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Groups and Combinatorics — in memory of Michio Suzuki
pp. 1–39

Michio Suzuki

Koichiro Harada

§1. Biographical Sketch

1926, October 2. Born in Chiba, Japan.

1942, April. Entered the Third High School of Japan located at Kyoto (Noboru Ito, Katsumi Nomizu, Hidehiko Yamabe were his seniors by one year and Singo Murakami was in the same class).

1945, April. Entered the University of Tokyo. Majored in mathematics. (Gaishi Takeuchi, Nagayoshi Iwahori, Tsuneo Tamagawa were friends of this period.)

1948, April. Entered the Graduate School of Tokyo University. Suzuki's supervisor was Shokichi Iyanaga. Kenkichi Iwasawa had a profound influence on Suzuki.

1948-'51. Received a special graduate fellowship from the Government of Japan.

1951, April to '52, January. Held a lecturership at Tokyo University of Education

1952, January to '52, May. Held a graduate fellowship at University of Illinois at Urbana-Champaign.

Received April 27, 1999.
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1952, May. Received the Doctor of Science Degree from the University of Tokyoin absentia.

1952. Spent two months in the summer at University of Michigan. R. Brauer was a professor of Mathematics at Michigan. J. Walter, W. Feit were graduate students there.

1952, September to '53, May. Held a post-doctoral fellowship at University of Illinois at Urbana-Champaign.

1952, November. Married to a daughter Naoko of Yasuo Akizuki (then Professor at Kyoto University).

1953, September to '55, May. Held a research associateship at University of Illinois.

1955, September. Promoted to an assistant professor at University of Illinois.

1956, September to '57, May. Held a research associateship at Harvard University.

1958, September. Promoted to an associate professor at University of Illinois.

1959, September. Promoted to a full professor at University of Illinois.

1960. Discovered a new series of finite simple groups $Sz(q)$.

1960-'61. Held a visiting appointment at the University of Chicago.

1962. Invited to speak at the International Congress of Mathematicians in Stockholm.

1962-'63. Held a Guggenheim Fellowship.

1962, September to '63, May. Held a membership at the Institute for Advanced Study, Princeton.

1967. Discovered a sporadic simple group *Suzuki* of order 448,345,497,600.

1968-'69. Held a visiting appointment at the Institute for Advanced Study, Princeton, NJ.

1970. Invited to speak at the International Congress of Mathematicians in Nice, France.

1974. Received the Academy Prize from the Japan Academy.

1987. The conference of group theory and combinatorics for the occasion of Suzuki's 60th birthday was held in Kyoto, Japan.

1991. Awarded an honorary doctoral degree from the University of Kiel, Germany.

1997. The conference of group theory and combinatorics for the occasion of Suzuki's 70th birthday was held in Tokyo, Japan.

1998, May 31. Died at the age of 71. (Evariste Galois died on May 31, 1832.)

(A cancer was discovered in his liver early in February, 1998. Left for Japan. Received the same diagnosis. Stayed in the hospital (February 12 - March 13), at a Guest House of the International Christian University (March 14 - April 17). Back to hospital on April 18.)

June 7: Funeral Service at the International Christian University, Mitaka, Japan.

September 18: Memorial Service at the University of Illinois, Urbana, Illinois.

§2. The early work of Michio Suzuki

Among Suzuki's earliest research papers are:

[2] On the finite group with a complete partition, 1950.

[5] A characterization of simple groups $LF(2,p)$, 1951.

[6] On finite groups with cyclic Sylow subgroups for all odd primes, 1955.

In [2], Suzuki investigates the structure of a finite group G having a partition by its subgroups $H_i, i = 1, \dots, n$:

$$G = \bigcup_{i=1}^{n} H_i, H_i \cap H_j = 1 \text{ if } i \neq j.$$

A partition of G is called complete if H_i is cyclic for all $i = 1, \dots, n$. The research on groups having a complete partition goes back at least to P. Kontorovich [Sur la representation d'un groupe fini sous la forme d'une somme directe de sous-groupes, I. Rec. Math. (Mat. Sbornik), 5 (47) (1939), 283–296].

In [2] Suzuki considers groups having a complete partition. Examples of such groups are $PGL(2,q)$ and $PSL(2,q)$ where q is a power of a prime. In [2], however, Suzuki determines only nonsimple groups having a complete partition. It is shown first that if G is a nonsimple, nonsolvable finite group with a complete partition then a minimal normal subgroup N of G is of index 2. The proof proceeds by induction on the order of G, since the complete partitionability carries over to its subgroups and even to its factor groups as Suzuki shows.

Suzuki next shows that the Sylow 2-subgroups of G are dihedral, and that for any odd prime p, any two distinct Sylow p-subgroups of G have a trivial intersection. He then uses a counting argument to obtain a configuration in which the group G is a sharply triply transitive permutation group acting on the coset space G/M where M is a suitable subgroup of G obtained in the counting argument mentioned above. Therefore Suzuki is able to use the result of Zassenhaus [Kennzeichnung endlicher linearer Gruppen als Permutationsgruppe, Hamb. Abh., 11(1936), 17–40], who had classified, among other results, all such permutation groups, hence the theorem:

Theorem. *Let G be a nonsimple, nonsolvable finite group with a complete partition. Then G is isomorphic to the full linear fractional group $PGL(2, q)$ where q is a power of an odd prime.*

Character theory is not used in [2]. This paper shows that Suzuki was a young mathematician of foresight. He was able to recognize the importance of the groups $PSL(2, q)$ and Zassenhaus' work. The concept of a group having a partition does not appear to be very important on its own right, but it should be mentioned that the infinite series of new simple groups $Sz(q)$ discovered by Suzuki in 1960 does have a partition, though not a complete partition. Suzuki completes the classification of all (semi) simple groups with a partition in 1961 [18].

As for the paper [5], let us first observe that the subgroups of the simple groups $PSL(2, p)$ for a prime p are of the types: (1) metacyclic groups; (2) the alternating group A_4 of degree 4; (3) the symmetric group S_4 of degree 4; or (4) the alternating group A_5 of degree 5. In [5], Suzuki characterizes $PSL(2, p)$ by this property.

Let G be a finite simple group such that all of its subgroups are of types (1)-(4) mentioned above. Suzuki first shows that G possesses a complete partition in the sense of the paper [2]. Among all papers of Suzuki, the theory of exceptional characters first appeared here in [5]. Using this theory and Brauer's work on a group whose order is divisible by a prime to the first power, Suzuki was able to show that G possesses an irreducible character of degree $\frac{1}{2}(p \pm 1)$ for some prime p. He next applies a result of H.F. Tuan [On groups whose orders contains a prime number to the first power, Ann. of Math., 45(1944), 110–140] to complete the characterization of $PSL(2, p)$.

As he recognized the importance of studying the simple groups $PSL(2, q)$, he began doing research on them from various points of view : in [2] as groups having a partition, in [5] as groups having only a special set of isomorphism classes of subgroups, etc.

Although the papers [2] or [5] of Suzuki might perhaps not be among his better works, if they are considered as stand-alone papers, the line of research in this direction served him well and it culminated in the discovery of the simple groups $Sz(q)$ and the classification of all Zassenhaus groups (which was completed by a joint effort of Zassenhaus, Feit, Ito and Suzuki).

The paper [6] is also part of Suzuki's continuing efforts to understand the simple groups $PSL(2,p)$. Its content is fully explained in the title. Its introduction begins with 'The purpose of this paper is to determine the structure of some finite groups in which all Sylow subgroups of odd order are cyclic.The assumption on Sylow subgroups simplifies the structure of groups considerably, but the structure of 2-Sylow subgroups might be too complicated to make any definite statement on the structure of the groups. In this paper, therefore, we shall make another assumption on 2-Sylow subgroups, $\cdots$'.

In fact, he assumes that the Sylow 2-subgroups of G are either (a) dihedral or (b) generalized quaternion. The Sylow 2-subgroups of $SL(2,p)$ are, as is well known, generalized quaternion if p is odd. Suzuki shows that the group G contains a normal subgroup $G_1 = Z \times L$ of index at most 2 such that $L \cong PSL(2,p)$ if (a) holds, and $L \cong SL(2,p)$ if (b) holds. Moreover, Z is a group of odd order all of whose Sylow subgroups are cyclic. Frobenius and Burnside treated groups such that all of their Sylow subgroups are cyclic and showed that all such groups are solvable, in fact all such groups are metacyclic. Zassenhaus classified all solvable groups with the same assumption on Sylow subgroups for odd primes but with the weaker assumption for the prime 2 that a Sylow 2-subgroups has a cyclic subgroup of index 2.

§3. Theory of exceptional characters

'Perhaps the first mathematician of the post war generation who mastered Brauer's work in group theory was M. Suzuki. He came to the United States in the early fifties and he has made many significant contributions to the theory of simple groups (from W. Feit [R.D. Brauer, Bull (New Series). Amer. Math. Soc., 1(1979), 1–20])'.

Having begun his research on exceptional characters in [5], Suzuki wrote a couple of papers on the subject [13], [19], and several papers in which the theory played a crucial role [6], [8], [9], [10], [17].

In his work on the theory of modular representations, Brauer defined the concept of an exceptional character. Brauer and Suzuki independently extended this concept of exceptional characters at about the same time, around 1950. Although the basic assumption of the theory can be loosened from the one given below, we will show it in the simplest but most important setting.

We are typically interested in a finite group G having an abelian subgroup A such that the centralizer of every nonidentity element of A is contained in A (hence it is equal to A itself and A is a maximal abelian subgroup of G). The simple group $PSL(2,q)$ contains a couple of conjugacy classes of such abelian subgroups. For Suzuki, a motivation to extend the theory of exceptional characters must have come from his investigation of the simple group $PSL(2,q)$. Under this condition on G and on A, the following conditions hold:

(1) A is an abelian TI subgroup of G: i.e. $A \cap A^g = A$ or 1 for every element g of G.

(2) The normalizer $N = N_G(A)$ of A in G is a Frobenius group.

Let $l = [N : A]$ and $w = \frac{|A|-1}{l}$. Then G possesses exactly w conjugacy classes of elements represented by nonidentity elements of A.

The Frobenius group N possesses l irreducible characters of degree 1, all of which contain A in their kernels. In addition to those linear characters, N possesses w irreducible characters not containing A in their kernels, and all of them have degree l. Those are all the irreducible characters of N. Thus N possesses exactly $l + w$ irreducible characters.

We can actually obtain the irreducible characters of N of degree l as follows. Let $\{\psi_i, i = 1, \dots, w\}$ be the complete set of representatives of N-orbits (by conjugation) consisting of nonidentity irreducible characters of A and $\Psi_i = \psi_i^N$ be the corresponding induced character of ψ_i to N. By computing the inner product directly, we see that Ψ_i is an irreducible character of N for all i. We thus obtain w irreducible characters of N of degree l. The remaining irreducible characters of N (of degree 1) will appear as constituents of the induced character of the trivial character of A.

Let $\Psi_i^G, i = 1, \dots, w$ be the corresponding induced characters to G. We compute that $\Psi_i^G(g) = 0$ if g is not conjugate to an element of $A\backslash 1$ and $\Psi_i^G(g) = \Psi_i(a)$ if g is conjugate to an element a of $A\backslash 1$. Thus

$$\langle \Psi_i^G, \Psi_i^G \rangle_G - (\Psi_i^G(1))^2 = \langle \Psi_i, \Psi_i \rangle_N - (\Psi_i(1))^2.$$

Therefore, the norm $||\Psi_i^G||_G$ is almost determined by the norm $||\Psi_i||_N$, but not completely so since $\Psi_i^G(1)$ is an unknown number. If we can find a way to eliminate the ambiguity then it will be nice.

Now assume, in addition to (1) and (2) mentioned above:

(3) $w \geq 2$.

Consider the generalized character $\Psi_i - \Psi_j, i \neq j$, of N. Then we obtain

$$||\Psi_i^G - \Psi_j^G|| = 2$$

since $||\Psi_i^G - \Psi_j^G||_G = ||\Psi_i - \Psi_j||_N = 2$ holds. Therefore, $\Psi_i^G - \Psi_j^G = \epsilon_{ij}(\Theta_i - \Theta_j)$ where Θ_i, Θ_j are irreducible characters of G and $\epsilon_{ij} = \pm 1$. Actually ϵ_{ij} is independent of i, j and so

$$\Psi_i^G - \Psi_j^G = \epsilon(\Theta_i - \Theta_j), \ \epsilon = \pm 1.$$

This implies that

$$\Psi_i^G = \epsilon\Theta_i + \Delta$$

where Δ is a generalized character of G independent of $i = 1, \dots, w$.

The irreducible characters $\Theta_i, i = 1, \dots, w$ obtained above are called *exceptional characters* of G associated with A. (W. Feit was able to extend the exceptional character theory by dropping the condition that A is abelian. Feit still needed that A is nilpotent and is not isomorphic to a certain type of p-group. A further extension was obtained by D. Sibley.)

Exceptional characters satisfy the following properties. Let D be the set of all elements of G not conjugate to any element of $A \backslash 1$.

(I) $\Theta_i(\sigma) = \Theta_j(\sigma)$ if $\sigma \in D$ for every pair i, j. In particular all exceptional characters Θ_i have the same degree.

(II) The exceptional characters are linearly independent on the conjugacy classes $\{C_1, \dots, C_w\}$ of G represented by the elements of $A \backslash 1$: i.e. if $\sum_{i=1}^{w} a_i\Theta_i(\sigma) = 0$ for all $\sigma \in \cup_{i=1}^{w} C_i$, then $a_i = 0$ for all $i = 1, \dots, w$.

(III) If B is another abelian subgroup of G not conjugate to A but satisfying the same property as A does, then the exceptional characters for A are nonexceptional characters for B.

Therefore if G has many nonconjugate abelian subgroups of the same property, then the majority of the irreducible characters of G will be exceptional characters associated with some abelian subgroup A. Using those irreducible characters, one can obtain strong numerical conditions on the order of G.

§4. The CA-paper of Suzuki

Theorem ([8]). *Let G be a finite simple group such that the centralizer of every nonidentity element is abelian. Then the order of G is even.*

Let us quote Thompson first:

'A third strategy (or was it a tactic ?) in OOP (Odd Order Paper) attempted to build a bridge from Sylow theory to character theory. The far shore was marked by the granite of Suzuki's theorem on CA-groups, flanked by W. Feit, M. Hall, Jr. and J.G. Thompson [Finite groups in which the centralizer of any non-identity element is nilpotent, Math. Z., 74(1960), 1–17]. The bridge was built of tamely embedded subsets with their supporting subgroups and associated tau (τ) isometry. The near shore was dotted with the E-theorems and the uniqueness theorems.

... ...

Suzuki's CA-theorem is marvel of cunning. In order to have a genuinely satisfying proof of the odd order theorem, it is necessary, it seems to me, not to assume this theorem. Once one accepts this theorem as a step in a general proof, one seems irresistibly drawn along the path which was followed. To my colleagues who have grumbled about the tortuous proofs in the classification of simple groups, I have a ready answer: find another proof of Suzuki's theorem (from J.G. Thompson [Finite Non-Solvable Groups, in Group Theory:essays for Philip Hall, Academic Press, (1984), 1–12])'

Now let G be a finite group such that the centralizer of every non-identity element is abelian. Let us call such a group G a CA-group. Already in 1920's, it was known that every CA-group is either solvable or simple (L. Weisner [Groups in which the normalizer of every element except the identity is abelian, Bull. Amer. Math. Soc., 31(1925), 413–416]). So let us assume that our CA-group G is nonabelian and simple.

Let g be a nonidentity element of G. Then the centralizer $A = C_G(g)$ is a proper abelian subgroup of G. Let $1 \neq h \in A$. Then $C_G(h) \supset A$. The fact that $C_G(h)$ is abelian forces the equality $C_G(h) = A$, thus A is a maximal abelian subgroup of G, and A is a TI-set. The rudiments of group theory also show that A is a Hall subgroup of G, i.e. $\gcd(|G:A|, |A|) = 1$. If the normalizer $N = N_G(A)$ is equal to A itself, then $N_G(P) = C_G(P)$ for a Sylow p-subgroup P of A for some prime p. Since P is a Sylow p-subgroup of G also, Burnside's theorem implies that G is nonsimple. Thus $N > A$ and N is a Frobenius group. In order to apply the exceptional character theory effectively, we need one more condition : $w \geq 2$ where $w = \frac{|A|-1}{l}, l = [N : A]$. For this purpose, we henceforth assume that G is of odd order as this is the case Suzuki treats. Then $|A|$ and l are both odd, and so w can not be equal to 1. Hence $w \geq 2$ as desired.

Let $\{A_i, i = 1, \ldots, n\}$ be a complete set of representatives of the conjugacy classes of maximal abelian subgroups of G and we put $N_i = N_G(A_i)$. We have shown that $N_i > A_i$ and N_i is a Frobenius group for all i. Moreover, every element of $G \backslash 1$ has a representative in $\cup_{i=1}^n A_i$.

Since each A_i is a TI-set, we have

$$|G| = 1 + \sum_{i=1}^{n} [G : N_i](|A_i| - 1).$$

Each A_i gives rise to $w_i = +(|A_i| - 1/l_i)$ (where $l_i = [N_i : A_i]$) exceptional characters and so G has $\sum_{i=1}^{n} w_i$exceptional characters in total. On the other hand, G possesses precisely $1 + \sum_{i=1}^{n} w_i$ conjugacy classes. Therefore every nonidentity irreducible character of G is exceptional for some A_i. Suzuki puts all of this information together and starts a counting argument. In three pages, he is able to reach a contradiction.

This CA-paper of Suzuki was received by the editors on December 24, 1954 but was published in 1957. Suzuki knew who was the referee. It was none other than R. Brauer. Apparently Brauer did not understand some argument of Suzuki and left it there for a (great) while. Suzuki submitted a revised version two years later and the paper was published soon.

'At the time its importance was not fully grasped, either by him or by others, as it seemed to be simply an elegant exercise in character theory. However, the result and the methods used had a profound impact on much succeeding work (W. Feit [Obituary written for Michio Suzuki, Notices of Amer. Math. Sci., Vol. 46(1999)]).'

L. Redei [Ein Satz über die endlichen einfachen Gruppen, Acta. Math., 84(1950), 129–153] considered finite simple groups such that every proper subgroup of every maximal subgroup is abelian. He showed that the alternating group of degree 5 is the only such group of even order. One obtains, as a corollary to the main theorem of this paper, that there is no such group of odd order. Moreover, Suzuki proved that the word *abelian* in Redei's theorem can be replaced by *nilpotent* to assert the same conclusion.

Suzuki uses the assumption that G is of odd order only to assert $w \geq 2$ and so this method can go farther under a suitable assumption. In fact, R. Brauer, M. Suzuki, and G.E. Wall, more or less independently proved:

Theorem. *If the centralizer of every element of a finite group G is abelian then either G is solvable or G is isomorphic to $PSL(2, 2^n)$.*

In the published form of the Brauer-Suzuki-Wall Theorem [9], however, it is stated as follows:

Theorem. *Let G be a group of even order which satisfies the condition:*
(1) *If two cyclic subgroups A and B of even order of G have a nontrivial intersection then there exists a cyclic subgroup C of G that contains both A and B.*
(2) $G = [G,G]$.
Then $G \cong PSL(2,q)$ for some prime power q.

One of my colleagues, Ronald Solomon, and I studied the latter theorem but could not conclude that it implies the former. We wrote a letter of inquiry to G.E. Wall, who replied that they worked fairly independently with not a great deal of communication between them. He says also that the BSW paper (published version) was written by R. Brauer who did not have enough time to weld together three rather different versions and that the CA-groups of even order are not covered in any obvious way (in the published version), but they are covered in the 'behind scenes' BSW versions.

§5. Zassenhaus groups

Let V be a 2-dimensional vector space over a field K and let

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in GL(V)$$

be a 2×2 matrix of nonzero determinant with entries in K. The matrix A acts on V as a linear transformation and so the image of a line (1 dimensional subspace of V) is again a line. Since the structure of $GL(V)$ depends only on the dimension of V and the field K, we write $GL(2,K)$ for $GL(V)$ also.

Let $P_1(K)$ be the set of all lines of V. $GL(2,K)$ acts on $P_1(K)$. The scalar matrices $A = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}$ are the only matrices that act trivially on $P_1(K)$. Denote by Z the set of all scalar matrices of $GL(2,K)$. Then the factor group $PGL(2,K) = GL(2,K)/Z$ acts on $P_1(K)$ faithfully.

If $\{u_1,u_2\}$ and $\{v_1,v_2\}$ are any pairs of linearly independent vectors of V, then there is an element $g \in GL(2,K)$ such that $g(u_1) = v_1, g(u_2) = v_2$. This implies that $PGL(2,K)$ is doubly transitive on $P_1(K)$ since if $[u]$ denotes the line spanned by the vector $u \in V$, then $\bar{g}([u_1]) = [v_1], \bar{g}([u_2]) = [v_2]$ where $\bar{g}$ is the image of $g \in GL(2,K)$ in $PGL(2,K)$.

Put $SL(2,K) = \{g \in GL(2,K) \mid \det g = 1\}$ and $PSL(2,K) = SL(2,K)/Z \cap SL(2,K)$. As is easily seen, $PSL(2,K)$ is also doubly transitive on $P_1(K)$. Let us consider subgroups of $G = SL(2, K)$ that leave points of $P_1(K)$ invariant. Let $\{[u_1], [u_2]\}$ be a set of two arbitrary elements of $P_1(K)$. We want to know the structure of the two point stabilizer $G_{\alpha,\beta}; \alpha, \beta \in P_1(K), \alpha \neq \beta$. Since G is doubly transitive, we may assume $\{\alpha = [(1,0)], \beta = [(0,1)]\}$ and we find

$$G_{\alpha,\beta} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \alpha \in K \right\}.$$

In particular, $G_{\alpha,\beta}$ is cyclic. If in addition, $g \in G_{\alpha,\beta}$ fixes a third point, then $g = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ and so every three point stabilizer in $\bar{G} = PSL(2, K)$ is trivial.

Definition. A permutation group G acting on a finite set Ω is called a Zassenhaus group, if
(1) G is doubly transitive on Ω,
(2) the identity element is the only element of G that leaves three distinct points of Ω invariant; and,
(3) G does not have a regular normal subgroup.

Remark. Let G be a permutation group on a finite set Ω. If a subgroup H of G acts transitively on Ω and $|H| = |\Omega|$, then H is called a regular subgroup of G.

As shown above, $G = PSL(2, K)$ acting on $P_1(K)$ is an example of a Zassenhaus group if $|K| \geq 4$. Let G be a Zassenhaus group acting on Ω and let $\alpha \in \Omega$. Then the one point stabilizer $H = G_\alpha$ of α is a transitive but not regular permutation group on $\Omega \backslash \alpha$ such that a two point stabilizer of H on $\Omega \backslash \alpha$ is trivial and so H is a Frobenius group. By Frobenius' theorem, the identity element and the set of all elements of H that do not leave any letter of $\Omega \backslash \alpha$ invariant forms a normal subgroup K of H. Let $C = H_\beta = G_{\alpha,\beta}$. Then H is a semi-direct product of K and C.

Zassenhaus was the first person to study a group having the property described in the definition above. In the paper [op.cit.], Zassenhaus determined the structure of G under some additional assumptions (see below). What Zasssenhaus did was to define an algebraic structure called a *near field* from the one point stabilizer H of G. He then used the structure of G to show that the near field is almost a field. He next constructed a suitable geometry of projective lines over a field and determined the structure of G.

In the paper [Über endliche Fastkörper, Hamb. Abh., 11(1936), 187–220], Zassenhaus was able to determine all near fields of finite order.

This implies that he completely determined all sharply doubly transitive permutation groups.

The complete classification of all Zassenhaus groups was carried out by a combined effort of H.Zassenhaus, W. Feit, N. Ito, and M. Suzuki.

Theorem (H. Zassenhaus [op.cit.]). *Let G be a Zassenhaus group on Ω. Suppose that G is triply transitive on Ω. Then $G \cong PGL(2,q)$, or $PGL^*(2,q^2)$.*

Here $G = PGL^*(2,q^2)$ is a group which is uniquely defined as follows. G contains a normal subgroup of index 2 isomorphic to $PSL(2,q^2)$ and the Sylow 2-subgroups of G are semi-dihedral.

Theorem (H. Zassenhaus [op.cit.]). *Let G be a Zassenhaus group on Ω. Suppose $|G| \geq |\Omega|(|\Omega|-1)(|\Omega|-2)/2$, then $G \cong PGL(2,q)$, $PGL^*(2,q^2)$, or $PSL(2,q)$.*

Theorem (W. Feit [On a class of doubly transitive permutation groups, Ill. J. Math., 4(1960), 170–186]). *Let G be a Zassenhaus group on Ω. Then the Frobenius kernel K of a one point stabilizer $H = G_\alpha$ is a p-group for some prime p. Furthermore if K is abelian, then G is contained in $PGL(2,q)$ or $PGL^*(2,q^2)$ as a normal subgroup of index at most 2.*

With this theorem of Feit, every researcher of the time must have conjectured that every Zassenhaus group is isomorphic to $PSL(2,q)$, $PGL(2,q)$ or $PGL^*(2,q^2)$ where q is a power of a prime p. N. Ito soon settled the cases in which the Frobenius kernel K is a p-group for an odd prime p.

Theorem (N. Ito [On a class of doubly transitive permutation groups, Ill. J. Math., 6(1962), 341–352]). *Let G be a Zassenhaus group on a set of $n+1$ letters. If n is odd, then the Frobenius kernel K of a one point stabilizer $H = G_\alpha$ is abelian (and so the structure of G is determined by Zassenhaus and Feit).*

Therefore the Zassenhaus groups on an even number of letters are now completely classified. Namely, they are isomorphic to

$$PSL(2,q), PGL(2,q) \text{ or } PGL^*(2,q^2), q \text{ odd } > 3.$$

Note that if $q = 3$, then $PSL(2,3)$ and $PGL(2,3)$ have a regular normal subgroup. I should mention here that the theorems of Feit and Ito stated above both use the fundamental result proved by Thompson, who solved affirmatively the long standing conjecture: the Frobenius kernel is nilpotent.

We are now left with the case in which $p = 2$ or equivalently $|\Omega| = 1 + 2^n$ for some n. We, however, need a new section to describe this case.

§6. Suzuki's simple groups $Sz(2^n)$

The late '50s must have been an exciting period for young (and old) group theorists, although the competition among them must have been intense also. In 1955, C. Chevalley [Sur certains groupes simples, Tohoku J. Math., 7(1955), 14–66] announced the discovery of several series of new simple groups of finite order. These simple groups are defined using Lie algebras over the ring of integers. The paper of R. Steinberg [Variations on a theme of Chevalley, Pacific J. Math., 9(1959), 875–891] followed, in which he defined several twisted versions of the Chevalley groups and showed that these twisted groups are also simple except for a few cases. After these theorems of Chevalley and Steinberg, no new simple groups were expected to come out from Lie theory.

Suzuki surprised the world by discovering a new series of simple groups, which were soon identified as groups coming from Lie theory, though they were not initially defined as such. These are now known as Suzuki groups $Sz(q)$ where q (≥ 8) is an odd power of 2. $Sz(q)$ is an example of a Zassenhaus group but it was, according to Suzuki, not discovered as a Zassenhaus group.

Groups such that the centralizer of every nonidentity element of G is abelian were all determined by late in the '50s. Feit, M. Hall, and J.G. Thompson [op.cit.] showed, in 1960, that all simple CN-groups (Centralizer-Nilpotent) are of even order. The next problem that Suzuki decided to treat was the determination of all (simple) CN-groups. In doing so, he discovered a new series of simple groups, which turned out to be Zassenhaus groups.

Let $F = F_q$ be a finite field with $q = 2^{2n+1}$ $(n \geq 1)$ elements and set $r = 2^{n+1}$. We have $r^2 = 2q$ and the mapping

$$\theta : \alpha \to \alpha^r$$

is an automorphism of F and it satisfies $\theta^2 = 2$. In other words,

$$\alpha^{\theta^2} = \alpha^2, \quad \alpha \in F$$

holds. Moreover, we define, for arbitrary elements α, β of F, a 4×4 matrix (α, β) and a subset Q as follows:

$$(\alpha,\beta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \alpha & 1 & 0 & 0 \\ \alpha^{1+\theta}+\beta & \alpha^\theta & 1 & 0 \\ \alpha^{2+\theta}+\alpha\beta+\beta^\theta & \beta & \alpha & 1 \end{pmatrix},$$

$$Q = Q(q) = \{(\alpha,\beta) \mid \alpha,\beta \in F_q\}.$$

Since the product is

$$(\alpha,\beta)(\gamma,\delta) = (\alpha+\gamma,\ \alpha\gamma^\theta+\beta+\delta),$$

Q is a subgroup of order q^2. Let us define, for an element k of the multiplicative groups F^* of the field F, a matrix (k) by:

$$(k) = \begin{pmatrix} \zeta_1 & & & 0 \\ & \zeta_2 & & \\ & & \zeta_3 & \\ 0 & & & \zeta_4 \end{pmatrix},$$

where

$$\zeta_1^\theta = k^{1+\theta}, \quad \zeta_2^\theta = k, \quad \zeta_3 = \zeta_2^{-1}, \quad \zeta_4 = \zeta_1^{-1}.$$

If we set

$$K = K(q) = \{(k) \mid k \in F^*\},$$

then, K is a cyclic group of order $q-1$ and is isomorphic to F^*. Since

$$(k)^{-1}(\alpha,\beta)(k) = (\alpha k, \beta k^{1+\theta}),$$

the set theoretical product QK is a subgroup and Q is a normal subgroup of QK. If $k \neq 1$, then the conjugation by the matrix (k) does not leave any element of $Q\backslash 1$ invariant. Let us define another matrix τ as follows:

$$\tau = \begin{pmatrix} 0 & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & 0 \end{pmatrix}.$$

We have $\tau^2 = 1$ and, $(k)^\tau = (k)^{-1}$.

Denote the subgroup of $GL(4,F)$ generated by $Q(q)$, $K(q)$, τ by:

$$Sz(q) = \langle Q(q),\ K(q),\ \tau\rangle.$$

The family $Sz(q)$ are called Suzuki's simple groups and form an infinite series of new simple groups of finite order. $Sz(q)$ has the following properties:

(1) $|Sz(q)| = q^2(q-1)(q^2+1),\ \ q = 2^{2n+1},\ \ n \geq 1.$
(1′) $|Sz(q)|$ is not divisible by 3.
(2) $Sz(q)$ has cyclic subgroups A_+, A_- of order $q \pm r + 1$ respectively and

$$Sz(q) = \bigcup_{g \in G} (Q^g \cup K^g \cup A_+^g \cup A_-^g)$$

is a union of subgroups of $Sz(q)$ such that any pair of subgroups have trivial intersection unless they coincide. ($Sz(q)$ has a *partition.*)
(3) If g is an arbitrary nonidentity element of $Sz(q)$, then the centralizer of g in $Sz(q)$ is always nilpotent. ($Sz(q)$ is a CN-group.)
(4) The natural action of $Sz(q)$ on its factor space $Sz(q)/QK$ is doubly transitive and the identity of $Sz(q)$ is the only element that leaves three distinct points of $Sz(q)/QK$ invariant. ($Sz(q)$ is a Zassenhaus group.)

Apparently it was a great surprise to many that the order of $Sz(q)$ is not divisible by 3: it was believed that every nonabelian simple group has order divisible by 6. All the generators $\{(\alpha, \beta),\ (k),\ \tau\}$ of $Sz(q)$ given above leave the bilinear form

$$x_1y_4 + x_2y_3 + x_3y_2 + x_4y_1$$

invariant and so $Sz(q)$ is a subgroup of the 4 dimensional symplectic group $Sp(4,q) = B_2(q)$. The group $B_2(q)$ has a special involutory automorphism σ only if q is an odd power of 2, and

$$Sz(q) = \{g \in B_2(q) \bigm| g^\sigma = g\}$$

holds. Therefore, $Sz(q)$ could have been constructed naturally through Lie theory. It was, however, discovered by Suzuki in a process of classifying all CN-groups (an important step to determine all Zassenhaus groups), which is independent of Lie theory. W. Feit told me that he, when he was young, uttered the following words to a famous Lie theorist

"It is better to have a good mathematician than a good theory !"

Although the discovery was purely group theoretic, for classification purpose, however, $Sz(q)$ can better be accounted for as a simple group of Lie type and is often denoted by ${}^2B_2(q)$.

Just before Suzuki announced the discovery of a new series of simple groups, he published a two-part paper:

[11] On characterizations of linear groups, I, II, 1959.

Suzuki published two more papers on the same theme.

[23] On characterizations of linear groups, III, 1962.

[32] On characterizations of linear groups, IV, 1968.

In [11, Part I], Suzuki proves:

Theorem. *Let G be a simple group such that the centralizer of every involution is abelian. Then $G \cong PSL(2, 2^n)$.*

The assumption Suzuki actually uses is slightly more general so that he can use induction. The simple group $PSL(2, 2^n)$ does have this property. In fact, $PSL(2, 2^n)$ has the property that the centralizer of every involution is an abelian 2-group. In 1951, K.A. Fowler showed that this property characterizes $PSL(2, 2^n)$. There is a generalization of Fowler's result by Brauer, Suzuki, and Wall. Suzuki puts the characterization of $PSL(2, 2^n)$ in its final shape.

Already in 1900, Burnside gave the following characterization of $PSL(2, 2^n)$.

Theorem (Burnside). *$PSL(2, 2^n)$ is the only simple group of even order such that the order of every element is either odd or equal to 2.*

This result of Burnside had been completely forgotten and was rediscovered by K.A. Fowler half a century later. It is quite surprising that Burnside worked on this relatively modern problem, considering the fact that the line of research did not continue until it was taken up again much later.

In [11, Part II], Suzuki studies the structure of $G = PGL(3, q)$ where $q = 2^n$. G is simple if $3 \not| q-1$. If $3|q-1$, then G has a normal subgroup of index 3. For example $PSL(3, 4)$, which has the same order as A_8, is a normal subgroup of index 3 of $PGL(3, 4)$. Every involution of G is conjugate to

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and the centralizer of I in G has order $q^3(q-1)$. In $PSL(3, 4)$, the centralizer of I has order 2^6, hence it is a 2-group.

In this paper, Suzuki shows that $PGL(3, q)$ is characterized by the structure of $C_G(I)$ except for one case when $q = 2$, in which case we obtain $G \cong PGL(3, 2)$ or $G \cong A_6$. A similar characterization of $PGL(3, q)$ where q is a power of a prime satisfying $q \equiv -1 \pmod 3$ was obtained by R. Brauer. With the initial work of Brauer and Suzuki's work that followed, the characterizations of simple groups by the centralizers of involutions began in full force and continued until early in the 1970's.

$C_G(I)$ is isomorphic to the subgroup of G consisting of all matrices of the form:

$$M(\alpha,\beta,\gamma,\delta)=\begin{pmatrix}1&0&0\\ \alpha&\delta&0\\ \beta&\gamma&1\end{pmatrix}$$

where $\alpha,\beta,\gamma,\delta\neq 0$ are elements of a finite field F of characteristic 2.

The matrix product shows

$$M(\alpha,\beta,\gamma,\delta)M(\alpha',\beta',\gamma',\delta')=M(\alpha^*,\beta^*,\gamma^*,\delta^*)$$

where

$$\alpha^*=\alpha+\delta\alpha',\beta^*=\beta+\gamma\alpha'\beta',\gamma^*=\gamma\delta'+\gamma',\delta^*=\delta\delta',$$

Suzuki computes the order of the group G in [11, Part I]. In [11, Part II] he also obtains the order of the group G. Namely $|G|=q^3(q-1)(q+1)(q^3-1)$ in this case. Here also, he uses the exceptional character theory. It is shown that there are elementary abelian subgroups P and L of order q^2 in a Sylow 2-subgroup Q of G. P and L are not conjugate in G. Moreover, G acts doubly transitively on the set $\mathfrak{P}$ consisting of all conjugates of P and also on the set $\mathfrak{L}$ consisting of all conjugates of L. The normalizer $N_G(P)$ of P is of index q^2+q+1 in G and so $|\mathfrak{P}|=q^2+q+1$. The same assertion holds for $\mathfrak{L}$. Suzuki calls the elements of $\mathfrak{P}$ points and the elements of $\mathfrak{L}$ lines. An incidence relation can be defined on the pair $(\mathfrak{P},\mathfrak{L})$ by saying that $P_1\in\mathfrak{P}$ is incident to $L_1\in\mathfrak{L}$ if and only if $P_1\cap L_1\neq 1$. Suzuki next shows that the geometry so defined on $(\mathfrak{P},\mathfrak{L})$ is Desarguesian using Gleason's result. This completes the characterization.

At the time of writing [11], Suzuki was only a year away from discovering $Sz(q), q=2^n$. If we compare his notation for $M(\alpha,\beta,\gamma,\delta)$ of $PGL(3,2^n)$ and their product, and the corresponding quantities (α,β), etc. of $Sz(2^n)$ which is a subgroup of $PGL(4,2^n)$, it appears that Suzuki had good practice in $PGL(3,2^n)$ before he discovered his new simple groups.

§7. ZT-groups and related classification theorems

Suzuki proved several fundamental classification theorems. I will make comments on some of them.

[17] Finite groups with nilpotent centralizers, 1961.

Let us call a finite group G a CN-group, as Feit-Hall-Thompson and then Suzuki did, if the centralizer of every nonidentity element of

G is nilpotent. Let us review some of the results mentioned in the preceding sections. K.A. Fowler investigated the structure of nonsolvable groups with the property that the centralizer of every involution is an abelian 2-group and showed that $PSL(2, 2^n)$ is the only family of simple groups having the property. Suzuki and Wall independently showed that $PSL(2, 2^n)$ is the only family of nonsolvable CA-groups.

Suzuki, in one of his famous papers [8], showed that every simple CA-group is of even order and Feit-Hall-Thompson extended this result to CN-groups: every nonabelian simple CN-group is of even order. Therefore, Suzuki is able to assume that his CN-group G is of even order and so G contains an involution. To classify all CN-group of even order, Suzuki gives another definition: a group G is a CIT-group if the centralizer of every involution is a 2-group.

Suzuki shows that nonsolvable CN-groups are CIT-groups and devotes his efforts to classify all nonsolvable CIT-groups. The property that the group G satisfies CIT is obviously hereditary to all subgroups and even to all sections of G (though a bit of work is necessary to show it), so by using induction on the order of G one can assume that all proper subgroups are of known type.

Theorem ([17]). *A finite group G is a nonabelian simple CIT-group if and only if G is isomorphic to one of the following groups:*
(i) *a Zassenhaus group of odd degree (called a ZT-group by Suzuki),*
(ii) $PSL(2,p)$ *where p is a Fermat prime or Mersenne prime,*
(iii) $PSL(2,9)$,
(iv) $PSL(3,4)$.

Therefore, all CIT-groups will be classified if all Zassenhaus groups of odd degree are determined. Zassenhaus groups of even degree had already been classified by Zassenhaus, Feit and Ito. Suzuki himself completes the classification for the even degree cases. In this paper [17], Suzuki claims to have shown that if the order of a Zassenhaus group G of odd degree is divisible by 3, then G is isomorphic to $PSL(2, 2^n)$. As already remarked in §6, Suzuki's simple group $Sz(2^n)$ has order not divisible by 3. Later Thompson and Glauberman treated simple groups of order not divisible by 3 and showed that $Sz(q)$ is the only family of simple groups with this property. Therefore, apart from $Sz(q)$, all simple groups have order divisible by 6. Although he writes in the introduction of this paper [17] that only fragmentary results are known for the general Zassenhaus groups of odd degree, he himself finishes the problem before the paper actually went to press. If we use this result (published later), we obtain, as a corollary, that every nonsolvable CIT-group is a CN-group.

Skimming through the paper [17], we can see that all the important classification results that Suzuki later shows are already presented here in their preliminary mode. For example,

Theorem. *A Zassenhaus group of odd degree is a nonabelian simple CIT-group.*

Theorem. *A nonsolvable CN-group is a CIT-group.*

Theorem . *Let G be a CIT-group and S a Sylow 2-group of G. Assume that Sylow 2-groups of G are independent (i.e. a TI set). Then we have one of the following:*
(i) *S is normal,*
(ii) *S is cyclic,*
(iii) *S is a generalized quaternion group, or;*
(iv) *G is a Zassenhaus group of odd degree.*

[21] On a class of doubly transitive groups, 1962.

In this paper, the class of finite groups called Zassenhaus groups is completely determined. Classified also are all simple CN-groups. This paper published in the Annals of Mathematics is one of Suzuki's major results. It is memorable to me personally also. As a student at the University of Tokyo in the middle of 1960s, I read this paper in a series of group theory seminars.

Suzuki acknowledges in the introduction of [21] that G. Higman's result on 2-groups is essential for the completion of this work.

Theorem (G. Higman). *Let Q be a 2-group which admits a cyclic group of automorphisms transitive on the set of involutions. Assume that Q is not abelian and contains $q-1$ involutions. If $q > 2$, then Q satisfies the following properties:*
(i) *Q is of exponent* 4, (ii) *the order of Q is either q^2 or q^3, and;* (iii) *if the order of Q is q^2, then Q is isomorphic with one of the groups $S(q;x)$.*

Here the 2-group $S(q;x)$ is defined as follows. Let F be the field $\mathrm{GF}(q)$ of q elements where q is a power of 2 ; $q = 2^n$. Let x denote an automorphism of the field F such that $x \neq 1$ and $\alpha^{1+x} = 1$ implies $\alpha = 1$.

Consider the matrices over F of the form

$$(\alpha,\beta)=\begin{pmatrix}1 & & \\ \alpha^x & 1 & \\ \beta & \alpha & 1\end{pmatrix}.$$

The product of two matrices is written as

$$(\alpha,\beta)(\gamma,\delta)=(\alpha+\gamma,\alpha\gamma^x+\beta+\delta).$$

Now define

$$S(q;x)=\{(\alpha,\beta)|\alpha,\beta\in F\}.$$

Then $S(q;x)$ is a 2-group of order q^2. The mapping:

$$\psi(\zeta):(\alpha,\beta)\ \rightarrow\ (\zeta\alpha,\zeta^{1+x}\beta)$$

is an automorphism of $S(q;x)$ that fixes no nonidentity element of $S(q;x)$ unless $\zeta=1$. Therefore $S(q;x)$ admits a fixed-point-free automorphism group Z of order $q-1$. Since Z is isomorphic to the multiplicative group of F, Z is cyclic also.

Now assume
(i) G: a Zassenhaus group acting on Ω such that $|\Omega|=1+N$ with N odd,
(ii) $H=G_\alpha$: the subgroup of G consisting of elements fixing a symbol $\alpha\in\Omega$,
(iii) Q: a Sylow 2-subgroup of H,
(iv) K: the subgroup consisting of elements fixing two symbols α and β,
(v) τ: an involution in $N_G(K)$.

Q is a normal subgroup of H and H is a semi-direct product of Q and K. One can prove that τ inverts every element of K, and so K is abelian and hence cyclic.

Suzuki proves:

Proposition. *Q contains two elements σ and ρ such that σ is an involution, σ is a certain power of ρ and:*

$$\tau\sigma\tau=\rho^{-1}\tau\rho,$$

$$\rho^{-1}(\sigma\tau)\rho=(\sigma\tau)^2.$$

Moreover, σ and ρ are unique if H,K, and τ are chosen and fixed.

Suzuki calls the identity obtained in the proposition above the *structure identity* of G. Firstly the case: $\sigma = \rho$ is treated. By counting the number of real elements, Suzuki shows that $|G| = N(N+1)(N-1)$. Therefore G is a sharply triply transitive permutation group. That $G \cong PSL(2, 2^n)$ follows from a theorem of Zassenhaus.

Assuming $G \ncong PSL(2, N)$, Suzuki continues his counting argument for real elements. He shows that if $q-1$ is the number of involutions of Q then $|Q| = q^2 = N$ and $|G| = q^2(q-1)(q^2+1)$. The rest of the paper is devoted to the proof of the uniqueness of the structure of G and that $G \cong Sz(q)$. A very subtle argument involving the structure identity is necessary to show the required uniqueness.

[24] Two characteristic properties of (ZT)-groups, 1963.

In this paper, Suzuki raised the following question: Suppose a proper subgroup H of even order of a finite group G contains the centralizer of every nonidentity element. Then what can we say about the structure of G?

Suzuki shows that if G is not a Frobenius group, then G is a Zassenhaus group of odd degree and H is either a Sylow 2-subgroup or the normalizer of a Sylow 2-subgroup of G.

Note that G is a special case of a group having a *strongly embedded* subgroup. Suzuki had been faithful to Brauer's program, and characterized quite a few simple or almost simple groups by the centralizers of involutions. Suzuki, however, went farther and began to form a concept of a strongly embedded subgroup, which was to be taken up seriously by H. Bender soon.

'The name of Michio Suzuki was forever engraved in my mind when in 1964 Bernd Fischer, who had just become an assistant of Reinhold Baer at Frankfurt, handed me a paper by Suzuki to be studied and presented in Baer's seminar. That paper [23] lies at the intersection of two main streams of Suzuki's work:

(1) Characterize the known simple groups by the centralizer of an involution. (2) Determine doubly transitive permutation groups with a regular fixed point behavior, especially Zassenhaus groups, and Suzuki-transitive groups (the stabilizer of a point has a normal subgroup regular on the remaining points). (H. Bender [Obituary written for Michio Suzuki, Notices of Amer. Math. Soc., Vol.46(1999)).'

We have come to the paper:

[27] On a class of doubly transitive groups, II, 1964.

Having put an end to the classification of all Zassenhaus groups, Suzuki began extending his results to a larger class of simple groups.

He considers:

($*$) G is a permutation group on a finite set Ω and the one point stabilizer G_α, for every $\alpha \in \Omega$, contains a normal subgroup acting regularly on the remaining points $\Omega\backslash\alpha$.

If a group G satisfies the condition ($*$) (Bender calls such a group a Suzuki-transitive group), then G is doubly transitive on Ω. Zassenhaus groups satisfy the condition. In addition to Zassenhaus groups, there is another family of groups that satisfy ($*$). Let $SU(3,q^2)$ be the totality of all unitary matrices of determinant 1 defined over the field E with q^2 elements. We have $|SU(3,q^2)| = q^3(q^2-1)(q^3+1)$. The group $SU(3,q^2)$ can also be defined as the set of all matrices of determinant 1 that leave the following form invariant.

$$\psi(\vec{x},\vec{y}) = x_1 y_3^q + x_2 y_2^q + x_3 y_1^q.$$

If we define

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

then

$$SU(3,q^2) = \{A | \bar{A}^t J A = J, detA = 1\}.$$

The subgroup Z consisting of all scalar matrices of $SU(3,q^2)$ is a cyclic group of order $(3, q+1)$. Define $PSU(3,q^2) = SU(3,q^2)/Z$.

Let Ω be the set of all points $\vec{x} = [x_1, x_2, x_3]$ on the projective plane $P^2(q^2)$ such that $\psi(\vec{x},\vec{x}) = 0$. We have $|\Omega| = q^3+1$ and $PSU(3,q^2)$ acts doubly transitively on Ω. Moreover, the one point stabilizer has a normal subgroup Q acting regularly on the remaining points.

More precisely, the stabilizer $H = G_\alpha$ of $\alpha = [0,0,1] \in \Omega$ in $G = PSU(3,q^2)$ contains a normal subgroup Q of order q^3 consisting of the projective images of the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ b & 1 & 0 \\ d & -b^q & 1 \end{pmatrix}, \quad b^{1+q} + d + d^q = 0.$$

Q acts regularly on the remaining points $\Omega\backslash\alpha$. Equivalently, G is Suzuki-transitive on the coset space $G/N(Q)$.

To state the main theorem of [27] we need one more assumption:

($**$) $|\Omega|$ is odd and the order of the two point stabilizer $G_{\alpha,\beta}$ is odd (hence $G_{\alpha,\beta}$ is solvable).

Suzuki shows that if G satisfies $(*)$, $(**)$ and G is simple, then G is isomorphic to a ZT-group or $PSU(3, q^2)$. Nonsimple cases are also treated by Suzuki. With the completion of this paper [27], Suzuki began to shift his attention to a general classification theorem. But let me make a comment on the following first:

[18] On a finite group with a partition, 1961.

Although he published the paper [1] on the lattices of subgroups of a finite group a little earlier, Suzuki, early in the 1950s, began his mathematical career by investigating finite groups having a partition [2]. For example, the nonsolvable groups $PGL(2, q), PSL(2, q)$ admit a partition. This problem or its solution did not appear to have much impact on finite group theory itself. Suzuki, however, did not lose his interest in the subject. The discovery of $Sz(q)$ by himself and the fact that the Suzuki groups admit a partition must have given him an added impetus to pursue the subject.

I did not make any comments on this subject in §2 and so let us come back to Suzuki's first love again. Throughout this paper, the idea of R. Baer [Partitionen endlicher Gruppen, Math. Z., 75(1961), 333–372] is used and it is so acknowledged.

Repeating the definition given in §2, if a finite group G can be expressed as a union of subgroups U_i with the property $U_i \cap U_j = 1$ if $i \neq j$, then we say G admits a partition:

$$G = \bigcup_{i=1}^{n} U_i, \;\; U_i \cap U_j = 1 \text{ if } i \neq j.$$

In other words, we say G has a partition if every nonidentity element of G is contained in one and only one subgroup in the collection $\{U_i, i = 1, \dots, n\}$. Of course, we, in principle, exclude the cases in which $G = U_i$ or $U_j = 1$ for some i, j. For example, if $\{U_i\}$ is the totality of all maximal cyclic subgroups of $PSL(2, q)$, then it gives a partition of $PSL(2, q)$. The partition of a group G is not necessarily unique. For example, one can use the Sylow p-subgroups of $PSL(2, q)$ where $q = p^n$ for a prime p as members of the set $\{U_i\}$ instead of using cyclic subgroups of the Sylow p-subgroups.

Every subgroup H of G has an induced partition by taking $V_i = H \cap U_i$ and throwing away some unnecessary V_j, provided that $H \not\subset U_i$ for any i.

If $\{U_i\}$ and $\{W_i\}$ are both partitions of G and if for every j, there is an i such that $W_j \subset U_i$, then $\{W_i\}$ is called a *refinement* of $\{U_i\}$. If all conjugates of every member of a partition U_i are again members of $\{U_i\}$, then we say the partition $\{U_i\}$ is *normal.* If a partition $\{U_i\}$ admits no nontrivial refinements, then it is called *minimal.*

Let us see how things unfold.

Lemma. *Every partition has a refinement which is normal.*

In fact, every minimal partition is normal. All one has do is to refine a given partition until it becomes minimal.

Lemma. *If a nontrivial partition $\{U_i\}$ is normal, then the normalizer $N(U_i)$ of each component U_i is larger than U_i unless G is a Frobenius group.*

Suppose $N(U_i) = U_i$ and the partition $\{U_i\}$ is normal. Then the permutation representation of G on $\Omega = G/U_i$ gives rise to a Frobenius group.

Lemma. *If $N(U_i) \neq U_i$, then U_i is nilpotent.*

Let $U = U_i$, $N = N(U)$ and assume $N > U$. Let H be a subgroup of N containing U such that $[H : U] = p$ where p is a prime. The subgroup H admits an inherited partition from G. Thus $H = U \cup (\cup V_j)$ where j ranges over some index set. Since $U \cap V_j = 1$, we have $|V_j| = p$ and $H = V_j U$. In other words, every element of $H \backslash U$ is of order p. If U is a p-group then of course it is nilpotent. Suppose not. Then the subgroup $H_p(H)$ generated by the elements of H which do not have order p is a proper subgroup of H. Hence U is an H_p-group in the sense of Hughes-Thompson [The H_p-problem and the structure of the H_p-groups, Pacific J. Math., 9(1959), 1097–1102]. Hughes and Thompson proved that $H_p(G) = 1, H_p(G) = G$ or $[G : H_p(G)] = p$ and so $H_p(H) = U$ in our case. Kegel [Die Nilpotenz der H_p-Gruppen, Math. Z., 75(1961), 373–376] proved that all H_p-groups are nilpotent. Hence U is nilpotent.

Thus we only need to treat groups having nilpotent partitions, i.e. all components U_i are nilpotent. Baer has shown that if G possesses a nontrivial nilpotent normal subgroup and a nilpotent partition, then G is solvable.

Let N be the largest nilpotent normal subgroup of G. Suppose first that $|N|$ is divisible by two distinct primes. Then N must be contained in one of the components of the partition since every nilpotent group having a nilpotent partition is a p-group. Call the component U that contains N. Since our partition $\{U_i\}$ is normal, $U = N$ must hold. If no element of $G \backslash N$ commutes with any nonidentity element of $N \backslash 1$, then

G is a Frobenius group and N is the Frobenius kernel. In particular, N is a Hall subgroup of G and the centralizer in G of every nonidentity element of N is nilpotent. Therefore, if the centralizer of some element of N is not nilpotent, then some element x of $N \backslash 1$ commutes with an element of $G \backslash N$.

Suzuki analyses this case carefully and eventually proves:

Theorem. *Let G be a non-solvable group with a nilpotent partition. If the centralizer of some involution is not nilpotent, then G is isomorphic with $PGL(2, q)$, q odd.*

Theorem. *Let G be a non-solvable group with a nilpotent partition. If the centralizer of every involution is nilpotent, then G is isomorphic with either $PSL(2, q)$ or $Sz(q)$.*

The major portion of this paper is devoted to the proof of the following theorem.

Theorem. *If G is a nonsolvable group having a normal nilpotent partition, then $|G|$ is even.*

Had Suzuki used the result of Feit-Thompson (which was not available when Suzuki wrote this paper), then this 14 pages paper would have been less than half its length. As in his paper [8], exceptional character theory is the key tool to prove that $|G|$ is even.

Therefore Suzuki has shown that there is no semi-simple group having a partition other than the groups $PGL(2, q), PSL(2, q)$ or $Sz(q)$, thus fulfilling his 'Jugendtraum'. Let us again come back to the main stream of simple group theory.

[28] Finite groups of even order in which Sylow 2-subgroups are independent, 1964.

Theorem. *Suppose that G is a nonabelian simple group satisfying the property that any two distinct Sylow 2-subgroups have only the identity element in common. Then G is isomorphic to $PSL(2, q), PSU(3, q^2)$ or $Sz(q)$ where q is a power of 2.*

[30] Finite groups in which the centralizer of any element of order 2 is 2-closed, 1965.

Theorem. *Let G be a finite nonabelian simple group such that the centralizer of any element of order 2 has a normal Sylow 2-subgroup. Then G is isomorphic to one of the simple groups $PSL(2,p)$; p a Fermat or Mersenne prime, $PSL(2,9)$; $PSL(2,q)$, $Sz(q)$, $PSU(3,q^2)$, or $PSL(3,q)$, q a power of 2.*

The theorems stated above show Suzuki's path from the permutation group theoretic results proved in [21] and [27] to general results which can readily be used for the classification of all simple groups of finite order. In [28] Suzuki reduces to the case in which the group G satisfies the condition of Suzuki-transitive groups, and then applies the main result of [27]. Therefore the characterization method used in [28] was still via the permutation group theory.

In the bibliography of [30], however, something new, which Suzuki had never before used, appeared. [D.G. Higman and J.E. McLaughlin, Geometric ABA-groups, Illinois J. Math., 5(1961), 382–397] and [J. Tits, Theoreme de Bruhat et sous-groupes paraboliques, C.R. Acad. Sci. Paris, 254(1962), 2910–2912] were the new papers required.

In order to prove the theorem stated above, we can assume that there is a pair of Sylow 2-subgroups which have a nonidentity element in common, since otherwise all such simple groups have been treated in [28]. The rest of the proof of the main theorem of [30] divides into two cases.

(i) Sylow 2-subgroups have cyclic center,
(ii) Sylow 2-subgroups have noncyclic center.

If the case (i) holds, Suzuki shows that G is isomorphic to $PSL(2,p)$ where p is a Fermat or a Mersenne prime, or $PSL(2,9)$. If the case (ii) holds, Suzuki shows that G possesses a Bruhat decomposition with its Weyl group isomorphic to the symmetric group of degree three and applies Higman-McLaughlin [op.cit.] to conclude $G \cong PSL(3,q)$, here also q is a power of 2.

Suzuki states, in Introduction, that the main theorem of the paper [30] will give an independent proof of some of the results he obtained earlier. For example, his classification of the CIT-groups is not used in [30]. Not used also are the characterizations of $PSL(3,2^n)$ and of $PSU(3,2^n)$ in terms of the centralizer of an involution. Moreover, he makes a remark that this paper is entirely group theoretic and free from the theory of characters. It is as though Suzuki is announcing to the world that he has at last cut himself off from the bondage of character theory and found a new tool.

In his paper [30], one can see the path in its primitive form, which the classification of all finite simple groups later followed. Case (ii) lead

Suzuki to the groups with BN-pairs and Case (i) and (the case in which any pair of Sylow 2-subgroups have only the identity element in common) lead him to the groups where the prime 2 is more or less isolated (or 2-nonconnected).

This dichotomy was to be followed later for an odd prime p also. In one case, we have a proper subgroup H of G such that H contains a Sylow p-subgroup of G and all its p-local subgroups (the normalizers of p-subgroups). Therefore, H is isolated (with respect to the prime p) in the group G. In the other case, there are no such subgroups and so G is connected through p-local subgroups and their intersections. Hence, for example, some graph or geometry can be associated with G. Bender took up the case in which the prime 2 is nonconnected. He first classified all doubly transitive permutation groups in which no involution stabilizes a point, and then classified all transitive permutation groups in which every involution stabilizes exactly one point. This latter result had a far reaching application for the classification of all finite simple groups. Suppose we are in the latter case and let H be the stabilizer of a point $\alpha \in \Omega$ and let t be an involution of H. Then every element in $C_G(t)$ fixes α and so $C_G(t) \subset H$. In fact, one can show also that the normalizer of every nontrivial 2-subgroup of H is contained in H. Such a proper subgroup H was to be called a strongly embedded subgroup. Bender was able to classify all simple groups having a strongly embedded subgroup.

Although Suzuki must have had his own idea of classifying all finite simple groups, [30] was to become his last general classification theorem. The world of finite group theory was changing rapidly. The solvability of all groups of odd order (Burnside's Conjecture) was shown to hold by Feit and Thompson (1963). Janko found new sporadic simple groups, later named $Janko_1$, $Janko_2$, $Janko_3$ (1965, 1968). The simple groups $Conway_1$, $Conway_2$, $Conway_3$ and $Fischer_1$, $Fischer_2$, $Fischer_3$ were to be discovered soon. The signalizer functor method of Gorenstein-Walter was shaping up. A new generation of young group theorists was coming of age. Stars and superstars were emerging into the field. The middle to the late '60s (and perhaps to early in the '70s) was the period of turbulence for finite group theory. This was also the golden era of group theory.

Suzuki wrote a number of papers whose titles contain the phrase 'Characterization of Linear Groups'. Let us pick up another paper and discuss it briefly.

[35] Characterization of linear groups, 1969.

This is an expanded and improved version of Suzuki's one hour address delivered at one of the AMS meetings in 1967. The purpose is to characterize the simple group $PSL(n,q)$ in terms of the centralizer of an involution.

The theme of this research direction was initiated by Brauer's address at the International Congress of Mathematicians held in 1954. As for $PSL(n,q)$, Brauer himself did the characterization when $n = 2,3$ and with some restriction on q. A great many papers followed Brauer's. Suzuki himself treated a large number of cases in which q is even.

In this paper, Suzuki talks about its history, which is short but quite readable. He mentions that the following doublets or triplets share the isomorphic centralizer of an involution.

$$(PSL(2,7), A_6), (PSL(3,3), Mathieu_1), (A_{4m}, A_{4m+1}),$$

$$(Janko_2, Janko_3), (A_{12}, A_{13}, S_6(2)), (PSL(5,2), Mathieu_5, Held).$$

There are no examples of four or more simple groups that have isomorphic centralizers of an involution.

In [35], Suzuki proves:

Theorem. *The simple group $PSL(m, 2^n)$ is characterized by the centralizer of an involution in the center of a Sylow 2-subgroup if $m \geq 6$ or $n > 1$.*

The remaining cases not treated in Suzuki's theorem had already been taken care of by others and Suzuki himself.

I believe that I have covered most of his contributions to the theory of finite groups except for his work on subgroup lattices [1], [3], [4] and [7]. For these papers I have too limited a knowledge to make any reasonable comments. I do add that Bender cites Suzuki's work on subgroup lattices as one of the reasons for the honorary degree he received from Kiel University, Germany. Skimming through the list of publications of Suzuki again, I find, however, that there are a few more papers that I should make comments on.

[12] On finite groups of even order whose 2-Sylow subgroup is a quaternion group, 1959.

In this paper, Brauer and Suzuki prove: Let G be a group of finite even order. If the 2-Sylow group P of G is a quaternion group (ordinary or generalized), then G is not simple. The proof is (modular) character theoretic. Groups having a cyclic Sylow 2-subgroup cannot be simple either as had been known since the turn of the century. Therefore if P is a Sylow 2-subgroup of a simple group of even order, then P must contain a Klein's four group ($\cong Z_2 \times Z_2$). We say P is of 2-rank at least two. There are examples of 2-groups of rank two which can be a Sylow 2-subgroup of a simple group. The Brauer-Suzuki theorem was the modern starting point of the classification theorems that dealt with simple groups having Sylow 2-subgroups of low 2-rank.

[34] A simple group of order 448,345,497,600 (1969).

Suzuki made big news with the discovery of a sporadic simple group *Suzuki* of order 448,345,497,600, which was announced in 1967.

Janko's second group $Janko_2$ was constructed by M. Hall using the idea of transitive extensions of rank 3. Other constructions of rank 3 extensions followed. Sporadic simple groups *McLaughlin*, $Fischer_1$, $Fischer_2$, $Fischer_3$, and *Higman-Sims* are examples. Starting from the simple group of Lie type $H = G_2(4)$, Suzuki constructed a rank 3 transitive extension of H of degree 1782.

[38], [39], [41], [44] Gunron (Japanese), 1977, 1978; Group Theory (translation of [38], [39]), 1982, 1986.

Suzuki began writing this book late in the 1960s. Aschbacher, who was at Illinois as a postdoc, says that Suzuki was giving group theory lectures from a draft of that book. It was nearly a 20 year effort from the draft until the completion of its translation.

§8. Group theory in Japan before Suzuki

Michio Suzuki lists Shokichi Iyanaga as his adviser and says that Kenkichi Iwasawa also had a profound influence on him. Let me discuss group theory in Japan before Suzuki briefly.

Let k be a number field and K/k be its absolute class field: i.e. the Galois group of the abelian extension K/k is isomorphic to the ideal class group of k. It was conjectured by D.Hilbert that every ideal of k extends to a principal ideal of K. This is called the Principal Ideal Theorem. Artin reformulated it into a group theoretical problem (see below). Furtwängler (1930) solved the conjecture affirmatively after a complicated computation and Iyanaga gave a simple proof (1934). (I looked at the Furtwängler's proof. It was indeed complicated. Magnus also published a short proof in 1934. As for the proof of the Principal Ideal Theorem, see [Artin-Tate, Class Field Theory, Benjamin, Inc., 1974].)

Theorem (Principal Ideal Theorem). *Let G be a (not necessarily finite) group whose commutator subgroup $G' = [G, G]$ is of finite index in G and is finitely generated. Then the transfer map $G \to G'/G''$ is the zero map.*

Iwasawa is of course better known for his work in Lie groups, number theory, etc. But let me mention only the following:

Theorem ([K. Iwasawa, Über die endlicher Gruppen und die Verbände ihrer Untergruppen, J. Univ. Tokyo, 43(1941), 171–199.]). *The maximal subgroup chains of a finite group G all have the same length if and only if G is supersolvable.*

A finite group G is supersolvable if it possesses a normal series

$$G = G_0 \supset G_1 \supset \cdots \supset G_r = 1$$

in which each factor group G_{i-1}/G_i is cyclic of prime order. If a finite group G is supersolvable, it can be shown that any chain of subgroups

$$G = H_0 \supset H_1 \supset \cdots \supset H_s = 1$$

can be refined by inserting further subgroups:

$$H_{i-1} = H_{i-1,0} \supset H_{i-1,1} \supset \cdots \supset H_{i-1,t} = H_i, t = t(i), i = 1, \ldots, s$$

such that all indices $[H_{i,j} : H_{i,j+1}]$ are primes. This implies that all maximal chains of subgroups have the same length, which is the total number of primes, counting repetitions, dividing the order of G. Iwasawa's Theorem shows that the converse also holds. The converse is of course nontrivial and the most difficult step is to show that G has a proper normal subgroup.

A monomial representation of a group G is an induced representation Ψ^G where Ψ is a one-dimensional representation of a subgroup H of G. All irreducible representations of a nilpotent group are known to be monomial. The converse is false. We, however, have:

Theorem ([K. Taketa, Über die Gruppen, deren Darstellungen sich sämtlich auf monomiale Gestalt transformieren lassen, Proc. Jap. Imp. Acad., 6(1930), 31–33]). *If every irreducible representation of a finite group G is monomial, then G is solvable.*

As seen above, there were some roots of finite group theory in the prewar Japan. It appears, however, that nobody in Japan was doing serious research on simple groups such as $PSL(2,q)$. Perhaps some people were interested in them but it would be fair to say that no important results came out from their efforts. It is, therefore, quite surprising that in such an environment, Suzuki took up a hard problem, which eventually lead him into the heart and the top of simple group theory.

There must have been time for me ask Suzuki personally how and why he got into the problems concerning $PSL(2,q)$ when nobody else in Japan was doing it. But such an opportunity is now lost for good. I could have asked Iwasawa,who had a great influence on Suzuki, about it, but he too passed away several months after Suzuki died.

§9. Michio Suzuki, my teacher and my mentor

I met Michio Suzuki for the first time in the spring of 1966 when he visited Japan with his family, then 2 year old Kazuko-chan and his wife Naoko Suzuki. D.G.Higman of the University of Michigan came to Japan with the Suzukis also. I was a second year graduate student at the University of Tokyo. I had decided to do group theory as my special field of mathematics in the spring of 1964 when I was a college senior.

It was the time when group theory reached its height and the golden era was continuing. For my decision to do group theory, I was influenced greatly by the work of Suzuki, especially:

(1) Discovery of the new series of simple groups $Sz(q)$.
(2) The classification of Zassenhaus groups (Zassenhaus, Feit, Ito, and Suzuki).
(3) Classification theorems for certain types of simple groups.

Under the supervision of N. Iwahori, I, together with a few other students, began reading [Curtis-Reiner, Representation Theory of Finite Groups and Associative Algebras]. I remember that Iwahori, who had visited the USA a few times, talked enthusiastically about Suzuki's work, Thompson's proof that the Frobenius kernel is nilpotent, the Odd Order Paper of Feit-Thompson, etc. I soon joined in the group theory seminar organized under Iwahori. Among the participants were Takeshi Kondo and Hiroyoshi Yamaki.

I chose Suzuki's classification of Zassenhaus groups of odd degree [21] for my seminar presentation. I next chose Thompson's proof of the nilpotency of the Frobenius kernel [Normal p-complements for finite groups, Math. Z., 72(1960), 332–354]. I found it impossible to read and gave up. Soon afterward fortunately, a shorter proof was published [J.G. Thompson, Normal p-complement for finite groups, J. Alg., 1 (1964), 43–46]. Thompson's new paper was much easier to read than the first one.

Around 1965, Japan was still in a poor state of affairs economically. A Xerox copier was delivered to the department of mathematics but students had to pay all copying cost, which was rather expensive for them. The expenses to participate in symposiums and conferences had to be borne by the students. We students tried to be winners under those conditions, since all Japanese were under the same constraints.

Besides, those who were students in the 1940s and 50s would say that the 60s were far better than their times.

After Suzuki's paper and Thompson's and a few more papers, T. Kondo, H. Yamaki, I and others started reading Feit-Thompson's odd order paper. Soon the group theory seminar lost most of its members. Left in the group were Kondo, Yamaki and myself, just three of us. After 30 years, we three still talk about the struggles we had in reading Feit-Thompson's paper in the seminar room of the basement of a building of the University of Tokyo.

R. Baer visited Japan in the fall of 1965 and other foreign group theorists came to Japan also. H. Wielandt visited Japan at a similar time. But not too many people in Japan were doing group theory and not too many students were going into the theory either. It was still a field of mathematics which did not command too much respect in Japan. My classmates at the University of Tokyo, Shigeru Iitaka, Takushiro Shintani, Takuro Shintani, Takushiro Ochiai, Ryoshi Hotta, went into fields such as algebraic geometry, number theory, differential geometry, and representation theory. But I took up group theory as my field with confidence and enthusiasm, and I have not regretted the decision since.

Suzuki's visit in 1966 to Japan was a very timely event for me. I was a second year graduate student at the University of Tokyo, and Suzuki was only 39 years of age and the peak of his career was continuing. He gave talks for us one after another, all without any compensation. In fact, he had to spend nearly two hours one way in a train to come from his home to the university. We, young group theorists, asked him to give lectures on Bender, Glauberman, Alperin and others. Week after week, Suzuki did everything we asked for.

At the time of his visit, I was working on a research problem. I completed it just as Suzuki was leaving for the USA. Much to my surprise and delight, he suggested that I submit it to the Illinois Journal of Mathematics. In addition to submitting the paper to him, I wrote him letters regularly, to which he gave replies regularly. One of his letters, dated October 23, 1966, contains many unpublished results. At the end of the letter, he writes that he will find time to write more. Apparently I had complained to him that the news on group theory would arrive late in Japan and I wrote him I would like to know them sooner. The letter cited above was his reply.

It was then customary for a graduate student to seek employment after earning the master's degree. I was offered an assistantship at Nagoya University as I was finishing my master's degree. One year after I first met Suzuki and after I had already moved to Nagoya, I received a letter from him in which he said that there would be a special program on finite groups and algebraic groups for the academic year 1968-69 at the Institute for Advanced Study in Princeton, N.J., Suzuki suggested that I apply for a membership of the Institute. He added in the letter that he would write a letter of recommendation. This was an incredible opportunity for me. The Institute at Princeton occupied so high a place in my mind that I did not quite believe what I was reading in his letter.

I and my wife arrived at the Institute on the 10th of September, 1968. The Suzukis arrived shortly afterward. As soon as he arrived, he asked me if I knew the game of bridge. I said no. In fact, I had never heard the word before either. Suzuki then began teaching me and my wife the game of contract bridge. So instead of Gorenstein's group theory book, I had to read Goren's book on contract bridge. Suzuki and his wife invited us over to their place usually twice a week to play bridge until they left for Illinois the next spring.

The following year, Takeshi Kondo came to the Institute also. We played bridge many nights and sometimes days. At some point, number theory friends stopped coming to the games. The rumour had it that Goro Shimura scolded young number theorists who were visiting the Institute at that time. We group theorists kept playing. If Michio Suzuki likes the game so much then it must be a good thing to play.

Suzuki invited me to spend a year at the University of Illinois at Champaign-Urbana after my second year at the Institute. By then Daniel Gorenstein and I had written quite a few joint papers together and I had begun thinking that I would like to stay in the USA as long as possible. Suzuki's invitation to Illinois guaranteed a third year for me in the States and soon afterward Gorenstein and Janko secured a permanent position for me at the Ohio State University starting the academic year of 1971. Over 30 years has passed. It all started from Suzuki's visit to Japan in 1966.

For Michio Suzuki, mathematics came first and research was everything. Apparently, however, he watched football games or basketball games whenever he wanted to have a relaxation. He talked about how good Jonny Unitas and Wilt Chamberlain were. He liked to read mystery stories. Iwasawa also said to me that he liked to read mysteries. Suzuki did not appear to like traveling much. Maybe this is not very precise. He did not mind going out from his home. But apparently, as soon as he went out, he wanted to come back home as quickly as possible.

Suzuki did not write too many research papers after 1980, but he visited Japan quite often. Conferences and symposiums were organized concurrently with his visits. Suzuki gave talks most of the time. At the memorial conference held for Suzuki's 70th birthday in July of 1997, he gave a talk on his new research effort. People must have been surprised to learn of his fresh enthusiasm to do research.

I received a Christmas card from him for the last time in December of 1997, five months after the conference held in his honor. In the card he writes 'I have been learning amstex recently. I can at last print out as I please. I am having fun since the product is very neat.' I am still a beginner in TEX and so apparently he was younger in this respect than me. Continuing his card, he writes 'Take a good care of yourself and have a good new year.'

In February of 1998, the sad news of a cancer in his liver was communicated to me and to the mathematical community of the world. It was a shock to me and to all who knew him. The cancer was discovered early in the month and Suzuki left for Japan immediately. The same doctor who had found nothing wrong in him in the summer of 1997 gave the same diagnosis as the Illinois doctor. The Illinois doctor gave Suzuki three to six months, but the Japanese doctor only two to four months.

As I could not leave for Japan immediately, I wrote several letters to him. In the following month, March 19, I left for Japan as soon as I handed the grades to the math office for the courses that I taught in the winter quarter.

I visited his room, which was a guest room of the International Christian University at Mitaka, Tokyo, Japan. Hiroshi Suzuki (no relation) was a faculty member there and had been taking care of Michio Suzuki and his wife since their arrival in Japan.

'I am happy to be able to see you while I am still well' were his first words. With Mrs. Suzuki and Hiroshi, we talked about many things. Suzuki and I had 30 years of memories together. We would never be able to stop talking. It was hardly believable that Michio Suzuki had only a month or so of his life remaining. But when we were talking about lots of things, I did not think about it. Everything was just as natural. He spoke a lot, sometimes smiling and I did so also. The thought of his short remaining life was not on the surface of the conversation. But when the conversation came to a quiet moment, then I had to think that this happy moment would end soon, much too soon.

Suzuki's incomplete 140 page manuscript was sitting on the table. It was nearly complete and he had been enjoying putting it into the TEX format, but the work had to come to an abrupt stop. Mrs. Suzuki said that Suzuki, many a time, tried, in vain, to continue working on the paper in the hospital or the guest room. As I saw he might be tired for the day, I promised to come back and left the place. Suzuki came to the door of his room. As he bid good-bye, he had a small smile on his face. The cherry trees were visible from the window of his room.

Suzuki would be able to see the cherry blossoms once more very soon. Mrs. Suzuki came down to the front door of the building. She had tears in her eyes when I said good-bye to her. The whole thing was so totally unexpected. I promised I would come back again soon.

With some of my friends I visited Suzuki two more times during my stay of three weeks in Japan. The last one was on April 10. The spring term had already started at my university in the States. Suzuki looked a little weaker than when I first saw him three weeks before. After an hour or so, Suzuki with an apology went back to his bed. I was sorry that I stayed a little too long till he got tired, but I knew this might be the last time for me to see him. I left the room. At the bottom of the stairs, I looked up. Suzuki was there near the top of the stairs. He too knew that this might be the last time, got out from his bed and said good-bye to us. Mrs. Suzuki saw me off at the front door of the guest house. I said I would come back in June, July. She said it would be hard for him to wait that long. I searched for a word. But none came out. She was being as cheerful as she could in front of her husband, but tears began to come down from her eyes and came down profusely. I looked up towards the window of Suzuki's room. The cherry blossoms were changing into tiny green leaves.

Suzuki went back to the hospital on April 18. He was to survive 43 days more. A surprise visitor to the hospital was Helmut Bender. Prior to his visit, Bender did not say anything to anybody. Bender flew from Germany and stayed with Suzuki in the hospital for a few days starting May 18th. Suzuki, with all of his remaining energy, discussed his new research work with Bender. It must have been a beautiful sight, Helmut Bender and Michio Suzuki together talking mathematics, just days before his death.

I had already purchased a plane ticket back to Japan for a June 4th flight. Michio Suzuki, however, passed away May 31. On the same day, though 166 years earlier, Evariste Galois died of a gunshot wound from a duel. The group theory emerged as a respectable field of mathematics largely through the efforts of Galois, and Suzuki was one of those who made it flourish.

The funeral service for Michio Suzuki took place on June 7th and it was a memorable one. Fortunate for the occasion, if it had to happen, was that there was a conference on class field theory honoring Teiji Takagi in Tokyo. Among the people who got together for his funeral were Michio Suzuki's adviser, Shokichi Iyanaga, and Suzuki's friends, Ichiro Satake, Gaishi Takeuchi, Takashi Ono. All of them left Japan in the 50s or early 60s and came to the USA, as Suzuki did.

Longtime friends Noboru Ito and Takeshi Kondo made moving memorial speeches, and I added one too. Ito talked about their friendship during the war and right after the war. Kondo touched on Suzuki's mathematical contributions. Hymns were sung and lines from the Bible were read. Each and every one of us paid tribute to him with a branch of yellow rose, Suzuki's favorite flower.

On September 18th, the memorial service for Michio Suzuki took place at the Chapel of the University of Illinois. Eiichi Bannai, Ronald Solomon, and I attended the service from Columbus, Ohio. Walter Feit, George Glauberman, Henry Leonard, Richard Lyons, Paul Bateman, Everett Dade, and John Walter were present also. Having sent Eiichi off to Japan from the Champaign airport the following day, I went to Suzuki's home. I looked around with emotion. How many hours did I spent in this home during the last 30 years ?

In this room, Michio Suzuki and I listened to Bach and Mozart together. Out from this home, his family and mine went to a McDonald's and ate hamburgers. He talked about how bad their Fighting Illini football team was but how good it once had been, all those things. He lived in the area, Champaign, Illinois, for nearly 45 years. Mrs. Suzuki had never driven a car, never needed it since Michio Suzuki did not mind taking his wife grocery shopping, taking his daughter Kazuko to her nursery school, elementary school, etc. all the time.

One of my colleagues and a friend for nearly 30 years, Ronald Solomon posts in his office a letter he received from Suzuki concerning Gorenstein, Lyons and Solomon's work.

'Dear Ron,

I would like to congratulate you on the publication of the second volume of the classification series which I have just glanced through. It is very well organized and readable. I have an elated feeling that I may be able to understand the proof of the classification in my life time. ....'

To this letter, Solomon replies: Professor Suzuki, I am sorry we were too slow. But I suppose you know a better proof by now.
(R. Solomon [Obituary written for Michio Suzuki, Notices of Amer. Math. Soc., Vol. 46 (1999)])

Suzuki kept his enthusiasm for mathematics and warm interest in the work of his colleagues to the end of his days. He is now gone and will be missed by his family and by those of us who knew him. But his name will forever be with us for his pioneering work.

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