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Einstein Metrics in Complex Geometry: An Introduction

Toshiki Mabuchi

Based on Calabi's pioneering work, systematic studies of Einstein-Kähler metrics commenced in mid-seventies with the affirmative solution of Calabi's conjecture by Aubin [2] and Yau [23]. Nowadays, via efforts of Yau himself and also other mathematicians, such metrics are shown to play by far important roles not only in differential geometry but also in algebraic geometry. Moreover, in eighties, the concept of Einstein-Hermitian metrics was introduced by S. Kobayashi (cf. [14]), and the resulting analogy of Calabi's conjecture to vector bundle cases, known as Hitchin-Kobayashi's conjecture, was solved affirmatively by Donaldson [9], [10], Uhlenbeck and Yau [22]. This now allows us to make differential-geometric studies of moduli spaces of vector bundles.

In view of these facts, we shall devote Volume 18-II to surveys of recent progress on the study of these metrics. By organizing three working groups, we divided the whole subjects into three categories:

(1) "Einstein-Kähler metrics with positive Ricci curvature" by Futaki, Mabuchi and Sakane.
(2) "Einstein-Kähler metrics with non-positive Ricci curvature" by Bando, Enoki, R. Kobayashi and Sugiyama.
(3) "Yang-Mills connections and Einstein-Hermitian metrics" by Itoh and Nakajima.

All of these are intended to be highly of expository nature but I believe that some particular places therein are written as original works. Now, to provide an introduction to subsequent surveys, we shall briefly discuss basic facts on Einstein metrics in complex geometry.

§1. Einstein-Kähler metrics and Calabi's Conjecture

Let X be an n-dimensional compact connected complex manifold endowed with a Kähler form ω. Then the corresponding Ricci form

$$\mathrm{Ric}(\omega) = \sqrt{-1}\,\bar{\partial}\partial \log \omega^n$$

represents the de Rham cohomology class $2\pi c_1(X)_{\mathbb{R}}$. Recall that ω is called an *Einstein-Kähler* form if $\mathrm{Ric}(\omega) = \lambda\omega$ for some real constant λ. Moreover, $\mathrm{Ric}(\omega)$ does not vary even if we replace ω by its positive constant multiple. Therefore, for an Einstein-Kähler form ω, such a replacement enables us to assume that λ above is either -1 or 0 or 1. Hence the following three cases are possible.

(1) $\lambda = -1$: $-\omega$ represents $2\pi c_1(X)_{\mathbb{R}}$, so that $c_1(X)_{\mathbb{R}} < 0$.
(2) $\lambda = \ \ 0$: ω is Ricci-flat, so that $c_1(X)_{\mathbb{R}} = 0$.
(3) $\lambda = \ \ 1$: ω represents $2\pi c_1(X)_{\mathbb{R}}$, so that $c_1(X)_{\mathbb{R}} > 0$.

There arises naturally the question whether or not the converse is true. Recall the following well-known conjecture posed by Calabi:

Calabi's Conjecture.
(a) *If* $c_1(X)_{\mathbb{R}} < 0$, *then* X *admits a unique Einstein-Kähler form* ω *satisfying* $\mathrm{Ric}(\omega) = -\omega$.
(b) *If* $c_1(X)_{\mathbb{R}} = 0$, *then each Kähler class on* X *contains a unique Ricci-flat Kähler form.*
(c) *If* $c_1(X)_{\mathbb{R}} > 0$ *and* $h^0(X, \mathcal{O}(TM)) = 0$, *then* X *admits a unique Einstein-Kähler form* ω *satisfying* $\mathrm{Ric}(\omega) = \omega$.

Most of this conjecture has already been proved affirmatively. For instance, the uniqueness is always true; see Calabi [7] for (a) and (b), and see Bando and Mabuchi [4] for (c). The existence, except (c), is established by Aubin [2] and Yau [23]: Namely, Aubin solved (a) affirmatively by treating a Monge-Ampère equation, while a little afterwards, Yau gave a complete affirmative answer to (b), as well as (a), by making systematic studies of such equations. However, (c) is still open unless either $\dim_{\mathbb{C}} X \le 2$ or X admits a suitable finite symmetry (see, Siu [18], Tian [20], Tian and Yau [21]).

§2. Einstein-Kähler metrics for compact Riemann surfaces

Let us now give examples of Einstein-Kähler metrics. For simplicity, assume $\dim_{\mathbb{C}} X = 1$, so that X is a compact Riemann surface of some genus p. Put $\Gamma := \pi_1(X)$. Moreover, let γ denote the positive generator of the cohomology group $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$. Then X carries an Einstein-Kähler metric g_X, unique up to the action of $\mathrm{Aut}^0(X)$, such that the associated Einstein-Kähler form (denoted also by g_X) represents the class $(2p-2)\gamma$, γ, or 2γ, according as $c_1(X)_{\mathbb{R}}$ is negative, zero, or positive:

(1) $c_1(X)_{\mathbb{R}} < 0$, i.e., $X = \Delta/\Gamma$, where Γ acts biholomorphically and freely on $\Delta = \{z \in \mathbb{C};\ |z| < 1\}$. Then the pull-back of g_X to Δ

is nothing but the Poincaré metric

$$\frac{2}{(1-|z|^2)^2}\sqrt{-1}\,dz\wedge d\bar{z}=\sqrt{-1}\,\bar{\partial}\partial\left\{\log\frac{(1-|z|^2)^2}{2}\right\}.$$

(2) $c_1(X)_{\mathbb{R}}=0$, i.e., $X=\mathbb{C}/\Gamma$, where Γ is a lattice $\mathbb{Z}+\mathbb{Z}\tau$ for some $\tau\in\mathbb{C}$ with $\operatorname{Im}\tau>0$. Then the pull-back of g_X to $\mathbb{C}$ is the translation-invariant flat metric

$$(2\ \operatorname{Im}\tau)^{-1}\sqrt{-1}\,dz\wedge d\bar{z}.$$

(3) $c_1(X)_{\mathbb{R}}>0$, i.e., $X=\mathbb{P}^1(\mathbb{C})=\{(z_1:z_2)\}$. By setting $z:=z_1/z_2$, we can express g_X as the Fubini-Study metric

$$\frac{2}{(1+|z|^2)^2}\sqrt{-1}\,dz\wedge d\bar{z}=\sqrt{-1}\partial\bar{\partial}\log\left\{\frac{(1+|z|^2)^2}{2}\right\}.$$

Thus, we see that $\operatorname{Ric}(g_X)$ is nothing but $-g_X$, 0, or g_X, according as $c_1(X)_{\mathbb{R}}$ is negative, zero, or positive.

§3. Birational moduli of algebraic surfaces of general type

Let $\mathcal{M}$ be the set of all connected nonsingular projective algebraic surfaces (defined over $\mathbb{C}$) of general type. For X, $Y\in\mathcal{M}$, we write $X\sim Y$ if X is birational to Y. Then the set of equivalence classes

$$\mathcal{M}_{\text{bir}}:=\mathcal{M}/\sim$$

is nothing but the birational moduli space of algebraic surfaces of general type. For each X in $\mathcal{M}$, let $[X]$ denote the associated equivalence class in $\mathcal{M}_{\text{bir}}$. Note that X is called a canonically polarized RDP surface if X has ample canonical divisor K_X and is in addition free from singularities other than rational double points. Now, let $\mathcal{M}_{\text{RDP}}$ be the set of all canonically polarized RDP surfaces (modulo isomorphisms). We then have a natural bijection

$$\mathcal{M}_{\text{bir}}\simeq\mathcal{M}_{\text{RDP}},\qquad [X]\leftrightarrow X_{\text{can}},$$

where X_{can} denotes the canonical model of $X\in\mathcal{M}$ defined by

$$X_{\text{can}}=\operatorname{Proj}\left(\oplus_{\nu=0}^{\infty}H^0(X,\mathcal{O}(\nu K_X))\right).$$

Let $\{\sigma_0,\sigma_1,\dots,\sigma_N\}$ be a $\mathbb{C}$-basis for $H^0(X,\mathcal{O}(5K_X))$. Then, more geometrically, X_{can} is nothing but the image of (cf. Bombieri [6])

$$\Phi_5:X\to\mathbb{P}^N(\mathbb{C}),\qquad x\mapsto\Phi_5(x):=(\sigma_0(x):\sigma_1(x):\cdots:\sigma_N(x)).$$

Now by R. Kobayashi [12], the results of Aubin [2] and Yau [23] on Calabi's Conjecture are valid also for orbifolds. Recall that rational double singularities are isolated quotient singularities. Hence, every $X \in \mathcal{M}_{\mathrm{RDP}}$ carries an Einstein-Kähler orbifold metric g_X uniquely characterized by the identity $\mathrm{Ric}(g_X) = -g_X$. Let $\mathcal{M}_{\mathrm{EK}}$ be the set of all Einstein-Kähler orbifolds (X, g), modulo holomorphic isometries, such that X is in $\mathcal{M}_{\mathrm{RDP}}$ and that $\mathrm{Ric}(g) = -g$. Then the mapping

$$\mathrm{pr}_1 : \mathcal{M}_{\mathrm{EK}} \to \mathcal{M}_{\mathrm{bir}}, \qquad (X, g_X) \mapsto X,$$

is bijective. The injectiveness of pr_1 is straightforward from the uniqueness of Einstein-Kähler metrics, while the surjectiveness of pr_1 follows from the existence of Einstein-Kähler metrics for all X in $\mathcal{M}_{\mathrm{RDP}}$. We thus have an identification

$$\mathcal{M}_{\mathrm{bir}} \simeq \mathcal{M}_{\mathrm{EK}}$$

between algebraic-geometric $\mathcal{M}_{\mathrm{bir}}$ and differential-geometric $\mathcal{M}_{\mathrm{EK}}$. This identification now allows us to look at $\mathcal{M}_{\mathrm{bir}}$ from differential-geometric viewpoints.

§4. Uniformization by Miyaoka-Van de Ven-Yau's inequality

We shall next study another aspect of Einstein-Kähler metrics. For simplicity, let X be a nonsingular irreducible projective algebraic surface defined over $\mathbb{C}$. Recall that Yau's affirmative solution of Calabi's Conjecture gave us some information on the extremal case of Miyaoka-Van de Ven-Yau's inequality:

$$c_1(X)^2 \leq 3c_2(X) \qquad \text{if } c_1(X)_{\mathbb{R}} < 0.$$

Namely, the equality holds if and only if the complex surface obtained as the universal cover of X is just the open unit ball $B^2 = \{\, |z_1|^2 + |z_2|^2 < 1 \,\}$ in $\mathbb{C}^2$. This uniformization result, for instance, enabled Yau to show the uniqueness of complex structure on $\mathbb{P}^2(\mathbb{C})$, which solved Severi's conjecture affirmatively. Another important consequence can be obtained by generalizing the uniformization result to ramified cases (see for instance R. Kobayashi [13]). Let Y be a compact complex connected normal surface with a finite number of irreducible reduced curves D_i, $i \in I$, sitting in Y. Associating, to each i, either an integer $b_i \geq 2$ or $b_i = \infty$, we have an effective Weil $\mathbb{Q}$-divisor

$$D = \Sigma_i \, (1 - b_i^{-1}) D_i \in \mathrm{Div}_{\mathbb{Q}}(Y).$$

Let Y_{sing}, D_{sing} be the (possibly empty) set of the singular points in Y, $\operatorname{Supp}(D)$, respectively. In particular, $D_i \cup D_j \subset D_{\text{sing}}$ for all $i, j \in I$ with $i \neq j$. Assume that all points in $Y_{\text{sing}} \cup D_{\text{sing}}$ are log-canonical singularities of (Y, D), i.e., we have a birational morphism $\mu\colon \tilde{Y} \to Y$ of a nonsingular projective algebraic surface $\tilde{Y}$ onto Y such that

(1) the set $\mu^{-1}(Y_{\text{sing}} \cup \operatorname{Supp}(D))$ has only simple normal crossings in the sense that its irreducible components are all nonsingular intersecting transversally.

(2) Let $\tilde{D}$ be the Weil $\mathbb{Q}$-divisor obtained from D by replacing each D_i with its proper transform $\tilde{D}_i$ in $\tilde{Y}$. Write the exceptional set $\mu^{-1}(Y_{\text{sing}} \cup D_{\text{sing}})$ as a union $\cup_{\alpha=1}^{k} E_\alpha$ of irreducible components. The rational numbers a_α defined by the equations

$$(K_{\tilde{Y}} + \tilde{D} + \Sigma_\alpha\, a_\alpha E_\alpha,\, E_\beta) = 0, \qquad 1 \leq \beta \leq k.$$

satisfy $a_\alpha \leq 1$ for all α.

Then by setting $E := \Sigma_\alpha\, a_\alpha E_\alpha$, we have $\mu^*(K_Y + D) = K_{\tilde{Y}} + \tilde{D} + E$, and $K_Y + D$ is $\mathbb{Q}$-Cartier. Let m_0 be the smallest positive integer such that $m_0(K_Y + D)$ is a Cartier divisor on Y. Assume now that $\bar{\kappa}(Y, D) = 2$, i.e., the log-canonical model $Y_{\text{can}} := \operatorname{Proj} \bar{R}(Y, D)$ is a projective algebraic surface, where $\bar{R}(Y, D)$ denotes the graded ring

$$\bar{R}(Y, D) = \bigoplus_{m=0}^{\infty} H^0(Y, \mathcal{O}(mm_0(K_Y + D))).$$

Note here that Y_{can} is obtained from Y by contracting successively log-exceptional curves of the first and second kinds. Let $\psi\colon Y \to Y_{\text{can}}$ be the contraction. Take the Zariski decomposition $K_Y + D = (K_Y + D)^+ + (K_Y + D)^-$. Then $(K_Y + D)^+$ is nothing but the divisor $\psi^*\psi_*(K_Y + D)$. Let D_∞ be the union of all D_i ($i \in I$) such that $b_i = \infty$. Moreover, let S be the set of singular points of Y_{can} other than quotient sigularities. Then every point in S is a finite cyclic quotient of either a simple elliptic singularity or a cusp singularity. We now put

$$Y^0 := Y_{\text{can}} - (\psi(D_\infty) \cup S),$$
$$D^0 := \psi(\operatorname{Supp}(D)) \cap Y^0.$$

We further put $D_i^0 := \psi(D_i) \cap Y^0$ for each i, and let d_i be the set-theoretical number of the points in $D_i^0 \cap (Y_{\text{sing}}^0 \cup D_{\text{sing}}^0)$. For each point q in $Y_{\text{sing}}^0 \cup D_{\text{sing}}^0$, let G_q be the local fundamental group at q of the orbifold pair (Y^0, D^0). Let r_q be the order $|G_q|$ of the group G_q. Then by setting $P := \psi_*(K_Y + D)$, we have the inequality (cf. [13])

$$(P^2) \leq 3\,\{\, e(Y^0) - \Sigma_i\,(1 - b_i^{-1})(e(D_i^0) - d_i) - \Sigma_q\,(1 - r_q^{-1})\}\,,$$

where $e(\cdot)$ denotes the Euler number. The equality (called the extremal case of the inequality) holds if and only if there exists a discrete subgroup Γ of the group $\mathrm{Aut}(B^2)$ of holomorphic automorphisms of B^2 such that

(a) Y^0 is biholomorphic to the ball quotient B^2/Γ;
(b) outside the set $(B^2/\Gamma)_{\mathrm{sing}} \cup D^0_{\mathrm{sing}}$, the covering $\rho\colon B^2 \to B^2/\Gamma$ is branched exactly over $\{\, D_i^0\,;\; b_i \neq \infty \,\}$ with branch indices $\{\, b_i \,\}$.

Interesting examples of ball quotients $Y^0 = B^2/\Gamma$ such that

(1) $Y^0 = Y_{\mathrm{can}} = \mathbb{P}^2(\mathbb{C})$;
(2) $\ell_i := \psi(D_i)$ is a line in $\mathbb{P}^2(\mathbb{C})$ for all i with $b_i \neq \infty$

were studied by many mathematicians (see for instance Hirzebruch [11]). An essential point in constructing such examples is to find out configurations of lines $\ell_i \subset \mathbb{P}^2(\mathbb{C})$ with suitable branch indices b_i such that the extremal case of the above inequality holds. Once such an example is constructed, the following question arises: Can one find a Fuchsian differential equation on $\mathbb{P}^2(\mathbb{C})$ with regular singular points exactly along the lines ℓ_i? Suppose yes. Let $\omega_0, \omega_1, \ldots, \omega_N$ be the set of all linearly independent solutions for the equation. Consider the multi-valued map

$$\tau\colon \mathbb{P}^2(\mathbb{C}) \to \mathbb{P}^N(\mathbb{C}), \qquad x \mapsto \tau(x) := (\omega_0(x) : \omega_1(x) : \cdots : \omega_N(x)).$$

Let Γ be the image of the projective monodromy representation

$$\pi_1(\mathbb{P}^2(\mathbb{C}) - \cup_i \ell_i) \to \mathrm{PGL}(N+1, \mathbb{C}).$$

Then we have a generically well-defined map

$$\tilde{\tau}\colon \mathbb{P}^2(\mathbb{C}) \to \mathrm{Image}(\tau)/\Gamma$$

induced by τ. If $\tilde{\tau}$ is nothing but a bijection satisfying $\mathrm{Image}(\tau) \cong B^2$, then the mapping τ is regarded as the inverse of $\rho\colon B^2 \to \mathbb{P}^2(\mathbb{C}) = B^2/\Gamma$. This construction is actually possible for several series of examples (see for instance Yoshida [24]). We finally note that examples of Terada [19], Deligne and Mostow [8] are easily understood from a differential-geometric point of view by checking that they satisfy just the extremal cases of the above inequality.

§5. Einstein-Hermitian metrics and stable vector bundles

Let E be a holomorphic vector bundle, of rank r, over a compact connected n-dimensional Kähler manifold (X, ω). For a Hermitian metric h on E, the corresponding curvature form is given by

$$\mathrm{R}(h) = \sqrt{-1}\,\bar{\partial}(h^{-1}\partial h).$$

We then put

$$\Lambda\,\mathrm{R}(h) := \frac{n\,\mathrm{R}(h) \wedge \omega^{n-1}}{\omega^n} \in C^\infty(\mathrm{End}\,E).$$

Recall that h is called an *Einstein-Hermitian* metric if $\Lambda\,\mathrm{R}(h) = c_0\,\mathrm{id}_E$, where $c_0 = (r\int_X \omega^n)^{-1}(n\int_X c_1(E) \wedge \omega^{n-1})$. Moreover, E is said to be stable if the inequality

$$\frac{\int_X c_1(\mathcal{S}) \wedge \omega^{n-1}}{\mathrm{rank}\,\mathcal{S}} < \frac{\int_X c_1(\mathcal{E}) \wedge \omega^{n-1}}{r}$$

holds for every coherent subsheaf $\mathcal{S}$ of $\mathcal{O}(E)$ with $0 < \mathrm{rank}\,\mathcal{S} < r$. Let us now assume that E is indecomposable, i.e., E cannot be written as a direct sum of two holomorphic vector bundles of lower ranks. A theorem of S. Kobayashi [14] and Lübke [17] shows that, if E admits an Einstein-Hermitian metric, then E is stable. Hitchin and S. Kobayashi conjectured its converse.

Hitchin-Kobayashi's Conjecture. *If E is stable, then E admits an Einstein-Hermitian metric (unique up to constant multiple).*

This can be regarded as a vector-bundle version of Calabi's conjecture, and was solved affirmatively by Donaldson [9], [10], Uhlenbeck and Yau [22]. This affirmative solution not only provided us with new methods in the study of moduli spaces of stable vector bundles, but also yielded numerous remarkable results on the differential structures of algebraic surfaces.

§6. Hyperkähler manifolds

A complete Riemannian manifold (X, g) is said to be *hyperkähler* if there exists global differentiable sections I, J, K of $\mathrm{End}(TX)$ such that

(a) $I^2 = J^2 = K^2 = -\,\mathrm{id}_{TX}$,
(b) $IJ = -JI = K,\ \ JK = -KJ = I,\ \ KI = -IK = J$,
(c) $\nabla I = \nabla J = \nabla K = 0$,

where ∇ denotes the covariant derivative naturally induced by the Riemannian metric g. Note that every hyperkähler manifold is Ricci-flat (cf. Berger [5]). For instance, every K3-surface S admits a hyperkähler structure (I, J, K) such that

(a) I is the complex structure of S,

(b) $\sqrt{-1}\,\omega_{\mathrm{J}} - \omega_{\mathrm{K}} \in H^0(S, \mathcal{O}(K_S))$, and ω_{I} is a Ricci-flat Einstein-Kähler form on S, where ω_{I}, ω_{J}, ω_{K} are the real symplectic 2-forms on S defined respectively by the identities

$$\omega_{\mathrm{I}}(u,v) = g(Iu,v),\ \omega_{\mathrm{J}}(u,v) = g(Ju,v),\ \omega_{\mathrm{K}}(u,v) = g(Ku,v),$$

with tangent vectors u, $v \in (T_sS)_{\mathbb{C}}$, $s \in S$.

An open version of a K3-surface is an asymptotically locally Euclidean hyperkähler 4-manifold. Kronheimer [15], [16] recently proved the following:

Theorem. *Every asymptotically locally Euclidean hyperkähler* 4-*manifold is diffeomorphic to the minimal resolution of the Kleinian singularity* $\mathbb{C}^2/\Gamma$ *for some finite subgroup* Γ *of* $\mathrm{SU}(2)$.

His new viewpoint relates Kleinian singularities to hyperkähler geometry via Mckay's observation. Moreover, recent works of Anderson [1], Bando, Kasue and Nakajima [3] show that, in Gromov's compactification of the moduli space of Ricci-positive Einstein-Kähler surfaces, every boundary point corresponds to a Ricci-positive Einstein-Kähler orbifold. Namely, in the degeneration procedure, a finite number of finite cyclic quotients of asymptotically locally Euclidean hyperkähler 4-orbifolds always bubble off producing the singular points of the limit orbifold. It is expected that some good theory exists also for more complicated singularities such as simple elliptic singularities or simple K3-singularities.

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Department of Mathematics,
College of General Education,
Osaka University,
Toyonaka, Osaka, 560
Japan