_{i=1}^{n} \Phi_{\kappa_i}(x_i) = \sum_{i=1}^{n} \Phi_{\kappa_1 + \cdots + \kappa_n}(x_i) \prod_{\substack{j=1 \\ j \neq i}}^{n} \Phi_{\kappa_j}(x_j - x_i) \;.$$

Extending this identity to $n + 1$ variables, including a κ_0 and x_0, and subsequently taking the limit $x_0 = x_1 + \varepsilon$, with $\varepsilon \to 0$, we obtain the following identity (after some obvious relabelling of parameters and changes of variables):

(C.2)

$$(-1)^{n-1}\Phi_{\kappa_0+\kappa_1+\cdots+\kappa_n}(x_1+\cdots+x_n)\,\frac{\sigma(x_1+\cdots+x_n)}{\prod_{j=1}^{n}\sigma(x_j)}$$

$$\times\left[\zeta(\kappa_0)+\sum_{j=1}^{n}(\zeta(\kappa_j)+\zeta(x_j))-\zeta(\kappa_0+\kappa_1+\cdots+\kappa_n+x_1+\cdots+x_n)\right]$$

$$=\sum_{i=1}^{n}\Phi_{\kappa_0+\kappa_1+\cdots+\kappa_n}(x_1+\cdots+\not x_i+\cdots+x_n)$$

$$\times\frac{\sigma(x_1+\cdots+\not x_i+\cdots+x_n)\,\sigma^{n-1}(x_i)}{\prod_{\substack{j=1\\j\neq i}}^{n}\sigma(x_i-x_j)}\prod_{j=0}^{n}\Phi_{\kappa_j}(x_i)\ .$$

These identities, which express sums of even numbers of ζ-functions, are associated with the famous Frobenius-Stickelberger (i.e., an elliptic van der Monde) determinantal formula, [16], which is given by:

(C.3)
$$\begin{vmatrix} 1 & \wp(x_0) & \wp'(x_0) & \cdots & \cdots & \wp^{(n-1)}(x_0) \\ 1 & \wp(x_1) & \wp'(x_1) & \cdots & \cdots & \wp^{(n-1)}(x_1) \\ \vdots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & \ddots & \vdots \\ 1 & \wp(x_n) & \wp'(x_n) & \cdots & \cdots & \wp^{(n-1)}(x_n) \end{vmatrix}$$

$$=\ (-1)^{\frac{1}{2}n(n-1)}1!2!\cdots n!\,\frac{\sigma(x_0+x_1+\cdots+x_n)\prod_{i<j=0}^{n}\sigma(x_i-x_j)}{\sigma^{n+1}(x_0)\,\sigma^{n+1}(x_1)\cdots\sigma^{n+1}(x_n)}\ .$$

A particular example of such an identity is the following one generalizing (A.4) to a 6-term relation:

$$\begin{aligned}\text{(C.4)}\quad &\sigma(\kappa+x)\,\sigma(\lambda+x)\,\sigma(\mu+x)\sigma(\kappa+\lambda+\mu+y)\,\sigma^2(y)\\ &\quad-\sigma(\kappa+y)\,\sigma(\lambda+y)\,\sigma(\mu+y)\sigma(\kappa+\lambda+\mu+x)\,\sigma^2(x)\\ &=\sigma(\kappa)\,\sigma(\lambda)\,\sigma(\mu)\,\sigma(x)\,\sigma(y)\,\sigma(\kappa+\lambda+\mu+x+y)\,\sigma(y-x)\\ &\quad\times[\zeta(\kappa)+\zeta(\lambda)+\zeta(\mu)+\zeta(x)+\zeta(y)-\zeta(\kappa+\lambda+\mu+x+y)]\end{aligned}$$

which derives from:

$$\begin{aligned}\text{(C.5)}\quad &\zeta(\kappa)+\zeta(\lambda)+\zeta(\mu)+\zeta(x)+\zeta(y)-\zeta(\kappa+\lambda+\mu+x+y)\\ &=\frac{\Phi_\kappa(x)\Phi_\lambda(x)\Phi_\mu(x)\Phi_{\kappa+\lambda+\mu}(y)-\Phi_\kappa(y)\Phi_\lambda(y)\Phi_\mu(y)\Phi_{\kappa+\lambda+\mu}(x)}{\Phi_{\kappa+\lambda+\mu}(x+y)\,(\wp(x)-\wp(y))}\ .\end{aligned}$$

In the same vein, we have the following 8-term ζ-function relation:

(C.6)
$$\Phi_{\kappa+\lambda+\mu+\nu}(x+y+z)\,\frac{\sigma(x+y+z)\,\sigma(x-y)\,\sigma(x-z)\,\sigma(y-z)}{\sigma^3(x)\,\sigma^3(y)\,\sigma^3(z)}$$
$$\times\Big[\zeta(\kappa)+\zeta(\lambda)+\zeta(\mu)+\zeta(\nu)+\zeta(x)+\zeta(y)+\zeta(z)$$
$$-\,\zeta(\kappa+\lambda+\mu+\nu+x+y+z)\Big]$$
$$=\ \ \Phi_\kappa(x)\Phi_\lambda(x)\Phi_\mu(x)\Phi_\nu(x)\,(\wp(z)-\wp(y))\,\Phi_{\kappa+\lambda+\mu+\nu}(y+z)$$
$$+\,\Phi_\kappa(y)\Phi_\lambda(y)\Phi_\mu(y)\Phi_\nu(y)\,(\wp(x)-\wp(z))\,\Phi_{\kappa+\lambda+\mu+\nu}(x+z)$$
$$+\,\Phi_\kappa(z)\Phi_\lambda(z)\Phi_\mu(z)\Phi_\nu(z)\,(\wp(y)-\wp(x))\,\Phi_{\kappa+\lambda+\mu+\nu}(x+y)\ .$$

In the treatment of the main text of the paper we also need suitable formulae for multiple products of $\Phi_\kappa(x)$-functions carrying the same index κ. By expansion of the Frobenius formula (B.4) we find the following identity for products of $\Phi_\kappa(x)$ functions with different arguments but with the same label κ:

(C.7)
$$\prod_{j=1}^{n}\Phi_\kappa(x_j)=\frac{(-1)^{n-1}}{n-1}\Phi_\kappa(x_1+\cdots+x_n)$$
$$\times\,\frac{\left|\mathbf{1}\,,\ \wp(\boldsymbol{x})\,,\ \wp'(\boldsymbol{x})\,,\ \cdots\,,\ \wp^{(n-2)}(\boldsymbol{x})\right|}{\left|\mathbf{1}\,,\ \frac{1}{2}\frac{\wp'(\boldsymbol{x})-\wp'(\kappa)}{\wp(\boldsymbol{x})-\wp(\kappa)}\,,\ \wp(\boldsymbol{x})\,,\ \wp'(\boldsymbol{x})\,,\ \cdots\,,\ \wp^{(n-3)}(\boldsymbol{x})\right|}\ ,$$

where the r.h.s. contains a ratio of two $n\times n$ Frobenius-Stickelberger determinants and where each $f(\boldsymbol{x})$ stands for a function $f(x)$ denotes a column with entries $f(x_j)$ with $j=1,\ldots,n$.

Particular examples of such identities, generalizing (B.1) are the following higher-order relations:

(C.8)
$$\Phi_\kappa(x)\Phi_\kappa(y)\Phi_\kappa(z)$$
$$=\tfrac{1}{2}\Phi_\kappa(x+y+z)\Big[\left(\zeta(\kappa)+\zeta(x)+\zeta(y)+\zeta(z)-\zeta(\kappa+x+y+z)\right)^2$$
$$+\,\wp(\kappa)-\left(\wp(x)+\wp(y)+\wp(z)+\wp(\kappa+x+y+z)\right)\Big]\ .$$

involving products of three Φ_κ functions, and the next higher one reads:

$$(C.9) \qquad \Phi_\kappa(x)\Phi_\kappa(y)\Phi_\kappa(z)\Phi_\kappa(w) = \tfrac{1}{6}\Phi_\kappa(x+y+z+w)$$

$$\times \left\{ \Big(\zeta(\kappa) + \zeta(x) + \zeta(y) + \zeta(z) + \zeta(w) - \zeta(\kappa+x+y+z+w) \Big)^3 \right.$$

$$- 3\Big(\zeta(\kappa) + \zeta(x) + \zeta(y) + \zeta(z) + \zeta(w) - \zeta(\kappa+x+y+z+w) \Big)$$

$$\times \Big(\wp(x) + \wp(y) + \wp(z) + \wp(w) + \wp(\kappa+x+y+z+w) - \wp(\kappa) \Big)$$

$$\left. - \Big(\wp'(\kappa) + \wp'(x) + \wp'(y) + \wp'(z) + \wp'(w) - \wp'(\kappa+x+y+z+w) \Big) \right\} .$$

The salient feature of the identities (C.8) and (C.9) is the way in which the label variable κ appears in the expressions between brackets on the right-hand sides: the κ appears on its own or in combination with the sums of all the arguments. The general structure of how these relations develop for higher and higher products is follows:

Lemma: *The general form of identities of the type of products of the form*

$$\prod_{j=1}^{n} \Phi_{\kappa_j}(x) =: \frac{1}{(n-1)!} \mathscr{F}(\kappa_1, \ldots, \kappa_n; x) ,$$

is as follows. The function $\mathscr{F}$ is given by the expansion of the $(n-1)^{\text{th}}$ derivative of the Weierstrass σ-function divided by σ in terms of ζ-functions and the $\wp$-function and its derivative, where whenever we have an odd function in this expansion (namely ζ and $\wp'$) we replace it by a combination of the form

$$\zeta(x) + \sum_{j=1}^{n} \zeta(\kappa_j) - \zeta\left(\sum_{j=1}^{n} \kappa_j + x \right) ,$$

(and similar for $\wp'$) and when we encounter an even function ($\wp$ and powers of it) we replace it by

$$\sum_{j=1}^{n} \wp(\kappa_j) + \wp\left(\sum_{j=1}^{n} \kappa_j + x \right) - \wp(x) .$$

In fact, if we inspect the first few cases of $\sigma^{(n-1)}(x)/\sigma(x)$:

$$\frac{\sigma'(x)}{\sigma(x)} = \zeta(x), \quad \frac{\sigma''(x)}{\sigma(x)} = \zeta^2(x) - \wp(x), \quad \frac{\sigma'''(x)}{\sigma(x)} = \zeta^3(x) - 3\zeta(x)\wp(x) - \wp'(x), \ldots$$

expanded in ζ, $\wp$ and $\wp'$, we see that they correspond exactly to the terms in the identities (B.1), (C.8) and (C.9).

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