Advanced Studies in Pure Mathematics 24, 1996
Progress in Algebraic Combinatorics
pp. 1–80

Combinatorial Cell Complexes

Michael Aschbacher

We define and discuss a category of combinatorial objects we call *combinatorial cell complexes* and a functor T from this category to the category of topological spaces with cell structure, whose image is closely related to the category of CW-complexes. This formalism was developed to study finite group actions on topological spaces. In order to make effective use of our detailed knowledge of the finite simple groups, it seems necessary to make such a translation from a purely topological setting to the language of geometric combinatorics.

Our functor T assigns to each combinatorial cell complex X its geometric realization $T(X)$. We show the functor T defines an equivalence of categories between the category of combinatorial cell complexes whose cell boundaries are spheres, and a certain subcategory of CW-complexes we call normal CW-complexes.

We often concentrate on a subcategory of combinatorial cell complexes we call restricted combinatorial cell complexes; the restricted CW-complexes are the CW-complexes corresponding to the restricted combinatorial cell complexes under our equivalence of categories. Restricted CW-complexes include regular CW-complexes but also many other classical examples like the torus, the Klein bottle, and the Poincaré dodecahedron, which are discussed here as illustrations.

We associate to each restricted combinatorial cell complex X, a simplicial complex $K(X)$ and a canonical triangulation of $T(X)$ by $K(X)$. The geometric realization of a general combinatorial cell complex can also be canonically triangulated, but by a more complicated simplicial complex than $K(X)$. However we do not supply a proof of this last fact here.

We define cellular homology combinatorially, and show that if X is restricted and the boundary of each cell is homologically spherical, then

Received November 3, 1994.

Revised February 22, 1995.

This work was partially supported by NSF DMS–9101237 and BSF 92-00320

the homology of $T(X)$ is the cellular homology of X. We define a duality operator on restricted complexes which will be used in a later paper to establish a version of Poincaré duality for homology manifolds with cell structure which is more concrete than the usual version.

Now some specifics. Let $\mathcal{P}$ be the category whose objects are the posets P such that each $a \in P$ is of finite height, and whose morphisms are the maps preserving order and height. Let $\mathcal{P}^*$ consist of those members of $\mathcal{P}$ with a greatest element. A *combinatorial cell complex* consists of a poset $X \in \mathcal{P}$, a function $f : X \to \mathcal{P}^*$, a map $\zeta : V = \coprod_{x \in X} f(x) \to X$, and maps $f_v : f(x)(\leq v) \to f(\zeta(v))$ for each $v \in V$, such that

(i) For each $x \in X$, $\zeta : f(x) \to X(\leq x)$ is a map of posets preserving height.

(ii) For each $x \in X$ and $v \in f(x)$, $f_v : f(x)(\leq v) \to f(\zeta(v))$ is an isomorphism of posets.

(iii) If $u, v \in f(x)$ with $u \leq v$ then $f_u = f_{f_v(u)} \circ f_v$.

(iv) For $v \in f(x)$, $\zeta = \zeta \circ f_v$ on $f(x)(\leq v)$.

(v) For each $x \in X$, $f_{\infty_x} : f(x) \to f(x)$ is the identity map and $\zeta(\infty_x) = x$, where ∞_x is the greatest element of $f(x)$.

The posets $f(x)$, $x \in X$, are the *cells* of X and the *boundary* of the cell $f(x)$ is $\dot{f}(x) = f(x) - \{\infty_x\}$. The *faces* of the cell are its subposets $f(x)(\leq v)$, $v \in f(x)$. A combinatorial cell complex is *restricted* if ζ is injective on $f(x)(\geq v)$ for each $v \in f(x)$. Equivalently, for each $x \in X$ and $y \leq x$, the faces $f(x)(\leq v)$ with $v \in \zeta^{-1}(y) \cap f(x)$ are pairwise disjoint.

Intuitively a combinatorial cell complex consists of a collection $f(x)$, $x \in X$, of cells, with the poset structure on X corresponding to inclusions among the cells. The maps ζ and f_v, $v \in V$, keep track of identifications of cells with faces of larger cells, and glue cells together at their boundaries. Extra structure in some category $\mathcal{C}$ can be adjoined to each combinatorial cell to obtain $\mathcal{C}$-cells and a $\mathcal{C}$-cell complex.

The triangulating complex $K(X)$ of a combinatorial cell complex is defined in Section 5 and the geometric realization $T(X)$ of X is defined in Section 10. Our major results are Theorem 10.6, which provides the canonical triangulation of $T(X)$ by $K(X)$ when X is restricted, Theorem 12.16, which shows the cellular homology of X is isomorphic to the homology of $T(X)$ when X is restricted with homologically spherical cell boundaries, and Theorem 15.15, which establishes the equivalence of categories between combinatorial cell complexes whose cell boundaries are spheres and normal CW- complexes. The definitions of normal and restricted CW-complexes appear in Section 15. The reader may wish to refer to Sections 13 and 16 for various examples such as the torus, the Klein bottle, and the Poincaré dodecahedron.

§1. Posets and typed simplicial complexes

Let X be a poset. For $x \in X$ let $h(x)$ be the *height* of x in X. That is $h(x)$ is the maximum length of a chain in X with greatest element x, if the length of such chains is bounded, and ∞ otherwise. Write $X(\leq x)$ for the set of elements $y \in X$ such that $y \leq x$ and define $X(< x)$, $X(\geq x)$, etc. similarly.

Denote by $\mathcal{P}$ the category of posets X such that each $x \in X$ is of finite height. The morphisms in $\mathcal{P}$ are the maps of posets which preserve height. Let $\mathcal{P}^*$ be the subcategory of those $X \in \mathcal{P}$ such that X has a unique maximal member ∞_X.

We regard X as a category whose objects are the members of X and with $\mathrm{Mor}(x, y) = \{(x, y)\}$ if $x \leq y$ and $\mathrm{Mor}(x, y) = \varnothing$ otherwise.

Recall the *order complex* of a poset X is the simplicial complex $\mathcal{O}(X)$ whose vertices are the members of X and whose simplices are the finite chains. Often we write X for the order complex $\mathcal{O}(X)$ of X.

Example (1). If K is a simplicial complex its simplices form a poset under the inclusion relation and the *barycentric subdivision* $\mathrm{sd}(K)$ of K is the order complex of this poset. Thus the vertices of $\mathrm{sd}(K)$ are the finite simplices of K and the simplices of $\mathrm{sd}(K)$ are the chains of simplices of K.

A *typed simplicial complex* over an index set I is a simplicial complex $K = (V, \Sigma)$ together with a type function $h : V \to I$ such that h is injective on simplices. The morphisms of typed complexes over I are the simplicial maps which preserve type.

Example (2). The order complex of a poset is a typed complex where $h(x)$ is the height of x.

We will use the following notational conventions in discussing the homology of a typed simplicial complex. Let K be a typed simplicial complex with type function $h : V \to I$ and pick a total ordering of I. Given a k-simplex s in K, write

$$s = \prod_{v \in s} v = v_0 \wedge \cdots \wedge v_k \in \bigwedge^k (V)$$

for the generator of $C_k(K) \leq \bigwedge^k(V)$ corresponding to s, where $s = \{v_0, \ldots, v_k\}$ with $h(v_0) < \cdots < h(v_k)$. Then our boundary map becomes

$$\partial(s) = \sum_{i=0}^{k}(-1)^i s^i$$

where $s^i = v_0 \wedge \cdots \wedge v_{i-1} \wedge v_{i+1} \wedge \cdots \wedge v_k$.

If $s = s_1 \cup \cdots \cup s_r$ is a partition of s we write $s = \prod_{i=1}^r s_i$. More generally if J is some subset of I we can consider

$$\Sigma(J) = \{s \in \Sigma : h(s) = J\}.$$

Then if $J \subseteq L \subseteq I$ and $c = \sum_{s\in\Sigma(L)} a_s s \in C_*(K)$ with $a_s \in \mathbf{Z}$, then $s = s_J s_{L-J}$, where $s_U = \{v \in s : h(v) \in U\}$ for $U = J$ or $L - J$, and $c = \sum_{t\in\Sigma(J)} tc_t$, where $c_t = \sum_{t\subseteq s} a_s s_{L-J}$. Indeed

(1.1) *Let K be a typed simplicial complex with type function $h : V \to I$, $J \subseteq L \subseteq I$, and*

$$c = \sum_{s\in\Sigma(L)} a_s s = \sum_{t\in\Sigma(J)} tc_t \in C_*(K)$$

with $a_s \in \mathbf{Z}$. Then

(1) $\partial(c) = \sum_t (-1)^N \partial(t) c_t + (-1)^M t\partial(c_t)$

(2) *If $\partial(c) = 0$ then $\partial(c_t) = 0$ for each $t \in \Sigma(J)$ and $\sum_{t\in\Sigma(J)} \partial(t)c_t$ $=0$.*

(3) *If $L - J = \{l\}$ is of order 1 then*

$$c_t = \sum_{u\in V_l(t)} a_{t,u} u$$

where $V_l(t) = \{v \in \mathrm{Link}_K(t) : h(v) = l\}$, $a_{t,u} = a_{t\cup\{u\}}$, and $\partial(c_t) = \sum_{u\in V_l(t)} a_{t,u}$.

(4) *If $|L| = k+1$ and $J = \{j\}$ with j the maximal member of L then $\partial(c) = \sum_t t\partial(c_t) + (-1)^k c_t$, so $\partial(c) = 0$ if and only if $\partial(c_t) = 0$ for each $t \in \Sigma(J)$ and $\sum_t c_t = 0$.*

Proof. Take $L = \{0, \ldots, k\}$. We first prove (1). Since

$$v_0 \wedge \cdots \wedge v_k = \mathrm{sgn}(\pi)(v_{\pi(0)} \wedge \cdots \wedge v_{\pi(k)})$$

for $\pi \in \mathrm{Sym}(L)$, changing our ordering of I by a suitable permutation π, we may assume $J = \{0, \ldots, j\}$ and the ordering of J and $L - J$ are induced from L. Subject to this choice of ordering, we prove

$$(*) \qquad \partial(c) = \sum_t (\partial(t)c_t + (-1)^{j+1} t\partial(c_t)).$$

As ∂ is linear, it suffices to take $c = s$. Let $t = s_J$; then $c_t = s_{L-J}$. Now for $i \le j$, $s^i = t^i c_t$, while for $i > j$, $s^i = tc_t^{i-j-1}$. Therefore

$$\begin{aligned}\partial(c) = \partial(s) &= \sum_{i=0}^{k} (-1)^i s^i = \sum_{i=0}^{j} (-1)^i t^i c_t + \sum_{i=0}^{k-j-1} (-1)^{i+j+1} tc_t^i \\ &= \partial(t)c_t + (-1)^{j+1} t\partial(c_t).\end{aligned}$$

So (1) is established. Further under the hypotheses of (4), the sign of a permutation π mapping k to the first member of L and preserving the order on $L - J$, is $(-1)^k$ and of course $\partial(t) = 1$, so by (*), $\partial(c) = \sum_t (-1)^k c_t + t\partial(c_t)$, establishing the first statement in (4).

Suppose $\partial(c) = 0$. Then by (1), $\partial(c) = A + (-1)^M \sum_t t\partial(c_t)$ with $A = \sum_{r \in S_A} b_r r$ and $(-1)^M t\partial(c_t) = \sum_{r \in S_t} b_r r$, where S_A and $S_t, t \in \Sigma(J)$ are suitable subsets of $\Sigma^{k-1}(K)$ and S_t is the set of simplices $s \in S_A \cup \bigcup_t S_t$ with $t \subseteq s$. In particular the sets S_A, S_t, $t \in \Sigma(J)$, are pairwise disjoint. As $\partial(c) = 0$, $b_r = 0$ for all r, so as our index sets are pairwise disjoint, $(-1)^M t\partial(c_t) = \sum_{r \in S_t} b_r r = 0$ and hence $\partial(c_t) = 0$. As this holds for all t, also $0 = \sum_t \partial(t)c_t$.

Thus (2) is established. Finally (2) completes the proof of (4), since under the hypotheses of (4), $\partial(t)c_t = c_t$. The proof of (3) is straightforward.

§2. Cells

Let $\mathcal{C}$ be a category. Define $Cell(\mathcal{C})$ to be the category of covariant functors $F : X \to \mathcal{C}$, where $X \in \mathcal{P}^*$ is regarded as a category as in Section 1. In addition we almost always impose extra conditions on the cells.

Regard $(X, F) \in Cell(\mathcal{C})$ as a category whose objects are the pairs $(x, F(x))$, $x \in X$, with

$$\mathrm{Mor}((x, F(x)), (y, F(y))) = \{(x, y), F(x, y)\} \text{ if } x \le y$$

where $F(x, y) : F(x) \to F(y)$ is the $\mathcal{C}$-morphism associated to (x, y) by F and

$$\mathrm{Mor}((x, F(x)), (y, F(y))) = \varnothing \text{ otherwise.}$$

Notice that as F is a functor, $F(x,z) = F(y,z) \circ F(x,y)$ whenever $x \leq y \leq z$.

The morphisms in $Cell(\mathcal{C})$ are the covariant functors $\phi : (X,F) \to (Y,G)$ together with isomorphisms $\phi_x : F(x) \to G(\phi(x))$ such that for all $x \leq z$ in X, the following diagram commutes:

$$\begin{tikzcd}
F(z) \arrow[r, "\phi_z"] & G(\phi(z)) \\
F(x) \arrow[u, "F(x,z)"] \arrow[r, "\phi_x"] & G(\phi(x)) \arrow[u, "G(\phi(x),\phi(z))"']
\end{tikzcd}$$

Here for $x \in X$, $(x, F(x))$ is an object of (X,F), so $\phi(x, F(x))$ is an object of (Y,G) which must then be of the form $(\phi(x), G(\phi(x)))$ for some $\phi(x) \in Y$. Further as ϕ is a functor, if $x_1 \leq x_2$ then $\mathrm{Mor}((x_1, F(x_1)),(x_2, F(x_2))) \neq \varnothing$, so $\mathrm{Mor}((\phi(x_1), G(\phi(x_1))), (\phi(x_2), G(\phi(x_2)))) \neq \varnothing$ and hence $\phi(x_1) \leq \phi(x_2)$, so $\phi : X \to Y$ is a map of posets.

The category $Cell(\mathcal{C})$ is the category of $\mathcal{C}$-*cells*. Intuitively a $\mathcal{C}$-cell is an object $F(\infty_X)$ in $\mathcal{C}$ together with a distinguished family $(F(x) : x \in X)$ of subobjects, called the *faces* of $F(\infty_X)$, indexed by the poset X, with the inclusion relation on these subobjects corresponding to the partial order on X. In the case of combinatorial and topological cells this intuition is made precise in the following two examples:

Example. (1) Let $\mathcal{C} = \mathcal{P}^*$. Then $Cell(\mathcal{P}^*)$ is the category of *combinatorial cells.* We also require that a combinatorial cell (X,F) satisfy $F(x) = X(\leq x)$ and $F(x,y) : F(x) \to F(y)$ be inclusion for all $x, y \in X$ with $x \leq y$. So a combinatorial cell is nothing more than a poset X in $\mathcal{P}^*$ together with its faces $X(\leq x)$, $x \in X$.

(2) Let $\mathcal{C} = Top$ be the category of topological spaces with morphisms closed injections. In this case $Cell(Top)$ is the category of *topological cells.* We also demand that a topological cell (X,F) satisfy the requirements that if $x, y \in X$ then

(i) $F(x,\infty)(F(x)) \cap F(y,\infty)(F(y)) = \bigcup_{z \leq x,y} F(z,\infty)(F(z))$.

(ii) If $x \neq y$ then $F(x,\infty)(F(x)) \neq F(y,\infty)(F(y))$.

(iii) For each $Y \subseteq X$, $\bigcup_{y \in Y} F(y,\infty)(F(y))$ is closed in T.

Since for $x \in X$, $F(x,\infty) : F(x) \to F(\infty)$ is a closed injection, we can identify $F(x)$ with its image in $F(\infty)$. Condition (i) says that if $x \leq y$ then $F(x) \subseteq F(y)$, and then condition (ii) says the map $x \mapsto F(x)$ is an isomorphism of the poset X with the poset $\{F(x) : x \in X\}$ of distinguished closed subspaces of $F(\infty)$.

§3. Cell complexes

Again let $\mathcal{C}$ be a category. Let $Complex(\mathcal{C})$ be the category whose objects consist of

(1) Some $X \in \mathcal{P}$.

(2) A function $f : X \to Cell(\mathcal{C})$.

(3) A map $\zeta : V = \coprod_{x \in X} f(x) \to X$ such that for each $x \in X$, $\zeta : f(x) \to X(\leq x)$ is a morphism in $\mathcal{P}$.

(4) For each $x \in X$ and $v \in f(x)$, an isomorphism $f_v : f(x)(\leq v) \to f(\zeta(v))$ of $\mathcal{C}$-cells satisfying:

(5) If $u, v \in f(x)$ with $u \leq v$ then $f_u = f_{f_v(u)} \circ f_v$.

(6) For $v \in f(x)$, $\zeta = \zeta \circ f_v$ on $f(x)(\leq v)$.

(7) For each $x \in X$, $f_{\infty_x} : f(x) \to f(x)$ is the identity map and $\zeta(\infty_x) = x$, where ∞_x is the greatest element of $f(x)$.

Formally $f(x)$ is a pair (X_x, F_x) where X_x is a poset and $F_x : X_x \to \mathcal{C}$ is a functor, but we usually write $f(x)$ for the poset X_x and F for F_x. In particular this is the convention in axioms (3) and (7). However the isomorphism f_v of axiom (4) is an isomorphism of cells, so it consists of a covariant functor $f_v : f(x)(\leq v) \to f(\zeta(v))$ and isomorphisms $f_{v,u} : F_x(u) \to F_{\zeta(v)}(f_v(u))$ for each pair $u, v \in f(x)$ with $u \leq v$, and these isomorphisms satisfy $F_{\zeta(v)}(f_v(u), f_v(w)) \circ f_{v,w} = f_{v,w} \circ F_x(w, u)$ for each $w \leq u \leq v$.

The morphisms $\psi : (X, f) \to (Y, g)$ are morphisms $\psi : X \to Y$ in $\mathcal{P}$ together with morphisms $\psi_x : f(x) \to g(\psi(x))$ in $Cell(\mathcal{C})$ for each $x \in X$ such that

(a) For each $x \in X$ and $v \in f(x)$, the following diagram commutes:

$$\begin{array}{ccc} f(x)(\leq v) & \xrightarrow{\psi_x} & g(\psi(x))(\leq \psi_x(v)) \\ f_v \downarrow & & \downarrow g_{\psi_x(v)} \\ f(\zeta(v)) & \xrightarrow{\psi_{\zeta(v)}} & g(\psi(\zeta(v))) \end{array}$$

(b) For each $x \in X$ the following diagram commutes:

$$\begin{array}{ccc} f(x) & \xrightarrow{\zeta} & X(\leq x) \\ \psi_x \downarrow & & \downarrow \psi \\ g(\psi(x)) & \xrightarrow{\zeta} & Y(\leq \psi(x)) \end{array}$$

Further we define composition in our category so that if $\phi : (Y, g) \to (Z, h)$ is a morphism then $(\phi \circ \psi)_x = \phi_{\psi(x)} \circ \psi_x$ for each $x \in X$.

The members $X = (X, f)$ of *Complex*$(\mathcal{C})$ are called *$\mathcal{C}$-cell complexes.* X is a *combinatorial cell complex* if the cells are combinatorial cells. X is a *topological cell complex* if the cells of X are topological.

Intuitively a cell complex consists of cells indexed by the poset X together with identifications of the faces of cells accomplished by the maps ζ and f_v, $v \in V$.

Example (1). Let $X \in \mathcal{P}$ and for $x \in X$ and $v \in X(\leq x)$ let $f(x) = X(\leq x)$ and let ζ and f_v be the appropriate identity maps. Then X is a combinatorial cell complex. We call this cell complex the *simplicial cell complex* of the poset X.

Remarks. (1) Given any $\mathcal{C}$-cell complex (X, f, F) we can suppress the $\mathcal{C}$-structure supplied by the functor F and obtain the *combinatorial cell complex* (X, f) of (X, f, F). This gives us a forgetful functor from $\mathcal{C}$-cell complexes to combinatorial cell complexes.

(2) The combinatorial cell complexes are the simplest cell complexes, and we can give a somewhat simpler definition of this category equivalent to the specialization of the general definition above to the case of combinatorial cells: A *combinatorial cell complex* consists of a poset $X \in \mathcal{P}$, a function $f : X \to \mathcal{P}^*$, a map $\zeta : V = \coprod_{x \in X} f(x) \to X$, and maps $f_v : f(x)(\leq v) \to f(\zeta(v))$ for each $v \in V$, such that

(i) For each $x \in X$, $\zeta : f(x) \to X(\leq x)$ is a map of posets preserving height.

(ii) For each $x \in X$ and $v \in f(x)$, $f_v : f(x)(\leq v) \to f(\zeta(v))$ is an isomorphism of posets.

(iii) If $u, v \in f(x)$ with $u \leq v$ then $f_u = f_{f_v(u)} \circ f_v$.

(iv) For $v \in f(x)$, $\zeta = \zeta \circ f_v$ on $f(x)(\leq v)$.

(v) For each $x \in X$, $f_{\infty_x} : f(x) \to f(x)$ is the identity map and $\zeta(\infty_x) = x$.

Moreover a *morphism* $\psi : (X, f) \to (Y, g)$ of combinatorial cell complexes consists of a height preserving map $\psi : X \to Y$ of posets together with height preserving maps $\psi_x : f(x) \to g(\psi(x))$ of posets for each $x \in X$ such that ψ and ψ_x satisfy the commutative diagrams (a) and (b) for morphisms of cell complexes given earlier in this section.

Define a cell complex (X, f) to be *regular* if $\zeta : f(x) \to X(\leq x)$ is an isomorphism for each $x \in X$. For example the simplicial cell complex of a poset (defined in Example (1)) is a regular cell complex. We will see in a moment that, up to isomorphism, all regular combinatorial cell complexes are simplicial cell complexes. Define (X, f) to be *restricted* if ζ is injective on $f(x)(\geq v)$ for each $v \in f(x)$. For example regular cell complexes are restricted, but the converse is certainly not true. The combinatorial cell complexes of the torus and Klein bottle, discussed in Section 13, are examples of restricted cell complexes which are not regular, as is the complex of the Poincaré dodecahedron, discussed in Section 16.

The *cells* of the cell complex (X, f, F) are the $\mathcal{C}$-cells $f(x) = (X_x, F_x)$. The *boundary* of the combinatorial cell $f(x)$ is $\dot{f}(x) = f(x) - \{x\}$ and if $f(x)$ is a $\mathcal{C}$-cell with extra structure supplied by $F(x)$ then $\dot{f}(x)$ is the $\mathcal{C}$-cell complex with extra structure $\dot{F}(x) = F(x)_{|\dot{f}(x)}$. We say (X, f, F) is of *height* n if X is of height n.

Example (2). Let $0 \leq n \in \mathbf{Z}$ and let $X(n)$ be the poset $\{0, 1, \ldots, n\}$ under the usual order. For $k \in X(n)$ define $f(k) = \{(k, i) : 0 \leq i \leq k\}$ and order $f(k)$ so that the map $\zeta : f(k) \to X(n)$ defined by $\zeta(k, i) = i$ preserves order. Define $f_{(k,i)} : f(k)(\leq (k, i)) \to X(n)(\leq i)$ by $f_{(k,i)}(k, j) = (i, j)$. Then $(X(n), f)$ is a combinatorial cell complex isomorphic to the simplicial cell complex of the poset $X(n)$. As $X(n)$ is an n-simplex, we call $X(n)$ the *simplicial cell complex of the n-simplex.*

(3.1) *Let $X = (X, f)$ be a regular combinatorial cell complex. Then (X, f) is isomorphic to the simplicial cell complex of the poset X.*

Proof. Let $\bar{X} = (\bar{X}, \bar{f})$ be the simplicial cell complex of X. Thus $\bar{X} = X$, $\bar{f}(x) = X(\leq x)$ for each $x \in X$, and $\bar{\zeta}$ and $\bar{f}_v$ are the appropriate identity maps. Define

$$\psi : (X, f) \to (\bar{X}, \bar{f})$$

and

$$\bar{\psi} : (\bar{X}, \bar{f}) \to (X, f)$$

to be the morphisms with $\psi : X \to \bar{X}$ and $\bar{\psi} : \bar{X} \to X$ the identity maps, $\psi_x : f(x) \to \bar{f}(x)$ the restriction of ζ to $f(x)$, and $\bar{\psi}_x = \psi_x^{-1}$. It is essentially immediate from the definition of ψ and $\bar{\psi}$ and from axiom 6 for cell complexes that each of these maps *is* a morphism of cell complexes. Of course $\bar{\psi} = \psi^{-1}$, so ψ is an isomorphism.

(3.2) *Let (X, f) be a combinatorial cell complex and $x \in X$. Then the poset $f(x)$ is of height $h(x)$ and ∞_x is the unique member of $f(x)$ of height $h(x)$.*

Proof. As $f(x) \in \mathcal{P}^*$, ∞_x is the unique element of $f(x)$ of maximal height. Then as $\zeta : f(x) \to X(\leq x)$ preserves height, $h(f(x)) = h(\infty_x) = h(\zeta(\infty_x)) = h(x)$ by axiom 7 for cell complexes.

§4. Topological cell complexes

Let X, f, F be a topological cell complex. That is $X \in \mathcal{P}$, for $x \in X$, $f(x)$ is a topological cell with $F(v)$ the topological space associated to $v \in f(x) \in \mathcal{P}^*$, etc.

Write x for ∞_x. As in Example (2) in Section 2, we regard each cell $f(x)$ as a topological space $F(x)$ together with a distinguished class $\{F(v) : v \in f(x)\}$ of closed subspaces. Namely, for $v \in f(x)$, we have a closed injection $F(v,x) : F(v) \to F(x)$, and we identify $F(v)$ with its image under this injection and regard it as a closed subspace of $F(x)$. Because F is a functor, these identifications are compatible with the ordering on $f(x)$; that is if $u < v < x$ then $F(u) \subseteq F(v) \subseteq F(x)$ and the identification of $F(u)$ with a subspace of $F(x)$ factors through the identification of $F(u)$ with a subspace of $F(v)$. Subject to these conventions, $F(x)$ is a topological space with a poset of distinguished closed subspaces, and that poset is isomorphic to $f(x)$.

Recall from Section 2 that topological cells are required to satisfy the property that if $u, v \in f(x)$ then $F(u,x)(F(u)) \cap F(v,x)(F(v)) = \bigcup_{w \leq u,v} F(w,x)(F(w))$, which under our new notational conventions translates into the statement that $F(u) \cap F(v) = \bigcup_{w \leq u,v} F(w)$. Also if $u \neq v$ then $F(u,x)(F(u)) \neq F(v,x)(F(v))$, which in our new language reads if $u \neq v$ then $F(u) \neq F(v)$. In particular it follows that

(4.1) *For each $x \in X$ and $a \in F(x)$ there exists a unique $v \in f(x)$ of minimal height such that $a \in F(v)$.*

Next let $v \in f(x)$. Then we have an isomorphism $f_v : f(x)(\leq v) \to f(\zeta(v))$ of topological cells. The identifications above identify $f(x)(\leq v)$ with $F(v)$ and the subspaces determined by the poset $f(x)(\leq v)$, and identify $f(\zeta(v))$ with the space $F(\zeta(v))$ and its family of subspaces. As f_v is an isomorphism of topological cells, it induces an isomorphism $F_v : F(v) \to F(\zeta(v))$ such that if $u \leq v$ then $F_v(F(u)) = F(f_v(u))$. Further as F is a functor, $F_u = F_{f_v(u)} \circ F_v$.

Let $F_n = \coprod_{\dim(x)=n} F(x)$ be the disjoint union of the spaces associated to the n-cells of X, and $\hat{A}_n = \coprod_{m \leq n} F_n$. Thus for each $a \in \hat{A}_n$ there is a unique $x(a) \in X$ with $a \in F(x(a))$ and by 4.1 there is a unique $v(a) \in f(x)$ of minimal height such that $a \in F(v(a))$. Let $y(a) = \zeta(v(a))$ and observe that $F_{v(a)}(a) \in F(y(a))$ with $x(F_{v(a)}(a)) = y(a)$.

For $x \in X$ of dimension n, $a \in F(x)$, and $b \in \hat{A}_{n-1}$, define $a \searrow b$ if $F_v(a) = b$ for some $v \in f(x)$ with $a \in F(v)$ and $b \in F(\zeta(v))$.

We construct a topological space A_n by factoring out a suitable equivalence relation $\sim_n$ from $\hat{A}_n$. The definition is recursive. Namely the equivalence relation $\sim_n$ on $\hat{A}_n$ is defined to be the equivalence relation on $\hat{A}_n$ generated by $\searrow$, regarded as a relation on $F_n \cup \hat{A}_{n-1}$, and $\sim_{n-1}$. We first observe that:

(4.2) *For $a \in \hat{A}_{n-1}$, $[a]_n \cap \hat{A}_{n-1} = [a]_{n-1}$, where $[a]_k$ is the equivalence class of a with respect to $\sim_k$.*

Proof. It suffices to show that if $a \in F_n$ and $b, c \in \hat{A}_n$ with $a \searrow b$ and $a \searrow c$ then $[b]_{n-1} = [c]_{n-1}$. Let $x = x(a)$, $r = x(b)$, and $s = x(c)$. Then $a \in F(u) \cap F(w)$ where $u \in \zeta^{-1}(r)$ and $w \in \zeta^{-1}(s)$. By an earlier remark, $F(u) \cap F(w) = \bigcup_{v \le u,w} F(v)$, so $a \in F(v)$ for some $v \le u, w$. Now $F_{f_u(v)}(b) = (F_{f_u}(v) \circ F_u)(a) = F_v(a) = F_{f_w(v)}(c)$, so by induction on n, $[b]_{n-1} = [c]_{n-1}$.

Let $\hat{A} = \bigcup_n \hat{A}_n$ and $\sim = \bigcup_n \sim_n$. By 4.2, $\sim$ is an equivalence relation on $\hat{A}$. Write $\tilde{a}$ for the equivalence class of $a \in \hat{A}$ and let $A = A(X) = \hat{A}/\sim$. We conclude from 4.2 that

(4.3) *For each $a \in \hat{A}$ and nonnegative integer n, $\tilde{a} \cap \hat{A}_n = [a]_n$.*

For $x \in X$, $u \in f(x)$, define $I(u) = F(u) - \bigcup_{u > v \in f(x)} F(v)$. Further define

$$\begin{aligned} \lambda_x : F(x) &\to A \\ a &\mapsto \tilde{a} \end{aligned}$$

and let $\tilde{F}(x)$, $\tilde{I}(x)$ be the image of $F(x)$, $I(x)$ in A under the map λ_x.

(4.4) (1) *The map $\lambda_x : I(x) \to A$ is an injection.*

(2) *For $\tilde{a} \in A$ there exists a unique $y(\tilde{a}) \in X$ such that $\tilde{a} \cap I(y(\tilde{a})) \neq \varnothing$.*

(3) *There exists a unique element $\xi(\tilde{a}) \in F(y(\tilde{a}))$ with $\xi(\tilde{a}) \in \tilde{a}$.*

(4) *If $x \in X$ and $b \in \tilde{a} \cap F(x)$, then there exists $v \in f(x) \cap \zeta^{-1}(y(\tilde{a}))$ with $b \in F(v)$ and $F_v(b) = \xi(\tilde{a})$. Further $\tilde{F}(y(\tilde{a})) = \widetilde{F(v)} \subseteq \tilde{F}(x)$.*

Proof. Let $n = h(x)$ and $a, b \in I(x)$. If $\tilde{a} = \tilde{b}$ then by 4.3, $[a]_n = [b]_n$. But by definition of $\sim_n$, $[a]_n = \{a\}$ for $a \in I(x)$ as $h(x) = n$. This establishes (1) and (3).

Let $x \in X$ and suppose x is minimal subject to $\tilde{a} \cap F(x) \neq \varnothing$. Let $b \in F(x) \cap \tilde{a}$; claim $b \in I(x)$. For if not then $b \in F(v)$ for some $x \neq v \in f(x)$ and then $b \sim F_v(b) \in F(y)$ with $y = \zeta(v)$. Now $\tilde{F}(y) = \widetilde{F(v)} \subseteq \tilde{F}(x)$, and the minimality of x is contradicted. In particular there exists some $y \in X$ with $\tilde{a} \cap I(y) \neq \varnothing$.

On the other hand suppose $a_i \in \tilde{a}$ with $a_i \in I(y_i)$, for $i = 1, 2$, and let $n = \max\{h(y_1), h(y_2)\}$. Then, as we saw in paragraph one, $|\tilde{a} \cap \hat{A}_n| = 1$, so $a_1 = a_2$ and hence as a_1 is in $F(y)$ for a unique $y \in X$, $y_1 = y_2$. This establishes (2).

We saw in paragraph two that if $x \in X$ and $b \in \tilde{a} \cap F(x)$, then either $b \in I(x)$ or there is $y < x$ and $v \in \zeta^{-1}(y)$ with $b \in F(v)$ and $\tilde{F}(y) = \tilde{F}(v) \subseteq \tilde{F}(x)$. In the former $x = y(a)$ by (2), so that (4) holds, and in the latter (4) holds by induction on $h(x)$.

(4.5) (1) *The sets* $\tilde{I}(x)$, $x \in X$, *partition* A.
(2) $\tilde{F}(x) \cap \tilde{F}(y) = \bigcup_{z \le x,y} \tilde{F}(z)$.

Proof. Part (1) follows from 4.4.2. Let $\tilde{a} \in \tilde{F}(x) \cap \tilde{F}(y)$. Then by 4.4.4, $\tilde{a} \in \tilde{F}(y(\tilde{a})) \subseteq \tilde{F}(x) \cap \tilde{F}(y)$ and $y(a) \le x, y$. Thus (2) holds.

We topologize $\tilde{F}(x)$ by defining a subset C of $\tilde{F}(x)$ to be closed if and only if $\lambda_x^{-1}(C)$ is closed in $F(x)$. Then we topologize A by decreeing that $C \subseteq A$ is closed in A if and only if $C \cap \tilde{F}(x)$ is closed in $\tilde{F}(x)$ for each $x \in X$.

(4.6) (1) $\tilde{F}(x)$ *is closed in* A *so a subset* C *of* $\tilde{F}(x)$ *is closed in* $\tilde{F}(x)$ *if and only if* C *is closed in* A.
(2) $\lambda_x : F(x) \to A$ *is continuous.*
(3) *If* $\zeta : f(x) \to X(\le x)$ *is injective then* $\lambda_x : F(x) \to \tilde{F}(x)$ *is a homeomorphism.*
(4) *For each* $Y \subseteq X$, $\bigcup_{y \in Y} \tilde{F}(y)$ *is closed in* A.

Proof. Let C be closed in A. Then by definition of the topology on A, C is closed in $\tilde{F}(x)$. Conversely if $\tilde{F}(x)$ is closed in A and C is closed in $\tilde{F}(x)$ then C is closed in A, so to prove (1) it remains to show $\tilde{F}(x)$ is closed in A. We must show $\tilde{F}(x) \cap \tilde{F}(y)$ is closed in $\tilde{F}(y)$ for all $y \in X$. By 4.5.2,

$$\tilde{F}(x) \cap \tilde{F}(y) = \bigcup_{z \le x,y} \tilde{F}(z)$$

Further

$$\lambda_y^{-1}(\tilde{F}(z)) = \bigcup_{v \in \zeta^{-1}(z) \cap f(y)} F(v)$$

so

$$\lambda_y^{-1}(\tilde{F}(x) \cap \tilde{F}(y)) = \bigcup_{\substack{z \le x,y \\ v \in \zeta^{-1}(z) \cap f(y)}} F(v)$$

is closed in $F(y)$ by axiom (iii) for topological cells in Example (2) of Section 2. Now by definition of the topology on A, $\tilde{F}(x) \cap \tilde{F}(y)$ is closed in $\tilde{F}(y)$, completing the proof of (1). A similar argument establishes (4).

By definition of the topology on $\tilde{F}(x)$, $\lambda_x : F(x) \to \tilde{F}(x)$ is continuous, so (2) follows from (1). Assume $\zeta : f(x) \to X(\leq x)$ is injective. Claim $\lambda_x : F(x) \to \tilde{F}(x)$ is bijective. For if $a, b \in F(x)$ with $\tilde{a} = \tilde{b}$ then by 4.4.4, there is $u, v \in f(x) \cap \zeta^{-1}(y(\tilde{a}))$ with $a \in F(u)$, $b \in F(v)$, and $F_u(a) = F_v(b) = \xi(\tilde{a})$. As ζ is injective, $u = v$. Then as F_u is injective, $a = b$. So λ_x is bijective. Now for $D \subseteq F(x)$ closed, $D = \lambda_x^{-1}(\lambda_x(D))$ is closed, so $\lambda_x(D)$ is closed in $\tilde{F}(x)$ by definition of the topology on $\tilde{F}(x)$. This proves (3).

(4.7) *Let $\varphi : (X, f, F) \to (Y, g, G)$ be a morphism of topological cell complexes. Then*

(1) *φ induces a continuous map $A(\varphi) : A(X) \to A(Y)$ via $A(\varphi)(\tilde{a}) = \widetilde{\varphi(a)}$.*

(2) *A is a covariant functor from the category of topological cell complexes to the category of topological spaces.*

Proof. Let $\tilde{\varphi} = A(\varphi)$. Observe first that $\tilde{\varphi}$ is well defined, since if $a_i \in \tilde{a}$, $i = 1, 2$, then $a_i \in F(v_i) \subseteq F(x_i)$ with $v_i \in \zeta^{-1}(y)$, where $y = y(\tilde{a})$ and $F_{v_i}(a_i) = \xi(\tilde{a}) \in F(y)$. Then $G_{\varphi(v_i)}(\varphi(a_i)) = \varphi(\xi(\tilde{a}))$, so $\varphi(a_1) \sim \varphi(\xi(\tilde{a})) \sim \varphi(a_2)$.

Next claim $\tilde{\varphi} : \tilde{F}(x) \to \tilde{G}(\varphi(x))$ is continuous for each $x \in X$. For if C is a closed subset of $\tilde{G}(\varphi(x))$ then $\lambda_{\varphi(x)}^{-1}(C)$ is closed in $G(\varphi(x))$ and then as $\varphi : F(x) \to G(\varphi(x))$ is continuous, $\varphi^{-1}(\lambda_{\varphi(x)}^{-1}(C))$ is closed in $F(x)$. So as $\lambda_{\varphi(x)} \circ \varphi = \tilde{\varphi} \circ \lambda_x$, $\lambda_x^{-1}(\tilde{\varphi}^{-1}(C))$ is closed in $F(x)$. Therefore $\tilde{\varphi}^{-1}(C)$ is closed in $\tilde{F}(x)$, so indeed $\tilde{\varphi} : \tilde{F}(x) \to \tilde{G}(\varphi(x))$ is continuous. Therefore by 4.6 and the definition of the topology on $A(X)$ and $A(Y)$, $\tilde{\varphi} : A(X) \to A(Y)$ is continuous. Hence part (1) of the lemma is established. Part (2) is straightforward.

§5. The triangulating complex of a combinatorial cell complex

Let (X, f) be a combinatorial cell complex. Let $V = V(X) = \coprod_{x \in X} f(x)$ be the disjoint union of the posets $f(x)$, $x \in X$. So for each $v \in V$ there exists a unique $\hat{\zeta}(v) \in X$ with $v \in f(\hat{\zeta}(v))$.

For $v \in V$ define

$$L(v) = \{u \in V : f_w(v) \geq u \text{ for some } w \in f(\hat{\zeta}(v))(\geq v)\}.$$

(5.1) *Let $x \in X$, $v \in f(x)$, $u \in L(v)$, and $w \in f(x)(\geq v)$ with $f_w(v) \geq u$. Then*

(1) $\hat{\zeta}(u) = \zeta(w)$.

(2) *If (X, f) is restricted then w is the unique $z \in f(x)$ $(\geq v)$ with $\zeta(z) = \hat{\zeta}(u)$ and we denote w by $\hat{f}_u(v)$.*

Proof. As $f_w(v) \geq u$, $u \in f(\zeta(w))$, so $\hat{\zeta}(u) = \zeta(w)$. If X is restricted then for each $y \in X$ there is at most one $z \in f(x)$ $(\geq v)$ with $\zeta(z) = y$, so (2) holds.

Define the *graph* $\Delta = \Delta(X)$ of the cell complex X to be the graph with vertex set V and u adjacent to v if $u \in L(v)$ or $v \in L(u)$. The *clique complex* of a graph Γ is the simplicial complex with vertex set Γ and simplices the cliques of Γ. Denote by $K(X)$ the clique complex of $\Delta(X)$. We call $K(X)$ the *triangulating complex* of X.

(5.2) *If $u \in L(v)$ then $\zeta(u) \leq \zeta(v) \leq \hat{\zeta}(u) \leq \hat{\zeta}(v)$.*

Proof. As $u \in L(v)$, there is $w \in f(x)(\geq v)$ with $f_w(v) \geq u$. Then $\hat{\zeta}(u) = \zeta(w) \leq \hat{\zeta}(v)$, $\zeta(v) = \zeta(f_w(v)) \geq \zeta(u)$, and $\zeta(v) \leq \zeta(w) = \hat{\zeta}(u)$.

(5.3) *If $u, v \in V$ are adjacent with $\hat{\zeta}(u) = \hat{\zeta}(v)$ and $\zeta(u) = \zeta(v)$ then $u = v$.*

Proof. We may take $u \in L(v)$. Let $x = \hat{\zeta}(v)$ and $w \in f(x)(\geq v)$ with $f_w(v) \geq u$. Then $\zeta(w) = \hat{\zeta}(u) = \hat{\zeta}(v) = x$, so by 3.2, $w = \infty_x = x$. Then $v = f_x(v) \geq u$, so as $\zeta(v) = \zeta(u)$ and ζ preserves height, $v = u$.

For $s \subseteq \Delta$ define $X(s) = \{\hat{\zeta}(v) : v \in s\}$ and $\zeta(s) = \{\zeta(v) : v \in s\}$.

(5.4) *Let s be a simplex in $K(X)$. Then*

(1) *There is a unique ordering $v_0, \ldots, v_k$ of the vertices of s such that $v_i \in L(v_j)$ for $0 \leq i \leq j \leq k$.*

(2) $\zeta(v_0) \leq \cdots \leq \zeta(v_k) \leq \hat{\zeta}(v_0) \leq \cdots \leq \hat{\zeta}(v_k)$.

(3) *Assume (X, f) is restricted and let $w_i = \hat{f}_{v_i}(v_k)$. Then w_i is the unique $w \in f(\hat{\zeta}(v_k))(\geq v_k)$ with $\zeta(w) = \hat{\zeta}(v_i)$. Moreover $w_0 \leq \cdots \leq w_k = \hat{\zeta}(v_k)$ and $f_{w_j}(w_i) \geq v_j$ for $j \geq i$.*

Proof. Induct on the dimension k of s. The case $k = 0$ is trivial, so take $k > 0$. By 5.2, $X(s)$ and $\zeta(s)$ are chains, so pick $v_k \in s$ with $\hat{\zeta}(v_k)$ maximal, and subject to this constraint, with $\zeta(v_k)$ maximal. Let $t = s - \{v_k\}$. By induction on k, there is a unique ordering $v_0, \ldots, v_{k-1}$ of t satisfying the conditions of the lemma. Let $i < k$. By 5.3 and the choice of v_k, either $\hat{\zeta}(v_i) < \hat{\zeta}(v_k)$ or $\zeta(v_i) < \zeta(v_k)$. Then by 5.2, $v_i \in L(v_k)$, and $\zeta(v_i) \leq \zeta(v_k) \leq \hat{\zeta}(v_i) \leq \hat{\zeta}(v_k)$, establishing (1) and (2).

Thus it remains to prove (3), so we may assume X is restricted. By 5.1.2, w_i is the unique $w \in f(\hat{\zeta}(v_k))(\geq v_k)$ with $\zeta(w) = \hat{\zeta}(v_i)$. By induction, $z_i = \hat{f}_{v_i}(v_{k-1})$ is the unique $z \in f(\hat{\zeta}(v_{k-1}))(\geq v_{k-1})$ with $\zeta(z) = \hat{\zeta}(v_i)$ for $i < k$. Further $z_0 \leq \cdots \leq z_{k-1}$ and $f_{z_i}(z_j) \geq v_i$. Now $f^{-1}_{w_{k-1}}(z_i) \geq f^{-1}_{w_{k-1}}(v_{k-1})$ and $w_i \geq v_k \geq f^{-1}_{w_{k-1}}(v_{k-1})$, so as X is restricted, $w_i = f^{-1}_{w_{k-1}}(z_i)$. So as $z_j \geq z_i$ for $j \geq i$, $w_j \geq w_i$. Then $f_{w_j}(w_i) \geq f_{w_j}(v_k) \geq v_j$, completing the proof of (3).

For s a simplex of $K(X)$, Lemma 5.4 says that $X(s)$ has a greatest element $\hat{\zeta}(s)$.

(5.5) *If s is a simplex of $K(X)$ then* $\dim(s) \leq h(\hat{\zeta}(s))$ *and* $\dim(s) \leq |X(s)| + |\zeta(s)| - 2$.

Proof. Let $s = \{v_0, \ldots, v_k\}$ be ordered as in 5.4. By 5.3, for each $1 \leq i \leq k$, $\hat{\zeta}(v_{i-1}) < \hat{\zeta}(v_i)$ or $\zeta(v_{i-1}) < \zeta(v_i)$. Let $\phi(i) = \hat{\zeta}(v_{i-1})$ or $\zeta(v_{i-1})$ in the respective case. Then the map $\phi : \{i : 1 \leq i \leq k\} \to X(s) - \{\hat{\zeta}(s)\} \cup \zeta(s) - \{y\}$ is an injection, where $y = \zeta(v_k)$. Therefore the second remark in the lemma holds. Also by 5.4, $\{\phi(i) : i\}$ is a chain of length $k-1$ in $X(< \hat{\zeta}(s))$, so the first remark holds.

(5.6) *Let (X, f) be of height n and $V(n) = \{(m,k) : 0 \leq k \leq m \leq n\}$. Define $\tau : V \to V(n)$ by $\tau(v) = (h(\hat{\zeta}(v)), h(\zeta(v)))$. Then $K(X)$ is a typed simplicial complex over $V(n)$ with type function τ.*

Proof. This follows from 5.3 and 5.4.

Remark 5.7. Observe we have a covariant functor K from the category of combinatorial cell complexes to the category of typed simplicial complexes. We have already associated a typed simplicial complex $K(X)$ to X. Suppose $\alpha : X \to \bar{X}$ is a morphism of combinatorial cell complexes. Then for $x \in X$, we have a map $\alpha_x : f(x) \to \bar{f}(\alpha(x))$ of posets which induces a map $K(\alpha) : V(X) \to V(\bar{X})$ defined by $K(\alpha)(v) = \alpha_{\hat{\zeta}(v)}(v)$. If $u \in L(v)$ there is $w \in f(x)(\geq v)$ with $f_w(v) \geq u$. Then as

$$\begin{array}{ccc} f(x)(\leq w) & \xrightarrow{\alpha_x} & \bar{f}(\alpha(x))(\leq \alpha_x(w)) \\ f_w \downarrow & & \downarrow \bar{f}_{\alpha_x(w)} \\ f(\zeta(w)) & \xrightarrow{\alpha_{\zeta(w)}} & \bar{f}(\alpha(\zeta(w))) \end{array}$$

commutes,

$$K(\alpha)(u) = \alpha_{\hat{\zeta}(u)}(u) \le \alpha_{\hat{\zeta}(u)}(f_w(v))$$
$$= \bar{f}_{\alpha_x(w)}(\alpha_x(v)) = \bar{f}_{K(\alpha)(w)}(K(\alpha)(v)),$$

with $K(\alpha)(v) = \alpha_x(v) \le \alpha_x(w) = K(\alpha)(w)$, so $K(\alpha)(u) \in L(K(\alpha)(v))$. Thus $K(\alpha) : \Delta(X) \to \Delta(\bar{X})$ is a map of graphs, and thus induces a simplicial map from $K(X)$ to $K(\bar{X})$. As α preserves height, $K(\alpha)$ also preserves the type function τ of Lemma 5.6. It is easy to check that $K(\alpha \circ \beta) = K(\alpha) \circ K(\beta)$, so K is indeed a functor.

Example 5.8. Consider the simplicial cell complex $(X(n), f)$ of the n-simplex defined in Example (2) in Section 3. The set $V(n) = \{(k,i) : 0 \le i \le k \le n\}$ is $V(X(n))$. Denote by $K(n)$ the triangulating complex $K(X(n))$ of $X(n)$. Then for $(a,b), (\alpha,\beta) \in V(n)$, $(a,b) \in L(\alpha,\beta)$ if and only if $b \le \beta \le a \le \alpha$. The complex $X(n)$ is regular and hence restricted. Observe that if $(a,b) \in L(\alpha,\beta)$ then $\hat{f}_{(a,b)}((\alpha,\beta)) = (\alpha,a)$.

In the remainder of this section we discuss the triangulating complex $K(n)$ of the simplicial cell complex $X(n)$ of the n-simplex. Observe first that we may regard $V(n)$ as the lower diagonal elements in an $n+1$ by $n+1$ square array. From this point of view, for $(\alpha,\beta) \in V(n)$, $L(\alpha,\beta)$ is the set of entries in $V(n)$ living in the rectangle with corners $(\beta,0)$, (β,β), $(\alpha,0)$, (α,β) sitting directly above and to the left of (α,β). Similarly those (a,b) with $(\alpha,\beta) \in L(a,b)$ form the rectangle with corners (α,β), (α,α), (n,β), (n,α), sitting directly below and to the right of (α,β).

Next 5.4 translates into the statement:

(5.9) *A subset s of $V(n)$ is in the set $\Sigma(n)$ of simplices of $K(n)$ if and only if we can order s so that $s = \{(\alpha_i,\beta_i) : 0 \le i \le k\}$ with $\beta_0 \le \cdots \le \beta_k \le \alpha_0 \le \cdots \le \alpha_k$.*

For $(\alpha,\beta) \in V(n)$ define

$$l(\alpha,\beta) = \{(a,b) : \alpha = a \text{ and } b = \beta - 1 \text{ or } \beta = b \text{ and } a = \alpha - 1\}$$

and a directed graph structure on $V(n)$ by $e \to f$ if $e \in l(f)$. Notice if $e \to f$ then e is adjacent to f in the graph $\Delta(n)$ of $X(n)$. Also $e \to f$ if e and f are adjacent lattice points in the array $V(n)$.

For $s \in \Sigma(n)$, let $\alpha^*(s) = \max\{\alpha : (\alpha,\beta) \in s\}$ and $\alpha_*(s) = \min\{\alpha : (\alpha,\beta) \in s\}$. Define $\beta^*(s)$ and $\beta_*(s)$ similarly.

(5.10) *The maximal simplices $\Sigma^*(n)$ of $K(n)$ are precisely the directed paths $p = p_0 \cdots p_n$ of length n in the directed graph $(V(n), \rightarrow)$ such that $p_i \rightarrow p_{i-1}$, $p_0 = (\alpha_*(p), 0)$, $p_n = (n, \beta^*(p))$, and $\alpha_*(p) = \beta^*(p)$. In particular $\alpha^*(p) = n$ and $\beta_*(p) = 0$.*

Proof. Let p be a maximal path and order p as in 5.9. Notice $\beta_0 = \beta_*(p)$, $\alpha_0 = \alpha_*(p)$, $\beta_k = \beta^*(p)$, and $\alpha_k = \alpha^*(p)$. Now $p \cup \{(n, \beta_k), (\alpha_0, 0)\} \in \Sigma(n)$, so by maximality of p, $n = \alpha_k = \alpha^*(p)$ and $0 = \beta_0 = \beta_*(p)$. If $p_i \notin l(p_{i+1})$ then $\alpha_{i+1} - \alpha_i > 1$ or $\beta_{i+1} - \beta_i > 1$, or $\alpha_{i+1} - \alpha_i = \beta_{i+1} - \beta_i = 1$, and we adjoint $(\alpha_i + 1, \beta_i)$, $(\alpha_i, \beta_i + 1)$, or $(\alpha_i + 1, \beta_i)$ to p in the respective case to contradict the maximality of p. Thus p is a directed path in $(V(n), \rightarrow)$. Notice the length of the path p is the number N of changes down and to the right as the path proceeds from p_0 to p_k, since at each step there is exactly one such change. Finally $p \cup \{(n, \alpha_0)\} \in \Sigma(n)$, so $(n, \alpha_0) = p_k$ by maximality of p. Thus $\beta^*(p) = \alpha_0 = \alpha_*(p)$. This implies that $n = N$, so p is of length n, completing the proof.

Remark 5.11. Lemma 5.10 says the maximal simplices p of $\Sigma(n)$ are all of dimension n and are the paths in the directed graph $(V(n), \rightarrow)$ within rectangles $R(k)$ with corners $(k, 0)$, (k, k), $(n, 0)$, (n, k) running from the upper left hand corner $(k, 0)$ to the lower right hand corner (n, k), where $k = \beta^*(p) = \alpha_*(p)$.

§6. Affine space, convex sets, and triangulations

Let $\mathbf{R}^n$ be n-dimensional Eulidean space. An *affine subspace* of $\mathbf{R}^n$ is a coset $U + x$ of a linear subspace U of $\mathbf{R}^n$. The *dimension* of the affine subspace $U + x$ is $\dim(U)$, with the empty set of dimension -1.

A subset C of $\mathbf{R}^n$ is *convex* if for each $x, y \in C$ and each real number t with $0 \leq t \leq 1$, $tx + (1 - t)y \in C$. The intersection of any family of convex sets is convex, so for each subset S if $\mathbf{R}^n$ there is a smallest convex subset $[S]$ of $\mathbf{R}^n$ containing S. We call $[S]$ the *convex closure* of S.

Define the *affine dimension* of a subset S of $\mathbf{R}^n$ to be the smallest dimension of an affine subspace containing S. Thus the affine dimension of S is $\dim(U(S))$, where $U(S) = \langle x - y : x, y \in S \rangle$, since $U(S) + s$ is the smallest affine subspace containing S for any $s \in S$. In particular $\dim(S) \leq |S| - 1$ and we say S is *affine independent* if $\dim(S) = |S| - 1$ achieves this bound.

The next lemma is well known and easy to prove:

(6.1) (1) *For* $S \subseteq \mathbf{R}^n$,

$$[S] = \{ \sum_{x \in X} a_x x : 0 \leq a_x \in \mathbf{R}, \sum_x a_x = 1, \text{ and } X \text{ is a finite subset of } S \}.$$

(2) *Let* $S = \{x_0, \ldots, x_k\}$ *be an affine independent subset of* $\mathbf{R}^n$. *Then each* $x \in [S]$ *can be written uniquely as* $x = \sum_i a_i x_i$ *with* $0 \leq a_i$ *and* $\sum_i a_i = 1$.

(6.2) *Let* $X, Y \subseteq \mathbf{R}^n$ *be convex,* $X = [x, X \cap Y]$, $Y = [y, X \cap Y]$, *and* $[x, y] \subseteq X \cup Y$. *Then* $X \cup Y$ *is convex.*

Proof. $X \cup Y \subseteq Z = [X, Y] = [X \cap Y, x, y]$. Let $z \in Z - [x, y]$. Then $z = ax + by + (1 - a - b)v$ for some $v \in X \cap Y$ and $0 \leq a, b \in \mathbf{R}$ with $a + b < 1$. Then $w = (ax + by)/(a + b) \in [x, y] \subseteq X \cup Y$, so we may take $w \in X$. Hence $z = (1 - a - b)v + (a + b)w \in X$, so that $Z \subseteq X \cup Y$ as desired.

(6.3) *Let* $x, y, z, w \in \mathbf{R}^n$, $0 < \varepsilon < 1$, $p = \varepsilon z + (1 - \varepsilon)y$, $q = \varepsilon x + (1 - \varepsilon)w$, $X = [p, e, f]$, *and* $Y = [q, e, f]$, *where either*
(1) $e = \varepsilon x + (1 - \varepsilon)y$ *and* $f = \varepsilon z + (1 - \varepsilon)w$, *or*
(2) $e = \varepsilon z + (1 - \varepsilon)x$ *and* $f = \varepsilon y + (1 - \varepsilon)w$.
Then $[p, q] \subseteq X \cup Y$ *and* $X \cup Y$ *is convex.*

Proof. We assume (1) holds; the proof of when (2) holds is essentially the same. Notice if $[p, q] \subseteq X \cup Y$ then $X \cup Y$ is convex by 6.4, so it remains to show $[p, q] \subseteq X \cup Y$.

Let $0 \leq t \leq 1$ and $v = tp + (1 - t)q$. Suppose first $t \geq 1/2$ and let $a = b = 1 - t$. Then $1 - a - b = 2t - 1$ and $0 \leq 2t - 1 \leq 1$ as $1/2 \leq t \leq 1$. Then by 6.1, $v = ae + bf + (1 - a - b)p \in X$. So let $t \leq 1/2$ and this time take $a = b = t$, so that $1 - a - b = 1 - 2t$ and $0 \leq 1 - 2t \leq 1$ because $0 \leq t \leq 1/2$. Now $v = ae + bf + (1 - a - b)q \in Y$.

In (2) take $a = (1 - t)\varepsilon/(1 - \varepsilon)$ and $b = 1 - t$ if $t \geq \varepsilon$, while if $t \leq \varepsilon$ take $a = t$ and $b = t(1 - \varepsilon)/\varepsilon$.

Let $K = (V, \Sigma)$ be a finite dimensional simplicial complex with vertex set V and simplices Σ. A *triangulation* of a topological space T by K is a map φ of Σ into the set of closed subspaces of T together with homeomorphisms

$$\varphi_s : \varphi(s) \to \hat{\varphi}(s) = [u(s, v) : v \in s] \subset \mathbf{R}^k$$

for each k-simplex s of K such that

(T1) For $s, t \in \Sigma$, $\varphi(s) \cap \varphi(t) = \varphi(s \cap t)$, where $\varphi(\varnothing) = \varnothing$.

(T2) $T = \bigcup_{s \in \Sigma} \varphi(s)$ and $C \subseteq T$ is closed in T if and only if $C \cap \varphi(s)$ is closed in $\varphi(s)$ for all $s \in \Sigma$.

(T3) For each k-simplex s of K and $t \subseteq s$, $\hat{\varphi}(s) = [u(s,v) : v \in s]$ is of affine dimension k and $\varphi_{t,s} = \varphi_s \circ \varphi_t^{-1}$ acts on $\hat{\varphi}(t) = [u(t,v) : v \in t]$ via $\varphi_{t,s} : \sum_{v \in t} a_v u(t,v) \mapsto \sum_{v \in t} a_v u(s,v)$.

A *morphism of topological spaces* $\varphi^i : K^i \to T^i$, $i = 1, 2$, *with triangulation* is a pair (α, β) where $\alpha : K^1 \to K^2$ is a simplicial map, $\beta : T^1 \to T^2$ is continuous, and for each $s \in \Sigma^1$,

(T_1) $\beta(\varphi^1(s)) \subseteq \varphi^2(\alpha(s))$, and

(T_2) $\alpha_s \circ \varphi^1_s = \varphi^2_{\alpha(s)} \circ \beta$, where $\alpha_s : \hat{\varphi}^1(s) \to \hat{\varphi}^2(\alpha(s))$ is defined by

$$\alpha_s : \sum_{v \in s} a_v u(s,v) \mapsto \sum_{v \in s} a_v u(\alpha(s), \alpha(v)).$$

Example 6.4. Let I be an index set and for $J \subseteq I$ let $T_J = [u_j : j \in J]$ be a convex subset of $\mathbf{R}^k$ of affine dimension $k = |J| - 1$. Let $K = (X_0, \Sigma)$ be a typed simplicial complex with type function $h : X_0 \to I$ (cf. Section 1) and let $X = \mathrm{sd}(K)$ be the barycentric subdivision of K.

We now construct a topological cell complex $\chi(K)$. We begin by defining X to be the poset of $\chi(K)$. (Notice $X \in \mathcal{P}$.) Then for $x \in X$ we form a topological cell $f(x) = (f(x), F(x))$ by letting $f(x) = X(\leq x)$ and for $u \leq v \leq x$, defining $F(v) = T_{h(v)}$, (where $h(v) = \{h(z) : z \in v\} \subseteq I$, keeping in mind that v is a simplex of K) and defining $F(u,v) : F(u) \to F(v)$ to be the inclusion map. The map ζ is defined to be the identity map on each $f(x)$, and for $v \in f(x)$, $f_v : f(x)(\leq v) \to f(\zeta(v))$ is also the identity map. It is straightforward to check that $\chi(K)$ *is* a topological cell complex. Notice that the combinatorial cell complex of the topological cell complex $\chi(K)$ is the simplicial cell complex of $\mathrm{sd}(K)$. In particular $\chi(K)$ is a regular complex.

We next extend χ to a covariant functor from the category of typed simplicial complexes over I to the category of topological cell complexes. Namely if $\alpha : K \to \bar{K}$ is a morphism of typed complexes over I, then α extends to a map $\chi(\alpha) : \mathrm{sd}(K) \to \mathrm{sd}(\bar{K})$ of posets via $\chi(\alpha)(x) = \{\alpha(v) : v \in x\}$. Next for $x \in X$, define $\chi(\alpha)_x : F(x) \to F(\chi(\alpha)(x))$ to be the identity map. This makes sense, since as α preserves the type function h, $h(x) = h(\chi(\alpha)(x))$, so $F(x) = F(\chi(\alpha)(x))$. It is easy to check that $\chi(\alpha)$ is a morphism of topological cell complexes and that $\chi(\alpha \circ \beta) = \chi(\alpha) \circ \chi(\beta)$, so that χ is indeed a covariant functor.

Form the topological space $A = A(\chi(K))$ as in Section 4. We next construct a triangulation $\varphi : K \to A$. Namely for $x \in X$ define $\varphi(x) = \tilde{F}(x) \subseteq A$. Then for $v \in x$, define $u(x,v) = u_{h(v)} \in T_{h(x)}$ and define $\varphi_x : \varphi(x) \to T_{h(x)}$ by $\varphi_x : \tilde{a} \mapsto a$. The map φ_x is just the inverse of $\lambda_x : F(x) \to \tilde{F}(x)$ defined in Section 4 by $\lambda_x(a) = \tilde{a}$. As ζ is injective, 4.6.3 says the map λ_x is a homeomorphism, so φ_x is a well defined homeomorphism.

By definition of the space A, $\varphi(x) = \tilde{F}(x)$ is a closed subspace of A for each $x \in X$ and axiom (T2) for triangulation holds. By 4.5, for $x, y \in X$, $\varphi(x) \cap \varphi(y) = \tilde{F}(x) \cap \tilde{F}(y) = \bigcup_{z \leq x,y} \tilde{F}(z) = \tilde{F}(x \cap y) = \varphi(x \cap y)$, so axiom (T1) holds. Finally if $y \leq x$ then

$$\varphi_{y,x} = \varphi_x \circ \varphi_y^{-1} = \lambda_x^{-1} \circ \lambda_y : \sum_i a_i u_i \to \lambda_x^{-1}(\widetilde{\sum_i a_i u_i}) = \sum_i a_i u_i,$$

so axiom (T3) is satisfied.

Let $T(K) = A(\chi(K))$ and $\varphi^K : K \to T(K)$ the triangulation just constructed. The space $T(K)$ is the *geometric realization* of the simplicial complex K.

To complete our discussion in this example, we extend T to a covariant functor from the category of typed simplicial complexes over I to the category of topological spaces with triangulation by essentially viewing T as the composition $T_0 = A \circ \chi$. We have just seen that χ is a functor and by 4.7, A is a functor, so $T_0 = A \circ \chi$ is a functor from typed complexes to topological spaces. Suppose $\alpha : K^1 \to K^2$ is a morphism of typed simplicial complexes over I. Define $T(\alpha) = (\alpha, T_0(\alpha))$. As T_0 is a covariant functor, $T(\alpha \circ \beta) = T(\alpha) \circ T(\beta)$, so it remains to check that $T(\alpha)$ is a morphism of triangulated spaces. We leave that as an exercise for the reader.

§7. Polyhedral cell complexes

A *polyhedral cell complex* is a $\mathcal{C}$-cell complex (X, f) where $\mathcal{C}$ is the category of triangulated topological spaces. Thus for $x \in X$ and $v \in f(x)$, $F(v)$ is a topological space together with a triangulation $B^v : f(x)(\leq v) \to F(v)$, where $f(x)(\leq v)$ is regarded as an order complex. Moreover for each simplex s of $f(x)(\leq v)$, $\hat{B}^v(s) = \hat{B}^{\zeta(v)}(f_v(s))$ and if $u \leq v$ and $s \subseteq f(x)(\leq u)$ then $\hat{B}^u(s) = \hat{B}^v(s)$ and $F(u,v) : F(u) \to F(v)$ satisfies $F(u,v)(B^u(s)) = B^v(s)$ and $B_s^v \circ F(u,v) = B_s^u$ on $B^u(s)$. Finally $f_v : f(x)(\leq v) \to f(\zeta(v))$ as an isomorphism of polyhedral cells satisfies $F_v(B^v(s)) = B^{\zeta(v)}(f_v(s))$ and $B_{f_v(s)}^{\zeta(v)} \circ F_v = B_s^v$ on $B^v(s)$.

A *morphism of polyhedral cell complexes* $\alpha : (X, f, F, B) \to (\bar{X}, \bar{f}, \bar{F}, \bar{B})$ is a morphism $\alpha : (X, f, F) \to (\bar{X}, \bar{f}, \bar{F})$ of topological cell complexes such that for each $x \in X$ and each simplex s of $f(x)$, $\alpha_x(B^x(s)) \subseteq B^{\alpha(x)}(\alpha(s))$ and $\alpha_s \circ B^x_s = B^{\alpha(x)}_{\alpha(s)} \circ \alpha_x$ on $B^x(s)$.

Example 7.1. We proceed as in Example 6.4. In particular let $I = \{0, 1, 2, \dots\}$ and for $J \subseteq I$ let $T_J = [u_j : j \in J]$ be a convex subset of $\mathbf{R}^k$ of affine dimension $k = |J| - 1$.

Let $X = (X, f)$ be a combinatorial cell complex. We associate a polyhedral cell complex $\mathcal{P}(X) = (X, f, F, B)$ to X. For $x \in X$ the polyhedral cell associated to x is obtained using the construction of Example 6.4. Namely if $h(x) = n$ then $f(x)$ is a typed complex over I with respect to the height function $h : f(x) \to I$, so we can apply the geometric realization functor T of Example 6.4 to $f(x)$ and obtain a topological space $F(x) = T(f(x))$ (the geometric realization of the order complex of $f(x)$) and a triangulation $B^x : f(x) \to F(x)$. Then for $v \in f(x)$, $F(v)$ is the subspace $\bigcup_{u \leq v} B^x(u) \cong T(f(x)(\leq v))$ and $B^v : f(x)(\leq v) \to F(v)$ is the restriction of B^x to $f(x)(\leq v)$. For $u \leq v$, $F(u, v) : F(u) \to F(v)$ is the inclusion map. Notice as B^v and B^u are restrictions of B^x, $B^v_s = B^u_s$ on $B^u(s)$ for each simplex s of $f(x)(\leq v)$.

Axioms (i) and (iii) for topological cells (given in Example (2) of Section 2) are satisfied by 4.5.2 and 4.6.4.

Next as T is a functor, the isomorphism f_v of posets induces an isomorphism $T(f_v) = (f_v, F_v)$ of spaces with triangulation from $B^v : f(x)(\leq v) \to F(v)$ to $B^{\zeta(v)} : f(\zeta(v)) \to F(\zeta(v))$. In particular $F_v(B^v(s)) = B^{\zeta(v)}(f_v(s))$ for each simplex s of $f(x)(\leq v)$ and $B^{\zeta(v)}_{f_v(s)} \circ F_v = B^v_s$ on $B^v(s)$.

We next extend $\mathcal{P}$ to a functor from combinatorial cell complexes to polyhedral cell complexes. Let $\alpha : X \to \bar{X}$ be a morphism of combinatorial cell complexes. Our morphism $\mathcal{P}(\alpha) : \mathcal{P}(X) \to \mathcal{P}(\bar{X})$ is defined so that its image under the forgetful functor is α. Further for $x \in X$, $\alpha_x : f(x) \to \bar{f}(\alpha(x))$ as a map of posets is a morphism of typed complexes, so applying our functor T we get a morphism $T(\alpha_x) = (\alpha_x, T_x)$ of spaces with triangulation from $T(f(x)) = F(x)$ to $T(f(\alpha(x))) = F(\alpha(x))$, where $T_x = A(\chi(\alpha_x))$. We let $\mathcal{P}(\alpha)_x = T(\alpha_x)$.

Check that $\mathcal{P}(\alpha)$ is a morphism of polyhedral cell complexes. Moreover as $T(\alpha \circ \beta) = T(\alpha) \circ T(\beta)$ we have $\mathcal{P}(\alpha \circ \beta) = \mathcal{P}(\alpha) \circ \mathcal{P}(\beta)$.

In the remainder of this section assume (X, f, F, B) is a polyhedral cell complex. We adopt the notational conventions of Section 4. In particular for $x \in X$ and $v \in f(x)$ we regard $F(v)$ as a subspace of

$F(x)$, so that $F(u,v)$ becomes inclusion for each $u \leq v$. Similarly for s a simplex of $f(x)(\leq v)$ we can regard $B^v(s) = B^x(s)$ and write both as $B(s)$. Already by the definition of polyhedral cell complex we have $\hat{B}^x(s) = \hat{B}^v(s)$, and as $F(v,x)$ is inclusion, $B_s^v = B_s^x \circ F(v,x) = B_s^x$, and we denote both by B_s. That is $B : f(x) \to F(x)$ is a triangulation and for each $v \in f(x)$, B restricts to a triangulation $B : f(x)(\leq v) \to F(v)$.

Let S be a simplex in $f(x)$. We have a homeomorphism $B_S : B(S) \to \hat{B}(S) = [u_j : j \in J]$, where J is the set of heights of vertices in S and by axiom (T3), $B_S(v) = u_{h(v)}$. Now if $U \subseteq S$ and $a_v \geq 0$ with $\sum_{v\in U} a_v = 1$, $\sum_v a_v u_{h(v)}$ is a well defined element of $\hat{B}(S)$ and we define $\sum_{v\in U} a_v B(v) = B_S^{-1}(\sum_v a_v u_{h(v)})$. Notice by axiom (T3) that this definition is independent of the choice of S containing U. Further

(7.2) *If $w \in f(x)$ and U is a simplex in $f(x)(\leq w)$ then $Z = f_w(U)$ is a simplex in $f(\zeta(w))$, $F_w(B(U)) = B(Z)$, $\sum_{v\in U} a_v B(v) \in B(U) \subseteq F(w)$, and*

$$F_w\big(\sum_{v\in U} a_v B(v)\big) = \sum_{z\in Z} a_z B(z).$$

Proof. Let $q = \sum_{v\in U} a_v B(v)$. As $U \subseteq f(x)(\leq w)$, $S = U \cup \{w\}$ is a simplex in $f(x)$ and by the discussion above $\sum_v a_v u_{h(v)} \in \hat{B}(w)$ so $q \in B(w) \subseteq F(w)$. As f_w is a map of posets, $Z = f_w(U)$ is a simplex in $f(\zeta(w))$. Finally by definition of polyhedral cell complex, $B_Z \circ F_w = B_U$, so $F_w(q) = F_w(B_U^{-1}(\sum_v a_v u_{h(v)})) = B_Z^{-1}(\sum_v a_v u_{h(v)}) = \sum_z a_z u_{h(z)}$.

Next we associate to each $x \in X$ a graph $\Gamma = \Gamma(x)$ called the *residual graph of X at x*. Let

$$V(x) = \{(w,v) : w \in f(x) \text{ and } v \in f(\zeta(w))\}.$$

For $(w,v) \in V(x)$ define

$$L(w,v) = \{(w',v') \in V(x) : w' \leq w,\ f_w(w') \geq v, \text{ and } f_{f_w(w')}(v) \geq v'\}$$

Finally let $\Gamma(x)$ be the graph with vertex set $V(x)$ and (w,v) adjacent to (w',v') if $(w',v') \in L(w,v)$ or $(w,v) \in L(w',v')$.

Let $K(x)$ be the clique complex of $\Gamma(x)$. We call $K(x)$ the *residual complex of X at x*. Observe

(7.3) *Let $x \in X$, $(f(x),g)$ the simplicial cell complex of the poset $f(x)$, and for $w \in f(x)$ and $u \leq w$, write (w,u) for u regarded as an* *element of $f(x)(\leq w)$ in $V(f(x), g)$. Then the map $(w, v) \mapsto (w, f_w^{-1}(v))$ is an isomorphism of $K(x)$ with $K(f(x), g)$.*

Next pick a real number $0 < \varepsilon < 1$. For $v \in f(x)$ define

$$P(v) = \varepsilon B(x) + (1 - \varepsilon)B(v) \in F(x).$$

As x is the greatest element of $f(x)$, $\{x, v\}$ is a simplex of $f(x)$ and so the notation is well defined.

Next for $(w, v) \in V(x)$ define $P(w, v) \in F(x)$ by

$$P(w, v) = F_w^{-1}(P(v)) = \varepsilon B(w) + (1 - \varepsilon)B(f_w^{-1}(v)).$$

Thus $P(w, v) \in F(w)$. Indeed

(7.4) *For $w \in f(x)$, $P(w) \in I(x)$ and for $(w, v) \in V(x)$, $P(w, v) \in I(w)$.*

Proof. Recall from Section 4 that $I(w) = F(w) - \bigcup_{u \in f(x)(<w)} F(u)$. Now B is a triangulation of $F(x)$ with $F(w)$ the union of the topological simplices $B(S)$, $S \subseteq f(x)(\leq w)$, so

$$\dot{F}(x) = F(x) - I(x) = \bigcup_{x \neq w \in f(x)} F(w) = \bigcup_{x \notin S} B(S).$$

So as $P(w) = \varepsilon B(x) + (1 - \varepsilon)B(w)$ with $0 < \varepsilon < 1$, $P(w)$ is contained only in $B(S)$ for simplices S containing x, and hence $P(w) \in I(x)$. Similarly $P(v) \in I(\zeta(w))$, so $P(w, v) = F_w^{-1}(P(v)) \in F_w^{-1}(I(\zeta(w))) = I(w)$.

(7.5) *If $s = \{(w_0, v_0), \ldots, (w_k, v_k)\}$ is a simplex in $K(x) = K(\Gamma(x))$ then*

(1) *We can order s so that $\bar{v}_0 \leq \bar{v}_1 \leq \cdots \leq \bar{v}_k \leq w_0 \leq \cdots \leq w_k$, where $\bar{v}_i = f_{w_i}^{-1}(v_i)$.*

(2) *$(w_j, v_j) \in L(w_i, v_i)$ for $j \leq i$.*

(3) *$S(s) = \{\bar{v}_i, w_i : 0 \leq i \leq k\}$ is a simplex of $f(x)$ with $P(w_i, v_i) \in B(S(s))$ for each i.*

(4) *$P(w, v) \in B(S)$ for some simplex S of $f(x)$ if and only if w and $f_w^{-1}(v) \in S$.*

Proof. Parts (1) and (2) follow from 7.3 and 5.4. Notice (1) implies $S(s)$ is a simplex of $f(x)$. Then (4) completes the proof of (3), so it remains to prove (4). But (4) holds as $P(w, v) = \varepsilon B(w) + (1 - \varepsilon)B(f_w^{-1}(v))$ and $B : f(x) \to F(x)$ is a triangulation.

Let s be a simplex in $K(x)$. By 7.5, there is a smallest simplex $S(s)$ of $F(x)$ such that $P(e) \in B(S(s))$ for each vertex $e \in s$. For S a simplex of $f(x)$ containing $S(s)$, identifying $B(S)$ with $\hat{B}(S)$ we can consider the convex closure $\varphi(s) = [P(e) : e \in s]$ of the points $P(e)$ in $B(S)$. Because B is a triangulation, these identifications are independent of S. We let $u(s,e) = P(e)$, $\hat{\varphi}(s) = \varphi(s)$, and $\varphi_s : \varphi(s) \to \hat{\varphi}(s)$ be the identity map. The remainder of this section and all of the next section are devoted to showing:

Theorem 7.6. *$\varphi : K(x) \to F(x)$ is a triangulation of $F(x)$.*

As $\varphi(s)$ is a convex subset of $B(S(s))$, $\varphi(s)$ is closed in $B(S(s))$, and then as $B(S(s))$ is closed in $F(x)$, we conclude $\varphi(s)$ is closed in $F(x)$. If $t \subseteq s$ then by 7.5, $S(t) \subseteq S(s)$, so $\varphi(t) \subseteq \varphi(s)$ and then as φ_s and φ_t are identity maps, $\varphi_{t,s} = \varphi_s \circ \varphi_t^{-1} : \varphi(t) \to \varphi(s)$ is the inclusion map. Therefore axiom (T3) in the definition of triangulation is satisfied by φ *if* $\{P(e) : e \in s\}$ is affine independent for each s. Hence

(7.7) *To establish Theorem* 7.6 *is suffices to verify:*
(1) *For $s, t \in \Sigma(x)$ the set of simplices of $K(x)$, $\varphi(s) \cap \varphi(t) = \varphi(s \cap t)$.*
(2) $F(x) = \bigcup_{s \in \Sigma(x)} \varphi(s)$.
(3) *$\{P(e) : e \in s\}$ is affine independent of order* $\dim(s) + 1$ *for each $s \in \Sigma(s)$.*
(4) *$C \subseteq F(x)$ is closed in $F(x)$ if and only if $C \cap \varphi(s)$ is closed in $\varphi(s)$ for each $s \in \Sigma(x)$.*

For 7.7.1 is axiom (T1), while 7.7.2 and 7.7.4 are axiom (T2). Finally we have seen that 7.7.3 implies axiom (T3).

So it remains to verify 7.7.1 through 7.7.4.

(7.8) (1) *We may assume $X = \{x_0 < \cdots < x_n = x\}$ is a chain and $\zeta : f(x_i) \to X(\leq x_i)$ is an isomorphism for each i.*
(2) *Under this assumption on X,* 7.7.2 *implies* 7.7.4.

Proof. Consider the polyhedral cell complex $\bar{X} = (\bar{X}, \bar{f}, \bar{F}, \bar{B})$, where $\bar{X} = f(x)$, $\bar{f}(v) = f(v)$, $\bar{F}(v) = F(v)$, $\zeta : f(v) \to f(x)$ is inclusion, $\bar{f}_v$ is the identity map, and $\bar{B}(v) = B(v)$ for each $v \in f(x)$. Notice that the combinatorial cell complex $(\bar{X}, \bar{f})$ of this polyhedral complex is just the simplicial cell complex of $\mathcal{O}(f(x))$; in particular by 7.3, there is a natural isomorphism of $K(x)$ with $K(\bar{X}, \bar{f})$. Then if Theorem 7.6 holds for $\bar{X}$ it also holds for X, so replacing X by $\bar{X}$, it suffices to take $X = f(x)$ and $\zeta : f(x) \to X$ an isomorphism.

Let S be a simplex of X, $\Sigma_S = \{s \in \Sigma(x) : B(s) \subseteq S\}$, $s \in \Sigma(x)$, and $s_S = \{(w,v) \in s : w, f_w^{-1}(v) \in S\}$. Then $\varphi(s) \cap B(S) \subseteq B(S(s)) \cap$

$B(S) = B(S(s) \cap S)$ as B is a triangulation. Also $\varphi(s) \cap B(S(s) \cap S) = \varphi(s_S)$ as the elements of $\varphi(s)$ are of the form

$$\sum_i a_i(\varepsilon B(w_i) + (1-\varepsilon)B(f_{w_i}^{-1}(v_i)))$$

while the elements $B(w_i), B(f_{w_i}^{-1}(v_i))$, are affine independent in $B(S(s))$. Therefore $\varphi(s) \cap B(S) = \varphi(s_S)$. In particular for $s, t \in \Sigma(x)$, $\varphi(s) \cap \varphi(t) = \varphi(s) \cap B(S(t)) \cap \varphi(t) \cap B(S(s)) = \varphi(s_{S(t)}) \cap \varphi(t_{S(s)}) = \varphi(s_T) \cap \varphi(t_T)$, where $T = S(s) \cap S(t)$, since $s \subseteq S(s)$ so $S_{s(t)} = s_T$ and similarly $t_{S(s)} = t_T$. Therefore if for each simplex T and each $e, f \in \Sigma_T$, we have $\varphi(e) \cap \varphi(f) = \varphi(e \cap f)$, then 7.7.1 holds. Similarly if $B(T) = \bigcup_{e \in \Sigma_T} \varphi(e)$, for each T, then 7.7.2 holds as does 7.7.4. The latter holds because B is a triangulation so C is closed in $F(x)$ if and only if $C \cap B(T)$ is closed in $B(T)$ for each T, and because 7.7.4 holds when $T = X$ and $B(T) = \bigcup_{e \in \Sigma_T} \varphi(e)$, since in that case $\Sigma(x)$ is finite. So it suffices to show for each simplex S of X that $\varphi : \Sigma_S \to B(S)$ satisfies 7.7.1 through 7.7.3. Hence replacing X by S, we may assume X is a chain. That is (1) holds, and we have already observed that (2) holds.

In the remainder of the proof we assume X is as described in Lemma 7.8. Therefore X is an n-simplex with greatest element x and ζ is injective, so by 3.1, (X, f) is the simplicial cell complex of the n- simplex. That is (X, f) is isomorphic to the complex $(X(n), f)$ of Example (2) in Section 3, so $K = K(X)$ is isomorphic to the complex $K(n) = K(X(n))$ discussed in Example 5.8 and subsequent lemmas in Section 5. Further by 7.3, the residual complex $K(x)$ is isomorphic to K. Thus without loss we may take $X = X(n)$, $V(x) = V(X) = V(n)$, etc. We adopt the notational conventions of Section 5 used to discuss $X(n)$.

Next $F(x) = [B_i : 0 \leq i \leq n]$ is the convex affine subspace of $\mathbf{R}^n$ generated by the affine independent set of vectors $B_i = B(x_i)$, $0 \leq i \leq n$. Further for $(\alpha, \beta) \in V(n)$, $P(\alpha, \beta) = \varepsilon B_\alpha + (1-\varepsilon)B_\beta$.

(7.9) *For each* $s \in \Sigma(x)$, $\{P(e) : e \in s\}$ *is affine independent of order* $\dim(s) + 1$.

Proof. Without loss s is a maximal simplex, so $s = \{s_0, \ldots, s_n\}$ is described in 5.10. Translating, we may take $B_0 = 0$, so it remains to show $Y = \{P(s_i) : 0 \leq i \leq n\}$ contains a basis for $\mathbf{R}^n$ as a linear space. As $\{B_0, \ldots, B_n\}$ is affine independent and generates $F(x)$ of affine dimension n, $\{B_1, \ldots, B_n\}$ contains a basis for $\mathbf{R}^n$ so it suffices to show $B_i \in \langle Y \rangle = U$. Assume not; then as $i = \alpha$ or β for some $(\alpha, \beta) \in s$, we can pick k minimal subject to B_{α_k} or $B_{\beta_k} \notin U$, where $s_k = (\alpha_k, \beta_k)$.

As s is a path in $(V(n), \to)$, $s_k \in l(s_{k-1})$ so without loss $\alpha_k = \alpha_{k-1}$ and $\beta_k = \beta_{k-1} + 1$. Now $P(s_k) = \varepsilon B_{\alpha_k} + (1-\varepsilon)B_{\beta_k} \in U$. By minimality of k, $B_{\alpha_{k-1}} \in U$, so as $\alpha_k = \alpha_{k-1}$, $B_{\beta_k} \in U$, contrary to the choice of k.

Remark 7.10. Observe that if (α, β) and $(\gamma, \delta) \in V(n)$ with $\alpha \geq \gamma$ then one of the following holds:

(0) $\gamma \geq \beta \geq \delta$ and $y \in x^{\perp}$.

(I) $\beta > \gamma$ and $y \notin x^{\perp}$.

(II) $\beta < \delta$ and $y \notin x^{\perp}$.

Define a subset θ of $V(n)$ to be *convex* if whenever $u = (\alpha, \beta)$ and $v = (\gamma, \delta)$ are in θ with $\alpha \geq \gamma$ and $u \notin v^{\perp}$ then

(i) (α, γ) and (β, δ) are in θ if $\beta > \gamma$, and

(ii) (γ, β) and (α, δ) are in θ if $\beta < \delta$.

Theorem 7.11. *If $\theta \subseteq V(n)$ is convex then $D(\theta) = C(\theta)$ is convex, where $D(\theta) = \bigcup_{s \subseteq \theta, s \in \Sigma} \varphi$ and $C(\theta) = [P(x) : x \in \theta]$.*

Proof. First if θ is a clique then $D(\theta) = \varphi(\theta) = C(\theta)$ by definition of $\varphi(\theta)$. So we may assume θ is not a clique. In particular as we prove Theorem 7.11 by induction on $|\theta|$, the induction is anchored.

Let $N = \min\{\alpha : (\alpha, \beta) \in \theta\}$ and $M = \max\{\beta : (\alpha, \beta) \in \theta\}$. Assume first that $N \geq M$. Then whenever $(\alpha, \beta), (\gamma, \delta) \in \theta$ with $\alpha \geq \gamma$, we have $\beta \leq M \leq N \leq \gamma$, so by Remark 7.10, either $(\gamma, \delta) \in (\alpha, \beta)^{\perp}$ or $\beta < \delta$, $\gamma < \alpha$, and as θ is convex, (γ, β) and (α, δ) are in θ.

Let $\theta^* = \{(\alpha, \beta) \in \theta : \theta \nsubseteq (\alpha, \beta)^{\perp}\}$. As θ is not a clique, $\theta^* \neq \varnothing$. Define

$$\beta_0 = \min\{\beta : (\alpha, \beta) \in \theta^*\},$$
$$\alpha_0 = \max\{\alpha : (\alpha, \beta_0) \in \theta^*\},$$
$$\alpha_1 = \min\{\alpha : (\alpha, \beta) \in \theta^*\},$$
$$\beta_1 = \max\{\beta : (\alpha_1, \beta)\theta^*\},$$

$v_i = (\alpha_i, \beta_i)$, and $\theta_i = \theta - \{v_i\}$.

By definition of v_0, there exists $(\gamma, \delta) \in \theta - v_0^{\perp}$, so by definition of β_0, we have $\delta \geq \beta_0$. Thus by an earlier remark, $\beta_0 < \delta$ and $\gamma < \alpha_0$. Then by definition of v_1, $\alpha_1 \leq \gamma < \alpha_0$. By symmetry, $\beta_0 < \beta_1$.

Claim θ_0 is convex. For if $(\alpha, \beta), (\gamma, \delta) \in \theta$ with $\alpha \geq \gamma$ and $(\gamma, \delta) \notin (\alpha, \beta)^{\perp}$, then by an earlier remark, $\beta < \delta$, $\gamma < \alpha$, and $(\gamma, \beta), (\alpha, \delta) \in \theta$. As $\beta_0 \leq \beta < \delta$, $(\alpha, \delta) \neq v_0$, while if $\beta = \beta_0$ then $\alpha_0 \geq \alpha > \gamma$, so $(\gamma, \beta) \neq v_0$. Hence $(\gamma, \beta), (\alpha, \delta) \in \theta_0$ and θ_0 is indeed convex. Similarly θ_1 is convex.

Next let $X = C(\theta_0)$, $Y = C(\theta_1)$, $q = P(v_0)$, and $p = P(v_1)$. Then $X = [p, C(\theta_0 \cap \theta_1)]$ and $C(\theta_0 \cap \theta_1) \subseteq X \cap Y$, so $X = [p, X \cap Y]$ and similarly $Y = [q, X \cap Y]$. By induction on the order of θ, $X = D(\theta_0)$ and $Y = D(\theta_1)$.

Also $p = P(v_1) = \varepsilon B_{\alpha_1} + (1-\varepsilon)B_{\beta_1}$ and $q = P(v_0) = \varepsilon B_{\alpha_0} + (1-\varepsilon)B_{\beta_0}$. Further $\alpha_0 > \alpha_1$ and $\beta_1 > \beta_0$, so $v_0 \notin v_1^\perp$ and hence by an earlier remark, $(\alpha_0, \beta_1), (\alpha_1, \beta_0) \in \theta$, and indeed as neither is v_0 nor v_1, each is even in $\theta_0 \cap \theta_1$. Therefore

$$e = P(\alpha_0, \beta_1) = \varepsilon B_{\alpha_0} + (1-\varepsilon)B_{\beta_1} \in X \cap Y$$

and

$$f = \varepsilon B_{\alpha_1} + (1-\varepsilon)B_{\beta_0} \in X \cap Y.$$

Thus $X \cup Y$ is convex by 6.3.1 and 6.2. Finally observe that $X \cup Y = D(\theta)$, so that Theorem 7.11 holds in this case. For as $v_0 \notin v_1^\perp$, each $s \in \Sigma$ with $s \subseteq \theta$ is contained in θ_0 or θ_1.

This leaves the case $N < M$. This time let $\alpha_0 = N$, $\beta_0 = \max\{\beta : (\alpha_0, \beta) \in \theta\}$, $\beta_1 = M$, $\alpha_1 = \min\{\alpha : (\alpha, \beta_1) \in \theta\}$, $v_i = (\alpha_i, \beta_i)$, and $\theta_i = \theta - \{v_i\}$. This time $\beta_0 \leq \alpha_0 = N < M = \beta_1 \leq \alpha_1$, so $v_0 \notin v_1^\perp$. Again θ_i is convex. For if $(\alpha, \beta), (\gamma, \delta) \in \theta$ with $\alpha \geq \gamma$ and $(\gamma, \delta) \notin (\alpha, \beta)^\perp$ then by Remark 7.10, either (i) $\beta > \gamma$ and $(\alpha, \gamma), (\beta, \delta) \in \theta$, or (ii) $\beta < \delta$ and $(\gamma, \beta), (\alpha, \delta) \in \theta$. In case (i), $\alpha \geq \beta > \gamma \geq N = \alpha_0$, so $v_0 \neq (\alpha, \gamma)$ or (β, δ). Similarly $\beta_1 = M \geq \beta > \gamma \geq \delta$, so $v_1 \neq (\alpha, \gamma)$ or (β, δ). Thus $(\alpha, \gamma), (\beta, \delta) \in \theta_0 \cap \theta_1$ in case (i). On the other hand in case (ii), $\beta < \delta$, so $\alpha > \gamma \geq \alpha_0$ and if $\gamma = \alpha_0$ then $\beta_0 \geq \delta > \beta$, so $v_0 \neq (\gamma, \beta)$ or (α, δ). Also $\beta_1 \geq \delta > \beta$ and if $\beta_1 = \delta$ then $\alpha_1 \leq \gamma < \alpha$, so $v_1 \neq (\gamma, \beta)$ or (α, δ). So again $(\alpha, \gamma), (\beta, \delta) \in \theta_0 \cap \theta_1$.

So θ_0 and θ_1 are convex. Again let $X = C(\theta_0)$, $Y = C(\theta_1)$, $q = P(v_0)$, and $p = P(v_1)$. As before, $X = [p, X \cap Y]$ and $Y = [q, X \cap Y]$, and by induction, $X = D(\theta_0)$ and $Y = D(\theta_1)$. As $v_0 \notin v_1^\perp$, as $\alpha_1 > \alpha_0$, and as $\beta_1 = M < N = \alpha_0$, it follows from Remark 7.10 and the convexity of θ that $(\alpha_1, \alpha_0), (\beta_1, \beta_0) \in \theta$, and then even are in $\theta_0 \cap \theta_1$. So

$$e = P(\alpha_1, \alpha_0) = \varepsilon B_{\alpha_1} + (1-\varepsilon)B_{\alpha_0}$$

and

$$f = P(\beta_1, \beta_0) = \varepsilon B_{\beta_1} + (1-\varepsilon)B_{\beta_0}$$

are in $D(\theta_0 \cap \theta_1) \subseteq X \cap Y$. Hence by 6.3.2 and 6.2, $X \cup Y$ is convex. Finally as above, $X \cup Y = D(\theta)$, completing the proof.

Corollary 7.12. $A = \bigcup_{s \in \Sigma} \varphi(s)$.

Proof. Notice $V(n)$ is convex with $\Sigma = \{s \in \Sigma : s \subseteq V(n)\}$, so $D(V(n)) = \bigcup_{s \in \Sigma} \theta(s)$ is convex by Thoerem 7.11. But $B_\alpha = P(\alpha, \alpha) \in C(V(n)) = D(V(n))$ for each α, $0 \leq \alpha \leq n$. Therefore $A = [B_\alpha : 0 \leq \alpha \leq n] \subseteq D(V(n))$, completing the proof.

§8. The proof of Theorem 7.6 is completed

In this section we complete the proof of Theorem 7.6. By 7.8 and the discussion following the proof of that lemma, we have reduced to the case where $X = X(n)$ is an n-simplex with maximal member $x = n$, $A = F(x)$, $\Sigma = \Sigma(x)$, and $K = K(X) = K(x) = K(n)$. By 7.7, 7.8, 7.9, and 7.12, it suffices to prove:

(8.1) *For $s, t \in \Sigma$, $\varphi(s) \cap \varphi(t) = \varphi(s \cap t)$.*

Recall we have

$$V(n) = \{(\alpha, \beta) : 0 \leq \beta \leq \alpha \leq n\}$$

with $(\alpha', \beta') \in L(\alpha, \beta)$ if and only if $\beta' \leq \beta \leq \alpha' \leq \alpha$.

Let $y \in A$. Recall from Section 7 that $A = [B_i : 0 \leq i \leq n]$ is the polytope in $\mathbf{R}^n$ generated by the affine independent set of vectors B_i and for $(\alpha, \beta) \in V(n)$, $P(\alpha, \beta) = \varepsilon B_\alpha + (1 - \varepsilon) B_\beta$. In particular $y = \sum_{i=0}^{n} a_i B_i$ with $a_i \geq 0$ and $\sum_i a_i = 1$, and that this expression is unique as the B_i are affine independent. Define

$$\sup(y) = \{i \in X : a_i \neq 0\}$$

and for $s \in \Sigma$ let

$$\sup(s) = \{\alpha, \beta : (\alpha, \beta) \in s\}$$

(8.2) *If $s, t \in \Sigma$ with $\sup(s) \cup \sup(t) \neq X$ then $\varphi(s) \cap \varphi(t) = \varphi(s \cap t)$.*

Proof. Let $m \notin \sup(s) \cup \sup(t)$. Replace X by $\bar{X} = X - \{m\} \cong X(n-1)$, $V(n)$ by

$$\bar{V}(n) = \{(\alpha, \beta) \in V(n) : \alpha, \beta \in \bar{X}\} \cong V(n-1),$$

and A by $\bar{A} = [B_i : i \in \bar{X}]$. Then by induction on n, $\bar{\varphi} : K(\bar{V}(n)) \to \bar{A}$ is a triangulation, where $\bar{\varphi}$ is the restriction of φ to $K(\bar{V}(n))$. As $\sup(s) \cup \sup(t) \subseteq \bar{X}$, s and t are simplices of $K(\bar{V}(n))$, so $\varphi(s) \cap \varphi(t) = \bar{\varphi}(s) \cap \bar{\varphi}(t) = \bar{\varphi}(s \cap t) = \varphi(s \cap t)$.

From now on pick $s, t \in \Sigma$ such that $\varphi(s) \cap \varphi(t) \neq \varphi(s \cap t)$. If $r \subseteq s$ then $\varphi(r) = [P(v) : v \in r] \subseteq [P(v) : v \in s] = \varphi(s)$, so $\varphi(s \cap t) \subseteq \varphi(s) \cap \varphi(t)$. Thus we can pick $y \in \varphi(s) \cap \varphi(t) - \varphi(s \cap t)$. By a remark above there is a unique expression, $y = \sum_i a_i B_i$ with $0 \leq a_i \in \mathbf{R}$ and $\sum_i a_i = 1$. Also by 7.9, $\{P(v) : v \in s\}$ is affine independent, so there is a unique expression $y = \sum_{v \in s} b_v P(v)$ and similarly a unique expression $y = \sum_{v \in t} c_v P(v)$. For $r \in \Sigma$ and $\gamma \in X$, let

$$I_r(\gamma) = \{\alpha : (\alpha, \gamma) \in r\} \text{ and } J_r(\gamma) = \{\beta : (\gamma, \beta) \in r\}$$

(8.3) $\sup(y) = \sup(s) = \sup(t) = X$.

Proof. If $\sup(y) = X$ then as $\sup(y) \subseteq \sup(s) \cap \sup(t)$, the lemma holds. So it remains to prove $\sup(y) = X$. So assume not. Let $X_s = \{\alpha, \beta : b_{(\alpha,\beta)} \neq 0\}$ and $s_0 = \{v \in s : b_v \neq 0\}$. Define X_t and t_0 similarly. Then $X \neq \sup(y) = X_s = X_t$, $X_s = \sup(s_0)$, and $X_t = \sup(t_0)$. By 8.2, $y \in \varphi(s_0) \cap \varphi(t_0) = \varphi(s_0 \cap t_0) \subseteq \varphi(s \cap t)$, a contradiction.

(8.4) (1) *For* $\alpha > \beta^*(s)$, $a_\alpha/\varepsilon = \sum_{\beta \in J_s(\alpha)} b_{\alpha,\beta}$.
(2) *For* $\beta < \alpha_*(s)$, $a_\beta/(1-\varepsilon) = \sum_{\alpha \in I_s(\beta)} b_{\alpha,\beta}$.
(3) *If* $\kappa = \alpha_*(s) = \beta^*(s)$ *then*

$$a_\kappa = \varepsilon\Big(\sum_{\beta \in J_s(\kappa)} b_{\kappa,\beta}\Big) + (1-\varepsilon) \sum_{\alpha \in I_s(\kappa)} b_{\alpha,\kappa}.$$

Proof. This follows as $P(\alpha,\beta) = \varepsilon B_\alpha + (1-\varepsilon)B_\beta$ and $\beta^*(s) \leq \alpha_*(s)$.

Recall from Section 5 that we can think of the members of $V(n)$ as the lower diagonal elements in an $n+1$ by $n+1$ square array. Notice we have a duality on statements concerning $V(n)$ and K corresponding to the involution on X interchanging i and $n-i$ for each $i \in X$, and this corresponds to reflecting $V(n)$ about the "diagonal" $\{(n,0),(n-1,1),\ldots,(0,n)\}$. In applying this duality, one must also interchange the roles of ε and $1-\varepsilon$. We use this duality frequently from now on. In particular for $k \in X$ define

$$R^*(k) = \Big\{z = \sum_i z_i B_i \in A : \sum_{i \geq k} z_i \geq \varepsilon\Big\}$$

and define $R_*(k)$ dually. That is

$$R_*(k) = \Big\{z = \sum_i z_i B_i : \sum_{i \leq k} z_i \geq 1-\varepsilon\Big\}$$

Define $\alpha_*(s,y)$ to be the minimum α such that $b_{\alpha,\beta} \neq 0$ for some $(\alpha,\beta) \in s$ and define $\beta^*(s,y)$ dually.

(8.5) *Let* $k \in X$. *Then*
(1) $y \in R^*(k)$ *if and only if* $k \leq \alpha_*(s,y)$,
(2) $y \in R_*(k)$ *if and only if* $k \geq \beta^*(s,y)$.

Proof. We prove (1); then (2) follows by duality. Suppose first that $k \leq \alpha_*(s,y)$. Then by 8.3, $\sum_{j \geq k} a_j = \varepsilon(\sum_{v \in s} b_v) + (1-\varepsilon)\sum_{\alpha \in I_s(k)} b_{\alpha,k}$

$= \varepsilon + (1-\varepsilon)b$, where $b \geq 0$, since $\sum_{v\in s} b_v = 1$ and $b_v \geq 0$. Thus if $k \leq \alpha_*(s,y)$, $y \in R^*(k)$.

On the other hand if $k > \beta^*(s,y)$ then by 8.4, $\sum_{j\geq k} a_j = \varepsilon(\sum_{\alpha(v)\geq k} b_v) \leq \varepsilon$ with equality if and only if $k \leq \alpha_*(s,y)$, where $v = (\alpha(v), \beta(v))$. So as $\alpha_*(s,y) \geq \beta^*(s,y)$, we conclude that if $y \in R^*(k)$ then $k \geq \alpha_*(s,y)$.

(8.6) (1) *$\varphi(s) \cap R^*(k) = \{z \in \varphi(s) : \alpha_*(s,z) \geq k\} = \varphi(s^*(k))$, where $s^*(k) = \{(\alpha,\beta) \in s : \alpha \geq k\}$.*

(2) *$\varphi(s) \cap R_*(k) = \{z \in \varphi(s) : \beta^*(s,z) \leq k\} = \varphi(s_*(k))$, where $s_*(k) = \{(\alpha,\beta) \in s : \beta \leq k\}$.*

Proof. As usual (2) is the dual of (1) so it suffices to prove (1). But (1) follows from 8.5.1.

(8.7) $\alpha_*(s) \leq \beta^*(t) + 1$.

Proof. Let $\alpha_*(s) = k$. By 8.6, $\varphi(s) \subseteq R^*(k)$ and $\varphi(t) \cap R^*(k) = \varphi(t^*(k))$. Similarly setting $j = \beta^*(t)$, we have $\varphi(t) \subseteq R_*(j)$ and $\varphi(s) \cap R_*(j) = \varphi(s_*(j))$. Therefore

$$\varphi(s) \cap \varphi(t) = \varphi(s) \cap R_*(j) \cap \varphi(t) \cap R^*(k) = \varphi(s_*(j)) \cap \varphi(t^*(k))$$

Assume $j < k-1$. Then $k-1 \notin \sup(t^*(k))$ as $\alpha_*(t^*(k)) \geq k > k-1$ and $\beta^*(t^*(k)) \leq j < k-1$. Therefore by 8.3, $\varphi(s) \cap \varphi(t) = \varphi(s_*(j)) \cap \varphi(t^*(k)) = \varphi(s_*(j) \cap t^*(k)) \subseteq \varphi(s \cap t)$, a contradiction.

(8.8) *Suppose $v \in s \cap t$. Then*
(1) *If $\sup(s - \{v\}) \neq X$ then $b_v \neq c_v$.*
(2) *If $J_s(\alpha) = \{\beta\}$ then $J_t(\alpha) \neq \{\beta\}$.*
(3) *If $I_s(\beta) = \{\alpha\}$ then $I_t(\beta) \neq \{\alpha\}$.*

Proof. Assume $\sup(s - \{v\}) \neq X$ but $b_v = c_v = b$ and let $y' = (y - bP(v))/(1-b)$. As $P(v) \in \varphi(s \cap t)$, $y' \in \varphi(s) \cap \varphi(t) - \varphi(s \cap t)$. Then as $\sup(y') \subseteq \sup(s - \{v\}) \neq X$, 8.3 supplies a contradiction.

Thus (1) is established. Notice (3) is the dual of (2) so it remains to prove (2). Assume $J_s(\alpha) = \{\beta\}$. Then by 8.4, $a_\alpha/\varepsilon = b_v$ and $\alpha \notin \sup(s - \{v\})$. Thus by (1), $b_v \neq c_v$, so by symmetry, $J_t(\alpha) \neq \{\beta\}$.

We next observe that we can choose s, t to be maximal simplices of dimension n. For $s \subseteq s_0$ and $t \subseteq t_0$ where s_0 and t_0 are maximal simplices and hence of dimension n by Remark 5.11. Notice if $r \subseteq s_0$ then by definition of φ, $\varphi(r) \cap \varphi(s) = \varphi(r \cap s)$. Hence if $\varphi(s_0) \cap \varphi(t_0) = \varphi(s_0 \cap t_0)$ then $\varphi(s) \cap \varphi(t) = \varphi(s) \cap \varphi(t) \cap \varphi(s_0 \cap t_0) = \varphi(s) \cap \varphi(s_0 \cap t) = \varphi(s \cap t)$. Therefore replacing s, t by s_0, t_0 if necessary, we may assume s

and t are of dimension n. Hence s and t are described in 5.10 and 5.11. In particular $\alpha_*(s) = \beta^*(s)$ and $\alpha_*(t) = \beta^*(t)$. Let $\kappa = \alpha_*(s)$.

(8.9) $\alpha_*(t) = \alpha_*(s) = \kappa$.

Proof. Without loss $\kappa > \alpha_*(t)$. Then as $\alpha_*(t) = \beta^*(t)$, $\beta^*(t) = \kappa - 1$ by 8.7.

Now $\alpha_*(s,y) \geq \alpha_*(s) = \kappa$, so by 8.5, $y \in R^*(\kappa)$ and then by another application of 8.5, $\alpha_*(t,y) \geq \kappa$. Then as $X = \sup(y)$, $v = (0,\kappa)$ and $u = (n, \kappa - 1)$ are in t. By duality, $\beta^*(s,y) \leq \kappa - 1$ and $u, v \in s$.

Next by 8.4, either $I_s(0) = \{\kappa\}$ and $b_v = a_0/(1-\varepsilon)$ or $J_s(\kappa) = \{0\}$ and $b_v = a_\kappa/\varepsilon$. Similarly either $I_t(0) = \{\kappa - 1, \kappa\}$ and $c_v = a_0/(1-\varepsilon)$ or $J_t(\kappa) = \{0\}$ and $c_v = a_\kappa/\varepsilon$. Then by 8.6, we may asssume $I_s(0) = \{\kappa\}$, $b_v = a_0/(1-\varepsilon)$, $J_t(\kappa) = \{0\}$ and $c_v = a_\kappa/\varepsilon$. Now by 8.4, $a_0/(1-\varepsilon) = c_v + C$ and $a_\kappa/\varepsilon = b_v + B$, where $C = \sum_{\alpha \in I_t(0)} c_{\alpha,0} - c_v$ and $B = \sum_{\beta \in J_s(\kappa)} b_{\kappa,\beta} - b_v$. But then

$$b_v = a_0/(1-\varepsilon) = c_v + C = a_\kappa/\varepsilon + C = b_v + B + C.$$

We conclude $B = C = 0$ and hence $b_v = c_v$. Now 8.6 supplies a contradiction.

Let S be the set of maximal simplices m of K with $\alpha_*(m) = \kappa$. Thus $s, t \in S$. We partition S into four classes S_i, $1 \leq i \leq 4$, where

$$\begin{aligned}
S_1 &= \{m \in S : J_m(\kappa) = \{0\} \text{ and } I_m(\kappa) = \{n\}\},\\
S_2 &= \{m \in S : J_m(\kappa) = \{0\} \text{ and } J_m(n) = \{\kappa\}\},\\
S_3 &= \{m \in S : I_m(0) = \{\kappa\} \text{ and } I_m(\kappa) = \{n\}\},\\
S_4 &= \{m \in S : I_m(0) = \{\kappa\} \text{ and } J_m(n) = \{\kappa\}\}.
\end{aligned}$$

Notice that the duality map interchanges S_2 and S_3 and fixes S_1 and S_4. So anything we prove about S_2 establishes the dual statement for S_3 at the same time.

(8.10) *If* $s \in S_3 \cup S_4$ *then* $t \in S_1 \cup S_2$.

Proof. This follows from 8.8.3.

(8.11) *Let* $s \in S_3 \cup S_4$ *and define*

$$b = \sum_{\alpha \in I_s(\kappa)} b_{\alpha,\kappa}, \quad c = \sum_{\alpha \in I_t(\kappa)} c_{\alpha,\kappa}$$

$$\bar{b} = \sum_{0 < \beta \in J_s(\kappa)} b_{\kappa,\beta}, \quad c^* = \sum_{\kappa < \alpha \in I_t(0)} c_{\alpha,0}$$

Then

(1) $b < c$ *and* $\bar{b} + c^* \neq 0$,

(2) $s \in S_3$.

Proof. By 8.10, $t \in S_1 \cup S_2$. Thus $J_t(\kappa) = \{0\}$, so $c_{\kappa,0} = a_\kappa/\varepsilon - (1-\varepsilon)c/\varepsilon$ by 8.4.3. Similarly as $s \in S_3 \cup S_4$, $I_s(0) = \{\kappa\}$, so $b^* = \sum_{\kappa<\alpha\in I_s(0)} b_{\alpha,0} = 0$. Now by 8.4.2,

$$(*) \qquad a_0/(1-\varepsilon) = b_{\kappa,0} + b^* = c_{\kappa,0} + c^*$$

so $b_{\kappa,0} = c_{\kappa,0} + c^* = a_\kappa/\varepsilon - (1-\varepsilon)c/\varepsilon + c^*$. Next by 8.4.3, $a_\kappa/\varepsilon = b_{\kappa,0} + \bar{b} + (1-\varepsilon)b/\varepsilon$, so

$$b_{\kappa,0} = a_\kappa/\varepsilon - (1-\varepsilon)c/\varepsilon + c^* = b_{\kappa,0} + \bar{b} + (1-\varepsilon)(b-c)/\varepsilon + c^*$$

so that

$$(**) \qquad 0 = \bar{b} + c^* + (1-\varepsilon)(b-c)/\varepsilon.$$

Therefore to prove (1) it suffices to show $b < c$. Assume otherwise. Then $\bar{b}, c^*, (b-c) \geq 0$, so by (**), $\bar{b} = c^* = b - c = 0$. But then $b^* = c^* = 0$, so by (*), $b_{\kappa,0} = c_{\kappa,0}$ and applying 8.8.1 to $v = (\kappa, 0)$, we have a contradiction.

Therefore (1) is established and it remains to prove (2), so we may take $s \in S_4$. Therefore $J_s(n) = \{\kappa\}$, so by 8.4.1, $a_n/\varepsilon = b_{n,\kappa}$. Hence $b = b_{n,\kappa} + \tilde{b} = a_n/\varepsilon + \tilde{b}$, where $\tilde{b} = \sum_{n>\alpha\in I_s(\kappa)} b_{\alpha,\kappa}$. Again by 8.4.1, $\hat{c} + c_{n,\kappa} = a_n/\varepsilon$, where $\hat{c} = \sum_{n>\beta\in J_t(n)} c_{n,\beta}$. Therefore

$$b = a_n/\varepsilon + \tilde{b} = c_{n,\kappa} + \tilde{b} + \hat{c}.$$

Finally as $s \in S_4$, $t \notin S_2$ by the dual of 8.10, so $t \in S_1$. Thus $I_t(\kappa) = \{n\}$, so $c = c_{n,\kappa}$. Therefore $b = c_{n,\kappa} + \tilde{b} + \hat{c} = c + \tilde{b} + \hat{c} \geq c$, as $\tilde{b}, \hat{c} \geq 0$. This contradicts (1).

(8.12) *Up to a permutation of* $\{s, t\}$ *and duality,* $(s,t) \in S_1 \times S_1$, $S_3 \times S_1$, *or* $S_3 \times S_2$.

Proof. This follows from 8.10 and 8.11.

Define $v = (\alpha, \beta) \in s$ to be an *inflection point* of s if $v = (\kappa, 0)$ or (n, κ), or $|J_s(\alpha)| > 1 < |I_s(\beta)|$. By 5.10, $s = \{v_0, \ldots, v_n\}$ is a directed path in $(V(n), \to)$ with $v_0 = (\kappa, 0)$ and $v_n = (n, \kappa)$. Let $v_{i_0}, \ldots, v_{i_l}$ be the inflection points with $i_j < i_{j+1}$. Then $v_{i_0} = (\alpha_0, \beta_0) = (\kappa, 0)$ and $v_{i_l} = (n, \kappa)$. If $s \in S_1 \cup S_2$ then $J_s(\kappa) = \{0\}$, so $v_{i_1} = (\alpha_1, \beta_0)$

with $\kappa = \alpha_0 < \alpha_1$. Then proceeding recursively, $v_{i_{2r}} = (\alpha_r, \beta_r)$ and $v_{i_{2r+1}} = (\alpha_{r+1}, \beta_r)$ with $\alpha_i < \alpha_{i+1}$ and $\beta_i < \beta_{i+1}$. Further if $s \in S_1$ then $I_s(\kappa) = \{n\}$, so $l = 2N$ is even and $(n, \kappa) = (\alpha_N, \beta_N)$, while if $s \in S_2$ then $J_s(n) = \{\kappa\}$ so $l = 2N + 1$ is odd and $(n, \kappa) = (\alpha_{N+1}, \beta_N)$. Then dualizing the case $s \in S_2$ to get the answer when $s \in S_3$, we conclude:

(8.13) *The inflection points for s are:*
(1) (α_i, β_i), (α_{i+1}, β_i), $0 \le i < N$, *and* $(\alpha_N, \beta_N) = (n, \kappa)$, *if* $s \in S_1$.
(2) (α_i, β_i), (α_{i+1}, β_i), $0 \le i \le N$, *if* $s \in S_2$.
(3) (α_i, β_i), (α_i, β_{i+1}), $0 \le i \le N$, *if* $s \in S_3$.

(8.14) *For $s \in S_1 \cup S_2$ we have:*
(1) $b_{\alpha,\beta_i} = a_\alpha/\varepsilon$ *for* $\alpha_i < \alpha < \alpha_{i+1}$.
(2) $b_{\alpha_i,\beta} = a_\beta/(1-\varepsilon)$, *for* $\beta_{i-1} < \beta < \beta_i$.
(3) $b_{\alpha_i,\beta_i} = b_{\kappa,0} + 1/\varepsilon \sum_{\kappa<\alpha\le\alpha_i} a_\alpha - 1/(1-\varepsilon) \sum_{0\le\beta<\beta_i} a_\beta$

$$= 1 - 1/\varepsilon \textstyle\sum_{\kappa<\alpha\le n} a_\alpha - 1/(1-\varepsilon) \sum_{0\le\beta<\beta_i} a_\beta$$

(4) $b_{\alpha_i,\beta_{i-1}} = 1/(1-\varepsilon) \sum_{0\le\beta\le\beta_{i-1}} a_\beta - 1/\varepsilon \sum_{\kappa<\alpha<\alpha_i} a_\alpha - b_{\kappa,0}$

$$= 1/(1-\varepsilon) \textstyle\sum_{0\le\beta\le\beta_{i-1}} a_\beta + 1/\varepsilon \sum_{\alpha_i\le\alpha\le n} a_\alpha - 1$$

unless $s \in S_2$ *and* $i = N + 1$.
(5) $b_{\kappa,0} = 1 - 1/\varepsilon \sum_{\kappa<\alpha\le n} a_\alpha$.
(6) *If* $s \in S_2$ *then* $b_{n,\kappa} = a_n/\varepsilon$.

Proof. If $\alpha_j < \alpha < \alpha_{j+1}$ then $J_s(\alpha) = \{\beta_j\}$ because (α, β_j) is not an inflection point since $v_{i_{2j+1}} = (\alpha_{j+1}, \beta_j)$ is the next inflection point after $v_{i_{2j}} = (\alpha_j, \beta_j)$. Therefore (1) holds by 8.4.1. Similarly (2) holds. We prove the first equality in (3) and (4) by induction on i. To anchor the induction, recall $(\alpha_0, \beta_0) = (\kappa, 0)$, so (3) holds when $i = 0$. Then for $i > 0$, 8.4.1 says

$$b_{\alpha_i,\beta_i} = a_{\alpha_i}/\varepsilon - \sum_{\beta_i\ne\beta\in J_s(\alpha_i)} b_{\alpha_i,\beta} = a_{\alpha_i}/\varepsilon - \sum_{\beta_{i-1}\le\beta<\beta_i} b_{\alpha_i,\beta}$$

which by (2) and induction on i is equal to

$$a_{\alpha_i}/\varepsilon - \sum_{\beta_{i-1}<\beta<\beta_i} a_\beta/(1-\varepsilon) - 1/(1-\varepsilon) \sum_{0\le\beta\le\beta_{i-1}} a_\beta + 1/\varepsilon \sum_{\kappa<\alpha<\alpha_i} a_\alpha + b_{\kappa,0}$$

$$= b_{\kappa,0} + 1/\varepsilon \sum_{\kappa<\alpha\le\alpha_i} a_\alpha - 1/(1-\varepsilon) \sum_{0\le\beta<\beta_i} a_\beta$$

as claimed. A similar argument establishes (4), except when $s \in S_2$ and $i = N+1$, when $\beta_{i-1} = \kappa$ so that 8.4.3 must be used rather than 8.4.2, which is appropriate for smaller. In this case as $s \in S_2$, $J_s(n) = \{\kappa\}$ so $b_{n,\kappa} = a_n/\varepsilon$ by 8.4.1. But also by 8.4.3, a_κ is equal to

$$\begin{aligned}\varepsilon b_{\kappa,0} + (1-\varepsilon) \sum_{\alpha_{N-1} \le \alpha \le n} b_{\alpha,\kappa} =& \varepsilon b_{\kappa,0} + (1-\varepsilon)/\varepsilon \sum_{\alpha_{N-1} < \alpha < n} a_\alpha \\ &+ (1-\varepsilon)(b_{\alpha_{N-1},\kappa} + b_{n,\kappa})\end{aligned}$$

by (1), and then by induction on i, $b_{n,\kappa}$ is equal to

$$\begin{aligned}&a_\kappa/(1-\varepsilon) - \varepsilon b_{\kappa,0}/(1-\varepsilon) - 1/\varepsilon \sum_{\alpha_{N-1}<\alpha<n} a_\alpha - b_{\kappa,0}/(1-\varepsilon) \\ &- 1/\varepsilon \sum_{K<\alpha\le\alpha_{N-1}} a_\alpha + 1/(1-\varepsilon) \sum_{0\le\beta<\kappa} a_\beta \\ =&1/(1-\varepsilon) \sum_{0\le\beta\le\kappa} a_\beta - 1/\varepsilon \sum_{\kappa<\alpha<n} a_\alpha - b_{\kappa,0}/(1-\varepsilon).\end{aligned}$$

Then as $b_{n,\kappa} = a_n/\varepsilon$, we have

$$b_{\kappa,0} = \sum_{0\le\beta\le\kappa} a_\beta - (1-\varepsilon)/\varepsilon \sum_{\kappa<\alpha\le n} a_\alpha = 1 - 1/\varepsilon \sum_{\kappa<\alpha\le n} a_\alpha$$

as $\sum_{i=0}^n a_i = 1$. This gives (5) and (6) when $s \in S_2$.

Similarly when $s \in S_1$ we conclude from the first equality in (3) that

$$b_{n,\kappa} = b_{\kappa,0} + 1/\varepsilon \sum_{\kappa<\alpha\le n} a_\alpha - 1/(1-\varepsilon) \sum_{0\le\beta<\kappa} a_\beta$$

But also by 8.4.3,

$$\begin{aligned}a_\kappa &= \varepsilon b_{\kappa,0} + (1-\varepsilon) b_{n,\kappa} \\ &= \varepsilon b_{\kappa,0} + (1-\varepsilon) b_{\kappa,0} + (1-\varepsilon)/\varepsilon \sum_{\kappa<\alpha\le n} a_\alpha - \sum_{0\le\beta<\kappa} a_\beta \\ &= b_{\kappa,0} + 1/\varepsilon \sum_{\kappa<\alpha\le n} a_\alpha - \sum_{i\ne\kappa} a_i = b_{\kappa,0} + 1/\varepsilon \sum_{\kappa<\alpha\le n} a_\alpha - (1-a_\kappa)\end{aligned}$$

as $\sum_i a_i = 1$. Therefore $b_{\kappa,0} = 1 - 1/\varepsilon \sum_{\kappa<\alpha\le n} a_\alpha$, so that (5) holds in this case too. Finally substituting (5) into the first inequality in (3) and (4) gives the second inequality.

The dual of 8.14 in the case $s \in S_2$ is:

(8.15) *For $s \in S_3$ we have:*
(1) $b_{\alpha,\beta_i} = a_\alpha/\varepsilon$ *for* $\alpha_{i-1} < \alpha < \alpha_i$.
(2) $b_{\alpha_i,\beta} = a_\beta/(1-\varepsilon)$, *for* $\beta_i < \beta < \beta_{i+1}$.
(3) $b_{\alpha_i,\beta_{i+1}} = 1 - 1/(1-\varepsilon)\sum_{0<\beta<\beta_{i+1}} a_\beta - 1/\varepsilon \sum_{\alpha_i<\alpha\leq n} a_\alpha$.
(4) $b_{\alpha_i,\beta_i} = 1/\varepsilon \sum_{\alpha_i\leq\alpha\leq n} a_\alpha + 1/(1-\varepsilon)\sum_{0\leq\beta\leq\beta_i} a_\beta - 1$ *unless* $i = 0$.
(5) $b_{n,\kappa} = 1 - 1/(1-\varepsilon)\sum_{0\leq\beta<\kappa} a_\beta$.
(6) $b_{\kappa,0} = a_0/(1-\varepsilon)$.

(8.16) $(s,t) \notin S_1 \times S_1$.

Proof. Assume $s, t \in S_1$. Let $v = (\kappa, 0)$ and $u = (n, \kappa)$. Then by 8.14.5, $b_v = c_v$, while by 8.14.3, $b_u = c_u = 1 - 1/(1-\varepsilon)\sum_{0\leq\beta<\kappa} a_\beta$. Let

$$y' = (y - b_v P(v) - b_u P(u))/(1 - b_v - b_u).$$

Then $y' \in \varphi(s) \cap \varphi(t) - \varphi(s \cap t)$, while $\kappa \notin \sup(y')$, contradicting 8.3.

(8.17) $(s,t) \notin S_3 \times S_j$ *for* $j = 1$ *or* 2.

Proof. Assume otherwise. Recall the definition of $\bar{b}$ and c^* from 8.11. Then as $s \in S_3$, 8.15 says

$$\begin{aligned} c^* &= \sum_{\kappa<\alpha\in I_t(0)} c_{\alpha,0} = \sum_{\kappa<\alpha\leq\alpha_1} c_{\alpha,0} \\ &= 1/\varepsilon \sum_{\kappa<\alpha<\alpha_1} a_\alpha + a_0/(1-\varepsilon) + 1/\varepsilon \sum_{\alpha_1\leq\alpha\leq n} a_\alpha - 1 \\ &= 1/\varepsilon \sum_{\kappa<\alpha\leq n} a_\alpha - 1 + a_0/(1-\varepsilon). \end{aligned}$$

Similarly as $t \in S_1 \cup S_2$, 8.14 says

$$\begin{aligned} \bar{b} &= \sum_{0<\beta\in J_s(\kappa)} b_{\kappa,\beta} = \sum_{0<\beta\leq\beta_1} b_{\kappa,\beta} \\ &= 1/(1-\varepsilon)\sum_{0<\beta<\beta_1} a_\beta + 1 - 1/(1-\varepsilon)\sum_{0\leq\beta<\beta_1} a_\beta - 1/\varepsilon \sum_{\kappa<\alpha\leq n} a_\alpha \\ &= 1 - a_0/(1-\varepsilon) - 1/\varepsilon \sum_{\kappa<\alpha\leq n} a_\alpha. \end{aligned}$$

That is $\bar{b} = -c^*$, contradicting 8.11.1.

Notice 8.12, 8.16, and 8.17 constitute a contradiction. Therefore the proof of Theorem 8.1, and hence also of Theorem 7.6, is at last complete.

§9. Triangulating the space of a restricted polyhedral cell complex

In this section we continue to assume (X, f, F, B) is a polyhedral cell complex and also continue the notational conventions of Section 7. Let $K = K(X)$ be the triangulating complex of X and let Σ be the set of simplices of K.

Recall for $x \in X$ we have the residual complex $K(x)$ of X at x with vertex set

$$V(x) = \{(w, v) : w \in f(x) \text{ and } v \in f(\zeta(w))\}$$

with simplex set $\Sigma(x)$. Define $\eta : V(x) \to V$ by $\eta(w, v) = v$.

(9.1) (1) *$\eta(L(w, v)) \subseteq L(v)$ for each $(w, v) \in V(x)$.*

(2) *$\eta : V(x) \to V$ induces a morphism $\eta : \Gamma(x) \to \Delta$ of graphs and a morphism $\eta : K(x) \to K$ of simplicial complexes.*

(3) *If (X, f) is resticted then each $S \in \Sigma$ with $\hat{\zeta}(S) = x$ is the image under η of a unique $s \in \Sigma(x)$. Indeed if $S = \{v_0, \ldots, v_k\}$ with $v_i \in L(v_j)$ for $i \leq j$, and $w_i = \hat{f}_{v_i}(v_k)$ then $s = s(S) = \{(w_i, v_i) : 0 \leq i \leq k\}$.*

Proof. Let $(y, u) \in L(w, v)$. Then $y \leq w$, $z = f_w(y) \geq v$, and $f_z(v) \geq u$. As $f_z(v) \geq u$, $\eta(y, u) = u \in L(v)$. Thus (1) holds. As $v = \eta(w, v)$, (1) implies (2).

Under the hypotheses and notation of (3), 5.4 says $w_0 \leq \cdots \leq w_k$ with $f_{w_j}(w_i) \geq v_j$ for $j \geq i$. Also as $v_i \in L(v_j)$, $\hat{f}_{v_i}(v_j)$ is the unique $w \in f(x_j)$ with $\zeta(w) = x_i$ and $w \geq v_j$. Thus $\hat{f}_{v_i}(v_j) = f_{w_j}(w_i)$, so $f_{f_{w_j}(w_i)}(v_j) = v_i$. Therefore $(w_i, v_i) \in L(w_j, v_j)$, so $s \in \Sigma(x)$. By construction, $\eta(s) = S$. Finally if $s' = \{(w_i', v_i') : 0 \leq i \leq k\} \in \Sigma(x)$ with $\eta(s') = S$ then $v_i' = \eta(w_i', v_i') = v_i$. Next $x = \hat{\zeta}(S) = \hat{\zeta}(v_k)$, so $v_k \in f(x)$, while as $(w_k', v_k) \in V(x)$, $v_k \in f(\zeta(w_k'))$, so $w_k' = x = w_k$. As $(w_i', v_i) \in L(x, v_k)$, $w_i' = f_x(w_i) \geq v_k$ and $f_{w_i'}(v_k) \geq v_i$, so $w_i' = \hat{f}_{v_i}(v_k) = w_i$. That is $s = s'$ is unique, completing the proof of (3).

Recall from Theorem 7.6, we have a triangulation $\varphi_x : K(x) \to F(x)$. Recall by construction that if $s \in \Sigma(x)$ then $\varphi_x(s) = \hat{\varphi}_x(s) = [P(e) : e \in s]$, where $P(w_i, v_i) = \varepsilon B_{h(w_i)} + (1 - \varepsilon) B_{h(\bar{v}_i)}$, and φ_s is the identity map.

In the remainder of this section we assume (X, f) is restricted and use the triangulations φ_x, $x \in X$, to construct a triangulation $\varphi : K \to A$, where $A = A(X)$ is the topological space of X constructed in Section 4. Recall from Section 4 that we have a map

$$\lambda : \coprod_{x\in X} F(x) \to A$$

$$a \mapsto \tilde{a}$$

For $S \in \Sigma$ define

$$\varphi(S) = \lambda(\varphi_{\hat{\zeta}(S)}(s(S)))$$

define

$$\hat{\varphi}(S) = \varphi_{\hat{\zeta}(S)}(s(S))$$

with $u(S, v_i) = u(S(s), (w_i, v_i)) = P(w_i, v_i)$ and define $\varphi_S : \varphi(S) \to \hat{\varphi}(S)$ by $\varphi_S = \lambda_S^{-1}$, where $\lambda_S : \varphi_{\hat{\zeta}(S)}(s(S)) \to \varphi(S)$ is the restriction of λ to $\varphi_{\hat{\zeta}(S)}(s(S))$.

(9.2) *For $S \in \Sigma$, $x = \hat{\zeta}(S)$, and $s = s(S)$, $\lambda_S : \varphi_x(s) \to \varphi(S)$ is a homeomorphism and $\varphi(S)$ is closed in A.*

Proof. Let $s = \{(w_i, v_i) : 0 \le i \le k\}$, $W = \{w_i : i\}$, $D = \varphi_x(s)$, and $D_w = I(w) \cap D$ for $w \in W$. Then $D = [P(w_i, v_i) : i]$, so by 6.3 each $d \in D$ can be written uniquely in the form $d = \sum_i d_i P(w_i, v_i)$ with $0 \le d_i$ and $\sum_i d_i = 1$. By 7.4, $P(w_i, v_i) \in I(w_i) \cap F(w_j)$ for $j \ge i$, so

$$D_w = \{d \in D : d_j = 0 \text{ for } j > i(w) \text{ and } d_i \ne 0 \text{ for some } i \in \mathcal{I}(w)\}$$

where $\mathcal{I}(w) = \{i : w_i = w\}$ and $i(w) = \max\{i : i \in \mathcal{I}(w)\}$. In particular the D_w partition D. Next $F_w : D_w \to F(\zeta(w))$ is a homeomorphism and by 4.4, $\lambda : I(\zeta(w)) \to A$ is an injection, so as $\lambda \circ F_w = \lambda$ on $F(w)$, $\lambda : D_w \to A$ is an injection. Further by 4.5, the sets $\tilde{I}(y)$, $y \in X$, partition A, so as $w \mapsto \zeta(w)$ is an injection on W, $\tilde{I}(w) \cap \tilde{I}(u) = \varnothing$ for distinct $u, w \in W$, and hence $\lambda : D \to A$ is an injection. Thus $\lambda_S : D \to \varphi(S)$ is bijective and continuous.

Finally D is closed in $F(x)$ and $C \subseteq \tilde{F}(x)$ is closed in A if and only if $\lambda^{-1}(C)$ is closed in $F(x)$. Now as $\lambda : D \to \varphi(S)$ is a bijection, if $E \subseteq D$ is closed then $E = \lambda^{-1}(\lambda(E))$ is closed in $F(x)$, so $\lambda(E)$ is closed in A. Thus λ_S is a homeomorphism and $\varphi(S) = \lambda(S)$ is closed in A.

(9.3) *Let $T \in \Sigma$ and $S \subseteq T$. Then $\varphi(S) \subseteq \varphi(T)$ and $\varphi_{S,T} : \sum_v a_v u(S, v) \to \sum_v a_v u(T, v)$.*

Proof. Let $x = \hat{\zeta}(T)$, $t = s(T) = \{v_0, \ldots, v_k\}$, $s = s(S)$, $r = \eta^{-1}(S) \subseteq t$, $l = \max\{i : v_i \in r\}$, $w_i = \hat{f}_{v_i}(v_k)$, $w = w_l$, and $y = \zeta(w)$.

Then $u(T,v_i) = P(w_i,v_i) = \varepsilon B(w_i) + (1-\varepsilon)B(f_{w_i}^{-1}(v_i))$, so for $i \leq l$,

$$F_w(P(w_i,v_i)) = \varepsilon B(f_w(w_i)) + (1-\varepsilon)B(v_i) = P(f_w(w_i),v_i)$$

by 7.2. Then

$$\begin{aligned}F_w(\sum_i a_i u(T,v_i)) = F_w(\sum_i a_i P(w_i,v_i)) &= \sum_i a_i P(f_w(w_i),v_i)\\ &= \sum_i a_i u(S,v_i)\end{aligned}$$

by another application of 7.2. In particular $F_w(\varphi_x(r)) = \varphi_y(s)$. Also $\lambda_S \circ F_w = \lambda_T$ on $\varphi_x(r)$ as $F_w(a) \sim a$ for each $a \in F(w)$. So $\varphi(S) = \lambda_S(\varphi_y(s)) = \lambda_S(F_w(\varphi_x(r))) = \lambda_T(\varphi_x(r)) \subseteq \varphi(T)$. Also

$$\begin{aligned}\varphi_{S,T}(\sum_i a_i u(S,v_i)) &= (\varphi_T \circ \varphi_S^{-1})(\sum_i a_i u(S,v_i))\\ &= \varphi_T(\lambda_S(F_w(\sum_i a_i u(T,v_i))))\\ &= \varphi_T(\lambda_T(\sum_i a_i u(T,v_i))) = \sum_i a_i u(T,v_i),\end{aligned}$$

completing the proof.

(9.4) *$\varphi(S) \cap \varphi(T) = \varphi(S \cap T)$ for all $S, T \in \Sigma$.*

Proof. By 9.3, $\varphi(S \cap T) \subseteq \varphi(S) \cap \varphi(T)$, so it remains to show that if $e \in \varphi(S) \cap \varphi(T)$ then $e \in \varphi(S \cap T)$. Then $e = \tilde{e}_S = \tilde{e}_T$ for some $e_R \in \varphi_{\hat{\zeta}(R)}(s(R))$. Then $x = y(e_T) = y(\tilde{e}_T) = y(e) = y(e_S)$ in the notation of Section 4; that is $e \in \tilde{I}(x)$ and $e_R \in I(w_R)$ with $w_R \in f(\hat{\zeta}(R))$ and $\zeta(w_R) = x$. For $R = S, T$, let $\bar{R} = \{v \in R : \hat{\zeta}(v) \leq x\}$. Then as we saw during the proof of the previous lemma, $F_{w_R}(e_R) \in \varphi_x(s(\bar{R}))$, so replacing S, T by $\bar{S}, \bar{T}$, we may assume $x = \hat{\zeta}(S) = \hat{\zeta}(T)$.

Now $\varphi_x(s(R)) \subseteq F(x)$ and $e_R \in I(x)$. Then as $\lambda : I(x) \to A$ is injective, $e_S = e_T \in \varphi_x(s(T)) \cap \varphi_x(s(S)) = \varphi_x(s(S) \cap s(T))$ by Theorem 7.6. Also $e_T \in I(x)$, so $s(S) \cap s(T) = s(S \cap T)$ with $\hat{\zeta}(S \cap T) = x$, so $e = \lambda(e_S) \in \lambda(\varphi_x(s(S \cap T))) = \varphi(S \cap T)$, completing the proof.

Theorem 9.5. *If (X, f, F, B) is a restricted polyhedral cell complex then $\varphi : K \to A$ is a triangulation.*

Proof. By 9.2, for each $S \in \Sigma$, $\varphi(S)$ is a closed subspace of A and

$$\lambda_S : \varphi_{\hat{\zeta}(S)}(s(S)) \to \varphi(S)$$

is a homeomorphism. Thus

$$\varphi_S = \lambda_S^{-1} : \varphi(S) \to \hat{\varphi}(S) = \varphi_{\hat{\zeta}(S)}(s(S))$$

is a homeomorphism. Axiom (T3) for triangulations holds by 9.3 and axiom (T1) holds by 9.4.

As $\varphi_x : K(x) \to F(x)$ is a triangulation, $F(x) = \bigcup_{s\in\Sigma(x)} \varphi_x(s)$ and $C \subseteq F(x)$ is closed if and only $C \cap \varphi_x(s)$ is closed in $\varphi_x(s)$ for all $s \in \Sigma(x)$. Also by 4.5, $A = \bigcup_x \tilde{F}(x)$, so as $\varphi(S) = \tilde{\varphi}_{\hat{\zeta}(S)}(s(S))$, $A = \bigcup_{S\in\Sigma} \varphi(S)$. Further by definition of the topology on A, $D \subseteq A$ is closed in A if and only if $D_x = A \cap \tilde{F}(x)$ is closed in $\tilde{F}(x)$ for all $x \in X$. Then as $\tilde{F}(x) = \bigcup_{s\in\Sigma(s)} \varphi(\eta(s))$, if $D_x \cap \varphi(\eta(s))$ is closed in $\varphi(\eta(s))$ for all $s \in \Sigma(x)$ then $\lambda_x^{-1}(D_x \cap \varphi(\eta(s)))$ is closed in $\varphi_x(s)$, so as φ_x is a triangulation, $\lambda_x^{-1}(D_x) = \bigcup_s \lambda_x^{-1}(D_x \cap \varphi(\eta(s)))$ is closed in $F(x)$, so D_x is closed in $\tilde{F}(x)$. Thus D is closed in A if and only if $D \cap \varphi(S)$ is closed in $\varphi(S)$ for all $S \in \Sigma$. That is axiom (T2) is satisfied.

Corollary 9.6. *The homology and fundamental group of the space $A(X)$ of a restricted polyhedral cell complex X are isomorphic to the simplicial homology and fundamental group of $K(X)$.*

Remark 9.7. We can also triangulate the space A of a polyhedral cell complex (X, f, F, B) which is not restricted, but not by the triangulating complex $K(X)$. Instead consider the set $\hat{V}(X)$ of all pairs (S, ω), where $S = \{v_0, \ldots, v_k\}$ is a simplex of $K(X)$ with the standard ordering, and $\omega : S \to f(\hat{\zeta}(v_k))$ with $\omega(S)$ a simplex in $f(\hat{\zeta}(v_k))$ and $w_i = \omega(v_i)$ satisfying $\zeta(w_i) = \hat{\zeta}(v_i)$, $f_{w_j}(w_i) \geq v_j$, and $f_{f_{w_i}(w_i)}(v_j) \geq v_i$ for $j \geq i$. Partially order $\hat{V}(X)$ by $(S, \omega) \geq (T, \theta)$ if $T \subseteq S$ and $\theta(v) = f_w(\omega(v))$ for each $v \in T$ and $w = \max(\omega(T))$. Finally let $\hat{K}(X)$ be the order complex of the poset $\hat{V}(X)$.

Observe we have a map of posets from $\hat{V}(X)$ into $\mathrm{sd}(K(X))$ defined by $(S, \omega) \mapsto S$, and this map is an isomorphism of $\hat{K}(X)$ with $\mathrm{sd}(K(X))$ when X is restricted.

For $x \in X$, Theorem 7.6 supplies a triangulation $\varphi_x : K(x) \to F(x)$. For s a simplex of $K(x)$, let $P(s)$ be the barycenter of $\varphi(s)$. A simplex σ of $\mathrm{sd}(K(x))$ is a chain $\{s_0 \subset \cdots \subset s_k\}$ of simplices of $K(x)$ and we have the barycentric subdivision $\psi_x : \Sigma(\mathrm{sd}(K(x))) \to F(x)$ of the triangulation φ_x, which is also a triangulation of $F(x)$, and is defined by $\psi_x(\sigma) = [P(s_i) : 0 \leq i \leq k]$.

We can use the triangulation ψ_x in place of φ_x and argue as in this section to construct a triangulation $\psi : \hat{K} \to A$. Namely suppose $\sigma =$

$\{s_0 \subset \cdots \subset s_r\}$ is a simplex in $\mathrm{sd}(K(x))$ with $s_i = \{(w_0^i, v_0^i), \ldots, (w_{k_i}^i, v_{k_i}^i)\} \in \Sigma(x)$ ordered as in 7.5 and with $w_{k_r}^r = x$. Define $\eta(\sigma) = \{(S_i, \omega_i) : 0 \le i \le r\}$ a simplex of $\hat{K}(X)$ by

$$S_i = \{f_{w_j^i}(v_j^i) : 0 \le j \le k_i\},$$

$$\omega_i(f_{w_j^i}(v_j^i)) = f_{w_{k_i}^i}(w_j^i), \quad 0 \le j \le k_i.$$

The map η plays the role that the map η defined at the start of this section played for restricted complexes.

Conversely given $\sigma = \{(S_i, \omega_i) : 0 \le i \le r\}$ a simplex in $\hat{K}(X)$ with $S_i = \{v_0^i, \ldots, v_{k_i}^i\}$ in the standard ordering and $x = \hat{\zeta}(v_{k_r}^r) = \hat{\zeta}(\sigma)$, define $\nu(\sigma) = \{s_0, \ldots, s_r\}$ by $s_i = \{(w_0^i, \hat{v}_0^i), \ldots, (w_{k_i}^i, \hat{v}_{k_i}^i)\}$, where $w_j^i = \omega_k(v_j^i)$ and $\hat{v}_j^i = f_{w_j^i}^{-1}(v_j^i)$. The simplex $\nu(\sigma)$ plays the role of the simplex $s(S)$. Thus we define

$$\psi(\sigma) = \lambda(\psi_{\hat{\zeta}(\sigma)}(\nu(\sigma))),$$

$$\hat{\psi}(\sigma) = \psi_{\hat{\zeta}(\sigma)}(\nu(\sigma)),$$

for σ a simplex of $\hat{K}(X)$. We can now repeat the proofs of Lemmas 9.1 through 9.5 with some small variation, to establish the analogous statements for general combinatorial cell complexes and the triangulation $\psi : \hat{K}(X) \to A$.

§10. The triangulation functor

By Theorem 9.5, if $X = (X, f, F, B)$ is a restricted polyhedral cell complex then there exists a triangulation $\xi^X : K(X) \to A(X)$. We seek to extend ξ to a functor from the category of restricted polyhedral cell complexes to the category of triangulated topological spaces. Our triangulation ξ^X depends on a choice of real ε, $0 < \varepsilon < 1$. Fix some choice of ε and use it to define ξ^X for all choices of X. Further given a morphism $\alpha : X \to Y$ of poyhedral cell complexes, define $\xi^\alpha : \xi^X \to \xi^Y$ by $\xi^\alpha = (K(\alpha), A(\alpha))$, where A is the functor of 4.7 and K is the functor of 5.7. We prove the following two results at the same time:

(10.1) *ξ is a covariant functor from the category of restricted polyhedral cell complexes to the category of triangulated topological spaces.*

(10.2) *Let $\alpha : X \to \bar{X}$ be a morphism of restricted polyhedral cell complexes. Then*

(1) *For $x \in X$ and σ a simplex in $f(x)$,*

$$F(x) = \{ \sum_{v \in \sigma} a_v B(v) : 0 \leq a_v \in \mathbf{R} \text{ and } \sum_v a_v = 1 \}$$

and

$$\alpha_x \big(\sum_{v \in \sigma} a_v B(v) \big) = \sum_{v \in \sigma} a_v \bar{B}(\alpha_x(v))$$

(2) *For $S = \{v_0, \ldots, v_k\}$ a simplex in $K(X)$ ordered as in Lemma 9.1 and $x = \hat{\zeta}(S)$,*

$$\xi^X(S) = \{ \sum_i a_i \tilde{P}(w_i, v_i) : 0 \leq a_i \in \mathbf{R} \text{ and } \sum_i a_i = 1 \}$$

where $w_i = \hat{f}_{v_i}(v_k) \in f(x)$, and

$$A(\alpha) \big(\sum_i a_i \tilde{P}(w_i, v_i) \big) = \sum_i a_i \tilde{P}(\bar{w}_i, \bar{v}_i)$$

with $K(\alpha)(S) = \{\bar{v}_i : 0 \leq i \leq k\}$, $\bar{v}_i = \alpha_{\hat{\zeta}(v_i)}(v_i)$, and $\bar{w}_i = \alpha_x(w_i) = \hat{f}_{\bar{v}_i}(\bar{v}_k)$.

By Theorem 9.5, $\xi^X : K(X) \to A(X)$ and $\xi^{\bar{X}} : K(\bar{X}) \to A(\bar{X})$ are triangulations. By Remark 5.7, $K(\alpha) : K(X) \to K(\bar{X})$ is a simplicial map, while by 4.7, $A(\alpha) : A(X) \to A(\bar{X})$ is a continuous map. So to prove 10.1, we must show $\xi^{\alpha \circ \beta} = \xi^\alpha \circ \xi^\beta$ and for each simplex S of $K(X)$

(T_1) $A(\alpha)(\xi^X(S)) \subseteq \xi^{\bar{X}}(K(\alpha)(S))$, and

(T_2) $\alpha_S \circ \xi^X_S = \xi^{\bar{X}}_{K(\alpha)(S)} \circ A(\alpha)$, where

$$\alpha_S : \sum_{v \in S} a_v u(S, v) \mapsto \sum_{v \in S} a_v u(K(\alpha)(S), K(\alpha)(v))$$

The first remark follows from the fact that A and K are functors. The first statements in (1) and (2) of 10.2 follow from the definition of $F(x)$ and $\xi^X(S)$, respectively. Moreover if S is as in 10.2, by definition of ξ^X_S, $u(S, v_i) = P(w_i, v_i)$ and

$$\xi^X_S : \sum_i a_i \tilde{P}(w_i, v_i) \mapsto \sum_i a_i P(w_i, v_i)$$

Therefore 10.2.2 implies (T_1) and (T_2), so it remains to prove 10.2.

As α is a morphism of polyhedral complexes, $\alpha_x(B^x(\sigma)) \subseteq \bar{B}^{\alpha(x)}(\alpha(\sigma))$ and $\alpha_\sigma \circ B^x_\sigma = \bar{B}^{\alpha(x)}_{\alpha(\sigma)} \circ \alpha_x$ on $B^x(\sigma)$ for each $x \in X$ and each simplex σ

of $f(x)$. In particular in the notation of Section 7, $\alpha_x(B(v)) = \bar{B}(\alpha_x(v))$ for each $v \in f(x)$. Also for $\sigma = \{y_0, \ldots, y_m\}$, by definition

$$\sum_i a_i B(y_i) = B_\sigma^{-1}\big(\sum_i a_i u_{h(y_i)}\big)$$

so as $\alpha_\sigma \circ B_\sigma^x = \bar{B}_{\alpha(\sigma)}^{\alpha(x)} \circ \alpha_x$ on $B^x(\sigma)$, we have

$$\alpha_x\big(\sum_i a_i B(y_i)\big) = \alpha_x(B_\sigma^{-1}\big(\sum_i a_i u_{h(y_i)}\big)) = \bar{B}_{\alpha(\sigma)}^{\alpha(x)}(\alpha_\sigma\big(\sum_i a_i u(\sigma, y_i)\big))$$
$$= \bar{B}_{\alpha(\sigma)}^{\alpha(x)}\big(\sum_i a_i u(K(\alpha)(\sigma), K(\alpha)(y_i))\big) = \sum_i a_i \bar{B}(\alpha_x(y_i))$$

so that 10.2.1 is established.

Recall the definition of $V(x)$ and $P(w,v)$ from Section 7, and observe that as α is a morphism of cell complexes, for $(w,v) \in V(x)$, $(\alpha_x(w), \alpha_{\zeta(w)}(v)) \in V(\alpha(x))$ and $\alpha_x(B(f_w^{-1}(v))) = \bar{B}(\alpha_x(f_w^{-1}(v))) = \bar{B}(\bar{f}_{\alpha_x(w)}^{-1}(\alpha_x(v)))$. Therefore

$$\alpha_x(P(w,v)) = \alpha_x(\varepsilon B(w) + (1-\varepsilon)B(f_w^{-1}(v)))$$
$$= \varepsilon \bar{B}(\alpha_x(w)) + (1-\varepsilon)\bar{B}(\bar{f}_{\alpha_x(w)}^{-1}(\alpha_x(v))) = \bar{P}(\alpha_x(w), \alpha_{x(v)}(v))$$

by an earlier remark. Therefore

$$A(\alpha)(\tilde{P}(w,v)) = \tilde{\alpha}_x(P(w,v)) = \tilde{P}(\alpha_x(w), \alpha_{x(v)}(v)).$$

Let $S = \{v_0, \ldots, v_k\}$ be a simplex of $K(X)$ ordered as in 9.1, and $x = \hat{\zeta}(S)$. Then from the definition of ξ^X in Section 9 and 9.1,

$$\xi^X(S) = \lambda(\xi_x(s(S))) = [\tilde{P}(w_i, v_i) : 0 \le i \le k]$$

where $s(S) = \{(w_i, v_i) : 0 \le i \le k\}$ and $w_i = \hat{f}_{v_i}(v_k)$. Therefore

$$\alpha_x(s(S)) = \{(\bar{w}_i, \bar{v}_i) : 0 \le i \le k\} = s(K(\alpha)(S))$$

where $\bar{v}_i = \alpha_{x(v_i)}(v_i)$ and $\bar{w}_i = \alpha_x(w_i) = \hat{f}_{\bar{v}_i}(\bar{v}_k)$. Finally

$$A(\alpha)\big(\sum_i a_i \tilde{P}(w_i, v_i)\big) = \tilde{\alpha}_x\big(\sum_i a_i P(w_i, v_i)\big) = \sum_i a_i \tilde{\alpha}_x(P(w_i, v_i))$$

by 10.2.1, and then by an earlier observation this is equal to $\sum_i a_i \tilde{P}(\bar{w}_i, v_i)$, completing our proof.

We can now use the functor ξ to construct the triangulation functor from the category of restricted combinatorial cell complexes to the category of triangulated topological spaces. Namely we define the *triangulation functor* to be the covariant functor $T = \xi \circ \mathcal{P}$, where $\mathcal{P}$ is the functor from the category of combinatorial cell complexes to the category of polyhedral cell complexes constructed in Example 7.1. As the composition of covariant functors, T is a covariant functor. Given a combinatorial cell complex $X = (X, f)$, we write $T(X)$ for the topological space $A(\mathcal{P}(X))$ and when X is restricted we write ξ^X for the triangulation $\xi^{\mathcal{P}(X)} : K(X) \to T(X)$. If $\alpha : X \to \bar{X}$ is a morphism of combinatorial cell complexes, we write $T(\alpha)$ for the morphism $\xi^{\mathcal{P}(\alpha)} = (K(\alpha), A(\mathcal{P}(\alpha)))$.

We define $T(X)$ to be the *geometric realization* of the combinatorial cell complex X.

(10.3) *Let $\gamma : K^1 \to K^2$ be a morphism of typed simplicial complexes over I and $\varphi^i : K^i \to T^i$ be triangulations. Then*

(1) *γ extends to a morphism $(\gamma, \beta^\gamma(\varphi^1, \varphi^2)) : \varphi^1 \to \varphi^2$ of triangulated topological spaces.*

(2) *If $\delta : K^2 \to K^3$ is a morphism of typed simplicial complexes over I and $\varphi^3 : K^3 \to T^3$ is a triangulation then $\beta^{\delta\circ\gamma}(\varphi^1, \varphi^3) = \beta^\delta(\varphi^2, \varphi^3) \circ \beta^\gamma(\varphi^1, \varphi^2)$.*

(3) *$\beta^{id_{K^1}}(\varphi^1, \varphi^1) = id_{T^1}$.*

(4) *If γ is an isomorphism then so is $(\gamma, \beta^\gamma(\gamma^1, \varphi^2))$.*

Proof. For $s \in \Sigma^1$ define

$$\gamma_s : \hat{\varphi}^1(s) \to \hat{\varphi}^2(\gamma(s))$$
$$\sum_{v\in s} a_v u^1(s, v) \mapsto \sum_{v\in s} a_v u^2(\gamma(s), \gamma(v)).$$

Observe that γ_s is continuous. Now define $\beta_s : \varphi^1(s) \to \varphi^2(\gamma(s))$ by

$$\beta_s = (\varphi^2_{\gamma(s)})^{-1} \circ \gamma_s \circ \varphi^1_s$$

and then define

$$\beta = \bigcup_{s\in\Sigma^1} \beta_s.$$

That is for $a \in \varphi^1(s)$, $\beta(a) = \beta_s(a)$.

We first check that β is well defined. To begin, if $t \subseteq s$ then by axiom (T3) for triangulations, $\varphi^i_{t,s} = \hat{\varphi}^i_{t,s}$ as maps from $\hat{\varphi}^i(t)$ to $\hat{\varphi}^i(s)$, where $\varphi^i_{t,s} = \varphi^i_s \circ (\varphi^i_t)^{-1}$ and

$$\hat{\varphi}^i_{t,s} : \sum_{v\in t} a_v u^1(t, v) \mapsto \sum_{v\in t} a_v u^1(s, v).$$

Also by definition of γ_t and γ_s,

$$\hat{\varphi}^2_{\gamma(t),\gamma(s)} \circ \gamma_t = \gamma_s \circ \hat{\varphi}^1_{t,s}$$

as maps from $\hat{\varphi}^1(t)$ to $\hat{\varphi}^2(\gamma(s))$. Therefore the diagram

$$\begin{array}{ccccccc}
\varphi^1(t) & \xrightarrow{\varphi^1_t} & \hat{\varphi}^1(t) & \xrightarrow{\gamma_t} & \hat{\varphi}^2(\gamma(t)) & \xrightarrow{(\varphi^2_{\gamma(t)})^{-1}} & \varphi^2(\gamma(t)) \\
\iota^1\downarrow & & \hat{\varphi}^1_{t,s}\downarrow & & \hat{\varphi}^2_{\gamma(t),\gamma(s)}\downarrow & & \downarrow\iota^2 \\
\varphi^1(s) & \xrightarrow{\varphi^1_s} & \hat{\varphi}^1(s) & \xrightarrow{\gamma_s} & \hat{\varphi}^2(\gamma(s)) & \xrightarrow{(\varphi^2_{\gamma(s)})^{-1}} & \varphi^2(\gamma(s))
\end{array}$$

commutes, so that $\beta_s = \beta_t$ on $\varphi^1(t)$.

Further by axiom (T1), if $r, s \in \Sigma^1$ then $\varphi^1(r) \cap \varphi^1(s) = \varphi^1(r \cap s)$. So setting $t = r \cap s$, we have $\beta_r = \beta_t = \beta_s$ on $\varphi^1(r) \cap \varphi^1(s)$, completing the proof that β is well defined.

As γ_s, φ^1_s, and $(\varphi^2_{\gamma(s)})^{-1}$ are continuous, so is β_s. Then by axiom (T2), β is continuous. That is if C is closed in T^2 then $C \cap \varphi^2(\gamma(s))$ is closed in $\varphi^2(\gamma(s))$ for each $s \in \Sigma^1$. Therefore $\beta_s^{-1}(C \cap \varphi^2(\gamma(s))) = \beta^{-1}(C) \cap \varphi^1(s)$ is closed in $\varphi^1(s)$. Hence $\beta^{-1}(C)$ is closed in T^1.

By definition of β, $\beta(\varphi^1(s)) = \beta_s(\varphi^1(s)) \subseteq \varphi^2(\gamma(s))$ and $\varphi^1_s \circ \gamma_s = \varphi^2_{\gamma(s)} \circ \beta_s = \varphi^2_{\gamma(s)} \circ \beta$. Therefore $(\gamma, \beta) : \varphi^1 \to \varphi^2$ is a morphism of triangulated topological spaces, establishing (1).

Assume the hypotheses of (2). Then

$$\beta^\gamma = \bigcup_{s\in\Sigma^1} \beta^\gamma_s, \quad \beta^\delta = \bigcup_{t\in\Sigma^2} \beta^\delta_t, \quad \beta^{\delta\circ\gamma} = \bigcup_{s\in\Sigma^1} \beta^{\delta\circ\gamma}_s.$$

Further $(\delta \circ \gamma)_s = \delta_{\gamma(s)} \circ \gamma_s$, so

$$\beta^{\delta\circ\gamma}_s = (\varphi^3_{(\delta\circ\gamma)(s)})^{-1} \circ (\delta \circ \gamma)_s \circ \varphi^1_s$$

$$= ((\varphi^3_{\delta(\gamma(s))})^{-1} \circ \delta_{\gamma(s)} \circ \varphi^2_{\gamma(s)}) \circ ((\varphi^2_{\gamma(s)})^{-1} \circ \gamma_s \circ \varphi^1_s) = \beta^\delta_{\gamma(s)} \circ \beta^\gamma_s$$

and therefore $\beta^{\delta\circ\gamma} = \beta^\delta \circ \beta^\gamma$, establishing (2).

Part (3) follows as if $\varphi^1 = \varphi^2$ and $\gamma = id_K$ then $\gamma_s = id_{\hat{\varphi}^1(s)}$, so

$$\beta_s = (\varphi^1_s)^{-1} \circ \gamma_s \circ \varphi^1_s = id_{\varphi^1(s)}.$$

Finally (2) and (3) imply that

$$\beta^{\gamma^{-1}}(\varphi^2, \varphi^1) \circ \beta^\gamma(\varphi^1, \varphi^2) = \beta^{id_K}(\varphi^1, \varphi^1) = id_{T^1}$$

so $\beta^{\gamma^{-1}}(\varphi^2, \varphi^1) = \beta^{\gamma}(\varphi^1, \varphi^2)^{-1}$, and hence $(\gamma, \beta^{\gamma}(\varphi^1, \varphi^2))$ is an isomorphism.

(10.4) *Let K be a typed simplicial complex. Then each triangulation of K is isomorphic in the category of triagulated topological spaces to the geometric realization $\varphi^K : K \to T(K)$ of K.*

Proof. Apply 10.3.4 to $K^1 = K^2 = K$, $\gamma = id_K$, $\varphi^1 = \varphi^K$, and any triangulation $\varphi^2 : K \to T^2$ of K.

Remark 10.5. It is well known that if T is the geometric realization of a simplicial complex K, then the singular homology $H_*(T)$ is isomorphic to the simplicial homology $H_*(K)$ and the fundamental group $\pi_1(T)$ is isomorphic to the fundamental group $\pi_1(K)$ of K.

As recalled in Remark 10.5, the homology and fundamental group of the geometric realization of a simplicial complex can be defined in a purely combinatorially manner in terms of the simplicial complex. We seek to do the same for combinatorial cell complexes. We know that if X is a restricted combinatorial cell complex, then $\xi^X : K(X) \to T(X)$ is a triangulation of the geometric realization $T(X)$ of X, so by Remark 10.5, $H_*(T(X)) \cong H_*(K(X))$ and $\pi_1(T(X)) \cong \pi_1(K(X))$. Thus we define $H_*(X) = H_*(K(X))$ and $\pi_1(X) = \pi_1(K(X))$. Thus we have our combinatorial definition of the homology and fundamental group of a restricted combinatorial cell complex, and from the discussion above we have:

Theorem 10.6. *The triangulation functor T is a covariant functor from the category of restricted combinatorial cell complexes to the category of triangulated topological spaces, which assigns to a restricted combinatorial cell complex (X, f) its geometric realization $T(X)$ and the triangulation $\xi^X : K(X) \to T(X)$ of the geometric realization by the triangulating complex of X. Moreover $H_*(T(X)) \cong H_*(X)$ and $\pi_1(T(X)) \cong \pi_1(X)$.*

(10.7) *Let (X, f) be the simplicial cell complex of the poset X. Then the geometric realization of the cell complex (X, f) is homeomorphic to the geometric realization of the order complex $\mathcal{O}(X)$ of the poset X.*

Proof. Let $\mathcal{P}(X, f) = (X, f, F, B)$ and $A = T(X, f)$ the topological space of this polyhedral cell complex. Thus A is the geometric realization of the simplicial cell complex (X, f). We show there exists a triangulation $\varphi : \mathcal{O}(X) \to A$. Then by 10.4, A is homeomorphic to the geometric realization of $\mathcal{O}(X)$.

The triangulation is defined by $\varphi(s) = \tilde{B}^z(s)$, $\hat{\varphi}(x) = \hat{B}^z(s)$, and $\varphi_s = B_s^z \circ \lambda_z^{-1}$, where z is the greatest element of s. As (X, f) is the simplicial cell complex for the poset X, ζ is injective, so by 4.6.3, $\lambda_x : F(x) \to \tilde{F}(x)$ is a homeomorphism and hence φ_s makes sense. Check that if $x \geq z$ then $\tilde{B}_x(s) = \tilde{B}_z(s)$, $\hat{B}^x(s) = \hat{B}^z(s)$, and $B_s^x \circ \lambda_x^{-1} = B_s^z \circ \lambda_z^{-1}$. Then use this fact to prove φ is a triangulation. We leave the details to the reader.

(10.8) *Assume $X = (X, f)$ is a regular combinatorial cell complex. Let $\mathcal{O}(X)$ be the order complex of the poset X. Then $H_*(\mathcal{O}(X)) \cong H_*(X)$ and $\pi_1(\mathcal{O}(X)) \cong \pi_1(X)$.*

Proof. By 3.1, X is isomorphic to the simpicial cell complex Y of the poset X. Thus $H_*(X) \cong H_*(Y)$ and $\pi_1(X) \cong \pi_1(Y)$. But by 10.7, the geometric realizations A of Y and T of $\mathcal{O}(X)$ are homeomorhic, so $H_*(Y) \cong H_*(A) \cong H_*(T) \cong H_*(\mathcal{O}(X))$. Similarly the fundamental groups are isomorphic.

§11. Homology in $K(X)$

Let (X, f) be a combinatorial cell complex, $\Delta = \Delta(X)$ the graph of X, and $K = K(X)$ the triangulating complex of X. Define

$$V^n = \{v \in V : h(\hat{\zeta}(v)) \leq n\}, \quad V_n^n = \{v \in V : h(\hat{\zeta}(v)) = h(\zeta(v)) = n\}$$

and $V_n = V^n - V_n^n$. Denote by Δ^n and Δ_n the graphs on V^n and V_n induced by Δ, respectively. Let $K^n = K(\Delta^n)$ and $K_n = K(\Delta_n)$ be the clique complexes of Δ^n and Δ_n, respectively. Define the n-skeleton of X to be the combinatorial cell complex $X^n = (X^n, f_{|X^n})$, where $X^n = \{x \in X : h(x) \leq n\}$. Thus $K^n = K(X^n)$ is the triangulating complex of the n-skelton of X.

By 5.6, K is a typed simplicial complex over $I \times I$, $I = \{0, 1, \ldots\}$, with type function $\tau(v) = (h(\hat{\zeta}(v)), h(\zeta(v)))$. Order $I \times I$ lexiographically; that is $(a, b) < (i, j)$ if $a < i$ or $a = i$ and $b < j$. We use this ordering to define our boundary operator on the simplicial chain complex $C_*(K)$ of K as in Section 1. We also adopt the notional conventions established in that section.

By 5.4, if s is a k-simplex of K then there is a unique ordering $v_0, \ldots, v_k$ of s such that $v_i \in L(v_{i+1})$ for each i. By 5.2, $\tau(v_0) \leq \cdots \leq \tau(v_k)$ also, so this is the ordering of s used to define the oriented simplex $s = v_0 \cdots v_k = v_0 \wedge \cdots \wedge v_k$ via the conventions of Section 1.

Example. Consider the simplicial cell complex $X(n)$ of the n-simplex, defined in Example (2) in Section 3. Recall from Section 5 that

$$V(X(n)) = V(n) = \{(\alpha, \beta) \in X(n) \times X(n) : \alpha \geq \beta\},$$

and for $(\alpha, \beta) \in V(n)$,

$$L(\alpha, \beta) = \{(a, b) \in V(n) : b \leq \beta \leq a \leq \alpha\}$$

Recall we write $K(n)$ for the triangulating complex $K(X(n))$ of the cell complex $X(n)$. Then $K(n)$ is a typed complex over $V(n)$ with type function the identity map. As $V(n)$ is ordered lexiographically, we have $(a, b) < (\alpha, \beta)$ if $a < \alpha$ or $a = \alpha$ and $b < \beta$. Notice $K(n)^m = \{(a, b) \in V(n) : a \leq m\} \cong K(m)$, $V(n)^m_m = \{(m, m)\}$, and $K(n)_m = K(n)^m - \{(m, m)\}$. By 5.9:

(11.1) *Let $s = \{(\alpha_i, \beta_i) : 0 \leq i \leq k\}$ be a subset of $V(n)$ with $(\alpha_i, \beta_i) \leq (\alpha_{i+1}, \beta_{i+1})$ for each i. Then s is in $\Sigma(n) = \Sigma(K(n))$ if and only if $\beta_0 \leq \cdots \leq \beta_k \leq \alpha_0 \leq \cdots \leq \alpha_k$.*

Observe next that

(11.2) *For $z \in V^n_n$, $\mathrm{Link}_{K^n}(z) \cong \mathcal{O}(\dot{f}(\hat{\zeta}(z)))$.*

Indeed $\mathrm{Link}_{K^n}(z) = \{u \in V_n : \hat{\zeta}(u) = \hat{\zeta}(z)\}$ and the identity map $\mathrm{Link}_{K^n}(z) \to \mathcal{O}(\dot{f}(\hat{\zeta}(z)))$ is an isomorphism. Observe also that under the notational conventions of Section 1, and by 1.1:

(11.3) *If $z \in V^n_n$ and*

$$\alpha = \sum_{s \in \Sigma^{k-1}(\mathrm{Link}_{K^n}(z))} a_s s \in C_{k-1}(\mathrm{Link}_{K^n}(z))$$

then

$$\alpha z = \sum_{s \in \Sigma^{k-1}(\mathrm{Link}_{K^n}(z))} a_s s z \in C_k(K^n)$$

where if $s = v_0 \cdots v_{k-1}$ then $sz = v_0 \cdots v_{k-1} z$. Further $\partial(\alpha z) = \partial(\alpha) z + (-1)^k \alpha$.

(11.4) *Let X be restricted of height at least $n+1$ and $\iota : K^n \to K_{n+1}$ the inclusion map. Then*

(1) ι is a homotopy equivalence.

(2) $\iota_* : H_*(K^n) \to H_*(K_{n+1})$ is an isomorphism.

Proof. We show for all simplices s of K_{n+1}, $\iota^{-1}(st_{K_{n+1}}(s))$ is contractible. Then by Theorem 1 in [3], ι is a homotopy equivalence.

Let $s = \{v_0, \ldots, v_k\}$ be a simplex in K_{n+1} ordered as in Lemma 5.4. If $v_0 \in K^n$ then $v_0 \in \iota^{-1}(st_{K_{n+1}}(s)) \subseteq st_{K^n}(v_0)$ and hence $\iota^{-1}(st_{K_{n+1}}(s))$ is contractible.

So assume $x = \hat{\zeta}(v_0)$ is of height $n+1$. Let $W = \{w \in f(x) : v_k \leq w < x\}$ and for $w \in W$ let $S(w) = f(\zeta(w))(\leq f_w(v_0))$, $T(w) = \bigcup_{u \in W(\leq w)} S(u)$, and for $U \subseteq W$ let $T(U) = \bigcup_{u \in U} T(u)$.

Observe that $\iota^{-1}(st_{K_{n+1}}(s)) = T(W)$. For if $a \in \iota^{-1}(st_{K_{n+1}}(s))$ then $f_{w_i}(v_i) \geq i$, where $w_i = \hat{f}_a(v_i)$. Then $w_i \geq v_i \geq v_0$ and $\zeta(w_i) = \hat{\zeta}(a)$, so as X is restricted, $w = w_i$ is independent of i. Thus $w = w_k \in W$ and $a \in S(w) \subseteq T(W)$.

Claim for each nonempty subset U of W, $T(U)$ is contractible. Observe $f_w(v_0) \in T(w) \subset L(f_w(v_0))$, so $T(w)$ is contractible. Further if Z is the set of maximal members of U then $C = \{T(z) : z \in Z\}$ is a cover of $T(U)$, so if $Z = \{z\}$ then $T(U) = T(z)$ is contractible. Next if $I \subseteq Z$, $T_I = \bigcap_{i \in I} T(i) = T(J)$, where $J = \bigcap_{i \in I} W(\leq i)$. So if $I = \{i\}$ is of order 1 then $T_I = T(i)$ is contractible, while if $|I| > 1$ then $M(J) < M(U) = \max\{h(u) : u \in U\}$, so by induction on $M(U)$, T_I is contractible. Here we use the fact that $J \neq \varnothing$ since v_k is the unique minimal element of J. It follows that $T(U)$ has the homotopy type of the nerve $N(C)$ of the cover C, and $N(C)$ has the homotopy type of the set of all nonempty subsets of C, so $N(C)$ is contractible. Thus the Claim is established. In particular $T(W) = \iota^{-1}(st_{K_{n+1}}(s))$ is contractible, completing the proof of (1). By (1), the inclusion $\iota : K^n \to K_{n+1}$ is a homotopy equivalence, so $\iota_* : H_*(K^n) \to H_*(K_{n+1})$ is an isomorphism.

Next some notation. If C_* and D_* are chain complexes, then write $\mathrm{Hom}(C_*, D_*)$ for the group of all maps $f = \bigcup_i f_i$, where $f_i \in \mathrm{Hom}(C_i, D_i)$ is a group homomorphism. Further if G is a group, denote by $\mathrm{Hom}(G, C_*)$ the chain complex whose ith term is $\mathrm{Hom}(G, C_i)$ and whose boundary map ∂ is defined by $\partial(x)(g) = \partial(\hat{\zeta}(g))$ for $g \in G$ and $x \in \mathrm{Hom}(G, C_i)$. Note that ∂ is a boundary map, since $\partial^2(x)(g) = (\partial(\partial(x)))(g) = \partial(\partial(x)(v)) = \partial(\partial(\hat{\zeta}(v)) = \partial^2(\hat{\zeta}(v)) = 0$, so $\partial^2 = 0$. Finally define $\hat{K}^n$ to be the full subcomplex of K^n consisting of those $v \in V$ with $h(\hat{\zeta}(v)) = n$.

Using these notions and the simplicial complex $K(n)$ discussed in the example earlier in this section, we now define

$$\phi \in \mathrm{Hom}(C_*(K(n)), \mathrm{Hom}(C_n(\hat{K}^n), C_*(K^n)))$$

For $c \in C_k(K(n))$ we write ϕ_c for the image of c under ϕ. Thus

$$\phi_c \in \mathrm{Hom}(C_n(\hat{K}^n), C_*(K^n))_k = \mathrm{Hom}(C_n(\hat{K}^n), C_k(K^n))$$

and as ϕ is a group homomorphism, it suffices to define ϕ on the generators $\sigma \in \Sigma^k(n)$ of $C_k(K(n))$. Similarly as $\phi_\sigma \in \mathrm{Hom}(C_n(\hat{K}^n), C_k(K^n))$ is a group homomorphism, it suffices to define ϕ_σ on the generators $s = \{v_0, \ldots, v_n\} \in \Sigma^n(\hat{K}^n)$, ordered so that $v_0 < \cdots < v_n$. We do so by decreeing that

$$\phi_\sigma(s) = \prod_{(i,j)\in\sigma} f_{v_i}(v_j).$$

Observe

(11.5) *For $\sigma = \{(\alpha_0, \beta_0), \ldots, (\alpha_k, \beta_k)\} \in \Sigma^k(n)$ and $s = \{v_0, \ldots, v_n\} \in \Sigma^n(\hat{K}^n)$, with $v_0 < \cdots < v_n$ and $(\alpha_0, \beta_0) < \cdots < (\alpha_k, \beta_k)$, we have $\phi_\sigma(s) \in \Sigma^k(K^n)$ with*

$$f_{v_{\alpha_i}}(v_{\beta_i}) \in L(f_{v_{\alpha_j}}(v_{\beta_j})) \text{ for } i < j,$$

and for $(a,b) \in \sigma$, $\hat{\zeta}(f_{v_a}(v_b)) = \zeta(v_a)$, and $\zeta(f_{v_a}(v_b)) = \zeta(v_b)$.

Proof. The last two remarks in the lemma follow from the definition of $f_w(v)$. As $s \in \Sigma^n(\hat{K}^n)$, $v_n = \infty_z$, where $z = \hat{\zeta}(v_n)$, and $v_r \in f(z)$ for each r. As $\beta_r \leq \alpha_r$, $v_{\beta_r} \leq v_{\alpha_r}$, so $f_{v_{\alpha_r}}(v_{\beta_r})$ is defined.

Let $i < j$. Then by 11.1, $\beta_i \leq \beta_j \leq \alpha_i \leq \alpha_j$. For $r \leq \alpha_j$ let $\bar{v}_r = f_{v_{\alpha_j}}(v_r)$. Then $\bar{v}_{\beta_j} \leq \bar{v}_{\alpha_i}$ with $f_{\bar{v}_{\alpha_i}}(\bar{v}_{\beta_j}) \geq f_{\bar{v}_{\alpha_i}}(\bar{v}_{\beta_i}) = f_{v_{\alpha_i}}(v_{\beta_i})$, so $f_{v_{\alpha_i}}(v_{\beta_i}) \in L(f_{v_{\alpha_j}}(v_{\beta_j}))$. In particular this shows $\phi_\sigma(s) \in \Sigma^k(K^n)$.

By 11.5, $\phi_\sigma(s) \in \Sigma^k(K^n)$, so ϕ is well defined. Thus we have proved:

(11.6) *The map ϕ defined above is in* $\mathrm{Hom}(C_*(K(n)), \mathrm{Hom}(C_n(\hat{K}^n), C_*(K^n)))$.

Next we have boundary maps ∂ on $C_*(K(n))$ and on the chain complex $\mathcal{H}(n,K) = \mathrm{Hom}(C_n(\hat{K}^n), C_*(K^n))$. We observe next that the map ϕ preserves these boundary maps:

(11.7) *$\partial(\phi_c) = \phi_{\partial(c)}$ for each $c \in C_*(K(n))$. Thus $\phi : C_*(K(n)) \to \mathcal{H}(n,K)$ preserves the boundary maps.*

Proof. Let $\sigma = \{(a_0, b_0), \ldots, (a_k, b_k)\} \in \Sigma^k(K(n))$ and

$$s = \{v_0, \ldots, v_n\} \in \Sigma^n(\hat{K}^n))$$

Then

$$\phi_{\sigma^i}(s) = \prod_{j \neq i} f_{v_{a_j}}(v_{b_j}) = \phi_\sigma(s)^i$$

Therefore if $c = \sum_\sigma a_\sigma \sigma \in C_k(K(n))$ then $\phi_{c^i}(s) = \sum_\sigma a_\sigma \phi_{\sigma^i}(s) = \sum_\sigma \phi_\sigma(s)^i = (\sum_\sigma a_\sigma \phi_\sigma)^i = \phi_c(s)^i$. Then

$$(\partial(\phi_c))(s) = \partial(\phi_c(s)) = \sum_i (-1)^i \phi_c(s)^i = \sum_i (-1)^i \phi_{c^i}(s) = \phi_{\partial(c)}(s)$$

Let $z \in V_n^n$ and $L = \mathrm{Link}_{K^n}(z)$. We can also regard

$$\phi \in \mathrm{Hom}(C_*(K(n)), \mathrm{Hom}(C_{n-1}(L), C_*(K^n)))$$

by composing ϕ with the map

$$C_{n-1}(L) \to C_n(K^n)$$
$$c \mapsto cz$$

using 11.3. That is $\phi_c = \phi_{cz}$ for $c \in C_{n-1}(L)$.

§12. Cellular homology

We begin by recalling a few standard facts from homological algebra and elementary algebraic topology.

(12.1) *If*

$$0 \to C \xrightarrow{\alpha} D \xrightarrow{\beta} E \to 0$$

is a short exact sequence of chain complexes, then there exists a map

$$\partial_* : H_*(E) \to H_*(C)$$

such that for $z \in H_n(E)$, $\partial_*(z) = [(\alpha^{-1} \circ \partial \circ \beta^{-1})(z)]$. That is if $z = e + B_n(E)$ with $e \in Z_n(E)$ then $\partial_*(z) = \alpha^{-1}(\partial(d)) + B_{n-1}(C)$ for each choice of $d \in D_n$ with $\beta(d) = e$.

Proof. See for example Lemma 4.5.3 in [5].

(12.2) *If*

$$0 \to C \xrightarrow{\alpha} D \xrightarrow{\beta} E \to 0$$

is a short exact sequence of chain complexes, then

$$\cdots \xrightarrow{\partial_*} H_n(C) \xrightarrow{\alpha_*} H_n(D) \xrightarrow{\beta_*} H_n(E) \xrightarrow{\partial_*} H_{n-1}(C) \to \cdots$$

is an exact sequence of groups.

Proof. See for example Theorem 4.5.4 in [5].

Recall if $\iota : L \to K$ is an inclusion of simplicial complexes then ι extends to a linear map $\iota : C_*(L) \to C_*(K)$ of simplicial chain groups, which induces a short exact sequence

$$(*) \qquad 0 \to C_*(L) \xrightarrow{\iota} C_*(K) \xrightarrow{j} C_*(K)/C_*(L) \to 0$$

of chain complexes. Let $H(K, L) = H_*(C_*(K)/C_*(L))$. Then by 12.1, we get a map

$$\begin{aligned} \partial_* : H(K, L) &\to H_*(L) \\ z &\mapsto [(\iota^{-1} \circ \partial \circ j^{-1})(z)] \end{aligned}$$

Let $C_n(K, L) = C_n(K)/C_n(L)$, $Z_n(K, L) = Z_n(C_*(K, L))$, and define $B_n(K, L) = B_n(C_*(K, L))$. Then $Z_n(K, L) = \partial^{-1}(C_{n-1}(L))/C_n(L)$ and $B_n(K, L) = (B_n(K) + C_n(L))/C_n(L)$. Now for $u \in Z_n(K, L)$ and $d \in C_n(K)$, $j(d) = u$ if and only if $u = d + C_n(L)$, in which case as $u \in Z_n(K, L)$, $\partial(d) \in C_{n-1}(L)$. Then by definition of ∂_*, $\partial_*([d]) = \partial(d) + B_{n-1}(L)$. Then composing ∂_* with $\iota_* : H_*(L) \to H_*(K)$ and $j_* : H_*(K) \to H_*(K, L)$, we can regard

$$\begin{aligned} \partial_* : H_n(K, L) &\to H_{n-1}(K, L) \\ [d] &\mapsto [\partial(d)] \end{aligned}$$

Notice $\partial_*^2([d]) = [\partial^2(d)] = 0$, so $\partial_*^2 = 0$. Therefore

(12.3) *If L is a subcomplex of the simplicial complex K then we have a chain complex $H_*(K, L)$ with boundary map ∂_* such that $Z_n(K, L) = \partial^{-1}(C_{n-1}(L))/C_n(L)$, $B_n(K, L) = (B_n(K)+C_n(L))/C_n(L)$, and $\partial_*([d]) = [\partial(d)]$*

(12.4) *If L is a subcomplex of the simplicial complex K containing no $(n-1)$-simplices of K then $H_n(K) \cong H_n(K, L)$.*

(12.5) *If L is a subcomplex of the simplicial complex K then*
(1) *We have an exact sequence of groups*:

$$\cdots \xrightarrow{\partial_*} H_n(L) \xrightarrow{\iota_*} H_n(K) \xrightarrow{j_*} H_n(K, L) \xrightarrow{\partial_*} H_{n-1}(L) \to \cdots$$

(2) *If $\iota_* : H_*(L) \to H_*(K)$ is an isomorphism then $H_*(K, L) = 0$.*
(3) *If $H_{n+1}(K, L) = H_n(K, L) = 0$ then $j_* : H_n(K) \to H_n(L)$ is an isomorphism.*

Proof. Applying 12.2 to the short exact sequence (*) above, we get (1). Then (1) implies (2) and (3).

(12.6) *Let $J \subseteq L \subseteq K$ be a chain of simplicial complexes. Then*

(1) *We have a short exact sequence of groups*:

$$\cdots \xrightarrow{\partial_*} H_n(L,J) \xrightarrow{\iota_*} H_n(K,J) \xrightarrow{j_*} H_n(K,L) \xrightarrow{\partial_*} H_{n-1}(L,J) \to \cdots$$

(2) *If* $H_{n-1}(L,J) = H_n(L,J) = 0$ *then* $j_* : H_n(K,J) \to H_n(K,L)$ *is an isomorphism.*

(3) *If* $H_n(L,J) = H_n(K,L) = 0$ *then* $H_n(K,J) = 0$.

(4) *Let* $\kappa : J \to L$ *be inclusion and assume* $\kappa_* : H_*(J) \to H_*(L)$ *is an isomorphism and* $H_n(K,L) = 0$. *Then* $H_n(K,J) = 0$.

Proof. We have the exact sequence

$$(**) \qquad 0 \to C_*(L)/C_*(J) \xrightarrow{\iota} C_*(K)/C_*(J) \xrightarrow{j} C_*(K)/C_*(L) \to 0$$

of chain complexes. Applying 12.2 to (**), we conclude that (1) holds. Of course (1) implies (2) and (3). Assume the hypotheses of (4). As $\kappa_* : H_*(J) \to H_*(L)$ is an isomorphism, 12.5.2 says $H_*(L,J) = 0$, so as $H_n(K,L) = 0$, (3) says $H_n(K,J) = 0$.

Now let (X,f) be a restricted combinatorial cell complex, $\Delta = \Delta(X)$ the graph of X, and $K = K(X)$ the triangulating complex of X. Define the subcomplexes K^n and K_n of K as in Section 11, and adopt the notational conventions of that section. An n-dimensional simplicial complex L is *homology spherical* if $\tilde{H}_i(L) = 0$ for $i \neq n$.

(12.7) *Let* $h(X) \geq n$ *and* $V_n^n = \{z_1, \ldots, z_r\}$. *Then*
(1) $Z_i(K^n, K_n) = \bigoplus_{j=1}^r Z_{ij}$ *and* $B_i(K^n, K_n) = \bigoplus_{j=1}^r B_{ij}$, *where*

$$Z_{ij} = \{\alpha z_j + C_i(K_n) : \alpha \in \tilde{Z}_i(\mathrm{Link}_{K^n}(z_j))\} \text{ and}$$

$$B_{ij} = \{\beta z_j + C_i(K_n) : \beta \in B_i(\mathrm{Link}_{K^n}(z_j))\}.$$

(2) $H_i(K^n, K_n) \cong \bigoplus_{j=1}^r \tilde{H}_i(\mathrm{Link}_{K^n}(z_j))$.
(3) $\partial_* : H_i(K^n, K_n) \to H_{i-1}(K_n)$ *acts via*

$$\partial_* : [\sum_{j=1}^i \alpha_j z_j] \mapsto (-1)^i \sum_{j=1}^i \alpha_j + B_{i-1}(K_n).$$

(4) *If* $\mathcal{O}(\dot{f}(x))$ *is homology spherical for all* $x \in X$ *of height* n *then* $H_i(K^n, K_n) = 0$ *for* $i \neq n$.

Proof. Without loss, X is of height n, so $K = K^n$. Let

$$V_{ij} = \{\alpha z_j + C_i(K_n) : \alpha \in C_i(\mathrm{Link}_K(z_j))\}.$$

Then $C_i(K, K_n) = \bigoplus_j V_{ij}$ and by 11.3, $\partial(\alpha z_j) = \partial(\alpha)z_j + (-1)^i\alpha \in \partial(\alpha)z_j + C_{i-1}(K_n)$, so

$$\partial(\alpha z_j + C_i(K_n)) = \partial(\alpha)z_j + C_{i-1}(K_n)$$

Therefore $\sum_j \alpha_j z_j \in \partial^{-1}(C_{i-1}(K_n))$ if and only if $\alpha_j \in \tilde{Z}_i(\mathrm{Link}_K(z_j))$ for all j. That is

$$Z_i(K, K_n) = \partial^{-1}(C_{i-1}(K_n))/C_i(K_n) = \bigoplus_j Z_{ij}.$$

Similarly the second statement of (1) holds. Of course (1) implies (2) and (2) and 11.2 imply (4). Also a typical element of $H_i(K, K_n)$ is of the form $[\sum_j \alpha_j z_j] = u + B_i(K, K_n)$, where $u = \sum_j \alpha_j z_j + C_i(K_n) \in Z_i(K, K_n)$. Then $\partial(\alpha_j) = 0$ so $\partial(\sum_j \alpha_j z_j) = (-1)^i \sum_j \alpha_j$ by 11.3, and hence

$$\partial([\sum_j \alpha_j z_j]) = \partial(\sum_j \alpha_j z_j) + B_{i-1}(K_n) = (-1)^i \sum_j \alpha_j + B_{i-1}(K_{n-1})$$

establishing (3).

(12.8) *Assume for all $x \in X$ of height n that $\mathcal{O}(\dot{f}(x))$ is homology spherical. Then $H_i(K^n) \cong H_i(K_n)$ for $i \leq n-2$.*

Proof. By 12.7.4, $H_i(K^n, K_n) = 0$ for $i \neq n$. Then by 12.5.3, $H_i(K_n) \cong H_i(K^n)$ for $i \neq n, n-1$.

(12.9) $H_*(K_{n+1}, K^n) = 0$.

Proof. By 11.4.2, $\iota_* : H_*(K^n) \to H_*(K_{n+1})$ is an isomorphism, so the lemma follows from 12.5.2.

(12.10) *Assume for all $m \geq n+2$ and for all $z \in X$ of height m that $\mathcal{O}(\dot{f}(x))$ is homology sphericial. Then $H_i(K) \cong H_i(K^{n+1})$ for $i \leq n$.*

Proof. It suffices to show $H_i(K^m) \cong H_i(K^{m-1})$ for each $m \geq n+2$ and $i \leq n$. As $m \geq n+2$, $\mathcal{O}(\dot{f}(z))$ is homology sphericial for each $z \in X$ of height m. So by 12.8, $H_i(K^m) \cong H_i(K_m)$ for $i \leq m-2$, and hence for $i \leq n$. Therefore by 11.4.2, $H_i(K^m) \cong H_i(K_m) \cong H_i(K^{m-1})$, as desired.

Recall the simplicial cell complex $X(n)$ of the n-simplex and its triangulating complex $K(n)$ discussed in earlier sections.

(12.11) (1) *$K(n)$ has the homotopy type of the n-simplex so it is contractible with trivial reduced homology and fundamental group.*

(2) *$K(n)^m \cong K(m)$ for $m \leq n$.*

(3) *$\tilde{H}_*(K(n)_m) = 0$ for all m, n.*

(4) *$H_*(K(n)^m, K(n)_m) = 0$ for $m > 0$.*

(5) *$H_*(K(n)_m, K(n)_{m-1}) = 0$ for $m \neq 1$.*

Proof. By Theorem 7.6, the n-simplex is triangulated by $K(n)$, so (1) holds. We observed in Section 11 that (2) holds. By 11.4.2, $H_*(K(n)_m) \cong H_*(K(n)^{m-1}) \cong H_*(K(m-1))$ by (2), so (1) implies (3).

To prove (4), we apply 12.5.1 with $K = K(m)$ and $L = K(m)_m$. By (3), $H_i(L) = 0$ for $i \neq 0$ and $\iota_* : H_0(L) \to H_0(K)$ is an isomorphism. As $H_i(L) = 0$ for $i \neq 0$, 12.5.1 says $H_i(K, L) = 0$ for $i \neq 0, 1$. Also

$$H_0(L) \xrightarrow{\iota_*} H_0(K) \xrightarrow{j_*} H_0(K, L) \xrightarrow{\partial_*} H_{-1}(L) = 0$$

is exact with ι_* an isomorphism, so $H_0(K, L) = 0$. Similarly

$$0 = H_1(K) \xrightarrow{j_*} H_1(K, L) \xrightarrow{\partial_*} H_0(L) \xrightarrow{\iota_*} H_0(K)$$

is exact with ι_* and isomorphism, so $H_1(K, L) = 0$. Thus (4) holds.

Finally we prove (5) by applying 12.6.3 with $J = K(n)_{m-1}$, $L = K(n)^{m-1}$, and $K = K(n)_m$. $H_*(L, J) = 0$ by (4) while $H_*(K, L) = 0$ by 12.9, so $H_*(K, J) = 0$ by 12.6.3.

(12.12) *Let $n \geq 1$, $B(n) = B_{n-1}(K(n)_n) + C_{n-1}(K(n)_{n-1})$, and let η and θ be the $(n-1)$-simplices of $K(n)$ defined by*

$$\eta = \{(n, i) : 0 \leq i < n\},$$

$$\theta = \{(n-1, i) : 0 \leq i < n\}.$$

Then $\eta \equiv \theta \mod B(n)$.

Proof. We prove the result by induction on n. Let $K = K(n)$. When $n = 1$, $\eta = \{(1, 0)\}$, $\theta = \{(0, 0)\}$, and $\sigma = \{(0, 0), (1, 0)\}$ is a simplex in K_1 with $\partial(\sigma) = \eta - \theta$. Then $\partial(\sigma) \in B_0(K_1) \leq B(1)$, so the lemma holds when $n = 1$ and our induction is anchored.

Let $0 \leq i < n - 1$ and

$$L(i) = \{(a, b) \in K : a \neq i \neq b\}$$

If we define $\pi : X(n) - \{i\} \to X(n-1)$ by $\pi(a) = a - 1$ for $a > i$ and $\pi(a) = a$ for $a < i$, then π induces an isomorphism $\pi : L(i) \to K(n-1)$ via $\pi(a, b) = (\pi(a), \pi(b))$. Moreover under this isomorphism, $\pi(\eta^i) =$

$\eta(n-1)$ and $\pi(\theta^i) = \theta(n-1)$ are the simplices playing the role of η and θ in $K(n-1)$. Thus by induction on n, $\eta(n-1) - \theta(n-1) = \partial(e) + f$ for some $e \in C_{n-1}(K(n-1)_{n-1})$ and $f \in C_{n-2}(K(n-1)_{n-2})$. Then $d_i = \pi^{-1}(e) \in C_n(K_n)$ and $c_i = \pi^{-1}(f) \in C_{n-1}(K_{n-1})$ with $\eta^i - \theta^i = \partial(d_i) - c_i$.

Next claim $\eta^{n-1} - \theta^{n-1} = \partial(d_{n-1}) + c_{n-1}$ for some $d_{n-1} \in C_n(K_n)$ and $c_{n-1} \in C_{n-1}(K_{n-1})$. When $n = 2$ take $d_{n-1} = \{(0,1),(0,2)\}$. Then observe that for $n > 2$ and $i < n-1$, $\pi(\eta^{n-1,i})$ and $\pi(\theta^{n-1,i})$ play the role in $K(n-1)$ of η^{n-1} and θ^{n-1}, so by induction on n there is

$$g \in C_{n-1}(K(n-1)_{n-1}) \text{ and } h \in C_{n-2}(K(n-1)_{n-2})$$

with $\pi(\eta^{n-1,i} - \theta^{n-1,i}) = \partial(g) + h$. Then $\delta_i = \pi^{-1}(g) \in C_n(K_n)$ and $\gamma_i = \pi^{-1}(h) \in C_{n-1}(K_{n-1})$ with $\eta^{n-1,i} - \theta^{n-1,i} = \partial(\delta_i) + \gamma_i$. Let $\delta = \sum_i (-1)^i \delta_i$ and $\gamma = \sum_i (-1)^i \gamma_i$. Then $\partial(\delta) + \gamma = \partial(\eta^{n-1} - \theta^{n-1})$, so $\eta^{n-1} - \theta^{n-1} - \delta \in Z_{n-1}(K_n, K_{n-1})$. However by 12.11.5, $H_{n-1}(K_n, K_{n-1}) = 0$, so there is $d_{n-1} \in C_n(K_n)$ and $c_{n-1} \in C_{n-1}(K_{n-1})$ with $\partial(d_{n-1}) = \eta^{n-1} - \theta^{n-1} - c_{n-1}$, completing the proof of the claim.

We now complete the proof of the lemma using the argument of the previous paragraph. Namely let $d = \sum_{i=0}^{n-1}(-1)^i d_i$ and $c = \sum_{i=0}^{n-1}(-1)^i c_i$. Then $\partial(d) + c = \partial(\eta - \theta)$, so as $H_{n-1}(K_n, K_{n-1}) = 0$, there is $\alpha \in C_n(K_n)$ and $\beta \in C_{n-1}(K_{n-1})$ with $\partial(\alpha) = \eta - \theta - \beta$.

In the next lemma we use the map

$$\phi \in \mathrm{Hom}(C_*(K(n)), \mathrm{Hom}(C_{n-1}(\mathrm{Link}_{K^n}(z)), C_*(K^n)))$$

defined in Section 11 for each $z \in V_n^n$.

(12.13) *Let θ be the $(n-1)$-simplex of $K(n)$ defined by*

$$\theta = \{(n-1, i) : 0 \le i < n\}$$

and let $B = B_{n-1}(K_n) + C_{n-1}(K_{n-1})$. Then for each $z \in V_n^n$ and each $\xi \in C_{n-1}(\mathrm{Link}_K(z))$,

$$\xi \equiv \phi_\theta(\xi) \mod B.$$

Moreover for $s = \{v_0, \ldots, v_{n-1}\} \in \Sigma^n(\mathrm{Link}_{K^n}(z))$, $\phi_\theta(s) = f_{v_{n-1}}(s)$.

Proof. The last statement of the lemma is just the definition of ϕ_θ. By 11.7,

$$\phi_{B_{n-1}(K(n)_n)} = \phi_{\partial(C_n(K(n)_n))} = \partial(\phi_{C_n(K(n)_n)}),$$

while $\phi_{C_n(K(n)_n)}(C_{n-1}(L)) \subseteq C_n(K_n)$. Therefore

$$\phi_{B_{n-1}(K(n)_n)}(C_{n-1}(L)) \leq B_{n-1}(K_n).$$

Similarly $\phi_{C_{n-1}(K(n)_{n-1})}(C_{n-1}(L)) \leq C_{n-1}(K_{n-1})$. Therefore the lemma follows from 12.12 and the fact that $\phi_\eta(\xi) = \xi$ for each $\xi \in C_{n-1}$ $(\mathrm{Link}_{K^n}(z))$, where η is the simplex of $K(n)$ defined in 12.12.

We now define the cellular homology of X. Let $D_n(X) = H_n(K^n, K_n)$. Equivalently for $z \in V_n^n$ let

$$D(z) = \{ \sum_{s\in\Sigma^{n-1}(\dot{f}(z))} d_{z,s}zs : \sum_s d_{z,s}s \in \tilde{Z}_{n-1}(\dot{f}(z))\} \leq C_n(f(z))$$

with $d_{z,s} \in \mathbf{Z}$, so that $D(z) \cong \tilde{H}_{n-1}(\dot{f}(z))$. Then by 12.7,

$$D_n(X) = \bigoplus_{z\in V_n^n} D(z) \cong \bigoplus_{z\in V_n^n} \tilde{H}_{n-1}(\dot{f}(z))$$

and this definition of $D_n(X)$ is usually easier to work with. Define the boundary map on the chain complex $D_*(X) = (D_n(X) : 0 \leq n \in \mathbf{Z})$ by

$$\begin{gathered}\partial_n : D_n(X) \to D_{n-1}(X)\\ \sum_{z\in V_n^n} zd_z \mapsto (-1)^n \sum_z \phi_\theta(d_z)\end{gathered}$$

where $d_z \in \tilde{Z}_{n-1}(\mathrm{Link}_{K^n}(z))$ and $\theta = \{(n-1, i) : 0 \leq i < n\} \in \Sigma_{n-1}(n)$. Recall from 12.13 that if $d_z = \sum_{s\in\Sigma^{n-1}(\mathrm{Link}_{K^n}(z))} d_{z,s}s$ with $d_{z,s} \in \mathbf{Z}$ then $\phi_\theta(d_z) = \sum_s d_{z,s}f_{w(s)}(s)$, where $w(s)$ is the greatest element of s. Therefore

$$\partial_n : \sum_{z,s} d_{z,s}zs \mapsto (-1)^n \sum_{z,s} d_{z,s}f_{w(s)}(s)$$

which is usually an easier definition to work with. We see in a moment that $\partial_{n-1} \circ \partial_n = 0$, so ∂ is indeed a boundary map. We call $D_*(X)$ the *cellular chain complex* of X and ∂ the *cellular boundary map*. The *cellular homology* of X is $H_*^c(X) = H_*(D_*(X))$.

Now we verify that $\partial_{n-1} \circ \partial_n = 0$ by showing $\partial_{n-1}(\partial_n(zd_z)) = 0$ for each $z \in V_n^n$ and $d_z \in \tilde{Z}_{n-1}(\mathrm{Link}_{K^n}(z))$. First $d_z = \sum_u ud_{z,u}$ and $d_{z,u} = \sum_t d_{z,u,t}t$, where

$$d_{z,u,t} \in \mathbf{Z}, \quad d_{z,u} \in \tilde{Z}_{n-2}(\mathrm{Link}_{K^n}(\{z, u\})),$$

and the sums are over all $u \in \text{Link}_{K^n}(z)$ of height $n-2$ and all $(n-2)$-simplices t in $\text{Link}_{K^n}(\{z,u\})$. As $\partial(d_z) = 0$, 1.1.4 says $0 = \sum_u d_{z,u}$. That is

$$0 = \sum_{u,t} d_{z,u,t}t = \sum_t \Big(\sum_{t<u<z} d_{z,u,t} \Big) t, \text{ so } \sum_{t<u<z} d_{z,u,t} = 0$$

for each t. Now

$$(\partial_{n-1} \circ \partial_n)(zd_z) = (-1)^n \partial_{n-1} \Big(\sum_{u,t} d_{z,u,t} f_u(u) f_u(t) \Big)$$

$$= -\sum_{u,t} d_{z,u,t} f_{w(f_u(t))}(f_u(t)) = -\sum_t \Big(\sum_{t<u<z} d_{z,u,t} \Big) f_{w(t)}(t) = 0.$$

So the cellular boundary map ∂ is indeed a boundary map on the cellular chain complex.

(12.14) *Let*

$$f_m : H_m(K^m, K^{m-1}) \to H_m(K^m, K_m) \text{ and}$$

$${}_{n+1}\partial_* : H_{n+1}(K^{n+1}, K^n) \to H_n(K^n, K^{n-1})$$

be the maps induced via 12.6 *by the inclusions*

$$K^{m-1} \subseteq K_m \subseteq K^m \text{ and}$$

$$K^{n-1} \subseteq K^n \subseteq K^{n+1}$$

respectively. Let

$$\partial_{n+1} : H_{n+1}(K^{n+1}, K_{n+1}) \to H_n(K^n, K_n)$$

be the cellular boundary map. Then the diagram

$$\begin{array}{ccc} H_{n+1}(K^{n+1}, K^n) & \xrightarrow{{}_{n+1}\partial_*} & H_n(K^n, K^{n-1}) \\ {\scriptstyle f_{n+1}}\downarrow & & \downarrow{\scriptstyle f_n} \\ H_{n+1}(K^{n+1}, K_{n+1}) & \xrightarrow[\partial_{n+1}]{} & H_n(K^n, K_n) \end{array}$$

commutes and f_n is an isomorphism for all n.

Proof. By 12.9, $H_*(K_m, K^{m-1}) = 0$ for all m, so applying 12.6.2 to the chain $K^{m-1} \subseteq K_m \subseteq K^m$, we conclude $f_m : H_m(K^m, K^{m-1}) \to H_m(K^m, K_m)$ is an isomorphism for each m.

To complete the proof it remains to show the following diagram commutes:

$$\begin{array}{ccccc} H_{n+1}(K^{n+1}, K^n) & \xrightarrow{\hat{\partial}_*} & H_n(K^n, K^{n-2}) & \xrightarrow{j_*} & H_n(K^n, K^{n-1}) \\ f_{n+1}\downarrow & & \iota_*\downarrow & & \downarrow f_n \\ H_{n+1}(K^{n+1}, K_{n+1}) & \xrightarrow[\partial'_*]{} & H_n(K_{n+1}, K^{n-2}) & \xrightarrow[\rho]{} & H_n(K^n, K_n) \end{array}$$

that ${}_{n+1}\partial_* = j_* \circ \hat{\partial}_*$, and that $\rho \circ \partial'_* = \partial_{n+1}$. Here ∂'_* and ι_* are the maps induced via 12.6 by the inclusions

$$K^{n-2} \subseteq K_{n+1} \subseteq K^{n+1}$$

$$K^{n-2} \subseteq K^n \subseteq K_{n+1}$$

respectively, $\rho = f_n \circ j_* \circ \iota_*^{-1}$, and

$$\hat{\partial}_* : H_{n+1}(K^{n+1}, K^n) \to H_n(K^n, K^{n-2}) \text{ and}$$

$$j_* : H_n(K^n, K^{n-2}) \to H_n(K^n, K^{n-1})$$

are the maps induced via 12.6 by the inclusions

$$K^{n-2} \subseteq K^n \subseteq K^{n+1} \text{ and}$$

$$K^{n-2} \subseteq K^{n-1} \subseteq K^n$$

respectively.

By 12.9, $H_*(K^n, K_{n+1}) = 0$, so applying 12.6 to $K^{n-2} \subseteq K^n \subseteq K_{n+1}$ we conclude $\iota_* : H_n(K^n, K^{n-2}) \to H_n(K_{n+1}, K^{n-2})$ is an isomorphism, so ρ is well defined. By definition of ρ, the right hand square in the diagram commutes, so to show the full diagram commutes, it remains to show the left hand square commutes.

From the discussion at the beginning of this section,

$$H_{m+1}(K^{m+1}, K^m) = Z_{m+1}(K^{m+1}, K^m)/B_{m+1}(K^{m+1}, K^m)$$

with

$$Z_{m+1}(K^{m+1}, K^m) = \partial^{-1}(C_m(K^m))/C_{m+1}(K^m)$$

and

$$B_{m+1}(K^{m+1}, K^m) = (B_{m+1}(K^{m+1}) + C_{m+1}(K^m))/C_{m+1}(K^m).$$

So as $C_{m+1}(K^m) = B_{m+1}(K^{m+1}) = 0$, we have

$$H_{m+1}(K^{m+1}, K^m) = \partial^{-1}(C_m(K^m)).$$

That is a typical element of $H_{m+1}(K^{m+1}, K^m)$ is some $d \in C_{m+1}(K^{m+1})$ with $\partial(d) \in C_m(K^m)$. Similarly

$$H_{m+1}(K^{m+1}, K^m) = \partial^{-1}(C_m(K_{m+1})/C_{m+1}(K_{m+1})$$

and $f_{m+1} : d \mapsto d + C_{m+1}(K_{m+1})$.

Also $H_n(K^n, K^{n-2}) = \tilde{Z}_n(K^n)$,

$$H_n(K_{n+1}, K^{n-2}) = H_n(K_{n+1}) = \tilde{Z}_n(K_{n+1})/B_n(K_{n+1}),$$

$\hat{\partial}_* : d \mapsto \partial(d)$, $\partial'_* : d + C_{n+1}(K_{n+1}) \mapsto \partial(d) + B_n(K_{n+1})$, and $\iota_* : z \mapsto z + B_n(K_{n+1})$. Therefore

$$(\iota_* \circ \hat{\partial}_*)(d) = \partial(d) + B_n(K_{n+1}) = (\partial'_* \circ f_{n+1})(d)$$

completing the proof that the diagram commutes.

Finally the proof that $\partial_{n+1} = \rho \circ \partial'_*$ and ${}_{n+1}\partial_* = j_* \circ \hat{\partial}_*$. As

$$\iota_* : H_n(K^n, K^{n-2}) \to H_n(K_{n+1}, K^{n-2})$$

is an isomorphism, $\tilde{Z}_n(K_{n+1}) = \tilde{Z}_n(K^n) \oplus B_n(K_{n+1})$. Thus if $\pi : \tilde{Z}_n(K_{n+1}) \to \tilde{Z}_n(K^n)$ is the projection with respect to this direct sum decomposition, we have $\rho = f_n \circ j_* \circ \iota_*^{-1} = f_n \circ j_* \circ \pi$. Further we have seen $H_n(K^n, K^{n-1}) = \partial^{-1}(C_{n-1}(K^{n-1}))$, $H_n(K^n, K_n) = \partial^{-1}(C_{n-1}(K_n)/C_n(K_n))$, and $f_n : d' \mapsto d' + C_n(K_n)$. Finally j_* is the identity map on $H_n(K^n, K^{n-1})$, so $\rho : e \mapsto \pi(e) + C_n(K_n)$. Therefore

$$\rho \circ \partial'_* : d + C_{n+1}(K_{n+1}) \mapsto \pi(\partial(d)) + C_n(K_n).$$

Also $(j_* \circ \hat{\partial}_*)(d) = \partial(d) = {}_{n+1}\partial_*(d)$, with the last equality following from the discussion at the beginning of this section.

Recall from 12.7 that we can choose are coset representative d so that $d = \sum_{z \in V_{n+1}^{n+1}} z d_z$, with $d_z \in \tilde{Z}_n(\mathrm{Link}_{K^{n+1}}(z))$, and that by 11.3, $\partial(z d_z) = (-1)^{n+1} d_z$. Thus it remains to show that if $d = z d_z$ then $\pi(d_z) \equiv \phi_\theta(z d_z) \mod C_n(K_n)$. Let $e = \phi_\theta(z d_z)$. By 12.13, $d_z = e + b + c$ for some $b \in B_n(K_{n+1})$ and $c \in C_n(K_n)$. Then

$$d_z - b = e + c \in \tilde{Z}_n(K_{n+1}) \cap C_n(K^n) \leq \tilde{Z}_n(K^n),$$

so $e + c = \pi(d_z)$ and therefore $\pi(d_z) = e + c \equiv e \mod C_n(K_n)$, completing the proof.

(12.15) *Assume for each $x \in X$ of height n that $\mathcal{O}(\dot{f}(x))$ is homology sphericial. Then $H_i(K^n, K^{n-1}) = 0$ for all $i \neq n$.*

Proof. By 12.9, $H_*(K_n, K^{n-1}) = 0$, so applying 12.6.2 to the sequence $K^{n-1} \subseteq K_n \subseteq K^n$, we conclude $H_*(K^n, K^{n-1}) \cong H_*(K^n, K_n)$. Then as $\mathcal{O}(\dot{f}(x))$ is homology spherical for each $x \in X$ of height n, 12.7.4 completes the proof.

Theorem 12.16. *Let X be a restricted combinatorial cell complex such that $\mathcal{O}(\dot{f}(x))$ is spherical for each $x \in X$. Then the ordinary homology $H_*(X)$ of X is isomorphic to the cellular homology $H^c_*(X)$.*

Proof. In Lemma 12.14 we defined maps

$$ {}_{n+1}j_* \circ {}_{n+1}\hat{\partial}_* = {}_{n+1}\partial_* : H_{n+1}(K^{n+1}, K^n) \to H_n(K^n, K^{n-1}) $$

and proved these maps are isomorphic to the cellular maps

$$ \partial_{n+1} : D_{n+1}(X) \to D_n(X). $$

Thus the chain complex $(H_n(K^n, K^{n-1}), {}_n\partial_* : 0 \le n \in \mathbf{Z})$ is isomorphic to the cellular chain complex $D_*(X)$. Therefore it suffices to show

$$ H_n(X) \cong ker({}_n\partial_*)/Im({}_{n+1}\partial_*) $$

To do so we use the standard proof for CW- complexes.

First the inclusions

$$ K^{n-2} \subseteq K^{n-1} \subseteq K^n $$

$$ K^{n-2} \subseteq K^n \subseteq K^{n+1} $$

induce via 12.6 the maps

$$ \begin{tikzcd} H_n(K^{n-1}, K^{n-2}) \arrow[dr] & & H_n(K^{n+1}, K^{n-1}) \arrow[r] & H_n(K^{n+1}, K^n) \\ & H_n(K^n, K^{n-2}) \arrow[ur, "\iota_*"] \arrow[dr, "j_*"'] & & \\ H_{n+1}(K^{n+1}, K^n) \arrow[ur, "\hat{\partial}_*"] & & H_n(K^n, K^{n-1}) \arrow[r, "{}_n\partial_*"] & H_{n-1}(K^{n-1}, K^{n-2}) \end{tikzcd} $$

By 12.15, $H_n(K^{n-1}, K^{n-2}) = 0$, so as the sequences of 12.6 are exact, $j_* = {}_{n+1}j_*$ is injective with $Im(j_*) = ker({}_n\partial_*)$. As $\mathcal{O}(\dot{f}(x))$ is homology spherical for each $x \in X$, $H_n(K^{n+1}, K^n) = 0$ by 12.15. Therefore ι_* is a surjection, so

$$ \iota_* \circ j_*^{-1} : ker({}_n\partial_*) \to H_n(K^{n+1}, K^{n-2}) $$

is a surjection with kernel

$$ j_*(ker(\iota_*)) = j_*(Im({}_{n+1}\partial_*)) = Im({}_{n+1}\partial_*) $$

That is

$$H_n(K^{n+1}, K^{n-2}) \cong ker({}_n\partial_*)/Im({}_{n+1}\partial_*)$$

Finally by 12.4, $H_n(K^{n+1}) \cong H_n(K^{n+1}, K^{n-2})$, while as $\mathcal{O}(\dot{f}(x))$ is homology spherical for each $x \in X$, $H_n(K^{n+1}) \cong H_n(K)$ by 12.10. Thus the Theorem is established.

(12.17) *Assume $\dot{f}(x)$ is simply connected for each $x \in X$ with $h(x) \geq 3$. Then*
(1) $\pi_1(K(X^{n+1})) \cong \pi_1(K(X^n))$ *if* $n \geq 2$,
(2) $\pi_1(K(X)) \cong \pi_1(K(X^n))$ *for* $n \geq 2$.

Proof. Evidently (1) implies (2), so we prove (1). Recall the definitions of X^n, K^n, and K_n from Section 11. In particular $K^n = K(X^n)$, so it suffices to show $\pi_1(K^{n+1}) \cong \pi_1(K^n)$.

Let $\iota : K^n \to K_{n+1}$ be the inclusion map. By 11.4.2, ι is a homotopy equivalence, so $\pi_1(K^n) \cong \pi_1(K_{n+1})$. Thus it remains to show $\pi_1(K_{n+1}) \cong \pi_1(K^{n+1})$.

Let $\pi : K_{n+1} \to K^{n+1}$ be the inclusion map and $s = \{v_0, \ldots, v_k\}$ be a simplex in K^{n+1} ordered as in Lemma 5.4. If $v_0 \in K_{n+1}$ then $v_0 \in P = \pi^{-1}(st_{K^{n+1}}(s)) \subseteq st_{K_{n+1}}(v_0)$ and hence P is contractible. Thus we may take $s = \{x\}$ with $h(x) = n+1$. In this case $P = \dot{f}(x)$ is simply connected by hypothesis. Thus by Theorem 1 in [3], $\pi_1(K_{n+1}) \cong \pi_1(K^{n+1})$, as desired.

§13. The torus and the Klein bottle

In this section we consider the torus and the Klein bottle as examples.

Let X be the poset of dimension 2 with a unique maximal element c, and unique minimal element d, and two elements a_1 and a_2 of height 1.

We associate a combinatorial cell $f(x)$ to each $x \in X$. Let $f(c)$ have 4 elements b_i of height 1 and 4 elements v_i of height 0, with $v_i, v_{i-1} < b_i$, where the indices are read modulo 4. Further we let $\zeta(b_i) = a_{[i]}$, where $[i] = i \mod 2$. This forces $f(a_i)$ to have 2 elements u_{ij}, $j = 1, 2$, of height 0.

It remains to describe $f_i = f_{b_i} : f(c)(\leq b_i) \to f(a_{[i]})$. We may choose notation so that $f_2(v_i) = u_{2,i}$ and $f_3(v_{i+1}) = u_{1,i}$. We say that b_2 and b_4 have the *same orientation* if $f_4(v_3) = f_2(v_2)$; here the pair v_2, v_3 is distinguished by $b_3 > v_2, v_3$, whereas no member of $f(c)$ of height 1 is greater than v_2 and v_4. Up to change of notation, we are left with 3 cases:

(i) b_2 and b_4 and b_1 and b_3 both have the same orientation.
(ii) Exactly one pair (say b_2 and b_4) has the same orientation.
(iii) Neither pair has the same orientation.

We will see that the geometric realization of (X, f) in case (i) corresponds to the torus, in case (ii) to the Klein bottle, and in case (iii) the realization is not a manifold.

We have defined our combinatorial cell (X, f). We next consider the polyhedral cell complex $\mathcal{P}(X) = (X, f, F, B)$ of (X, f) and the geometric realization $T(X)$ of (X, f). We can regard $F(c)$ as the unit square with vertices $F(v_i)$ arranged in order as in the diagram below. Then $F(b_i)$ is the edge $[v_{i-1}, v_i]$. Thus we have the picture

$$\begin{tikzcd} v_4 \arrow[d, "F(b_1)"'] & v_3 \arrow[l, "F(b_4)"] \\ v_1 \arrow[r, "F(b_2)"] & v_2 \arrow[u, "F(b_3)"'] \end{tikzcd}$$

Similarly $F(a_i)$ is the unit interval $[F(u_{i1}), F(u_{i2})]$ with initial point $F(u_{i1})$ and endpoint $F(u_{i2})$. Finally we have

$$F_{b_i} : F(b_i) \to F(a_{[i]})$$
$$tF(v_{i-1}) + (1-t)F(v_i) \mapsto tF(f_i(v_{i-1})) + (1-t)F(f_i(v_i))$$

In cases (i) and (ii), b_2 and b_4 have the same orientation, so in the geometric realization $T(X)$ the edges $F(b_2)$ and $F(b_4)$ are identified with the same orientation, resulting in a tube with ends $F(b_1)$ and $F(b_3)$. In case (i), these ends are identified with the same orientation, resulting in a torus, while in case (ii) they are identified with a twist, resulting in a Klein bottle. Finally in case (iii), $F(b_2)$ and $F(b_4)$ are identified with a twist, resulting in a Möbius strip, and then $F(b_1)$ and $F(b_3)$ are identified with a twist, yielding a space $T(X)$ which is not a manifold, since at a neighborhood of $\tilde{F}(d)$ we get two copies of the 2-ball glued at $\tilde{F}(d)$.

We next discuss the the homology of our cell complexes. First $\dot{f}(c)$ has the homotopy type of the 1-sphere and $\dot{f}(a_i)$ the type of the 0-sphere, so by Theorem 12.15, $H^c(X)_* = H_*(X)$. Further $H_1(\dot{f}(c)) = Z_1(\dot{f}(c))$ has a unique generator

$$\gamma = \sum_{i=1}^{4} v_i b_i - v_{i-1} b_i$$

where the indices are read modulo 4, so by the definition of $D_*(X)$, $D_2(X) = D(c)$ has a unique generator $c\gamma$. Similarly $D_1(X) = D(a_1) \oplus D(a_2)$ has two generators $a_i\alpha_i$, where

$$\alpha_i = u_{i,2} - u_{i,1}$$

and $D_0(X) = D(d)$ is 1-dimensional with generator d.

Next by definition of the cellular boundary map ∂,

$$\partial(a_i\alpha_i) = f_{u_{i,2}}(u_{i,2}) - f_{u_{i,1}}(u_{i,1}) = d - d = 0$$

so that $D_1(X) = Z_1(X)$ and $H_0(X) \cong D_0(X) \cong \mathbf{Z}$. Similarly

$$\partial(c\gamma) = \sum_{i=1}^{4} f_{b_i}(b_i(v_i - v_{i-1})) = \sum_{i=1}^{4} a_{[i]} f_i(v_i - v_{i-1})$$

$$= a_1(f_1(v_1 - v_4) + f_3(v_3 - v_1)) + a_2(f_2(v_2 - v_1) + f_4(v_4 - v_3)) = a_1A_1 + a_2A_2$$

Now if b_2 and b_4 have the same orientation then $f_2(v_2) = f_4(v_3)$, and hence also $f_4(v_4) = f_2(v_1)$, so that $A_2 = 0$. On the other hand if the orientation is opposite then $u_{2,2} = f_2(v_2) = f_4(v_4)$, so $u_{2,1} = f_2(v_1) = f_4(v_3)$ and hence $A_2 = 2\alpha_2$. A similar remark holds for A_1.

Now if X is the torus then both pairs have the same orientation, so $\partial(c\gamma) = 0$ and $B_1(X) = 0$. Thus $H_2(X) = D_2(X) \cong \mathbf{Z}$ and $H_1(X) \cong D_1(X) \cong \mathbf{Z} \oplus \mathbf{Z}$. On the other hand if X is the Klein bottle then $A_2 = 0$ and $A_1 = 2\alpha_1$, so $\partial(c\gamma) = 2\alpha_1$. Therefore for the Klein bottle, $H_2(X) = 0$ and $B_1(X) = 2\mathbf{Z}\alpha_1$, so $H_1(X) \cong \mathbf{Z} \oplus \mathbf{Z}_2$. Finally in case (iii), $\partial(c\gamma) = 2(\alpha_1 + \alpha_2)$, so again $H_2(X) = 0$ and $H_1(X) \cong \mathbf{Z} \oplus \mathbf{Z}_2$.

§14. The dual cell complex of a restricted cell complex

In this section (X, f) is a restricted combinatorial cell complex of finite height whose map ζ is *surjective*; that is $\zeta : f(x) \to X(\leq x)$ is surjective for each $x \in X$. The *dual complex* $(\hat{X}, \hat{f})$ of (X, f) is the combinatorial cell complex defined below. But first an example.

Example. Let (X, f) be a regular cell complex. Then (X, f) is isomorphic to the simplicial cell complex of the poset X, so without loss it is that complex. That is $f(x) = X(\leq x)$ and the maps ζ and f_v, $v \in f(x)$, are the appropriate identity maps. It will turn out that the dual complex for *this* complex is the simplicial cell complex of the dual poset $\hat{X}$ whose ordering $\lesssim$ is obtained by reversing the ordering on X. So the duality operation on cell complexes is a generalization of the duality on posets.

In general the poset of the dual complex is the dual poset $\hat{X}$. In addition to defining the poset of the dual complex we must define the cells $\hat{f}(x)$ for each $x \in \hat{X}$, the zeta map for $\hat{X}$, and the f-v maps for each $v \in \hat{V}$. In our example, $\hat{f}(x) = X(\geq x)$. It will turn out in general that the zeta map for the dual complex $\hat{X}$ is the hat-zeta map $\hat{\zeta}$ for the complex $\hat{X}$. So in our case $\hat{\zeta}(v) = v$ for $v \in V = \hat{V}$. Similarly it will turn out that the f-v map $\hat{f}_v : \hat{f}(x)(\lesssim v) \to \hat{f}(\hat{\zeta}(v))$ for $\hat{X}$ is the hat-f-v map for X, so in our case that map is $\hat{f}_v(u) = u$ for $u \leq v \in f(x)$.

Now the definition of the dual complex. First the poset $\hat{X}$ of the complex is the dual poset of X. That is X and $\hat{X}$ are the same as sets with the ordering $\lesssim$ on $\hat{X}$ defined by $x \lesssim y$ if and only if $y \leq x$. As X has finite height, $\hat{X}$ has the same finite height, so all elements of $\hat{X}$ are of finite height.

Second for $x \in \hat{X}$, define

$$\hat{f}(x) = \{v \in V : \zeta(v) = x\}$$

partially ordered by $u \lesssim v$ if and only if $v \in L(u)$. We check that this is a partial order: By definition the relation is reflexive and antisymmetric. Suppose $u \lesssim v \lesssim w$ in $\hat{f}(x)$. Then $v \in L(u)$ so $f_a(u) \geq v$ where $a = \hat{f}_v(u)$. Indeed as $u, v \in \hat{f}(x)$, $\zeta(u) = x = \zeta(v)$, so $\zeta(f_a(u)) = \zeta(u) = \zeta(v)$ and hence $f_a(u) = v$. Similarly as $v \lesssim w$, $f_b(v) = w$, where $b = \hat{f}_w(v)$. But now $c = f_a^{-1}(b) = \hat{f}_w(u)$ so $u \lesssim w$. For by axiom (ii) for combinatorial cell complexes,

$$f_c(u) = f_{f_a(c)}(f_a(u)) = f_b(v) = w.$$

Observe that by definition $\hat{V} = V$ and $\hat{f}(x)$ is the set of elements $v \in V$ such that $x_\infty \in L(v)$.

Recall the hat-zeta-function $\hat{\zeta}$ for X is defined by $\hat{\zeta}(v)$ is the unique $x \in X$ with $v \in f(x)$. Notice that the hat-zeta-function for $\hat{X}$ is the zeta-function for X. Conversely we define the zeta-function for $\hat{X}$ to be the hat-zeta-function for X. We verify $\hat{X}$ satisfies axiom (i) for combinatorial cell complexes; that is we check that $\hat{\zeta} : \hat{f}(x) \to \hat{X}$ is a map of posets preserving height. Namely if $u, v \in \hat{f}(x)$ with $u \lesssim v$ then $f_a(u) = v$ where $a = \hat{f}_v(u)$, and $\hat{\zeta}(u) \geq \zeta(a) = \hat{\zeta}(v)$, so $\hat{\zeta}$ preserves the order. Next as X is of finite height, there is a chain $y_0 > \cdots > y_m = \hat{\zeta}(u)$ of maximal length m and as $\zeta : f(y_i) \to X(\leq y_i)$ is surjective for each i, we can lift this chain to a chain $w_0 > \cdots > w_m$ in $f(y_0)$ with $\zeta(w_i) = y_i$. Let $a = f_{w_m}^{-1}(u)$ and $a_i = f_{w_i}(a)$. Then $a_0 \lesssim \cdots \lesssim a_m = u$ is a chain of length m in $\hat{f}(x)$, so $\hat{\zeta}$ preserves height. This completes the verification of axiom (i).

Next for $x \in X$ and $v \in \hat{f}(x)$,

$$\begin{aligned}\hat{f}(x)(\lesssim v) =&\{u \in V : \zeta(u) = \zeta(v) \text{ and } v \in L(u)\}\\ =&\{u \in V : v \in L(u) \text{ and } f_{\hat{f}_v(u)}(u) = v\}.\end{aligned}$$

As X is restricted, for each $v \in V$ we have the hat-f-v-function $\hat{f}_v$ defined on the set of $u \in V$ with $v \in L(u)$ by $\hat{f}_v(u)$ the unique $w \in f(\hat{\zeta}(u))$ with $w \geq u$ and $\zeta(w) = \hat{\zeta}(v)$. We define the f-v-function for $\hat{X}$ to be the restriction to $\hat{f}(x)(\lesssim v)$ of the hat-f-v-function $\hat{f}_v$ for X. We next verify that $\hat{X}$ satisfies axiom (ii) for combinatorial cell complexes. By definition $\zeta(\hat{f}_v(u)) = \hat{\zeta}(v) = x$, so

$$\hat{f}_v : \hat{f}(x)(\lesssim v) \to \hat{f}(\hat{\zeta}(v)) = \hat{f}(x).$$

We check that $\hat{f}_v$ is an isomorphism of posets. First if $\hat{f}_v(u_1) = \hat{f}_v(u_2) = w$ then $f_w(u_1) = v = f_w(u_2)$, so as f_w is injective, $u_1 = u_2$. That is $\hat{f}_v$ is injective. Next if $w \in \hat{f}(x)$ then $f_w^{-1}(v) = u \in \hat{f}(x)(\lesssim v)$ with $\hat{f}_v(u) = w$, so $\hat{f}_v$ is a bijection.

Suppose $a, b \in \hat{f}(x)(\lesssim v)$ with $a \lesssim b$. Then $b \in L(a)$ so $f_w(a) = b$ for $w = \hat{f}_b(a)$. Also $f_w(\hat{f}_v(a)) = \hat{f}_v(b)$ so $\hat{f}_v(b) \in L(\hat{f}_v(a))$ and therefore $\hat{f}_v(a) \lesssim \hat{f}_v(b)$. Similarly if $\hat{f}_v(a) \lesssim \hat{f}_v(b)$ then $a \lesssim b$. This completes the verification of axiom (ii).

We next verify axiom (iii); that is we prove that if $a, b \in \hat{f}(x)$ with $a \lesssim b$ then $\hat{f}_a = \hat{f}_{\hat{f}_b(a)} \circ \hat{f}_b$. For let $c \in \hat{f}(x)(\lesssim a)$. Then $\hat{f}_b(c) \lesssim \hat{f}_b(a)$ as $\hat{f}_b$ preserves order, so $w = \hat{f}_{\hat{f}_b(a)}(\hat{f}_b(c)) \geq \hat{f}_b(c) \geq c$ with $\zeta(w) = \hat{\zeta}(\hat{f}_b(a)) = \hat{\zeta}(a)$. Hence as X is restricted, $w = \hat{f}_b(c)$. So axiom (iii) is established.

Next axiom (iv). Let $v \in \hat{f}(x)$ and $a \in \hat{f}(x)(\lesssim v)$. Then $v \in L(a)$ with $\hat{\zeta}(\hat{f}_v(a)) = \hat{\zeta}(a)$ by definition of the hat-zeta map and the hat-f-a-map, so indeed $\hat{\zeta} = \hat{\zeta} \circ \hat{f}_v$, as required in axiom (iv).

Finally we check axiom (v). First $\hat{\infty}_x = \infty_x$ and for $v \in \hat{f}(x)$, $\hat{f}_{\infty_x}(a) = a$ as $\zeta(a) = x = \zeta(\infty_x)$. Thus $\hat{f}_{\infty_x}$ is the identity map. Also $\hat{\zeta}(\infty_x) = x$ again by definition of the hat-zeta-map.

We have shown

(14.1) *Let (X, f) be a combinatorial cell complex of finite height with ζ surjective. Then the dual complex $(\hat{X}, \hat{f})$ is a combinatorial cell* *complex.*

(14.2) $(\hat{X}, \hat{f})$ *is restricted.*

Proof. Let $u_1, u_2, v \in \hat{f}(x)$ with $v \lesssim u_1, u_2$ and $\hat{\zeta}(u_1) = \hat{\zeta}(u_2) = y$ say. Then $\zeta(u_1) = \zeta(u_2) = \zeta(v) = x$ and $u_1, u_2 \in L(v)$, so there exists $w_i \geq v$ with $f_{w_i}(v) = u_i$. But $\zeta(w_i) = \hat{\zeta}(u_i) = y$, so as X is restricted, $w_1 = w_2$ and hence $u_1 = f_{w_1}(v) = f_{w_2}(v) = u_2$.

(14.3) $u \in \hat{L}(v)$ *if and only if* $v \in L(u)$*, in which case the image of* v *under the hat-f-u-map for* $\hat{X}$ *is*

$$f_{f_{\hat{f}_v(u)}(u)}(v) = f_u(f^{-1}_{\hat{f}_v(u)}(v)).$$

Proof. Let $v \in L(u)$. Then $f_w(u) \geq v$ for $w = \hat{f}_v(u)$ and $u \lesssim f_w(u)$ in $\hat{f}(\zeta(u))$. Let $z = f_{f_w(u)}(v)$. Then $z \in L(v)$ with $\hat{f}_z(v) = f_w(u)$ and $\zeta(z) = \zeta(v)$ so $v \lesssim z$ in $\hat{f}(\zeta(v))$. Then as $\hat{f}_z(v) = f_w(u) \gtrsim u$ in $\hat{f}(\zeta(u))$, we conclude $u \in \hat{L}(v)$ and z is the image of v under the hat-f-u-map for $\hat{X}$.

Conversely suppose $u \in \hat{L}(v)$ and let z be the image of v under the hat-f-u-map for $\hat{X}$. That is $u \lesssim \hat{f}_z(v) = y$ say. Thus $y \geq v$ and as $u \lesssim y$ in $\hat{f}(\zeta(u))$, $f_w(u) = y$ for $w = \hat{f}_y(u)$. Therefore $f_w(u) = y \geq v$, so $v \in L(u)$.

(14.4) *Let* (X, f) *be a restricted combinatorial cell complex with* ζ *surjective. Then the dual complex* $(\hat{X}, \hat{f})$ *also satisfies these properties and* (X, f) *is the dual of* $(\hat{X}, \hat{f})$.

Proof. We have already observed that $\hat{X}$ is of finite height. Next if $y \in X(\lesssim x)$ then $x \in X(\geq y)$ so as $\zeta : f(y) \to X(\leq y)$ is surjective there is $v \in f(y)$ with $\zeta(v) = x$. Thus $v \in \hat{f}(x)$ and $\hat{\zeta}(v) = y$, so $\hat{\zeta} : \hat{f}(x) \to \hat{X}(\lesssim x)$ is a surjection.

Let (Y, g) be the dual of $(\hat{X}, \hat{f})$. Then $\hat{X} = X$ so $Y = \hat{X} = X$ and as $\lesssim$ is the dual of the ordering on X, the order on Y is the ordering $\leq$ dual to $\lesssim$. That is $Y = X$ as a partially ordered set. Similarly for $x \in X$,

$$g(x) = \{v \in V : \hat{\zeta}(v) = x\} = f(x)$$

and as we observed during the proof of 14.1, the hat-zeta-function for $\hat{X}$ is ζ. The ordering on $g(x)$ is given by $u \leqq v$ if and only if $v \in \hat{L}(u)$ if and only if $u \in L(v)$ (by 14.3) if and only if $u \leq v$ as $u, v \in f(x)$. That is $g(x) = f(x)$ as a partially ordered set.

Next the zeta-function for Y is the hat-zeta function for $\hat{X}$ which is the zeta-function for X. That is X and Y have the same zeta-function. Finally for $u \in V$, the f-u-function g_u for Y is the hat-f-u-function for $\hat{X}$, so by 14.3, $g_u(v) = f_{f_w(u)}(v)$, where $w = \hat{f}_v(u)$. But as $v \leq u$ in $f(x)$, $w = x$ and $f_w(u) = u$, so $g_u = f_u$, completing the proof.

(14.5) *$K(X) = K(\hat{X})$, so X and $\hat{X}$ have the same homology and fundamental group.*

Proof. As observed during the construction of $\hat{X}$, $V = \hat{V}$, so $K(X)$ and $K(\hat{X})$ have the same vertex set. Further $K(X)$ is the clique complex of the symmetric relation $*$ on V define by $u * v$ if and only if $u \in L(v)$ or $v \in L(u)$. Similarly $K(\hat{X})$ is the clique complex of the relation $\hat{*}$. But by 14.3, these two relations are the same.

§15. CW-complexes

Denote by B^n the unit n-ball

$$B^n = \{x \in \mathbf{R}^n : |x| \leq 1\}$$

in $\mathbf{R}^n$ and let

$$S^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}$$

be the $(n-1)$-sphere. We also write $\dot{B}^n$ for the boundary S^{n-1} of B^n and $I(B^n)$ for the interior $B^n - \dot{B}^n$ of B^n. In particular when $n = 0$, $I(B^0) = B^0$ and $\dot{B}^0 = \varnothing$.

(15.1) *Let X, Y be copies of B^n and $f : I(X) \to I(Y)$ be a homeomorphism. Then f extends to at most one homeomorphism $g : X \to Y$.*

Proof. Suppose $g, h : X \to Y$ are homeomorphisms extending f. Then $h^{-1} \circ g$ is a homeomorphism of X extending the identity map on $I(X)$, and it suffices to show $h^{-1} \circ g$ is the identity map on X. Thus it suffices to show that if $k : X \to X$ is a homeomorphism which is the identity on $I(X)$ then k is the identity. Let $x \in \dot{X}$, $y \in I(X)$ and consider the line segment $[x, y]$. Then $[x, y]$ is the closure of $(x, y]$ so $k([x, y])$ is the closure of $k((x, y]) = (x, y]$. That is $k(x) = x$, as desired.

Recall that a *CW-complex* is a triple (T, Λ, φ), where T is a topological space, Λ is a collection of subspaces of T, and

(CW1) T is Hausdorff.

(CW2) For each $\lambda \in \Lambda$, $\varphi_\lambda : F(\lambda) \to \bar{\lambda}$ is a continuous surjection such that $F(\lambda) = B^{n(\lambda)}$ and $\varphi_\lambda : I(F(\lambda)) \to \lambda$ is a homeomorphism.

(CW3) $\dot{\lambda} = \varphi_\lambda(\dot{F}(\lambda)) \subset \bigcup_{\alpha \in \Lambda(\lambda)} \alpha$ for some finite subset $\Lambda(\lambda)$ of Λ.

(CW4) T is the disjoint union of the subspaces $\lambda \in \Lambda$.

(CW5) A subset C of T is closed in T if and only if $C \cap \bar{\lambda}$ is closed in $\bar{\lambda}$ for all $\lambda \in \Lambda$, and if $C \subset \bar{\lambda}$ then C is closed in $\bar{\lambda}$ if and only if $\varphi_\alpha^{-1}(C)$ is closed in $F(\alpha)$ for all $\alpha \in \Lambda(\lambda) \cup \{\lambda\}$.

Define the CW-complex (T, Λ, φ) to be *normal* if

(R1) $\dot{\lambda} = \bigcup_{\alpha \in \Lambda(\lambda)} \alpha$ for each $\lambda \in \Lambda$.

(R2) For each $\lambda \in \Lambda$ and $\alpha \in \Lambda(\lambda)$, the set $\mathcal{C}_\lambda(\alpha)$ of connected components of $\varphi_\lambda^{-1}(\alpha)$ is finite and for each $C \in \mathcal{C}_\lambda(\alpha)$, the restriction φ_C of φ_λ to C is a homeomorphism of C with α such that $\varphi_\alpha^{-1} \circ \varphi_C$ extends to a homeomorphism of $\bar{C}$ with $F(\alpha)$.

Define the CW-complex (T, Λ, φ) to be *restricted* if $\bar{C}_1 \cap \bar{C}_2 = \varnothing$ for distinct $C_1, C_2 \in \mathcal{C}_\lambda(\alpha)$ and all $\lambda \in \Lambda$ and $\alpha \in \Lambda(\lambda)$.

Example. Recall a CW-complex (T, Λ, φ) is *regular* if Axiom (R1) holds and $\varphi_\lambda : F(\lambda) \to \bar{\lambda}$ is a homeomorphism for each $\lambda \in \Lambda$. Notice that if $\lambda \in \Lambda$ and $\alpha \in \Lambda(\lambda)$ then as φ_λ and φ_α are homeomorphisms, also $\varphi_\alpha^{-1} \circ \varphi_\lambda : \varphi_\lambda^{-1}(\bar{\alpha}) \to F(\alpha)$ is a homeomorphism, so Axiom (R2) is satisfied and (T, Λ, φ) is restricted. Therefore regular CW-complexes are restricted.

In the next few lemmas, assume (T, Λ, φ) is a normal CW-complex.

(15.2) *For each $\lambda \in \Lambda$ and $\alpha \in \Lambda(\lambda)$, $\Lambda(\alpha) \subseteq \Lambda(\lambda)$.*

Proof. As $\alpha \subseteq \dot{\lambda}$, $\bar{\alpha} \subseteq \bar{\lambda}$, so $\beta \subseteq \bar{\lambda}$ for each $\beta \in \Lambda(\alpha)$. Now by axiom (CW4), $\beta \cap \lambda = \varnothing$, so $\beta \subseteq \dot{\lambda}$. Then by Axioms (R1) and (CW4), $\beta \in \Lambda(\lambda)$.

(15.3) *For $\alpha, \beta \in \Lambda$, the following are equivalent:*

(1) *$\beta \in \Lambda(\alpha)$.*

(2) *$\bar{\beta} \subset \bar{\alpha}$.*

(3) *$\beta \subseteq \dot{\alpha}$.*

(4) *$\bar{\alpha} \cap \beta \neq \varnothing$ and $\beta \neq \alpha$.*

Proof. Clearly (1) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (4). Assume (4). By Axiom (CW4), $\alpha \cap \beta = \varnothing$, so $\dot{\alpha} \cap \beta \neq \varnothing$. Hence by axiom (R1), $\gamma \cap \beta \neq \varnothing$ for some $\gamma \in \Lambda(\alpha)$. Then by axiom (CW4), $\beta = \gamma$.

Remark 15.4. Let (T, Λ, φ) be a normal CW- complex. Partially order Λ by $\alpha \leq \beta$ if $\bar{\alpha} \subseteq \bar{\beta}$. Then by 15.3, $\Lambda(< \lambda) = \Lambda(\lambda)$ for each $\lambda \in \Lambda$. In particular $\Lambda(\leq \lambda)$ is finite, so $\Lambda \in \mathcal{P}$.

Next for $\lambda \in \Lambda$, define

$$f(\lambda) = \bigcup_{\alpha \leq \lambda} \mathcal{C}_\lambda(\alpha)$$

and partially order $f(\lambda)$ by $C \leq D$ if $\bar{C} \subseteq \bar{D}$. As $\Lambda(\leq \lambda)$ is finite and $\mathcal{C}_\lambda(\alpha)$ is finite for each $\alpha \leq \lambda$, $f(\lambda)$ is finite, so $f(\lambda) \in \mathcal{P}^*$.

Next let V be the disjoint union of the sets $f(\lambda)$, $\lambda \in \Lambda$, and define $\zeta : V \to \Lambda$ by $\zeta(C) = \alpha$ for $C \in \mathcal{C}_\lambda(\alpha)$. Then $\zeta : f(\lambda) \to \Lambda(\leq \lambda)$ is a morphism in $\mathcal{P}$.

Let $\alpha \in \Lambda(\lambda)$ and $C \in \mathcal{C}_\lambda(\alpha)$. By axiom (R2), $\varphi_C : C \to \alpha$ is a homeomorphism and $\varphi_\alpha^{-1} \circ \varphi_C$ extends to a homeomorphism of $\bar{C}$ with $F(\alpha)$. By 15.1, this homeomorphism is unique; denote it by F_C.

Finally $I(F(\lambda))$ is the unique member of $f(\lambda)$ mapping to λ under ζ; we define $F_{F(\lambda)} : f(\lambda) \to f(\lambda)$ to be the identity map.

(15.5) *For $\lambda \in \Lambda$, φ_λ is the unique extension of $\varphi_\lambda : I(F(\lambda)) \to \lambda$ to a continuous map of $F(\lambda)$ to $\bar{\lambda}$.*

Proof. Let $\alpha \in \Lambda(\lambda)$, $C \in \mathcal{C}_\lambda(\alpha)$, $x \in C$, and $y \in I(F(\lambda))$. Let $A = \varphi_\lambda([x,y])$ and $U = \varphi_\lambda((x,y])$. Then $\varphi_\lambda^{-1}(A) = (x,y] \cup \{x_1, \ldots, x_r\}$, where $\{x_i\} = C_i \cap \varphi_\lambda^{-1}(A)$ and $\mathcal{C}_\lambda(\alpha) = \{C_1, \ldots, C_r\}$. Further $[x,y]$ and each of the points x_i is closed in $F(\lambda)$, so $\varphi_\lambda^{-1}(A)$ is closed in $F(\lambda)$. Further if $\beta \in \Lambda(\lambda)$ then either $\varphi_\beta^{-1}(A) = \varnothing$ or $\alpha \leq \beta$ and $\varphi_\beta^{-1}(A) = \varphi_\beta^{-1}(\varphi_\lambda(x))$ is finite and hence closed in $F(\beta)$. So by axiom (CW5), A is closed in T.

Thus A is the closure in T of U. Therefore if $\psi : C^* = C \cup I(F(\lambda) \to \bar{\lambda}$ is a continuous extension of φ_λ then as $[x,y]$ is the closure of $(x,y]$ in C^*, $\psi(x)$ is contained in the closure A of U. Hence $\psi(x) \in \bigcap_{y \in I(F(\lambda))} \varphi_\lambda([x,y]) = \{\varphi_\lambda(x)\}$, so $\psi(x) = \varphi_\lambda(x)$. As this holds for each $C \in f(x)$ and $x \in C$, the lemma follows.

(15.6) *Let $\lambda \in \Lambda$ and $C, D \in f(\Lambda)$ with $C \leq D$. Then*
(1) $\varphi_\lambda = \varphi_{\zeta(C)} \circ F_C$ *on* $\bar{C}$.
(2) $F_D = F_{F_C(D)} \circ F_C$.
(3) $\zeta = \zeta \circ F_C$ *on* $f(\lambda)(\leq C)$.
(4) If T is restricted then ζ is injective on $f(\lambda)(\geq C)$.

Proof. Let $C \in \mathcal{C}_\lambda(\alpha)$ and $D \in \mathcal{C}_\lambda(\beta)$. By 15.5, φ_α is the unique extension to $F(\alpha)$ of φ_α restricted to $I(F(\alpha))$, so $\varphi_\alpha \circ F_C$ is the unique extension to $\bar{C}$ of $\varphi_\alpha \circ F_C$ restricted to C. But by construction of F_C, this restriction is equal to the restriction of φ_λ, so (1) holds.

Now by (1), $\varphi_\alpha^{-1}(\beta) = F_C(\varphi_\lambda^{-1}(\beta) \cap \bar{C}$, so if $D_1, \dots, D_r$ are the connected components of $\varphi_\lambda^{-1}(\beta) \cap \bar{C}$ then $F_C(D_1), \dots, F_C(D_r)$ are the connected components of $\varphi_\alpha^{-1}(\beta)$ and hence the members of $\zeta^{-1}(\beta)$. This establishes (3).

Next $F_{F_C(D)}$ is the unique extension of $\varphi_\beta^{-1} \circ \varphi_\alpha$ from $F_C(D)$ to $F_C(\bar{D})$ so $F_{F_C(D)} \circ F_C$ is the unique extension of $\psi = \varphi_\beta^{-1} \circ \varphi_\alpha \circ F_C$ from D to $\bar{D}$. But by (1), $\psi = \varphi_\beta^{-1} \circ \varphi_\lambda$ and by definition F_D is the extension of the latter map, so (2) is established.

Finally if $G, H \in \mathcal{C}_\lambda(\gamma)$ are distinct for some $\gamma \in \Lambda(\lambda)$ then $\bar{G} \cap \bar{H} = \varnothing$ if T is restricted. This proves (4).

(15.7) *Let $\lambda \in \Lambda$. Then*
(1) *If $S \subseteq \Lambda$ then $\bigcap_{\sigma \in S} \bar{\sigma} = \bigcup_{\alpha \leq S} \alpha$.*
(2) *$F(\lambda)$ is partitioned by $\Lambda_\lambda = \{C : C \in f(\lambda)\}$.*
(3) *For $S \subseteq f(\lambda)$, $\bigcap_{C \in S} \bar{C} = \bigcup_{D \leq S} D$.*
(4) *$(F(\lambda), \Lambda_\lambda, \psi)$ is a regular CW-complex, where $\psi = (\psi_C : C \in f(\lambda))$ and $\psi_C = F_C^{-1} : F(\zeta(C)) \to \bar{C}$.*

Proof. Part (1) follows from axiom CW4 and 15.3. Next $F(\lambda)$ is the disjoint union of $I(F(\lambda))$ and $\dot{F}(\lambda)$, so to prove (2) we must show $\dot{F}(\lambda)$ is partition by $\{C : C \in f(\lambda) - \{I(F(\lambda))\}\}$. Let $x \in \dot{F}(\lambda)$. Then $\varphi_\lambda(x) \in \alpha$ for a unique $\alpha \in \Lambda(\lambda)$, so $x \in C$ for some $C \in \mathcal{C}_\lambda(\alpha)$. Further as the members of $\mathcal{C}_\lambda(\alpha)$ are disjoint, $x \notin C'$ for $C \neq C' \in \mathcal{C}_\lambda(\alpha)$, while if $x \in D$ for $D \in f(x)$ then $\varphi_\lambda(x) \in \varphi_\lambda(D) \cap \alpha = \varphi_D(D) \cap \alpha = \zeta(D) \cap \alpha$, so $\zeta(D) = \alpha$. This establishes (2). Then (1) and (2) imply (3).

It remains to prove (4). As $F(\lambda) \cong B^{n(\lambda)}$, $F(\lambda)$ is a Hausdorff space. For $C \in f(\lambda)$, by definition $\bar{C} \cong F(C) = F(\zeta(C)) \cong B^{n(\zeta(C))}$ and $\psi_C : F(C) \to C$ is a homeomorphism. By (2) and (3), $\dot{C} = \bar{C} - C = \bigcup_{D \subset C} D$, and Λ_λ is a partition of $F(\lambda)$. Visibly axiom CW5 is satisfied. Thus (4) is established.

Remark 15.8. We can now associate a topological cell complex $\mathcal{T}(T)$ with the normal CW-complex $T = (T, \Lambda, \varphi)$. Namely let Λ be the poset of $\mathcal{T}(T)$; by Remark 15.4, $\Lambda \in \mathcal{P}$. For $\lambda \in \Lambda$ the cell $f(\lambda)$ associated to λ has as its poset the poset $f(\lambda)$; by Remark 15.4, $f(\lambda) \in \mathcal{P}^*$. The topological space associated to λ is of course $F(\lambda)$. For $C \in f(\lambda)$, $F(C) = \bar{C}$ is a closed subspace of $F(\lambda)$ and as usual with topological cells we take the maps $F(D, C)$, $D, C \in f(x)$, $D \leq C$, to be inclusions. Lemma 15.7 then says $f(\lambda)$ is a topological cell.

Our maps ζ and $f_C : f(\lambda)(\leq C) \to f(\zeta(C))$ were defined in Remark 15.4. That is $f_C(D) = F_C(D)$ for $D \in F(\lambda)(\leq C)$ and $F_C : \bar{C} \to F(\zeta(C))$ is our topological isomorphism. By 15.6, axioms 5 and 6 for cell complexes are satisfied. Axiom 7 for cell complexes holds by definition of $F_{\infty_\lambda} = F_{F(\lambda)}$ as the identity map in Remark 15.4. Thus we have checked that $\mathcal{T}(T)$ is a topological cell complex. By 15.6.4, $\mathcal{T}(T)$ is restricted if T is restricted.

We can extent the map $\mathcal{T}$ to a functor from the category of normal CW- complexes to the category of topological cell complexes. A *morphism of CW-complexes* from $(T_1, \Lambda_1, \varphi_1)$ to $(T_2, \Lambda_2, \varphi_2)$ is a continuous map $\phi : T_1 \to T_2$ such that $\phi(\Lambda_1) \subseteq \phi(\Lambda_2)$, together with homeomorphisms $\phi_\lambda : F(\lambda) \to F(\phi(\lambda))$, $\lambda \in \Lambda_1$, such that $\varphi_{\phi(\lambda)} \circ \phi_\lambda = \phi_{|\bar{\lambda}} \circ \varphi_\lambda$.

Given such a morphism ϕ, we define $\mathcal{T}(\phi) : \mathcal{T}(T_1) \to \mathcal{T}(T_2)$ by $\mathcal{T}(\phi)(\lambda) = \phi(\lambda)$ and $\mathcal{T}(\phi)_\lambda = \phi_\lambda$. Check that $\mathcal{T}(\phi)$ is a morphism of topological cell complexes.

(15.9) *For $\lambda \in \Lambda$ and $X \subseteq \bar{\lambda}$, X is closed in T if and only if $\varphi_\lambda^{-1}(X)$ is closed in $F(\lambda)$.*

Proof. By axiom CW5, X is closed in T if and only if $\varphi_\alpha^{-1}(X)$ is closed in $F(\alpha)$ for all $\alpha \leq \lambda$. Next

$$\varphi_\alpha^{-1}(X) = \coprod_{C \in \mathcal{C}_\lambda(\alpha)} \varphi_\alpha^{-1}(X) \cap \bar{C}$$

and as $\varphi_\lambda = \varphi_\alpha \circ F_C$ on $\bar{C}$, $\varphi_\alpha^{-1}(X)$ is closed in $F(\alpha)$ if and only if $\varphi_\lambda^{-1}(X) \cap \bar{C}$ is closed in $\bar{C}$ for all $C \in \mathcal{C}_\lambda(\alpha)$. Therefore X is closed in T if and only if $\varphi_\lambda^{-1}(X) \cap \bar{C}$ is closed in $\bar{C}$ for all $C \in f(\lambda)$ if and only if $\varphi_\lambda^{-1}(X)$ is closed in $F(\lambda)$.

(15.10) *Let $A = A(\mathcal{T}(T))$, where $\mathcal{T}(T)$ is the topological cell complex supplied by Remark* 15.8. *Then the map*

$$\begin{aligned} \phi : A &\to T \\ \tilde{x} &\mapsto \varphi_\alpha(x) \end{aligned}$$

for $x \in \alpha \in \Lambda$, is a homeomorphism.

Proof. By 4.5, the sets $\tilde{I}(\alpha)$, $\alpha \in \Lambda$, partition A. Further by 4.4 and 4.6, $\lambda_\alpha : F(\alpha) \to A$ is continuous with $\lambda_\alpha : I(\alpha) \to \tilde{I}(\alpha)$ a homeomorphism. Then as $\varphi_\alpha : I(\alpha) \to \alpha$ is a homeomorphism, so is

$\varphi_\alpha \circ \lambda_\alpha^{-1} : \tilde{I}(\alpha) \to \alpha$. But

$$\phi = \bigcup_{\alpha \in \Lambda} \varphi_\alpha \circ \lambda_\alpha^{-1}$$

so $\phi : A \to T$ is a bijection and for each $\alpha \in \Lambda$, $\phi \circ \lambda_\alpha = \varphi_\alpha$.

Next by 15.9, $X \subseteq \bar{\alpha}$ is closed if and only if $\varphi_\alpha^{-1}(X)$ is closed in $F(\alpha)$. But $\varphi_\alpha^{-1}(X) = \lambda_\alpha^{-1}(\phi^{-1}(X))$, so by definition of the topology on $\tilde{F}(\alpha)$, $\varphi_\alpha^{-1}(X)$ is closed in $F(\alpha)$ if and only if $\phi^{-1}(X)$ is closed in $\tilde{F}(\alpha)$. That is X is closed in T if and only if $\phi^{-1}(X)$ is closed in $\tilde{F}(\alpha)$ if and only if $\phi^{-1}(X)$ is closed in A by 4.6. That is ϕ is a homeomorphism.

(15.11) *If (T, Λ, φ) is a regular CW-complex then*

(1) *$\mathcal{T}(T)$ is isomorphic to $\mathcal{P}(\mathcal{X}(T))$, where $\mathcal{X}(T) = (\Lambda, f)$ is the combinatorial cell complex of $\mathcal{T}(T)$.*

(2) *T is homeomorphic to the geometric realization of $\mathcal{O}(\Lambda)$.*

Proof. Let $X = \mathrm{sd}(\Lambda)$ and for $\lambda \in \Lambda$ pick $P(\lambda) \in \lambda$. Let $x = \{\lambda_0, \ldots, \lambda_k\} \in X$ with $\lambda_i < \lambda_{i+1}$ for each i. Given $0 \leq a_i \in \mathbf{R}$ with $\sum_i a_i = 1$, we define $\sum_i a_i P(\lambda_i)$ recursively. Namely let $u = \sum_{i<k} a_i P(\lambda_i)/(1 - a_k)$, $v = P(\lambda_k)$, and define

$$\sum_I a_i P(\lambda_i) = \varphi_{\lambda_k}^{-1}(a_k \varphi_{\lambda_k}(v) + (1 - a_k)\varphi_{\lambda_k}(u))$$

This makes sense as $F(\lambda_k) \subseteq \mathbf{R}^{n(\lambda_k)}$ and $u \in \bar{\lambda}_{k-1} \subseteq \bar{\lambda}_k$ has been defined already via our recursive procedure.

Now define

$$G(x) = \{\sum_i a_i P(\lambda_i) : 0 \leq a_i \in \mathbf{R} \text{ and } \sum_i a_i = 1\}$$

Notice $G(x) \cap G(y) = G(x \cap y)$. Thus if we take (X, g) to be the simplicial cell complex of X, we can regard $G(x)$ as defining a topological cell on $g(x) = X(\leq x)$. Then we extend (X, g) to a topological cell complex by letting G_y be the identity map for $y \subseteq x$.

We next consider the topological cell complex $\chi = \chi(\mathcal{O}(\Lambda))$ and define an isomorphism $\phi : \chi \to (X, g, G)$. Recall from Example 6.4 that the combinatorial cell complex of χ is just (X, g); thus as a map of combinatorial cell complexes we take ϕ to be the identity. Also the topological cell associated to x by χ is the standard simplex $T_{h(x)} = [u_{h(v)} : v \in x]$, so it remains to define $\phi_x : T_{h(x)} \to G(x)$ by

$$\phi_x : \sum_i a_i u_{h(\lambda_i)} \mapsto \sum_i a_i P(\lambda_i)$$

which is of course a homeomorphism. Check that ϕ is a morphism, and then observe that as each ϕ_x is a homeomorphism, ϕ is an isomorphism. Therefore by 4.7, $A(\chi) \cong A(X, g, G)$. Recall also from Example 6.4 that $A(\chi)$ is the geometric realization of $\mathcal{O}(\Lambda)$. Further as (X, g) is the simplicial cell complex of X and $G(x) \subseteq T$ with the maps G_y identities, $G(x) \to \tilde{G}(x)$ for each x and $A(X, g, G) = \bigcup_x G(x) = T$. Thus (2) is established.

Next from Example 7.1, $\mathcal{P}(\mathcal{X}(T))$ is a topological cell complex with combinatorial cell complex $\mathcal{X}(T) = (\Lambda, f)$ and for $\lambda \in \Lambda$, the topological cell $H(\lambda)$ is just $T(f(\lambda))$ together with the subspaces $T(f(\alpha))$ for $\alpha \leq \lambda$. Applying our conclusions of the previous paragraph to $(\bar{\lambda}, \Lambda(\lambda) \cup \{\lambda\}, \varphi)$ in place of (T, Λ, φ), we see that $H(\lambda) \cong \bar{\lambda} \cong F(\lambda)$ via a homeomorphism ϕ_λ preserving the cell structure. Then the identity morphism on the combinatorial cell complex (Λ, f) together with the homeomorphisms ϕ_λ, $\lambda \in \Lambda$, define an isomorphism of $\mathcal{T}(T)$ with $\mathcal{P}(\mathcal{X}(T))$, establishing (1).

(15.12) (1) *$\mathcal{T}(T)$ is isomorphic to $\mathcal{P}(\mathcal{X}(T))$, where $\mathcal{X}(T) = (\Lambda, f)$ is the combinatorial cell complex of $\mathcal{T}(T)$.*

(2) *T is homeomorphic to the geometric realization $T(\mathcal{X}(T))$ of $\mathcal{X}(T)$.*

Proof. As $T(\mathcal{X}(T)) = A(\mathcal{P}(\mathcal{X}(T)))$, (1) and 15.10 imply (2), so it remains to prove (1). The last paragraph of the proof of 15.11 can be repeated virtually verbatim to prove (1).

Define a poset P to be a *n-sphere* if P is of height n and the geometric realization of $\mathcal{O}(P)$ is homeomorphic to the n-sphere S^n.

(15.13) *Let $\lambda \in \Lambda$ and $n = n(\lambda)$. Then $n = h(\lambda)$ and $\mathcal{O}(\dot{f}(\lambda))$ is an $(n-1)$-sphere.*

Proof. Let m be the height of $f(\lambda)$. By 15.7.4, $F(\lambda) = (F(\lambda), \Lambda_\lambda, \psi)$ is a regular CW-complex, so its $(n-1)$-skeleton $\dot{F}(\lambda) = (\dot{F}(\lambda), \Lambda(\lambda), \psi)$ is also a regular CW-complex. Then by 15.11, $S^{n-1} \cong \dot{F}(\lambda)$ is homeomorphic to the geometric realization of $\mathcal{O}(\Lambda(\lambda))$. But $\Lambda(\lambda) \cong \dot{f}(\lambda)$, so the geometric realization of $\mathcal{O}(\dot{f}(\lambda))$ is an $(n-1)$-sphere. In particular $H_{n-1}(\mathcal{O}(\dot{f}(n))) \neq 0$, so $h(\dot{f}(\lambda)) = \dim(\mathcal{O}(\dot{f}(\lambda))) \geq n-1$. Therefore $m = h(f(\lambda)) = h(\dot{f}(\lambda)) + 1 \geq n$.

So to complete the proof it remains to show $m \leq n$. We proceed by induction on m. If $m = 0$ the inequality is trivial. Now $m = h(\dot{f}(\lambda)) + 1$ and $h(\dot{f}(\lambda)) = \max\{h(\alpha) : \alpha \in \Lambda(\lambda)\}$. By induction on m, $h(\alpha) \leq n(\alpha)$, so if $n(\alpha) < n$, we are done. On the other hand $B^{n(\alpha)} \cong F(C)$ is a closed subspace of $\dot{F}(\lambda) \cong S^{n-1}$, and therefore $n(\alpha) < n$, (cf. Exercise H in Chapter 3 of [5]) completing the proof.

Example 15.14. Let $\mathcal{Q}$ be the category of combinatorial cell complexes (X, f) such that for each $x \in X$ of height n, $\dot{f}(x)$ is an $(n-1)$-sphere. Notice that as $T(\dot{f}(x))$ is compact, $\dot{f}(x)$ is finite, a fact we use below without comment several times. We associate to $X = (X, f) \in \mathcal{Q}$ a normal CW-complex $Q(X)$.

The topological space of $Q(X)$ is the geometric realization $A = T(X)$ of (X, f). The set Λ of open cells of A is the set of subspaces $\tilde{I}(x)$, $x \in X$, where we use the notation of Section 4. By definition of $T(X)$, $F(x)$ is the geometric realization of $f(x)$, and as $(X, f) \in \mathcal{Q}$, $\dot{f}(x)$ is an $(n-1)$-sphere, so $F(x)$ is homeomorphic to B^n and $\dot{F}(x)$ is homeomorphic to S^{n-1}. Thus we define

$$\varphi_x : F(x) \to \tilde{F}(x)$$
$$a \mapsto \tilde{a}$$

Our normal CW-complex is $Q(X) = (T(X), \lambda, \varphi)$.

By definition of $A = T(X)$, $\tilde{F}(x)$ is closed in A and φ_x is a continuous surjection. By 4.6, $C \subseteq \tilde{F}(x)$ is closed in A if and only if $\varphi_x^{-1}(C)$ is closed in $F(x)$, so as $F(x)$ is the closure of $I(x)$, $\tilde{F}(x)$ is the closure of $\tilde{I}(x)$. By 4.4, $\varphi_x : I(x) \to \tilde{I}(x)$ is a homeomorphism.

By definition of the topology on A and by remarks in the previous paragraph, axiom CW5 is satisfied. We have also seen that CW2 is satisfied. By 4.5, CW4 is satisfied. Also $\tilde{F}(x) - \tilde{I}(x) = \bigcup_{y<x} \tilde{I}(y)$, so axiom (R1) holds. For $y < x$,

$$\varphi_x^{-1}(\tilde{I}(y)) = \bigcup_{v \in f(x) \cap \zeta^{-1}(y)} I(v)$$

Also for distinct $u, v \in f(x) \cap \zeta^{-1}(y)$, $I(u) \cap I(v) = \varnothing$, so $\{I(v) : v \in f(x) \cap \zeta^{-1}(y)\}$ is the set of connected components of $\varphi_x^{-1}(\tilde{I}(y))$. As $F_v : F(v) \to F(y)$ is a homeomorphism with $\varphi_x = \varphi_y \circ F_v$ on $F(v)$, axiom (R2) holds. Also $F(u) \cap F(v) = \bigcup_{w \leq u,v} F(w)$ and if X is restricted, no such w exists, so $Q(X)$ is restricted. Therefore to complete our proof that $Q(X)$ is a normal CW-complex, it remains to show that $A = T(X)$ is a Hausdorff space, which we leave as an exercise.

So $Q(X)$ is a normal CW-complex. Now we extend Q to a covariant functor from $\mathcal{Q}$ into the category of normal CW-complexes. Namely if $\phi : X_1 \to X_2$ is a morphism of cell complexes in $\mathcal{Q}$, define $Q(\phi) : Q(X_1) \to Q(X_2)$ to be the morphism of CW-complexes whose map of topological spaces is $T(\phi) : T(X_1) \to T(X_2)$ and with $Q(\phi)_x : F(x) \to F(\phi(x))$ defined to be ϕ_x. Check that $Q(\phi)$ is indeed a morphism. Thus we have our functor.

Theorem 15.15. *Let $\mathcal{N}$ be the category of normal CW-complexes. Then we have a covariant functor $\mathcal{X}$ from $\mathcal{N}$ into $\mathcal{Q}$, where $\mathcal{X}(T)$ is the combinatorial cell complex of $\mathcal{T}(T)$. Further the functors $\mathcal{X} : \mathcal{N} \to \mathcal{Q}$ and $Q : \mathcal{Q} \to \mathcal{N}$ are equivalences of categories. Under these equivalences, restricted cell complexes correspond to restricted CW-complexes.*

Proof. First by Remark 15.8, $\mathcal{X}(T)$ is a combinatorial cell complex, and then by 15.13, $\mathcal{X}(T) \in \mathcal{Q}$. So $\mathcal{X}(T)$ is indeed a functor from $\mathcal{N}$ to $\mathcal{Q}$.

By definition of $Q(\mathcal{X}(T))$, the topological space of the CW-complex $Q(\mathcal{X}(T))$ is $T(\mathcal{X}(T))$, the set of open cells is $\Lambda' = \{\tilde{I}(\lambda) : \lambda \in \mathcal{X}(T) = \Lambda\}$, and the characteristic maps are $\varphi'_\lambda : F(\lambda) \to \tilde{F}(\lambda)$ taking a to $\tilde{a}$. By 15.12 and its proof, we have an isomorphism $\phi' : \mathcal{P}(\mathcal{X}(T)) \to \mathcal{T}(T)$ which is the identity map on the combinatorial cell complex $\mathcal{X}(T)$ and with homeomorphisms $\phi'_\lambda : F(x)' \to F(x)$, $\lambda \in \Lambda$. Then ϕ' induces the homeomorphism

$$A(\phi') : T(\mathcal{X}(T)) = A(\mathcal{P}(\mathcal{X}(T))) \to A(\mathcal{T}(T)).$$

We compose $A(\phi')$ with the homeomorphism $\phi : A(\mathcal{T}(T)) \to T$ of 15.10 to obtain a homeomorphism $\iota_T = \phi \circ A(\phi') : T(\mathcal{X}(T)) \to T$. Then we define $\iota_{T,\lambda} = \phi'_\lambda$. Then $\iota_T : Q(\mathcal{X}(T)) \to T = (T, \Lambda, \varphi)$ is an isomorphism of CW-complexes.

In the other direction, $\mathcal{X}(Q(X))$ is the combinatorial cell complex of $\mathcal{T}(Q(X))$, which is by definition (Λ', f'), where the poset Λ' is the set of open cells of $Q(X)$ and $f'(\lambda')$ is the set of connected components of $\varphi_{\lambda'}^{-1}(\alpha')$, where $\alpha' \leq \lambda'$ are open cells in $Q(X)$. Further by definition of $Q(X)$ in Example 15.14, $\Lambda' = \{\tilde{I}(x) : x \in X\}$ and $f'(\tilde{I}(x)) = \{I(v) : V \in f(x)\}$. Thus we define our isomorphism $\iota_X : X \to \mathcal{X}(Q(X))$ by defining $\iota_X : x \mapsto \tilde{I}(x)$ as a map of posets and defining

$$\iota_{X,x} : f(x) \to f'(\tilde{I}(x)) = \{I(v) : v \in f(x)\}$$

by $\iota_{X,x} : v \mapsto I(v)$. Check that ι_X is indeed an isomorphism of combinatorial cell complexes.

Finally check that if $\phi : X_1 \to X_2$ is a morphism in $\mathcal{Q}$ then $\mathcal{X}(Q(\phi)) \circ \iota_{X_1} = \iota_{X_2} \circ \phi$ and similarly if $\psi : T_1 \to T_2$ is a morphism in $\mathcal{R}$ then

$\psi \circ \iota_{T_1} = \iota_{T_2} \circ Q(\mathcal{X}(\psi))$. Therefore the functors $\mathcal{X}$ and Q are inverses of each other on equivalence classes of objects and maps in the two categories and hence equivalences of categories. To illustrate these last checks, observe that as a map of posets, $(\mathcal{X}(Q(\phi)) \circ \iota_{X_1})(x) = \tilde{I}(\phi(x)) = (\phi \circ \iota_{X_2})(x)$ while $(\mathcal{X}(Q(\phi)) \circ \iota_{X_1})_x(v) = I(\phi_x(v)) = (\phi \circ \iota_{X_2})_x(v)$.

§16. Group actions on posets and cell complexes

Let G be a group and $\mathcal{F} = (G_i : i \in I)$ a family of subgroups of G. Let

$$X = \coprod_{i \in I} G/G_i$$

be the disjoint union of the coset spaces G/G_i, $i \in I$. We consider relations defined on X which are preserved by the action of G via right multiplication. Let

$$\mathcal{E} = \bigcup_{(i,j) \in I \times I} \mathcal{E}_{i,j}$$

with $\mathcal{E}_{i,j} \subseteq G_i \backslash G / G_j$, where $G_i \backslash G / G_j$ is the set of double cosets $G_i x G_j$, $x \in G$. Define $\Gamma(G, \mathcal{F}, \mathcal{E})$ to be the relational structure on the set X with relation $G_i x$ related to $G_j y$ if and only if $G_i x y^{-1} G_j \in \mathcal{E}_{i,j}$. Check that this relation is well defined and preserved by the representation of G on X via right multiplication. Observe that there is a type function from X to I defined by $G_i x \mapsto i$ for each $i \in I$ and $x \in G$. Further this type function is preserved by the action of G; that is if h is our type function then $h(xg) = h(x)$ for each $x \in X$ and $g \in G$.

(16.1) *Let $\Gamma(G, \mathcal{F}, \mathcal{E})$ be a relational structure with relation R. Then*

(1) R is reflexive if and only if $G_i \in \mathcal{E}_{i,i}$ for each $i \in I$.

(2) R is symmetric if and only if $G_i u G_j \in \mathcal{E}_{i,j}$ implies $G_j u^{-1} G_i \in \mathcal{E}_{j,i}$ for all i, j.

(3) R is antisymmetric if and only if whenever $G_i u G_j \in \mathcal{E}_{i,j}$ and $G_j u^{-1} G_i \in \mathcal{E}_{j,i}$, then $i = j$ and $u \in G_i$, for all i, j.

(4) R is transitive if and only if whenever $i, j, k \in I$, $G_i u G_j \in \mathcal{E}_{i,j}$, and $G_j v G_k \in \mathcal{E}_{j,k}$, then $G_i u g v G_k \in \mathcal{E}_{i,k}$ for all $g \in G_j$.

Proof. First if $G_i x R G_j y$ and $G_j y R G_k z$, then $G_i x y^{-1} G_j \in \mathcal{E}_{i,j}$ and $G_j y z^{-1} G_k \in \mathcal{E}_{j,k}$. Then if $\mathcal{E}$ satisfies the hypotheses of (4), then

$$G_i x z^{-1} G_j = G_i (x y^{-1})(y z^{-1}) G_k \in \mathcal{E}_{i,k},$$

so $G_i x R G_k z$ and the relation R is transitive.

Conversely suppose R is transitive and let $G_iuG_j \in \mathcal{E}_{i,j}$ and $G_jvG_k \in \mathcal{E}_{j,k}$. Then G_iugRG_j for each $g \in G_j$ and $G_jRG_kv^{-1}$, so $G_iugRG_k v^{-1}$, and therefore $G_iugvG_k \in \mathcal{E}_{i,k}$.

Remark. Lemma 16.1 says that the relation defined by a relational structure $\Gamma(G, \mathcal{F}, \mathcal{E})$ is a partial order if and only if $\mathcal{E}$ satisfies the conditions of 16.1.1, 16.1.3, and 16.1.4. Define a *coset poset* over the index set I to be a relational structure $\Gamma(G, \mathcal{F}, \mathcal{E})$ over I such that

(CSP0) I comes equipped with a partial order $\leq$.

(CSP1) $\mathcal{E}_{i,j} = \varnothing$ unless $i \leq j$.

(CSP2) $\mathcal{E}_{i,i} = \{G_i\}$ for each i.

(CSP3) If $i \leq j \leq k$ in I, $G_iuG_j \in \mathcal{E}_{i,j}$, and $G_jvG_k \in \mathcal{E}_{j,k}$, then $G_iugvG_k \in \mathcal{E}_{i,k}$ for each $g \in G_j$.

Thus by Lemma 16.1, the relation defined by a coset poset is a partial order on X. Write $\leq$ for this partial order. Observe that our type function from X to I is a map of posets by axiom CSP1. Moreover the next lemma gives us a characterization of those posets which are coset posets. Notice that condition (*) of the lemma is always satisfied if the poset P is in $\mathcal{P}$, since morphisms in $\mathcal{P}$ preserve height.

(16.2) *Let G be represented as a group of automorphisms of the poset P and let $I = P/G$ be the orbit poset of this representation. That is $aG \leq bG$ if and only if $ag \leq b$ for some $g \in G$. Assume*
(*) *For each $a \in P$, $aG \cap P(< a) = \varnothing$.*
Let $(x_i : i \in I)$ be a set of representatives for the orbits of G on P with $i = x_iG$ and let $G_i = G_{x_i}$, $\mathcal{F} = (G_i : i \in I)$, and $\mathcal{E}_{i,j} = \{G_iuG_j : x_iu \leq x_j\}$. Then $\Gamma = \Gamma(G, \mathcal{F}, \mathcal{E})$ is a coset poset and the map $x_ig \mapsto G_ig$ is a G-equivariant isomorphism of posets.

Proof. By construction, Γ satisfies axioms CSP0 and CSP1. Axiom CSP2 follows from hypothesis (*). Finally if $G_iuG_j \in \mathcal{E}_{i,j}$ and $G_jvG_k \in \mathcal{E}_{j,k}$ then $x_iug \leq x_j$ for each $g \in G_j$ and $x_j \leq x_kv^{-1}$, so as P is a poset, $x_iug \leq x_kv^{-1}$, and hence $x_iugv \leq x_k$. Therefore $G_iugvG_k \in \mathcal{E}_{i,k}$, so axiom CSP3 is satisfied and Γ is a coset poset. As $G_i = G_{x_i}$, our map is a G-equivariant bijection. Finally $x_ig \leq x_jh$ if and only if $G_igh^{-1}G_j \in \mathcal{E}_{i,j}$ if and only if $G_ig \leq G_jh$, so the map and its inverse preserve order.

(16.3) *Let $\Gamma = \Gamma(G, \mathcal{F}, \mathcal{E})$ and $\bar{\Gamma} = \Gamma(\bar{G}, \bar{\mathcal{F}}, \bar{\mathcal{E}})$ be coset posets over I and $\bar{I}$, respectively, let $\beta : I \to \bar{I}$ be a map of posets, and let $\alpha : G \to \bar{G}$ a group homomorphism such that $\alpha(G_i) \subseteq \bar{G}_{\beta(i)}$ and $\alpha(\mathcal{E}_{ij}) \subseteq \bar{\mathcal{E}}_{\beta(i),\beta(j)}$ for each $i, j \in I$. Then the map $G_ix \mapsto \bar{G}_{\beta(i)}\alpha(x)$ is a map of posets from Γ into $\bar{\Gamma}$.*

Proof. As $\alpha(G_i) \subset \bar{G}_\beta(i)$, the map is well defined. As $\alpha(\mathcal{E}_{i,j}) \subseteq \bar{\mathcal{E}}_{\beta(i),\beta(j)}$, the map preserves order.

Let $\alpha : \tilde{P} \to P$ be a map of posets. We say α is a *lower covering* if for all $\tilde{a} \in \tilde{P}$, the restriction $\tilde{\alpha}_{\tilde{a}} : \tilde{P}(\leq \tilde{a}) \to P(\leq \alpha(\tilde{a}))$ is an isomorphism. The lower covering is *restricted* if α is injective on $\tilde{P}(\geq \tilde{a})$.

The *cone* of P is the poset $CP = P \cup \{x_*\}$ obtained by adjoining an element $x_* > a$ for all $a \in P$.

(16.4) *Assume $\alpha : \tilde{P} \to P$ is a lower covering of posets with $P \in \mathcal{P}$ of height n, and let (P, f) be the simplicial cell complex of P. Let $X = P \cup \{x_*\}$ be the cone of P. Then (X, f) is a combinatorial cell complex, where (X, f) has n-skeleton (P, f), $\dot{f}(x_*) = \tilde{P}$, $\zeta_{|\dot{f}(x_*)} = \alpha$, and $f_{\tilde{a}} = \alpha_{\tilde{a}}$ for each $\tilde{a} \in \dot{f}(x_*)$. If α is restricted, so is (X, f).*

Proof. Straightforward.

(16.5) *Assume (X, f) is a restricted combinatorial cell complex of height $n+1$, $\dot{f}(x)$ is homology spherical for each $x \in X$, and $H_n(X^n) = 0$. Then $\dim(H_{n+1}(X)) = \sum_{x \in X} \dim(\tilde{H}_n(\dot{f}(x)))$, the $(n+1)$st cellular boundary map ∂_{n+1} of X is 0, and $H_n(X) = 0$.*

Proof. As $\dot{f}(x)$ is homology spherical for each $x \in X$, $H_*^c(X) = H_*(X)$ by Theorem 12.16.

Suppose $\partial_{n+1} = 0$. Then $H_{n+1}^c(X) = Z_{n+1}^c(X) = D_{n+1}(X)$ has rank

$$\sum_{x \in X} \dim(\tilde{H}_n(\dot{f}(x)))$$

by 12.7. Also $H_n^c(X) = Z_n^c(X)/B_n^c(X) = Z_n^c(X)$ as $B_n^c(X) = \partial_{n+1}(D_{n+1}(X)) = 0$. Therefore $H_n(X) = H_n^c(X) = Z_n^c(X) = Z_n^c(X^n) = H_n^c(X^n) = H_n(X^n) = 0$.

So it remains to show $\partial_{n+1} = 0$. Assume not. Then $0 \neq B_n^c(X) \leq Z_n^c(X) = Z_n^c(X^n) = H_n^c(X^n) = H_n(X^n) = 0$, a contradiction.

Corollary 16.6. *Assume (X, f) is a restricted combinatorial cell complex of height $n+1$ such that $\dot{f}(x)$ is homology spherical for each $x \in X$ and X^n is acyclic. Then X is homology spherical with* $\dim(H_{n+1}(X)) = \sum_{x \in X} \dim(\tilde{H}_n(\dot{f}(x)))$.

Example. We consider the classical example of the Poincaré dodecahedron and the Poincaré dodecahedron disk. We regard the dodecahedron as the poset $\tilde{X}$ of faces partially ordered by inclusion. Then $\tilde{X}$

has one element x_* of height 3, 12 2-dimensional faces of height 2, 30 1-dimensional faces of height 1, and 20 vertices of height 0.

The Coxeter group W of type H_3 is isomorphic to $\mathbf{Z}_2 \times A_5$, where A_5 is the alternating group of degree 5, and we will see that the Coxeter complex of W is the order complex of the boundary $\tilde{X}(< x_*)$ of the dodecahedron. Namely W is regular on 3-chains in $\tilde{X}$ and if $c = (x_0 < \cdots < x_3)$ is a 3-chain and $c_i = c - \{x_i\}$ for $0 \leq i \leq 2$, then $W_{c_i} = \langle r_i \rangle$ is of order 2 and $R = \{r_0, r_1, r_2\}$ is the set of fundamental reflections making (W, R) a Coxeter system of type H_3. Further the stabilizer of the chain $c_J = c - \{x_j : j \in J\}$ is the parabolic $W_J = \langle R_J \rangle$, where $R_J = \{r_J : j \in J\}$. Let $I = \{0, 1, 2\}$ ordered as usual and $M_i = W_{I-\{i\}}$ be the stabilizer of x_i. The Coxeter complex of W is the order complex of the coset poset $\Gamma(W, \mathcal{F}^*, \mathcal{E}^*)$, where $\mathcal{F}^* = (M_i : i \in I)$ and $\mathcal{E}^*_{i,j} = \{M_i M_j\}$. By 16.2, $\tilde{X}(< x_*)$ is isomorphic to this coset poset and hence its order complex is isomorphic to the Coxeter complex.

Next the commutator group of W is the alternating group $G = A_5$ on $\{1, 2, 3, 4, 5\}$ and as $W_J \not\leq G$ for $J \subset I$, $W = W_J G$ for all such J so G is transitive on pairs (x, y) in $\tilde{X}$ with $y \leq x$, $h(x) = i$, and $h(y) = j$, for all $0 \leq j \leq i \leq 3$. On the other hand G has two orbits on 3-chains of $\tilde{X}$. Let $\tilde{G}_i = G \cap M_i = G_{x_i}$, and $\tilde{\mathcal{F}} = (\tilde{G}_i : 0 \leq i \leq 3)$. Then $\tilde{G}_3 = G$, $\tilde{G}_2 \cong \mathbf{Z}_5$, $\tilde{G}_1 \cong \mathbf{Z}_2$, and $\tilde{G}_0 \cong \mathbf{Z}_3$. By 16.2, $\tilde{X} \cong \Gamma(G, \tilde{\mathcal{F}}, \tilde{\mathcal{E}})$, where $\tilde{\mathcal{E}}_{j,i} = \{\tilde{G}_j \tilde{G}_i\}$ for $j \leq i$.

Let $\tilde{G}_i = \langle g_i \rangle$ for $0 \leq i \leq 2$. As the pair $\langle g_0 \rangle$, $\langle g_2 \rangle$ is determined up to conjugation in $Aut(G) = S_5$, we may take $g_2 = (1, 2, 3, 4, 5)$ and $g_0 = (2, 3, 5)$. As r_1 inverts g_0 and g_2, so does its projection p_1 on G, so $p_1 = (1, 4)(2, 3)$. Then we may take the projection p_2 of r_2 to be $p_2 = p_1 g_2 = (1, 5)(2, 4)$. Next the projection p_0 centralizes p_2 and inverts g_0, so $p_0 = (1, 4)(2, 5)$. Finally $g_1 = p_0 p_2 = (1, 2)(4, 5)$.

Next let G_0 be the stabilizer on G of the point 1, G_1 the global stabilizer of $\{1, 2\}$, and $G_2 = N_G(\tilde{G}_2)$. Let $\mathcal{F} = (G_i : 0 \leq i \leq 2)$ and consider the coset poset $P = \Gamma(G, \mathcal{F}, \mathcal{E})$, where $\mathcal{E}_{i,j} = \{G_i G_j\}$. Now $G_1 = G_{0,1} G_{1,2}$, so G is transitive on 2-chains of the poset P. The poset P is the *Poincaré dodecahedron disk.* It is well known that

(16.7) *Let P be the Poincaré dodecahedron disk. Then P is aspherical with $\pi_1(P) \cong SL_2(5)$.*

Proof. See for example [4].

Let $\bar{\mathcal{F}} = (\tilde{G}_i : 0 \leq i \leq 2)$ and $\tilde{P} = \Gamma(G, \bar{F}, \bar{\mathcal{E}})$ the coset poset with $\bar{\mathcal{E}} = \{\mathcal{E}_{j,i} : 0 \leq j \leq i \leq 2\}$. That is $\tilde{P}$ is the 2-skeleton or boundary of the dodecahedron. Thus the geometric realization of $\tilde{P}$ is the 2-sphere.

Define $\alpha : \tilde{P} \to P$ to be the map induced via the identity homomorphism on G and the identity map on $\{0, 1, 2\}$ as in Lemma 16.3. Observe that $\tilde{G}_i \leq G_i$ and $E \subseteq G_j G_i$ for all $j \leq i$ and all $E \in \bar{\mathcal{E}}_{i,j}$, so α is a map of posets by 16.2. Indeed α is a restricted lower covering. Therefore if we form the cone $X = \{x_*\} \cup P$ of P and make the construction of Lemma 16.4, we obtain a restricted combinatorial cell complex (X, f) whose 2-skeleton X^2 is the simplicial cell complex (P, f) of the Poincaré dodecahedron disc. We call X the *Poincaré dodecahedron*, since the geometric realization of X *is* the Poincaré dodecahedron.

By Lemma 16.7, $X^2 = P$ is acyclic. Also $\dot{f}(x_*) = \tilde{P}$ has the homotopy type of the 2-sphere, so $\dot{f}(x_*)$ is spherical with $H_2(\dot{f}(x_*)) \cong \mathbf{Z}$. Finally $\dot{f}(x)$ is isomorphic to the 5-gon, and the 0-sphere for x of height 2,1, respectively, so $\dot{f}(x)$ is homology spherical for all $x \in X$. Therefore by Corollary 16.6, X is homology spherical with $H_3(X) \cong \mathbf{Z}$; that is X is a homology 3-sphere. Also as $\dot{f}(x_*)$ has the homotopy type of the 2-sphere, it is simply connected, so by 12.17, $\pi_1(X) \cong \pi_1(X^2) \cong SL_2(5)$ by 16.7. We summarize this as:

(16.8) *Let X be the Poincaré dodecahedron. Then X is a homology 3-sphere with $\pi_1(X) \cong SL_2(5)$.*

Notice that G is a group of automorphisms of P transitive on 2-chains and hence also on 3-chains of X.

References

[1] M. Aschbacher, Simple connectivity of p-group complexes, Israel J. Math., **82** (1993), 1–43.

[2] M. Aschbacher and Y. Segev, Extending morphisms of groups and graphs, Ann. of Math., **135** (1992), 297–323.

[3] M. Aschbacher and Y. Segev, Locally connected simplical maps, Israel J. Math., **77** (1992), 285–303.

[4] M. Aschbacher and Y. Segev, A fixed point theorem for groups acting on finite 2-dimensional acyclic simplicial complexes, Proc. London Math. Soc., **67** (1993), 329–354.

[5] E. Spanier, "Algebraic Topology", McGraw-Hill, New York, 1966.

Department of Mathematics
California Institute of Technology
Pasadena, CA 91125-0001
U.S.A.