Advanced Studies in Pure Mathematics 76, 2018
Representation Theory, Special Functions and Painlevé Equations — RIMS 2015
pp. 469–485

Complex hypergeometric integrals

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Abstract.

We consider a complex version of the Gauss hypergeometric integral from the view point of the twisted de Rham theory. In particular, we give a formula to express the complex hypergeometric integral in terms of the hermitian form of the ordinary Gauss hypergeometric integrals.

§1. Introduction

The complex beta integral is

$$\frac{\sqrt{-1}}{2}\int\int_{\mathbb{C}}|t|^{2a}|t-1|^{2b}dt\wedge d\bar{t}=\frac{s(a)s(b)}{s(a+b)}\left(\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}\right)^2,$$

where $s(a)=\sin(\pi a)$ and $-a-b-1, a+1, b+1\notin\mathbb{Z}_{\leq 0}$. It is studied in [1] [2] [3] and [8]. In this paper, we consider its generalization to the case of the Gauss hypergeometric function and give a formula to express it in terms of the hermitian form of the ordinary Gauss hypergeometric integrals. The formula is the same one obtained by Strichartz in [8] by means of differential equations. Our way of derivation is by using the idea and terminology of the twisted de Rham theory.

We refer the reader to [4] and [5] for the terminology and technique of the twisted de Rham theory.

In this paper, we use the symbol

$$e(\lambda):=e^{\pi i\lambda}$$

for simplicity.

Received March 16, 2016.

2010 *Mathematics Subject Classification.* Primary 33C60; Secondary 33B15, 33C15.

Key words and phrases. complex hypergeometric integrals, complex beta integrals, twisted de Rham theory.

§2. Complex beta integrals

Before begining the study of the complex hypergeometric integrals, we consider the complex beta integrals.

As a complex version of the beta integral

$$\int_0^1 x^a(1-x)^b dx, \tag{2.1}$$

we consider the integral

$$\begin{aligned} &\frac{i}{2}\int\int_{\mathbb{C}\backslash\{0,1\}} t^{a^+}\bar{t}^{a^-}(1-t)^{b^+}(1-\bar{t})^{b^-}\,dtd\bar{t} \\ &= \int\int_{\mathbb{R}^2\backslash\{(0,0),(1,0)\}} (x+iy)^{a^+}(x-iy)^{a^-} \\ &\times(1-x-iy)^{b^+}(1-x+iy)^{b^-}\,dx\,dy. \end{aligned} \tag{2.2}$$

Here $t = x+iy$, $\bar{t} = x-iy$, the domain $\mathbb{C}\backslash\{0,1\} \simeq \mathbb{R}^2\backslash\{(0,0),(1,0)\}$ has a standard orientation, $a^+ - a^-$, $b^+ - b^- \in \mathbb{Z}$, and the branch of the integrand is determined by

x	$-\infty$	$\cdots\cdots$	$+\infty$
$\arg(x+iy)$	π	$\searrow$	0
$\arg(x-iy)$	$-\pi$	$\nearrow$	0
$\arg(1-x-iy)$	0	$\searrow$	$-\pi$
$\arg(1-x+iy)$	0	$\nearrow$	π

for $y > 0$, and

x	$-\infty$	$\cdots\cdots$	$+\infty$
$\arg(x+iy)$	$-\pi$	$\nearrow$	0
$\arg(x-iy)$	π	$\searrow$	0
$\arg(1-x-iy)$	0	$\nearrow$	π
$\arg(1-x+iy)$	0	$\searrow$	$-\pi$

for $y < 0$, and the branch for $y = 0$ is given by the analytic continuation of these two cases $y > 0$ and $y < 0$. The conditions $a^+ - a^- \in \mathbb{Z}$ and $b^+ - b^- \in \mathbb{Z}$ guarantee its well-definedness.

Theorem 1. *Suppose that*

$$\begin{aligned} &-1-a^- - b^-, 1+a^-, 1+b^- \notin \mathbb{Z}_{\leq 0}, \\ &\textit{and}\quad a^+ - a^-, b^+ - b^- \in \mathbb{Z}. \end{aligned} \tag{2.3}$$

Then we have

$$\int\int_{\{\mathbb{R}^2\backslash\{(0,0),(1,0)\}\}} (x+iy)^{a^+}(x-iy)^{a^-} \times (1-x-iy)^{b^+}(1-x+iy)^{b^-}\,dxdy = \frac{\sin(\pi a^+)\sin(\pi b^+)}{\sin(\pi(a^++b^+))}\frac{\Gamma(a^++1)\Gamma(b^++1)}{\Gamma(a^++b^++2)}\frac{\Gamma(a^-+1)\Gamma(b^-+1)}{\Gamma(a^-+b^-+2)}. \tag{2.4}$$

Remark 1. *Formula* (2.4) *is the same as* (1.5) *of* [8].

Remark 2. *In* [7], *the c-function for* $SL(2,\mathbb{C})/SO(2,\mathbb{C})$ *is given by*

$$\int_{\mathbb{R}^2}(1+t^2)^{\lambda+\frac{m}{2}}(1+\bar{t}^2)^{\lambda-\frac{m}{2}}\,dxdy$$

with $\lambda\in\mathbb{C}$, $m\in\mathbb{Z}$ *and* $t=x+iy$, *which is shown to be*

$$2^{4\lambda+2}\frac{\sin(\pi(\lambda+\frac{m}{2}))\sin(\pi(\lambda-\frac{m}{2}))}{\sin(\pi(2\lambda))}\frac{\{\Gamma(\lambda+\frac{m}{2}+1)\Gamma(\lambda-\frac{m}{2}+1)\}^2}{\Gamma(2\lambda+m+2)\Gamma(2\lambda-m+2)}$$

by applying formula (2.4) *with* $a^+=b^+=\lambda-\frac{m}{2}$ *and* $a^-=b^-=\lambda+\frac{m}{2}$ *after changing the variables* $x\mapsto 2y$ *and* $y\mapsto 2x-1$.

Corollary 1. *Suppose that*

$$1+a, 1+b, -1-a-b\notin\mathbb{Z}_{\leq 0}.$$

Then we have

$$\frac{\sqrt{-1}}{2}\int_{\mathbb{C}\backslash\{0,1\}}|t|^{2a}\,|1-t|^{2b}\,dt\,d\bar{t} = \frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi(a+b))}\left(\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}\right)^2. \tag{2.5}$$

Remark 3. *Formula* (2.5) *is the same as* (2.24) *of* [1], *where the complex Selberg integral is studied from the viewpoint of the twisted de Rham theory. See also* (3.63) *of* [2] *and p. 384 of* [3].

We start proving the formula.

First, consider x and y to be the complex coordinates, and the region of the integral $\mathbb{R}^2\backslash\{(0,0),(1,0)\}$ to be a subvariety of

$$\mathbb{C}^2\backslash D,$$

where

$$D=\{x+iy=0\}\cup\{x-iy=0\}\cup\{1-x-iy=0\}\cup\{1-x+iy=0\},$$

by the embedding

$$(x,y)\in\mathbb{R}^2\backslash\{(0,0),(1,0)\}\hookrightarrow(x,y)\in\mathbb{C}^2\backslash D.$$

Second, deform the region of the integral appropriately. As a preparatory lemma obtain the following:

Lemma 1. (1) *Consider the loaded chain with the standard orientation*

$$\{(x,y)\in\mathbb{R}^2\mid(x,y)\neq(0,0)\}\otimes(x+iy)^{a^+}(x-iy)^{a^-}, \tag{2.6}$$

where

$$\{(x,y)\in\mathbb{R}^2\mid(x,y)\neq(0,0)\}\subset\mathbb{C}^2\backslash\{x+iy=0\}\cup\{x-iy=0\}$$

and the arguments of $x+iy$ and $x-iy$ are fixed to be

x	$-\infty$	$\cdots$	$+\infty$
$\arg(x+iy)$	π	$\searrow$	0
$\arg(x-iy)$	$-\pi$	$\nearrow$	0

for $y>0$, and

x	$-\infty$	$\cdots$	$+\infty$
$\arg(x+iy)$	$-\pi$	$\nearrow$	0
$\arg(x-iy)$	π	$\searrow$	0

for $y<0$. Then (2.6) *is homologous to*

$$\begin{aligned}\{(x,y)\mid x+iy>0,\ x-iy>0\}&\otimes\ (x+iy)^{a^+}(x-iy)^{a^-}\\ +e(a^-)\{(x,y)\mid x+iy>0,\ x-iy<0\}&\otimes\ (x+iy)^{a^+}(iy-x)^{a^-}\\ +e(a^+-a^-)\{(x,y)\mid x+iy<0,\ x-iy<0\}&\otimes(-x-iy)^{a^+}(iy-x)^{a^-}\\ +e(a^+)\{(x,y)\mid x+iy<0,\ x-iy>0\}&\otimes(-x-iy)^{a^+}(x-iy)^{a^-},\end{aligned} \tag{2.7}$$

where the argument of each function is zero (*standard loading*) *and the orientation is standardly chosen.*

(2) *Consider the loaded chain with the standard orientation*

$$\{(x,y)\in\mathbb{R}^2\mid(x,y)\neq(1,0)\}\otimes(1-x-iy)^{b^+}(1-x+iy)^{b^-}, \tag{2.8}$$

where

$$\{(x,y)\in\mathbb{R}^2 \mid (x,y)\neq(1,0)\}\subset\mathbb{C}^2\backslash\{1-x-iy=0\}\cup\{1-x+iy=0\}$$

and the arguments of $1-x-iy$ *and* $1-x+iy$ *are fixed to be*

x	$-\infty$	$\cdots\cdots$	$+\infty$
$\arg(1-x-iy)$	0	$\searrow$	$-\pi$
$\arg(1-x+iy)$	0	$\nearrow$	π

for $y>0$, *and*

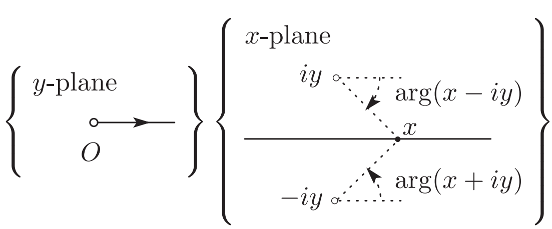
x	$-\infty$	$\cdots\cdots$	$+\infty$
$\arg(1-x-iy)$	0	$\nearrow$	π
$\arg(1-x+iy)$	0	$\searrow$	$-\pi$

for $y<0$. *Then* (2.8) *is homologous to*

$$\begin{aligned}(2.9)\qquad &\{(x,y)\mid 1-x-iy>0,\ 1-x+iy>0\}\\ &\qquad\otimes(1-x-iy)^{b^+}(1-x+iy)^{b^-}\\ &+e(b^-)\{(x,y)\mid 1-x-iy>0,\ 1-x+iy<0\}\\ &\qquad\otimes(1-x-iy)^{b^+}(-iy-1+x)^{b^-}\\ &+e(b^--b^+)\{(x,y)\mid 1-x-iy<0,\ 1-x+iy<0\}\\ &\qquad\otimes(-1+x+iy)^{b^+}(-iy-1+x)^{b^-}\\ &+e(b^+)\{(x,y)\mid 1-x-iy<0,\ 1-x+iy>0\}\\ &\qquad\otimes(-1+x+iy)^{b^+}(1-x+iy)^{b^-},\end{aligned}$$

where the loading and the orientation are standardly chosen.

Proof. (1) In the case $y>0$: Consider the loaded (twisted) chain



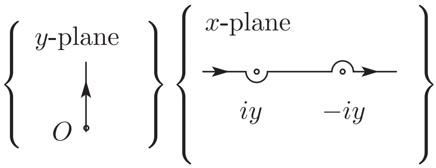
$\otimes\,(x+iy)^{a^+}(x-iy)^{a^-}$,

where the arguments of $x+iy$ and $x-iy$ are

x	$-\infty$	$\cdots$	$+\infty$
$\arg(x+iy)$	π	$\searrow$	0
$\arg(x-iy)$	$-\pi$	$\nearrow$	0

.

Rotate the path on the y-plane by $\pi/2$. Then we have



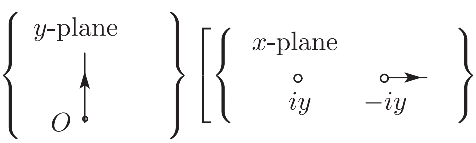
$$\otimes\,(x+iy)^{a^+}(x-iy)^{a^-},$$

where

x	∞	$\cdots$	(iy)	$\cdots$	$(-iy)$	$\cdots$	$+\infty$
$\arg(x+iy)$	π	$\cdots$	$\cdots$	π	$\searrow$	$\cdots$	0
$\arg(x-iy)$	$-\pi$	$\cdots$	$\nearrow$	0	$\cdots$	$\cdots$	0

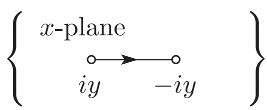
,

which is equal to



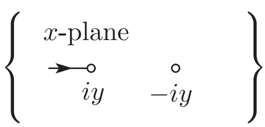
$$\Big[\ \cdots\ \otimes\,(x+iy)^{a^+}(x-iy)^{a^-}$$

$+e(a^+)$



$$\otimes\,(-x-iy)^{a^+}(x-iy)^{a^-}$$

$+e(a^+-a^-)$



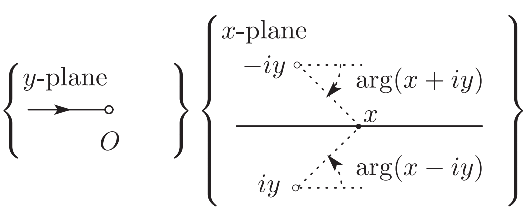
$$\otimes\,(-x-iy)^{a^+}(iy-x)^{a^-}\ \Big],$$

and

$$\begin{aligned}
&\{\,(x,y)\ |\ -iy<x,\ iy<0\ \}\otimes(x+iy)^{a^+}(x-iy)^{a^-}\\
&+e(a^+)\{\,(x,y)\ |\ iy<x<-iy,\ iy<0\ \}\otimes(-x-iy)^{a^+}(x-iy)^{a^-}\\
&+e(a^+-a^-)\{\,(x,y)\ |\ x<iy,\ iy<0\,\}\otimes(-x-iy)^{a^+}(iy-x)^{a^-},
\end{aligned}$$

where the loading and the orientation are standardly chosen.

In the case $y<0$: Rotate the path on the y-plane of



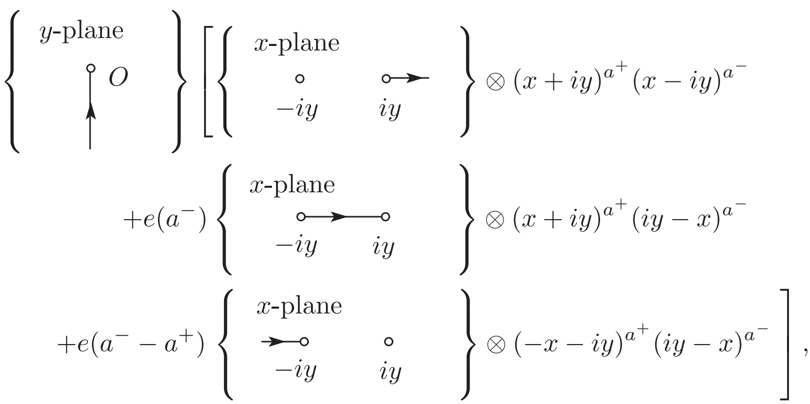
$$\otimes(x+iy)^{a^+}(x-iy)^{a^-}\,dxdy$$

by $\pi/2$, where

x	$-\infty$	$\cdots$	$+\infty$
$\arg(x+iy)$	$-\pi$	$\nearrow$	0
$\arg(x-iy)$	π	$\searrow$	0

.

Then we have



and

$$
\begin{aligned}
&\{\,(x,y)\ |\ iy < x,\ 0 < iy\,\} \otimes (x+iy)^{a^+}(x-iy)^{a^-}\\
&+e(a^-)\{\,(x,y)\ |\ -iy < x < iy,\ 0 < iy\ \} \otimes (x+iy)^{a^+}(iy-x)^{a^-}\\
&+e(a^- - a^+)\{\,(x,y)\ | < x < -iy,\ 0 < iy\,\} \otimes (-x-iy)^{a^+}(iy-x)^{a^-},
\end{aligned}
$$

where the loading and the orientation are standardly chosen.

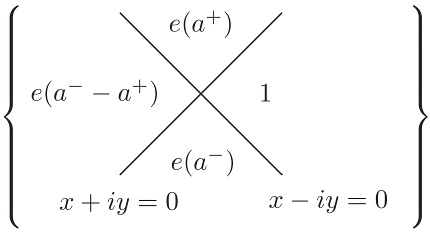
In the case $y = 0$: We have

$$
x^{a^+}x^{a^-} = \begin{cases} x^{a^+ + a^-}, & \text{if} \quad x > 0,\\ x^{a^+ + a^-}e^{\pi i(a^+ - a^-)}, & \text{if} \quad x < 0. \end{cases}
$$

Combining the three cases above, we reach the required result.

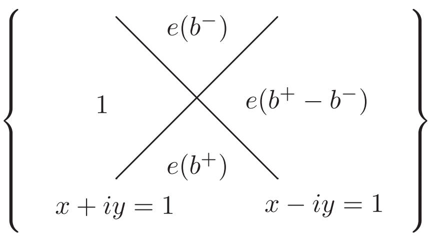
(2) The change of variables such that $x \mapsto -x, y \mapsto -y$ and $x \mapsto x-1$ in (1) implies the result. $\square$

We note that (2.7) and (2.9) can be described as



$$\otimes\, |x + iy|^{a^+} |x - iy|^{a^-}$$

and

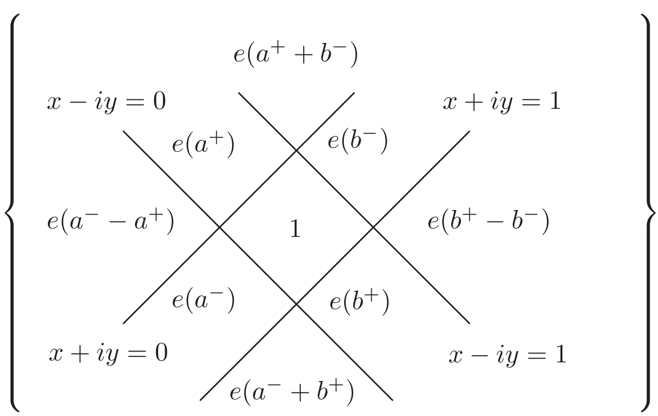


$$\otimes\, |1 - x - iy|^{b^+} |1 - x + iy|^{b^-},$$

respectively. By using this description, the combination of (1) and (2) in Lemma 1 implies that

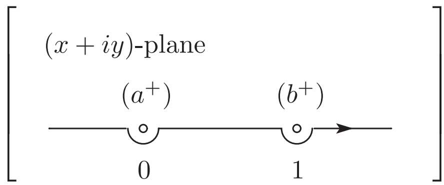
$$[\mathbb{R}^2 \backslash \{(0,0),(1,0)\}] \otimes (x+iy)^{a^+}(x-iy)^{a^-}(1-x-iy)^{b^+}(1-x+iy)^{b^-}$$

is homologous to



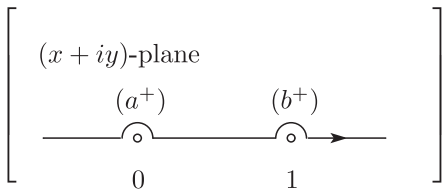
$$\otimes\, |x + iy|^{a^+} |x - iy|^{a^-} |1 - x - iy|^{b^+} |1 - x + iy|^{b^-}.$$

Here, if $x - iy < 0$, the path on the $(x + iy)$-plane is depicted as



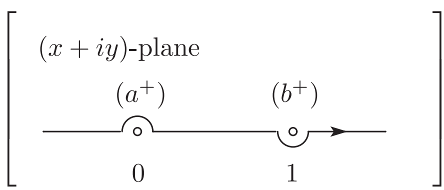
,

which is homologous to zero; if $x - iy > 1$, it is depicted as



,

which is homologous to zero; and if $0 < x - iy < 1$, it is depicted as



,

which is homologous to

$$\begin{aligned} &e(a^+)\overrightarrow{(-\infty, 0)} + \overrightarrow{(0,1)} + e(b^+)\overrightarrow{(1,\infty)} \\ &= \frac{(e(a^+) - e(-a^+))(e(b^+) - e(-b^+))}{e(a^+ + b^+) - e(-a^+ - b^+)}\overrightarrow{(0,1)} \\ &= 2i\,\frac{\sin(\pi a^+)\sin(\pi b^+)}{\sin(\pi(a^+ + b^+))}\overrightarrow{(0,1)}, \end{aligned}$$

because

$$\begin{aligned} e(-a^+)\overrightarrow{(-\infty, 0)} + \overrightarrow{(0,1)} + e(b^+)\overrightarrow{(1,+\infty)} = 0, \\ e(a^+)\overrightarrow{(-\infty, 0)} + \overrightarrow{(0,1)} + e(-b^+)\overrightarrow{(1,+\infty)} = 0. \end{aligned}$$

Here, $\overrightarrow{(-\infty, 0)}, \overrightarrow{(0,1)}, \overrightarrow{(1,+\infty)}$ stand for the standardly loaded cycles on the $(x + iy)$-space: For example, $\overrightarrow{(0,1)}$ means $\overrightarrow{(0,1)} \otimes (x + iy)^{a^+}(1 -$

$x-iy)^{b^+}$, where the arguments of $x+iy$, $1-x-iy$ are zero and the orientation is standardly fixed.

Consequently, we obtain

$$[\mathbb{R}^2\backslash\{(0,0),(1,0)\}]\otimes(x+iy)^{a^+}(x-iy)^{a^-}(1-x-iy)^{b^+}(1-x+iy)^{b^-}$$
$$=-2\sqrt{-1}\frac{\sin(\pi a^+)\sin(\pi b^+)}{\sin(\pi(a^++b^+))}\{0<x+iy<1\}\otimes(x+iy)^{a^+}(1-x-iy)^{b^+}$$
$$\times\{0<x-iy<1\}\otimes(x-iy)^{a^-}(1-x+iy)^{b^-},$$

which leads to

$$\int\int_{\mathbb{R}^2\backslash\{(0,0),(1,0)\}}(x+iy)^{a^+}(x-iy)^{a^-}(1-x-iy)^{b^+}(1-x+iy)^{b^-}dxdy$$
$$=-2\sqrt{-1}\frac{\sin(\pi a^+)\sin(\pi b^+)}{\sin(\pi(a^++b^+))}$$
$$\times\int\int_{\{0<x+iy<1\}\times\{0<x-iy<1\}}(x+iy)^{a^+}(x-iy)^{a^-}(1-x-iy)^{b^+}(1-x+iy)^{b^-}dxdy$$
$$=\frac{\sin(\pi a^+)\sin(\pi b^+)}{\sin(\pi(a^++b^+))}$$
$$\times\int_{\overrightarrow{\{0<x-iy<1\}}}(x-iy)^{a^-}(1-x+iy)^{b^-}d(x-iy)$$
$$\times\int_{\overrightarrow{\{0<x+iy<1\}}}(x+iy)^{a^+}(1-x-iy)^{b^+}d(x+iy)$$
$$=\frac{\sin(\pi a^+)\sin(\pi b^+)}{\sin(\pi(a^++b^+))}\frac{\Gamma(a^++1)\Gamma(b^++1)}{\Gamma(a^++b^++2)}\frac{\Gamma(a^-+1)\Gamma(b^-+1)}{\Gamma(a^-+b^-+2)}.$$

At the first stage before considering the integrals, we assume that

$$a^\pm,\ b^\pm,\ a^++b^+,\ a^-+b^-\notin\mathbb{Z},$$

and, at the second stage when we consider the integrals, we assume moreover that

$$\mathrm{Re}(1+a^\pm),\ \mathrm{Re}(1+b^\pm),\ \mathrm{Re}(-1-a^+-b^+),\ \mathrm{Re}(-1-a^--b^-)>0$$

to guarantee the existence of the integrals. Finally, however, as a result of analytic continuation with respect to the parameters $a^\pm$ and $b^\pm$, we relax the conditions into (2.3). This completes the proof of Theorem 1.

§3. Complex version of Gauss hypergeometric function

Let $\Phi(x,y;z)$ be the function defined by

$$\Phi(x,y;z)$$
$$=(x+iy)^{a^+}(x-iy)^{a^-}(1-x-iy)^{b^+}(1-x+iy)^{b^-}(z-x-iy)^{c^+}(\bar{z}-x+iy)^{c^-}.$$

Let D be

$$D=\{x\pm iy=0\}\cup\{x\pm iy=1\}\cup\{x+iy=z\}\cup\{x-iy=\bar{z}\}.$$

A complex version of the Gauss hypergeometric integral defined for $z\in\mathbb{C}\backslash\{0,1\}$ is

$$\int\int\Phi(x,y;z)dxdy, \tag{3.1}$$

where

$$a^+-a^-, b^+-b^-, c^+-c^-\in\mathbb{Z},$$

and the region of integral is

$$\mathbb{R}^2\backslash\{(0,0),(1,0),(\mathrm{Re}(z),\mathrm{Im}(z))\}$$

with the standard orientation. The region of the integral is considered as a subvariety of

$$\mathbb{C}^2\backslash D,$$

and the embedding

$$(x,y)\in\mathbb{R}^2\backslash\{(0,0),(1,0),(\mathrm{Re}(z),\mathrm{Im}(z))\}\hookrightarrow(x,y)\in\mathbb{C}^2\backslash D$$

is fixed.

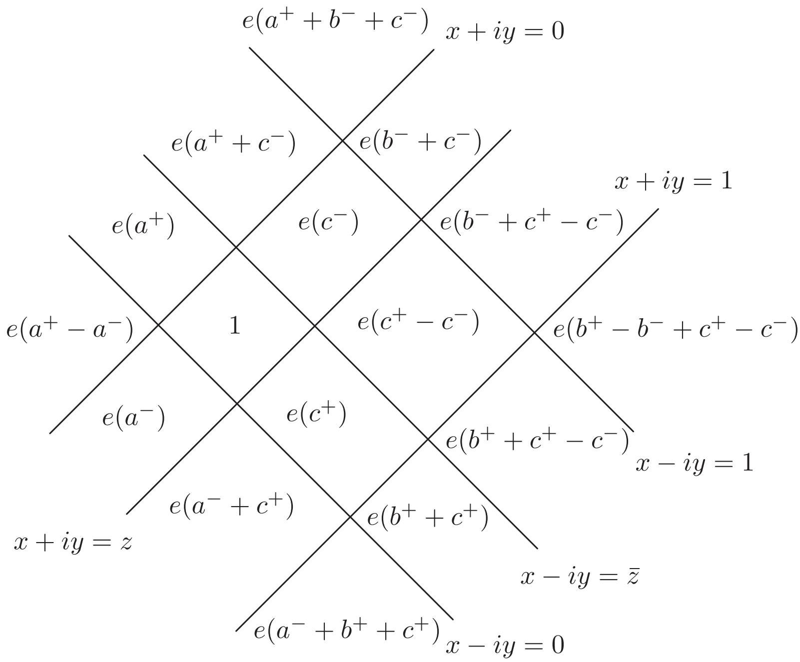
In this section, for simplicity, we temporarily assume that the complex variable z is real and $0<z<1$, and we load $\Phi(x,y)$ on $(0,z)\times(0,\bar{z})$ standardly.

Lemma 1 leads to the following.

Proposition 1. *The loaded chain with the standard orientation*

$$\{(x,y)\in\mathbb{R}^2\mid(x,y)\neq(0,0),(1,0),(\mathrm{Re}(z),\mathrm{Im}(z))\}\otimes\Phi(x,y) \tag{3.2}$$

is homologous to



$$\otimes\, |x+iy|^{a^+}|x-iy|^{a^-}|1-x-iy|^{b^+}|1-x+iy|^{b^-}|z-x-iy|^{c^+}|\bar{z}-x+iy|^{c^-},$$

where the orientation and the loading are standardly chosen.

The same argument as in the previous section shows that (3.2) is homologous to

$$\begin{aligned}(3.3)\qquad &\overrightarrow{(0,\bar{z})}^* \times \Big\{e(a^+)\overrightarrow{(-\infty,0)} + \overrightarrow{(0,z)} \\ &\qquad + e(c^+)\overrightarrow{(z,1)} + e(b^+ + c^+)\overrightarrow{(1,+\infty)}\Big\} \\ &+ \overrightarrow{(\bar{z},1)}^* \times \Big\{e(a^+ + c^-)\overrightarrow{(-\infty,0)} + e(c^-)\overrightarrow{(0,z)} \\ &\qquad + e(c^+ - c^-)\overrightarrow{(z,1)} + e(b^+ + c^+ - c^-)\overrightarrow{(1,+\infty)}\Big\}.\end{aligned}$$

Here, $\overrightarrow{(\infty,0)}, \overrightarrow{(0,z)}, \overrightarrow{(z,1)}, \overrightarrow{(1,+\infty)}$ stand for the standardly loaded cycles in the $(x+iy)$-space, and $\overrightarrow{(\infty,0)}^*, \overrightarrow{(0,z)}^*, \overrightarrow{(z,1)}^*, \overrightarrow{(1,+\infty)}^*$ the standardly loaded cycles in the $(x-iy)$-space.

On the other hand, when $a^- + c^- \notin \mathbb{Z}$, we have

$$\overrightarrow{(\bar{z},1)}^* = -\frac{1}{e(a^-+c^-)-e(-a^--c^-)}\Big\{(e(a^-)-e(-a^-))\overrightarrow{(0,\bar{z})}^* + (e(a^-+b^-+c^-)-e(-a^--b^--c^-))\overrightarrow{(1,+\infty)}^*\Big\}, \tag{3.4}$$

which follows from

$$\overrightarrow{(-\infty,0)}^* + e(a^-)\overrightarrow{(0,\bar{z})}^* + e(a^-+c^-)\overrightarrow{(\bar{z},1)}^* + e(a^-+b^-+c^-)\overrightarrow{(1,+\infty)}^* = 0,$$

$$\overrightarrow{(-\infty,0)}^* + e(-a^-)\overrightarrow{(0,\bar{z})}^* + e(-a^--c^-)\overrightarrow{(\bar{z},1)}^* + e(-a^--b^--c^-)\overrightarrow{(1,+\infty)}^* = 0.$$

Equality (3.4) makes (3.3) into

(3.5)

$$\frac{e(c^-)-e(-c^-)}{e(a^-+c^-)-e(-a^--c^-)}\overrightarrow{(0,\bar{z})}^* \times \Big\{e(a^+-a^-)\overrightarrow{(-\infty,0)} + e(-a^-)\overrightarrow{(0,z)} + e(a^-+c^+)\overrightarrow{(z,1)} + e(a^-+b^++c^+)\overrightarrow{(1,+\infty)}\Big\}$$
$$-\frac{e(a^-+b^-+c^-)-e(-a^--b^--c^-)}{e(a^-+c^-)-e(-a^--c^-)}\overrightarrow{(1,+\infty)}^* \times \Big\{e(a^++c^-)\overrightarrow{(-\infty,0)} + e(c^-)\overrightarrow{(0,z)} + e(c^+-c^-)\overrightarrow{(z,1)} + e(b^++c^+-c^-)\overrightarrow{(1,+\infty)}\Big\}.$$

Moreover, the equalities

$$e(a^--a^+)\overrightarrow{(-\infty,0)} + e(a^-)\overrightarrow{(0,z)} + e(a^-+c^+)\overrightarrow{(z,1)} + e(a^-+c^++b^+)\overrightarrow{(1,+\infty)} = 0,$$

$$e(a^++c^-)\overrightarrow{(-\infty,0)} + e(c^-)\overrightarrow{(0,z)} + e(c^--c^+)\overrightarrow{(z,1)} + e(c^--c^+-b^+)\overrightarrow{(1,+\infty)} = 0$$

and $e(c^+-c^-) = e(c^--c^+)$, which follows from $c^+ - c^- \in \mathbb{Z}$, make (3.5) into

$$
\begin{aligned}
&- e(a^+ - a^-)\,\frac{(e(c^-) - e(-c^-))(e(a^+) - e(-a^+))}{e(a^- + c^-) - e(-a^- - c^-)} \\
&\times \overrightarrow{(0,\bar{z})}^* \times \overrightarrow{(0,z)} \\
&- e(c^+ - c^-)\frac{(e(a^- + b^- + c^-) - e(-a^- - b^- - c^-))(e(b^+) - e(-b^+))}{e(a^- + c^-) - e(-a^- - c^-)} \\
&\times \overrightarrow{(1,+\infty)}^* \times \overrightarrow{(1,+\infty)} \\
= &- 2\sqrt{-1}\,e(a^+ - a^-)\frac{\sin(\pi c^-)\sin(\pi a^+)}{\sin(\pi(a^- + c^-))}\,\overrightarrow{(0,\bar{z})}^* \times \overrightarrow{(0,z)} \\
&- 2\sqrt{-1}\,e(c^+ - c^-)\frac{\sin(\pi(a^- + b^- + c^-))\sin(\pi(b^+))}{\sin(\pi(a^- + c^-))}\,\overrightarrow{(1,+\infty)}^* \times \overrightarrow{(1,+\infty)}.
\end{aligned}
$$

Therefore, we have the following.

Theorem 2. *Suppose that*

$$
\begin{aligned}
&-1 - a^+ - b^+ - c^+,\ -1 - a^- - b^- - c^-, \\
&1 + a^+,\ 1 + b^+,\ 1 + c^+,\ 1 + a^-,\ 1 + b^-,\ 1 + c^- \notin \mathbb{Z}_{\leq 0}\,, \\
&a^+ - a^-,\ b^+ - b^-,\ c^+ - c^- \in \mathbb{Z} \ \text{ and } \ a^- + c^- \notin \mathbb{Z}\,.
\end{aligned}
\tag{3.6}
$$

Then we have

(3.7)

$$\int\int_{\mathbb{R}^2\setminus\{(0,0),(1,0),(\mathrm{Re}(z),\mathrm{Im}(z))\}}(x+iy)^{a^+}(x-iy)^{a^-}$$
$$\times(1-x-iy)^{b^+}(1-x+iy)^{b^-}(z-x-iy)^{c^+}(\bar{z}-x+iy)^{c^-}\,dxdy$$
$$=e(a^+-a^-)\frac{\sin(\pi c^-)\sin(\pi a^+)}{\sin(\pi(a^-+c^-))}$$
$$\times\int_{\overrightarrow{(0,\bar{z})}}(x-iy)^{a^-}(1-x+iy)^{b^-}(\bar{z}-x+iy)^{c^-}d(x-iy)$$
$$\times\int_{\overrightarrow{(0,z)}}(x+iy)^{a^+}(1-x-iy)^{b^+}(z-x-iy)^{c^+}d(x+iy)$$
$$+e(c^+-c^-)\frac{\sin(\pi(a^-+b^-+c^-))\sin(\pi(b^+))}{\sin(\pi(a^-+c^-))}$$
$$\times\int_{\overrightarrow{(1,+\infty)}}(x-iy)^{a^-}(x-iy-1)^{b^-}(x-iy-\bar{z})^{c^-}d(x-iy)$$
$$\times\int_{\overrightarrow{(1,+\infty)}}(x+iy)^{a^+}(x+iy-1)^{b^+}(x+iy-z)^{c^+}d(x+iy)$$
$$=e(a^+-a^-)\frac{\sin(\pi c^-)\sin(\pi a^+)}{\sin(\pi(a^-+c^-))}$$
$$\times\int_{\overrightarrow{(0,\bar{z})}}u^{a^-}(1-u)^{b^-}(\bar{z}-u)^{c^-}du\int_{\overrightarrow{(0,z)}}u^{a^+}(1-u)^{b^+}(z-u)^{c^+}du$$
$$+e(c^+-c^-)\frac{\sin(\pi(a^-+b^-+c^-))\sin(\pi(b^+))}{\sin(\pi(a^-+c^-))}$$
$$\times\int_{\overrightarrow{(1,+\infty)}}u^{a^-}(u-1)^{b^-}(u-\bar{z})^{c^-}du\int_{\overrightarrow{(1,+\infty)}}u^{a^+}(u-1)^{b^+}(u-z)^{c^+}du.$$

Corollary 2. *Suppose that*

$$-1-a-b-c, 1+a, 1+b, 1+c, \notin \mathbb{Z}_{\leq 0} \text{ and } a+c \notin \mathbb{Z}.$$

Then we have

$$\begin{aligned}(3.8)\qquad &\frac{\sqrt{-1}}{2}\int_{\mathbb{C}\setminus\{0,1,z\}} |t|^{2a}\,|1-t|^{2b}\,|z-t|^{2c}\,dt\,d\bar{t} \\ &= \frac{\sin(\pi c)\sin(\pi a)}{\sin(\pi(a+c))}\left|\int_0^z u^a(1-u)^b(z-u)^c du\right|^2 \\ &+ \frac{\sin(\pi(a+b+c))\sin(\pi(b))}{\sin(\pi(a+c))}\left|\int_1^\infty u^a(u-1)^b(u-z)^c du\right|^2.\end{aligned}$$

Here the integrands are loaded standardly.

Remark 4. *At the first stage in this section before considering the integrals, we temporarily assume that*

$$a^\pm,\ b^\pm,\ c^\pm,\ a^+ + b^+ + c^+,\ a^- + b^- + c^- \notin \mathbb{Z},$$

and, at the second stage when we consider the integrals, we assume moreover that

$$\begin{aligned}&\mathrm{Re}(1+a^\pm) > 0,\ \mathrm{Re}(1+b^\pm) > 0,\ \mathrm{Re}(1+c^\pm) > 0, \\ &\mathrm{Re}(-1-a^+-b^+-c^+) > 0,\ \mathrm{Re}(-1-a^--b^--c^-) > 0\end{aligned}$$

to guarantee the existence of the integrals in (3.7). *Finally, however, the analytic continuation relaxes the conditions into* (3.6). *This completes the proof of Theorem 2.*

Remark 5. *Formula* (3.8) *is* (3.64) *of* [2], *which represents a correlation function of the basic operators in conformal field theory, and* (3.7) *is Theorem 2.2 of* [8], *which is obtained by considering the differential equations satisfied by the function* (2.1). *See also* (2) *of* [6] *for the relation with the correlation function of conformal field theory.*

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