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Quantum representations of braid groups and holonomy Lie algebras

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Abstract.

We review various aspects of representations of the holonomy Lie algebras and the associated monodromy representations of the fundamental groups for the complement of hyperplane arrangements. In particular, we describe the relation between monodromy representations of the KZ equation and homological representations of the braid groups by means of hypergeometric integrals.

§1. Introduction

The purpose of this article is to provide a review on developments concerning representations of the fundamental group of the complement of hyperplane arrangements obtained as iterated integrals of logarithmic 1-forms. In particular, we focus on the linear representations of the braid groups of appearing as the monodromy of the KZ equation.

In the 1970's K.-T. Chen developed the theory of iterated integrals of differential forms and gave a description of the de Rham cohomology of the loop spaces of simply connected manifolds. In the case the manifolds are non-simply connected the iterated integrals of 1-forms provide non-commutative information of the fundamental groups. More precisely, the iterated integrals of 1-forms determine the nilpotent completion of the fundamental group over **R**.

We apply the theory of iterated integrals in the case of the complement of complex hyperplane arrangements. We introduce the notion of the holonomy Lie algebra, which is determined by the codimension two intersections of the hyperplane arrangements. It turns out that the holonomy Lie algebra is isomorphic to the dual of Sullivan's 1-minimal model

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and describes the above nilpotent completion of the fundamental group of the complement of complex hyperplane arrangements. There is a universal holonomy map from the fundamental group to the completion of the universal enveloping algebra of the holonomy Lie algebra. Given a representation of the holonomy Lie algebra we obtain a linear representation of the fundamental group induced from the above universal holonomy map.

We deal with the case of the braid arrangements. In this case the universal holonomy map is a prototype of the Kontsevich integrals for knots (see [20]). There is a representation of the holonomy Lie algebra associated with a complex semi-simple Lie algebra and its representations. The corresponding connection is called the KZ connection. It was shown by Schechtman and Varchenko [28] that horizontal sections of the KZ connection are expressed by means of hypergeometric integrals. Based on this method we describe a relationship between linear representations of the braid groups obtained as the action on the homology of local systems and the monodromy representations of the KZ equation. This provides a relationship between quantum representations of the braid groups and homological representations, which were extensively investigated by Bigelow [1] and Krammer [21].

The paper is organized in the following way. In Section 2 we recall basic notions of the iterated integrals due to K.-T. Chen. We describe the bar complex of the Orlik–Solomon algebra and the holonomy Lie algebra for hyperplane arrangements. By means of the iterated integrals of logarithmic 1-forms we introduce the universal holonomy map. In Section 3 we deal with the homology of local systems on the complement of hyperplane arrangements. In Section 4 we focus on the case of the braid arrangements. We give a basis of the space of solutions of the KZ equation by means of hypergeometric integrals over the bounded chambers in the complement of discriminantal arrangements. Based on this expression we clarify the relation between the monodromy representations of the KZ equation and the action of the braid groups on the homology of local systems.

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§2. Iterated integrals and holonomy Lie algebras

2.1. Chen's iterated integrals

The theory of iterated integrals was developed by K.-T. Chen [5]. One of the main objects of the theory of iterated integrals was to describe the de Rham cohomology of loop spaces. First, we briefly recall basic definitions for iterated integrals. Let M be a smooth manifold and $\omega_1, \ldots, \omega_k$ be differential forms on M of positive degrees. We fix a base point $x_0 \in M$ and denote by ΩM the loop space of M based at x_0. Namely, ΩM is the space of piecewise smooth maps $\gamma\colon I \to M$ such that $\gamma(0) = \gamma(1) = x_0$. Let Δ_k denote the Euclidean simplex defined by

$$\Delta_k = \{(t_1, \ldots, t_k) \in \mathbf{R}^k; 0 \leq t_1 \leq \cdots \leq t_k \leq 1\}.$$

There is an evaluation map

$$\varphi\colon \Delta_k \times \Omega M \longrightarrow \underbrace{M \times \cdots \times M}_{k}$$

defined by $\varphi(t_1, \ldots, t_k; \gamma) = (\gamma(t_1), \ldots, \gamma(t_k))$. Let $\pi_i\colon \underbrace{M \times \cdots \times M}_{k} \to M$ be the projection on the i-th factor. We put

$$\omega_1 \times \cdots \times \omega_k = \pi_1^* \omega_1 \wedge \cdots \wedge \pi_k^* \omega_k.$$

We define the iterated integral $\int \omega_1 \cdots \omega_k$ by

$$\int_{\Delta_k} \varphi^*(\omega_1 \times \cdots \times \omega_k)$$

which is the integration along the fiber with respect to the projection $p\colon \Delta_k \times \Omega M \to \Omega M$.

The iterated integral $\int \omega_1 \cdots \omega_k$ is considered to be a differential form on the loop space ΩM with degree $p_1 + \cdots + p_k - k$, where p_j is the degree of the differential form ω_j.

In particular, in the case $\omega_1, \ldots, \omega_k$ are 1-forms on M, the expression $\int \omega_1 \cdots \omega_k$ is considered to be a function on the loop space ΩM. For a loop $\gamma\colon I \to M$ we write

$$\gamma^*(\omega_i) = f_i(t)\, dt, \quad 1 \leq i \leq n$$

and denote by

$$\int_\gamma \omega_1 \cdots \omega_k$$

the iterated line integral

$$\int_{0 \leq t_1 \leq \cdots \leq t_k \leq 1} f_1(t_1) \cdots f_k(t_k)\, dt_1 \cdots dt_k.$$

The above integral is the value of $\int \omega_1 \cdots \omega_k$ at the loop γ.

As a differential form on the loop space $d\int \omega_1 \cdots \omega_k$ is expressed as

$$\sum_{j=1}^{k}(-1)^{\nu_{j-1}+1}\int \omega_1 \cdots \omega_{j-1} d\omega_j\, \omega_{j+1}\cdots \omega_k$$
$$+\sum_{j=1}^{k-1}(-1)^{\nu_j+1}\int \omega_1 \cdots \omega_{j-1}(\omega_j \wedge \omega_{j+1})\omega_{j+2}\cdots \omega_k$$

where $\nu_j = p_1 + \cdots + p_j - j$.

We denote by $\mathcal{B}^q(M)$ the space of iterated integrals of degree q on the loop space ΩM obtained as the iterated integrals of differential forms of positive degrees on M in the above way. We have a complex

$$0 \longrightarrow \mathcal{B}^0(M) \longrightarrow \cdots \longrightarrow \mathcal{B}^q(M) \longrightarrow \mathcal{B}^{q+1}(M) \longrightarrow \cdots$$

which is a subcomplex of the de Rham complex of the loop space ΩM. We call the above complex $\mathcal{B}^*(M)$ the bar complex of the de Rham complex of M. A fundamental result due to Chen is stated in the following way.

Theorem 2.1 ([4]). *Let M be a simply connected manifold. Then the cohomology of the bar complex $\mathcal{B}^*(M)$ is isomorphic to the cohomology of the loop space $H^*(\Omega M; \mathbf{R})$.*

2.2. Bar complex of Orlik–Solomon algebras

We apply Chen's theory of iterated integrals to the case of the complement of hyperplane arrangements. We refer the reader to [16] for a more detailed account of this subject.

Let $\mathcal{A} = \{H_1, \ldots, H_\ell\}$ be a set of complex codimension one hyperplanes in $\mathbf{C}^n$. We call $\mathcal{A}$ an arrangement of hyperplanes. In this article we do not assume that $\mathcal{A}$ is central. Namely, we suppose that the hyperplanes $H_1, \ldots, H_\ell$ may not pass through the origin.

We consider the complement

$$M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H\in\mathcal{A}} H.$$

Let f_j be a linear form defining H_j and we set

$$\omega_j = d\log f_j, \quad 1 \le j \le \ell.$$

We denote by $A = \bigoplus_{q\ge 0} A^q$ the graded algebra generated by the logarithmic forms ω_j, $1 \le j \le \ell$. Here we consider A as a subalgebra of the algebra of differential forms on $M(\mathcal{A})$. We call A the the Orlik–Solomon algebra of the hyperplane arrangement $\mathcal{A}$. It is known by Brieskorn [3] that there is an isomorphism

$$A \cong H^*(M(\mathcal{A}); \mathbf{C}). \tag{1}$$

For more details about the Orlik–Solomon algebra we refer the reader to Orlik–Terao [26].

We define the reduced complex $\overline{A}$ by shifting the degrees by one as

$$\overline{A}^q = \begin{cases} 0, & q < 0, \\ A^{q+1}, & q \geq 0. \end{cases}$$

The reduced bar complex $\overline{B}^*(A)$ is the tensor algebra

$$\overline{B}^*(A) = \bigoplus_{k \geq 0} \left(\bigotimes^k \overline{A} \right)$$

generated by $\overline{A}$. Then $\overline{B}^*(A)$ has a natural structure of a graded algebra and we introduce the coboundary operator by

$$\begin{aligned} & d(\varphi_1 \otimes \cdots \otimes \varphi_k) \\ & = \sum_{j=1}^{k-1} (-1)^{\nu_j + 1} \varphi_1 \otimes \cdots \otimes (\varphi_j \wedge \varphi_{j+1}) \otimes \cdots \otimes \varphi_k \end{aligned}$$

where $\varphi_j \in \overline{A}^{q_j}$ and we set $\nu_j = q_1 + \cdots + q_j$. We define the iterated integral map $\mathcal{I}$ from the reduced bar complex to the space of differential forms on the loop space by

$$\mathcal{I}(\varphi_1 \otimes \cdots \otimes \varphi_k) = \int \varphi_1 \cdots \varphi_k.$$

In particular, the iterated integrals of 1-forms give functions on the loop space of $M(\mathcal{A})$. We define the filtration on the reduced bar complex by

$$\mathcal{F}^{-k}(\overline{B}^*(A)) = \bigoplus_{\ell \leq k} \left(\bigotimes^{\ell} \overline{A} \right)$$

For a group G we denote by $\mathbf{Z}G$ its group algebra. There is an augmentation map

$$\varepsilon \colon \mathbf{Z}G \longrightarrow \mathbf{Z}$$

defined by

$$\varepsilon\left(\sum_i a_i g_i \right) = \sum_i a_i, \quad a_i \in \mathbf{Z},\ g_i \in G$$

and we define the augmentation ideal J as $\operatorname{Ker}\varepsilon$.

We consider the case G is the fundamental group of $M(\mathcal{A})$. We obtain a pairing map

$$H^0(\overline{B}^*(A)) \times \mathbf{Z}\pi_1(M(\mathcal{A}), \mathbf{x}_0) \longrightarrow \mathbf{C}$$

given by the iterated integrals of logarithmic 1-forms. There is an induced filtration $\mathcal{F}^{-k}$ on $H^0(\overline{B}^*(A))$. We have a cochain map

$$i\colon \overline{B}^*(A) \longrightarrow \mathcal{B}^*(M(\mathcal{A}))$$

preserving the filtration, where $\mathcal{B}^*(M(\mathcal{A}))$ denotes the bar complex of the de Rham complex of $M(\mathcal{A})$. By applying a fundamental result due to Chen [5] relating the 0-dimensional cohomology of the bar complex and fundamental groups together with the isomorphism (1), we obtain the following theorem by an argument based on the spectral sequence.

Theorem 2.2 ([16])**.** *Let $M(\mathcal{A})$ be the complement of a hyperplane arrangement $\mathcal{A}$ and $\overline{B}^*(A)$ the reduced bar complex of the Orlik–Solomon algebra. Then the iterated integral map gives an isomorphism*

$$\mathcal{F}^{-k}H^0(\overline{B}^*(A)) \cong \operatorname{Hom}(\mathbf{Z}\pi_1(M(\mathcal{A}), \mathbf{x}_0)/J^{k+1}, \mathbf{C}).$$

We denote by $\mathbf{Z}\widehat{\pi}_1(M, \mathbf{x}_0)$ the completed group ring defined by

$$\varprojlim \mathbf{Z}\pi_1(M(\mathcal{A}), \mathbf{x}_0)/J^{k+1}.$$

By taking the limit $k \to \infty$ we observe that the above theorem implies the isomorphism

$$H^0(\overline{B}^*(A)) \cong \operatorname{Hom}(\mathbf{Z}\widehat{\pi}_1(M, \mathbf{x}_0), \mathbf{C}).$$

2.3. Holonomy Lie algebras for hyperplane arrangements

In general for a space M we define the holonomy Lie algebra $\mathfrak{h}(M)$ over a field k in the following way. Let

$$\eta\colon H_2(M; k) \longrightarrow H_1(M; k) \wedge H_1(M; k)$$

be the dual of the cup product homomorphism

$$\cup\colon H^1(M; k) \wedge H^1(M; k) \longrightarrow H^2(M; k).$$

We denote by $\mathcal{L}(H_1(M; k))$ the free Lie algebra over k generated by $H_1(M; k)$. We define the holonomy Lie algebra $\mathfrak{h}(M)$ as the quotient Lie algebra

$$\mathcal{L}(H_1(M; k))/\langle \operatorname{Im}\eta \rangle$$

where $\langle \operatorname{Im} \eta \rangle$ denotes the ideal generated by the image of η.

Let $\mathcal{A} = \{H_1, \ldots, H_\ell\}$ be an arrangement of hyperplanes in the sense of the previous section and we consider the complement $M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H \in \mathcal{A}} H$. In the case $M = M(\mathcal{A})$ the holonomy Lie algebra $\mathfrak{h}(M)$ has the following description. We set $k = \mathbf{C}$. Let $\mathcal{L}(X_1, \ldots, X_\ell)$ be the free Lie algebra whose generators are in one to one correspondence with the hyperplanes in $\mathcal{A}$. The holonomy Lie algebra is expressed as

$$\mathfrak{h}(M) = \mathcal{L}(X_1, \ldots, X_\ell)/\mathfrak{a}$$

where $\mathfrak{a}$ is the ideal generate by

$$[X_{j_p}, X_{j_1} + \cdots + X_{j_k}], \quad 1 \leq p < k \tag{2}$$

for the maximal family of hyperplanes $\{H_{j_1}, \ldots, H_{j_k}\}$ such that

$$\operatorname{codim}_{\mathbf{C}}(H_{j_1} \cap \cdots \cap H_{j_k}) = 2.$$

Let $\pi_1(M) = \Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_k \supset \cdots$ be the lower central series defined by

$$\Gamma_{k+1} = [\Gamma_1, \Gamma_k], \quad k \geq 1.$$

Then the direct sum

$$\bigoplus_{k \geq 1} [\Gamma_k/\Gamma_{k+1}] \otimes \mathbf{C}$$

has a structure of a graded Lie algebra by defining the Lie bracket by the commutator. On the other hand, the holonomy Lie algebra has a natural filtration

$$\mathfrak{h}(M) = \mathfrak{h}(M)_1 \supset \mathfrak{h}(M)_2 \supset \cdots \supset \mathfrak{h}(M)_k \supset \cdots$$

by defining

$$\mathfrak{h}(M)_{k+1} = [\mathfrak{h}(M)_1, \mathfrak{h}(M)_k], \quad k \geq 1.$$

We consider the completion

$$\widehat{\mathfrak{h}}(M) = \lim_{\leftarrow} \mathfrak{h}(M)/\mathfrak{h}(M)_k.$$

There is an extension of nilpotent groups

$$1 \longrightarrow \Gamma_k/\Gamma_{k+1} \longrightarrow \pi_1(M)/\Gamma_{k+1} \longrightarrow \pi_1(M)/\Gamma_k \longrightarrow 1.$$

We write $\pi_1(M)$ for $\pi_1(M, \mathbf{x}_0)$. By means of these extensions one can construct the nilpotent Lie algebra

$$[\pi_1(M)/\Gamma_k] \otimes \mathbf{C}, \quad k \geq 1$$

and the nilpotent completion

$$\widehat{\pi}_1(M) \otimes \mathbf{C} = \lim_{\leftarrow}[\pi_1(M)/\Gamma_k] \otimes \mathbf{C}$$

which is called the Malcev Lie algebra. It was shown in [12] that there is an isomorphism of Lie algebras

$$\widehat{\pi}_1(M) \otimes \mathbf{C} \cong \widehat{\mathfrak{h}}(M).$$

In particular, there is an isomorphism of graded Lie algebras

$$\bigoplus_{k \geq 1}[\Gamma_k/\Gamma_{k+1}] \otimes \mathbf{C} \cong \bigoplus_{k \geq 1} \mathfrak{h}(M)_k/\mathfrak{h}(M)_{k+1}.$$

2.4. Universal holonomy map

We deal with the case $M = M(\mathcal{A})$ for a complex hyperplane arrangement $\mathcal{A}$. We consider the expression

$$\omega = \sum_{j=1}^{m} \omega_j \otimes X_j$$

which is considered to be an element of $A^1 \otimes H_1(M; \mathbf{C})$. Here A^1 stands for the degree 1 part of the Orlik–Solomon algebra A.

We denote by $\mathbf{C}\langle\langle X_1, \ldots, X_\ell \rangle\rangle$ the algebra of non-commutative formal power series with indeterminates $X_1, \ldots, X_\ell$ and $\widehat{\mathfrak{a}}$ its ideal generated by the elements in the equation (2).

Then there is a universal holonomy map

$$\Theta \colon \pi_1(M, \mathbf{x}_0) \longrightarrow \mathbf{C}\langle\langle X_1, \ldots, X_\ell \rangle\rangle/\widehat{\mathfrak{a}} \tag{3}$$

defined by

$$\Theta(\gamma) = 1 + \sum_{k=1}^{\infty} \int_\gamma \underbrace{\omega \cdots \omega}_{k}.$$

Here the RHS of the above expression is explicitly given by

$$\int_\gamma \underbrace{\omega \cdots \omega}_{k} = \sum_{i_1, \ldots, i_k} \left(\int_\gamma \omega_{i_1} \cdots \omega_{i_k} \right) X_{i_1} \cdots X_{i_k}.$$

We denote by $\mathbf{C}\widehat{\pi}_1(M, \mathbf{x}_0)$ the completed group ring of $\pi_1(M, \mathbf{x}_0)$ over $\mathbf{C}$. The above universal holonomy map Θ induces an isomorphism

$$\mathbf{C}\widehat{\pi}_1(M, \mathbf{x}_0) \cong \mathbf{C}\langle\langle X_1, \ldots, X_\ell \rangle\rangle/\widehat{\mathfrak{a}}$$

of complete Hopf algebras. Here the coproduct structure of the RHS is induced from

$$\Delta(X_j) = X_j \otimes 1 + 1 \otimes X_j, \quad 1 \leq j \leq \ell.$$

By taking the primitive part with respect to the complete Hopf algebra structure, we obtain the nilpotent completion of the fundamental group over $\mathbf{C}$.

Proposition 2.3. *Let V be a complex vector space. For any representation of the holonomy Lie algebra $r\colon \mathfrak{h}(M) \to \mathrm{End}(V)$ there is a linear representation of the fundamental group*

$$\rho\colon \pi_1(M, \mathbf{x}_0) \longrightarrow \mathrm{GL}(V)$$

obtained by substituting the representation r to the universal holonomy map.

Proof. We set $r(X_j) = A_j$, $1 \leq j \leq \ell$. Since r is a homomorphism of Lie algebras we obtain that

$$\omega = \sum_j A_j \omega_j$$

satisfies the relation

$$\omega \wedge \omega = 0$$

This implies the above ω defines a flat connection for a trivial vector bundle over M with fiber V. It is not difficult show that the expression (3) with the substitution $X_j = A_j$ is convergent by using the fact that the volume of the simplex Δ_k is $1/k!$. This gives the holonomy of the connection ω, which gives a linear representation $\rho\colon \pi_1(M, \mathbf{x}_0) \to \mathrm{GL}(V)$. Q.E.D.

§3. Homology of local systems

3.1. Local systems for hyperplane arrangements

First, we recall some basic definition for local systems. Let M be a smooth manifold and V a complex vector space. Given a linear representation of the fundamental group

$$r\colon \pi_1(M, \mathbf{x}_0) \longrightarrow \mathrm{GL}(V)$$

there is an associated flat vector bundle E over M. The local system $\mathcal{L}$ associated to the representation r is the sheaf of horizontal sections of the flat bundle E. Let $\pi\colon \widetilde{M} \to M$ be the universal covering. We consider the chain complex

$$C_*(\widetilde{M}) \otimes_{\mathbf{Z}\pi_1} V$$

with the boundary map defined by $\partial(c \otimes v) = \partial c \otimes v$. Here $\mathbf{Z}\pi_1$ acts on $C_*(\widetilde{M})$ via the deck transformations and on V via the representation r. The homology of this chain complex is called the homology of M with coefficients in the local system $\mathcal{L}$ and is denoted by $H_*(M, \mathcal{L})$.

Let $\mathcal{L}$ be a complex rank one local system over $M(\mathcal{A})$ associated with a representation of the fundamental group

$$r\colon \pi_1(M(\mathcal{A}), x_0) \longrightarrow \mathbf{C}^*.$$

For an arrangement of complex hyperplanes $\mathcal{A} = \{H_1, \ldots, H_\ell\}$ we denote by f_j be a linear form defining the hyperplane H_j, $1 \leq j \leq \ell$. We associate a complex number a_j called an exponent to each hyperplane and consider a multivalued function

$$\Phi = f_1^{a_1} \cdots f_\ell^{a_\ell}.$$

The homology $H_1(M(\mathcal{A}); \mathbf{Z})$ is isomorphic to $\mathbf{Z}^{\oplus \ell}$, where each generator corresponds to a hyperplane. By associating to the generator of $H_1(M(\mathcal{A}); \mathbf{Z})$ corresponding to the hyperplane H_j the complex number $e^{2\pi\sqrt{-1}a_j}$, we obtain a homomorphism $H_1(M(\mathcal{A}); \mathbf{Z}) \to \mathbf{C}^*$. Combining with the abelianization map $\pi_1(M(\mathcal{A}), x_0) \to H_1(M(\mathcal{A}); \mathbf{Z})$ we obtain a homomorphism

$$\rho\colon \pi_1(M(\mathcal{A}), x_0) \longrightarrow \mathbf{C}^*.$$

The associated local system is denoted by $\mathcal{L}_\Phi$.

We shall investigate $H_*(M(\mathcal{A}), \mathcal{L})$, the homology of $M(\mathcal{A})$ with coefficients in the local system $\mathcal{L}$. For our purpose the homology of locally finite and possibly infinite chains denoted by $H_*^{lf}(M(\mathcal{A}), \mathcal{L})$ also plays an important role.

3.2. Vanishing theorem

In this section we prove a vanishing theorem for the homology of the complement of a complex hypersurface. Let M denote the complement of a complex hypersurface in $\mathbf{C}^n$. We choose a smooth compactification $i\colon M \to X$. Namely, M is written as $X \setminus D$, where X is a smooth projective variety and D is a divisor with normal crossings. For a rank one local system $\mathcal{L}$ on M we consider the Leray spectral sequence.

$$E_2^{p,q} = H^p(X, R^q i_* \mathcal{L}) \Longrightarrow H^{p+q}(M, \mathcal{L}).$$

We consider the condition

$$i_*\mathcal{L} \cong i_!\mathcal{L} \tag{4}$$

for the local system $\mathcal{L}$. Here i_* is the direct image and $i_!$ is the extension by 0. The condition (4) means that the eigenvalue of monodromy of $\mathcal{L}$ along any divisor at infinity is not equal to 1. The following theorem was shown in the case of the complement of a hyperplane arrangement in [14].

Theorem 3.1. *If the local system $\mathcal{L}$ satisfies the condition* (4), *then there is an isomorphism*

$$H_*(M, \mathcal{L}) \cong H_*^{lf}(M, \mathcal{L}).$$

We have $H_k(M, \mathcal{L}) = 0$ for any $k \neq n$.

Proof. By the hypothesis we have

$$R^q i_* \mathcal{L} = 0 \tag{5}$$

for $q > 0$. In fact, since D is a divisor with normal crossings the vanishing of the higher direct images in (5) follows from

$$H^q(\mathbf{C} \setminus \{0\}, \mathcal{L}) = 0, \quad q > 0$$

for a local system $\mathcal{L}$ over $\mathbf{C} \setminus \{0\}$ whose monodromy is non-trivial together with the Künneth formula. Hence the Leray spectral sequence degenerates at E_2-term and we have

$$E_2^{p,0} \cong E_\infty^{p,0} = H^p(M, \mathcal{L}),$$

where $E_2^{p,0} = H^p(X, i_*\mathcal{L})$. Thus we obtain an isomorphism

$$H^*(X, i_*\mathcal{L}) \cong H^*(M, \mathcal{L}).$$

On the other hand, there is an isomorphism

$$H^*(X, i_!\mathcal{L}) \cong H_c^*(M, \mathcal{L})$$

where H_c^* denotes cohomology with compact supports.

There are Poincaré duality isomorphisms:

$$H_k^{lf}(M, \mathcal{L}) \cong H^{2n-k}(M, \mathcal{L}),$$
$$H_k(M, \mathcal{L}) \cong H_c^{2n-k}(M, \mathcal{L}).$$

Since $i_*\mathcal{L} \cong i_!\mathcal{L}$ we obtain an isomorphism

$$H_k^{lf}(M, \mathcal{L}) \cong H_k(M, \mathcal{L}).$$

It follows from the above Poincaré duality isomorphisms and the Lefschetz theorem saying that that M has a homotopy type of a CW complex of dimension n (see [24]) we have

$$H_k^{lf}(M, \mathcal{L}) \cong 0, \quad k < n,$$
$$H_k(M, \mathcal{L}) \cong 0, \quad k > n.$$

Therefore we obtain $H_k(M, \mathcal{L}) = 0$ for any $k \neq n$. Q.E.D.

3.3. Bounded chambers

Let $\mathcal{A}$ be an essential hyperplane arrangement. Namely, we suppose that maximal codimension of a non-empty intersection of some subfamily of $\mathcal{A}$ is equal to n.

Let us suppose that each hyperplane in $\mathcal{A}$ is defined over $\mathbf{R}$. We set $M(\mathcal{A})_{\mathbf{R}} = M(\mathcal{A}) \cap \mathbf{R}^n$ and denote by Δ_ν, $1 \leq \nu \leq s$, the bounded chambers in $M(\mathcal{A})_{\mathbf{R}}$. We denote by $\overline{\Delta}_\nu$ the closure of Δ_ν in $X \setminus D$. Let

$$j \colon M(\mathcal{A}) \setminus \bigcup_\nu \Delta_\nu \longrightarrow X$$

be the inclusion map. We denote by $\mathcal{L}_0$ the restriction of the local system $\mathcal{L}$ on $M(\mathcal{A}) \setminus \bigcup_j \Delta_j$. In this situation we have the following theorem.

Theorem 3.2. *In addition to the condition* (4) *we suppose that there is an isomorphism*

$$j_*\mathcal{L}_0 \cong j_!\mathcal{L}_0.$$

Then the homology with locally finite chains $H_n^{lf}(M(\mathcal{A}), \mathcal{L})$ is spanned by the homology class of bounded chambers Δ_ν, $1 \leq \nu \leq s$.

Proof. We put $\overline{\Delta} = \bigcup_\nu \overline{\Delta}_\nu$. Let us consider the homology exact sequence of the triple $(X, D \cup \overline{\Delta}, D)$ with the local system coefficient $\mathcal{L}$:

$$\longrightarrow H_p(D \cup \overline{\Delta}, D) \longrightarrow H_p(X, D) \longrightarrow H_p(X, D \cup \overline{\Delta})$$
$$\longrightarrow H_{p-1}(D \cup \overline{\Delta}, D) \longrightarrow \cdots$$

where $\mathcal{L}$ is extended by 0 on $D \cup \overline{\Delta}$. Since we have

$$H_p(X, D) \cong H_p^{lf}(M(\mathcal{A}))$$

there is a long exact sequence

$$\longrightarrow H_p^{lf}\left(\bigcup_\nu \Delta_\nu\right) \longrightarrow H_p^{lf}(M(\mathcal{A})) \longrightarrow H_p^{lf}\left(M(\mathcal{A}) \setminus \bigcup_j \Delta_j\right)$$
$$\longrightarrow H_{p-1}^{lf}\left(\bigcup_\nu \Delta_\nu\right) \longrightarrow \cdots$$

with the local system coefficient $\mathcal{L}$. By the same argument as in the proof of Theorem 2.1 we have an isomorphism

$$H_k\left(M(\mathcal{A}) \setminus \bigcup_\nu \Delta_\nu, \mathcal{L}_0\right) \cong H_k^{lf}\left(M(\mathcal{A}) \setminus \bigcup_\nu \Delta_\nu, \mathcal{L}_0\right)$$

for any k and the vanishing

$$H_k\left(M(\mathcal{A}) \setminus \bigcup_\nu \Delta_\nu, \mathcal{L}_0\right) \cong 0$$

for k with $k \neq n$. Here we use the theorem of Zaslawsky saying that the number of bounded chambers is equal to the absolute value of the Euler–Poincaré characteristic $|\chi(M(\mathcal{A}))|$ to conclude that the above vanishing holds for any k. Combining with the above exact sequence, we obtain an isomorphism

$$H_n^{lf}\left(\bigcup_\nu \Delta_\nu, \mathcal{L}\right) \cong H_n^{lf}(M(\mathcal{A}), \mathcal{L}).$$

This leads to the statement of the theorem. Q.E.D.

§4. Holonomy Lie algebras and representations of the braid groups

4.1. Representations of holonomy Lie algebras

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $\{I_\mu\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan-Killing form. We set $\Omega = \sum_\mu I_\mu \otimes I_\mu$. Let $r_i \colon \mathfrak{g} \to \mathrm{End}(V_i)$, $1 \leq i \leq n$, be representations of the Lie algebra $\mathfrak{g}$. We denote by Ω_{ij} the action of Ω on the i-th and j-th components of the tensor product $V_1 \otimes \cdots \otimes V_n$. It is known that the Casimir element $c = \sum_\mu I_\mu \cdot I_\mu$ lies in the center of the universal enveloping algebra $U\mathfrak{g}$. Let us denote by $\Delta \colon U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$

the coproduct, which is defined to be the algebra homomorphism determined by $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$. Since Ω is expressed as $\Omega = \frac{1}{2}(\Delta(c) - c \otimes 1 - 1 \otimes c)$ we have the relation

$$[\Omega, x \otimes 1 + 1 \otimes x] = 0 \tag{6}$$

for any $x \in \mathfrak{g}$ in the tensor product $U\mathfrak{g} \otimes U\mathfrak{g}$. By means of the above relation we obtain the infinitesimal pure braid relations:

$$[\Omega_{ik}, \Omega_{ij} + \Omega_{jk}] = 0, \quad (i,\, j,\, k \text{ distinct}), \tag{7}$$

$$[\Omega_{ij}, \Omega_{k\ell}] = 0, \quad (i,\, j,\, k,\, \ell \text{ distinct}). \tag{8}$$

Let us briefly explain the reason why we have the above infinitesimal pure braid relations. For the first relation it is enough to show the case $i = 1$, $j = 3$ and $k = 2$. Since we have

$$[\Omega \otimes 1, (I_\mu \otimes 1 + 1 \otimes I_\mu) \otimes I_\mu] = 0$$

by the equation (6) we obtained the desired relation. The equation (4.3) in the infinitesimal pure braid relations is clear from the definition of Ω on the tensor product.

Let us denote by

$$X_n = \{(z_1, \ldots, z_n) \in \mathbf{C}^n ; z_i \neq z_j \text{ if } i \neq j\}$$

the configuration space of ordered distinct n points on the complex plane $\mathbf{C}$. The configuration space X_n is the complement of the hyperplane arrangement given by

$$X_n = \mathbf{C}^n \setminus \bigcup_{1 \le i < j \le n} H_{ij}$$

where H_{ij} is the big diagonal hyperplane defined by $z_i = z_j$. The holonomy Lie algebra $\mathfrak{h}(X_n)$ is isomorphic to the quotient of the free Lie algebra generated by indeterminates X_{ij}, $1 \le i \neq j \le n$ with relations $X_{ij} = X_{ji}$ together with the infinitesimal pure braid relations

$$[X_{ik}, X_{ij} + X_{jk}] = 0, \quad (i,\, j,\, k \text{ distinct}), \tag{9}$$

$$[X_{ij}, X_{k\ell}] = 0, \quad (i,\, j,\, k,\, \ell \text{ distinct}). \tag{10}$$

The structure of the above holonomy Lie algebra $\mathfrak{h}(X_n)$ was studied in [13]. We obtain a linear representation of the holonomy Lie algebra

$$r_\kappa \colon \mathfrak{h}(X_n) \longrightarrow \mathrm{End}(V_1 \otimes \cdots \otimes V_n)$$

by defining

$$r_\kappa(X_{ij}) = \frac{1}{\kappa}\Omega_{ij} \tag{11}$$

for any non-zero complex parameter κ.

In this article we deal with the case $\mathfrak{g} = sl_2(\mathbf{C})$. Let us recall basic facts about the Lie algebra $sl_2(\mathbf{C})$ and its Verma modules. As a complex vector space the Lie algebra $sl_2(\mathbf{C})$ has a basis H, E and F satisfying the relations:

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H. \tag{12}$$

For a complex number λ we denote by M_λ the Verma module of $sl_2(\mathbf{C})$ with highest weight λ. Namely, there is a non-zero vector $v_\lambda \in M_\lambda$ called the highest weight vector satisfying

$$Hv_\lambda = \lambda v_\lambda, \quad Ev_\lambda = 0 \tag{13}$$

and M_λ is spanned by $F^j v_\lambda$, $j \geq 0$. The elements H, E and F act on this basis as

$$\begin{cases} H \cdot F^j v_\lambda = (\lambda - 2j)F^j v_\lambda, \\ E \cdot F^j v_\lambda = j(\lambda - j + 1)F^{j-1} v_\lambda, \\ F \cdot F^j v_\lambda = F^{j+1} v_\lambda. \end{cases} \tag{14}$$

It is known that if $\lambda \in \mathbf{C}$ is not a non-negative integer, then the Verma module M_λ is irreducible.

For $\Lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n$ we put $|\Lambda| = \lambda_1 + \cdots + \lambda_n$ and consider the tensor product $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$. For a non-negative integer m we define the space of weight vectors with weight $|\Lambda| - 2m$ by

$$W[|\Lambda| - 2m] = \{x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}; Hx = (|\Lambda| - 2m)x\} \tag{15}$$

and consider the space of null vectors defined by

$$N[|\Lambda| - 2m] = \{x \in W[|\Lambda| - 2m]; Ex = 0\}. \tag{16}$$

Since the representation

$$r_\kappa \colon \mathfrak{h}(X_n) \longrightarrow \mathrm{End}(M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n})$$

defined as in (11) commutes with the diagonal action of $\mathfrak{g}$ it induces the representation

$$r_{\kappa,m} \colon \mathfrak{h}(X_n) \longrightarrow \mathrm{End}(N[|\Lambda| - 2m]). \tag{17}$$

4.2. Quantum representations of the braid groups

The fundamental group of the configuration space X_n is the pure braid group on n strings denoted by P_n. In view of Proposition 2.3 we obtain the linear representation

$$\theta_{\kappa,m}\colon P_n \longrightarrow \mathrm{GL}(N[|\Lambda|-2m]) \tag{18}$$

associated with the representation of the holonomy Lie algebra $r_{\kappa,m}$ defined in (11). This is the monodromy of the Knizhnik–Zamolodchikov (KZ) connection defined by

$$\omega = \frac{1}{\kappa}\sum_{1\le i<j\le n} \Omega_{ij}\, d\log(z_i - z_j)$$

with values in $\mathrm{End}(N[|\Lambda|-2m])$ for a non-zero complex parameter κ. We set $\omega_{ij} = d\log(z_i - z_j)$, $1 \le i, j \le n$. It follows from the infinitesimal pure braid relations among Ω_{ij} together with Arnold's relation

$$\omega_{ij}\wedge\omega_{jk} + \omega_{jk}\wedge\omega_{k\ell} + \omega_{k\ell}\wedge\omega_{ij} = 0$$

that $\omega \wedge \omega = 0$ holds. This implies that ω defines a flat connection for a trivial vector bundle over the configuration space X_n with fiber $N[|\Lambda|-2m]$.

A horizontal section of the above flat bundle is a solution of the total differential equation

$$d\varphi = \omega\varphi \tag{19}$$

for a function $\varphi(z_1,\dots,z_n)$ with values in $N[|\Lambda|-2m]$. This total differential equation can be expressed as a system of partial differential equations

$$\frac{\partial\varphi}{\partial z_i} = \frac{1}{\kappa}\sum_{j,j\ne i}\frac{\Omega_{ij}}{z_i - z_j}\varphi, \quad 1\le i\le n, \tag{20}$$

which is called the KZ equation. The KZ equation was first introduced in [11] as the differential equation satisfied by n-point functions in Wess–Zumino–Witten conformal field theory.

The symmetric group $\mathfrak{S}_n$ acts on X_n by permutations of coordinates. We denote the quotient space $X_n/\mathfrak{S}_n$ by Y_n. The fundamental group of Y_n is the braid group on n strings denoted by B_n. In the case $\lambda_1 = \cdots = \lambda_n = \lambda$, the symmetric group $\mathfrak{S}_n$ acts diagonally on the trivial vector bundle over X_n with fiber $N[n\lambda - 2m]$ and the connection

ω is invariant by this action. Thus we obtain a one-parameter family of linear representations of the braid group

$$\theta_{\kappa,m}\colon B_n \longrightarrow \mathrm{GL}(N[n\lambda - 2m]). \tag{21}$$

As is shown in [15] and in a more general situation by Drinfel'd [9] the above representations of the braid groups have symmetry of the quantum group $U_h\mathfrak{g}$. We call $\theta_{\kappa,m}$ quantum representations of the braid groups.

§5. Hypergeometric integrals and quantum representations of the braid groups

5.1. Discriminantal arrangements and hypergeometric integrals

Following Schechtman and Varchenko [28], we shall express the horizontal sections of the KZ connection ω in terms of hypergeometric integrals. Let

$$\pi_{n,m}\colon X_{m+n} \longrightarrow X_n$$

be the projection map defined by

$$(z_1, \dots, z_n, w_1, \dots, w_m) \longmapsto (z_1, \dots, z_n).$$

For $p \in X_n$ the fiber $\pi_{n,m}^{-1}(p)$ is denoted by $X_{n,m}$. Let $(z_1, \dots, z_n)$ be the coordinates for p. Then, $X_{n,m}$ is the complement of hyperplanes defined by

$$w_i = z_\ell, \quad 1 \le i \le m, \quad 1 \le \ell \le n, \quad w_i = w_j, \quad 1 \le i < j \le m. \tag{22}$$

We denote these hyperplanes by $H_{i\ell}$, $1 \le i \le m$, $1 \le \ell \le n$, and D_{ij}, $1 \le i < j \le m$. Such arrangement of hyperplanes is called a discriminantal arrangement.

Let us suppose that $m \ge 2$. For parameters κ and $\lambda_1, \dots, \lambda_n$ we consider the multi-valued function

$$\Phi_{n,m} = \prod_{1\le i<j\le n} (z_i - z_j)^{\frac{\lambda_i\lambda_j}{2\kappa}} \prod_{1\le i\le m, 1\le \ell\le n} (w_i - z_\ell)^{-\frac{\lambda_\ell}{\kappa}} \prod_{1\le i<j\le m} (w_i - w_j)^{\frac{2}{\kappa}}$$

defined over X_{n+m}. Let $\mathcal{L}$ denote the local system over $X_{n,m}$ associated to the multi-valued function $\Phi_{n,m}$.

The symmetric group $\mathfrak{S}_m$ acts on $X_{n,m}$ by the permutations of the coordinate functions $w_1, \dots, w_m$. The function $\Phi_{n,m}$ is invariant by the action of $\mathfrak{S}_m$. The local system $\mathcal{L}$ over $X_{n,m}$ defines a local system on

$Y_{n,m}$, which we denote by $\overline{\mathcal{L}}$. We denote by $\mathcal{L}^*$ the local system dual to $\mathcal{L}$.

We put $v = v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n}$ and for $J = (j_1, \dots, j_n)$ set $F^J v = F^{j_1} v_{\lambda_1} \otimes \cdots \otimes F^{j_n} v_{\lambda_n}$, where $j_1, \dots, j_n$ are non-negative integers. The weight space $W[|\Lambda| - 2m]$ has a basis $F^J v$ for each J with $|J| = j_1 + \cdots + j_n = m$. For the sequence of integers

$$(i_1, \dots, i_m) = (\underbrace{1, \dots, 1}_{j_1}, \dots, \underbrace{n, \dots, n}_{j_n})$$

we set

$$S_J(z, w) = \frac{1}{(w_1 - z_{i_1}) \cdots (w_m - z_{i_m})} \tag{23}$$

and define the rational function $R_J(z, w)$ by

$$R_J(z, w) = \frac{1}{j_1! \cdots j_n!} \sum_{\sigma \in \mathfrak{S}_m} S_J(z_1, \dots, z_n, w_{\sigma(1)}, \dots, w_{\sigma(m)}). \tag{24}$$

For example, we have

$$R_{(1,0,\dots,0)}(z, w) = \frac{1}{w_1 - z_1}, \quad R_{(2,0,\dots,0)}(z, w) = \frac{1}{(w_1 - z_1)(w_2 - z_1)},$$
$$R_{(1,1,0,\dots,0)}(z, w) = \frac{1}{(w_1 - z_1)(w_2 - z_2)} + \frac{1}{(w_2 - z_1)(w_1 - z_2)}$$

and so on.

Since $\pi_{n,m}\colon X_{m+n} \to X_n$ is a fiber bundle with fiber $X_{n,m}$ the fundamental group of the base space X_n acts naturally on the homology group $H_m(X_{n,m}, \mathcal{L}^*)$. Thus we obtain a representation of the pure braid group

$$\rho_{n,m}\colon P_n \longrightarrow \mathrm{Aut} H_m(X_{n,m}, \mathcal{L}^*) \tag{25}$$

which defines a local system on X_n denoted by $\mathcal{H}_{n,m}$. In the case $\lambda_1 = \cdots = \lambda_n$ there is a representation of the braid group

$$\rho_{n,m}\colon B_n \longrightarrow \mathrm{Aut} H_m(Y_{n,m}, \overline{\mathcal{L}}^*) \tag{26}$$

which defines a local system $\overline{\mathcal{H}}_{n,m}$ on Y_n. For any horizontal section $c(z)$ of the local system $\mathcal{H}_{n,m}$ we consider the hypergeometric type integral

$$\int_{c(z)} \Phi_{n,m} R_J(z, w)\, dw_1 \wedge \cdots \wedge dw_m \tag{27}$$

for the above rational function $R_J(z, w)$.

The twisted de Rham complex $(\Omega^*(X_{n,m}), \nabla)$ is a complex with differential $\nabla\colon \Omega^j(X_{n,m}) \to \Omega^{j+1}(X_{n,m})$ defined by

$$\nabla\omega = d\omega + d\log\Phi_{n,m} \wedge \omega.$$

for $\omega \in \Omega^j(X_{n,m})$. There is a pairing between the homology of the local system $\mathcal{L}^*$ and the cohomology of the twisted de Rham complex

$$H_m(X_{n,m}, \mathcal{L}^*) \times H^m(\Omega^*(X_{n,m}), \nabla) \longrightarrow \mathbf{C}$$

defined by

$$(c, \omega) \longmapsto \int_c \Phi_{n,m}\omega.$$

Such integrals are called hypergeometric integrals. We refer the reader to [27] for a detailed treatment of hypergeometric integrals in a more general situation of hyperplane arrangements.

Many works have been done on the solutions of the KZ equation by means of hypergeometric type integrals (see [7] and [28]). We have the following theorem.

Theorem 5.1 (Schechtman and Varchenko [28])**.** *The integral*

$$\sum_{|J|=m} \left(\int_{c(z)} \Phi_{n,m} R_J(z, w)\, dw_1 \wedge \cdots \wedge dw_m \right) F^J v$$

lies in the space of null vectors $N[|\Lambda| - 2m]$ and is a solution of the KZ equation.

The above theorem gives a period map

$$\phi\colon H_m(X_{n,m}, \mathcal{L}^*) \longrightarrow N[|\Lambda| - 2m]$$

defined by

$$\phi(c) = \sum_{|J|=m} \left(\int_c \Phi_{n,m} R_J(z, w)\, dw_1 \wedge \cdots \wedge dw_m \right) F^J v. \tag{28}$$

5.2. Burau and Gassner representations

In this section we describe the solutions of the KZ equation in the case $m = 1$ and compare the monodromy representations with the Burau and Gassner representations. We refer the reader to [2] for the original definition of the Burau and Gassner representations.

We start with the case $m = 0$. The space $N[|\Lambda|]$ is equal to $W[|\Lambda|]$ and is a one-dimensional vector space spanned by the tensor product of the highest weight vectors $v = v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n}$. We have

$$\Omega_{ij} v = \frac{\lambda_i \lambda_j}{2} v$$

and the solution of the KZ equation is spanned by

$$\Phi_{n,0} = \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{2\kappa}}.$$

As the monodromy we obtain a one-dimensional representation of the pure braid group

$$\theta_0 \colon P_n \longrightarrow \mathrm{GL}(1, \mathbf{C}).$$

Let us describe the case $m = 1$. We express the solutions of the KZ equation with values in $N[|\Lambda| - 2]$. We consider the fibration $\pi_{n,1} \colon X_{n+1} \to X_n$ defined by

$$\pi_{n,1}(z_1, \dots, z_n, w) = (z_1, \dots, z_n).$$

The fiber $X_{n,1}$ over $p = (z_1, \dots, z_n)$ is $\mathbf{C} \setminus \{z_1, \dots, z_n\}$. Let $\Phi_{n,1}$ be the multi-valued function on X_{n+1} defined by

$$\Phi_{n,1} = \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{2\kappa}} \prod_{1 \le \ell \le n} (w - z_\ell)^{-\frac{\lambda_\ell}{\kappa}}.$$

We denote by $\mathcal{L}$ the associated local system on $X_{n,1}$. We define the 1-forms η_j by

$$\eta_j = \Phi_{n,1} \frac{dw}{w - z_j}, \quad j = 1, \dots, n.$$

For a loop γ in $X_{n,1}$ such that $[\gamma]$ defines an element of $H_1(X_{n,1}, \mathcal{L}^*)$ we consider the integrals

$$I_j = \int_\gamma \eta_j, \quad j = 1, \dots, n.$$

Since we have

$$\lambda_1 \eta_1 + \cdots + \lambda_n \eta_n = -\kappa \, d\Phi_{n,1}$$

there is a linear relation

$$\lambda_1 I_1 + \cdots + \lambda_n I_n = 0 \tag{29}$$

by the Stokes theorem.

The weight space $W[|\Lambda| - 2]$ is a vector space of dimension n spanned by $F^{(j)} v$, $1 \le j \le n$, where $v = v_{\lambda_1} \otimes \cdots \otimes v_{\lambda_n}$ as in the case of $m = 0$ and $F^{(j)}$ stands for the action of F on the j-th component of the tensor product. The space of null vectors $N[|\Lambda| - 2]$ is given by

$$N[|\Lambda| - 2] = \left\{ \sum_{j=1}^{n} \alpha_j F^{(j)} v ; \sum_{j=1}^{n} \lambda_j \alpha_j = 0 \right\}$$

and we have $\dim N[|\Lambda|-2] = n-1$. The period map

$$\phi\colon H_1(X_{n,1}, \mathcal{L}^*) \longrightarrow N[|\Lambda|-2]$$

is defined by

$$\phi([\gamma]) = \sum_{j=1}^{n} I_j F^{(j)} v.$$

Here we notice that

$$E\phi([\gamma]) = \left(\sum_{j=1}^{n} \lambda_j I_j\right) v = 0$$

holds since we have the relation (29). We have a local system $\mathcal{H}_{n,1}$ on X_n whose fiber over p is $H_1(X_{n,1}, \mathcal{L}^*)$. Let $c(z)$ be a horizontal section of the local system $\mathcal{H}_{n,1}$. We can verify that

$$\sum_{j=1}^{n} \int_{c(z)} \eta_j F^{(j)} v$$

is a solution of the KZ equation with values in $N[|\Lambda|-2]$. This is classically known as the Jordan-Pochhammer system. As the holonomy of the KZ connection we obtain the linear representation

$$\theta_\kappa\colon P_n \longrightarrow \mathrm{GL}(N[|\Lambda|-2]).$$

The above representation can be written as $\theta_\kappa = \theta_0 \otimes \overline{\theta}_\kappa$. In the following we focus on the representation $\overline{\theta}_\kappa$.

We set $p = (1, 2, \ldots, n)$ and describe the homology $H_1(X_{n,1}, \mathcal{L}^*)$ for the fiber $X_{n,1}$ over p. Let γ_j be the open interval $(j, j+1)$, $1 \leq j \leq n-1$, and we consider the homology class $[\gamma_j] \in H_1^{lf}(X_{n,1}, \mathcal{L}^*)$. In the following we suppose

$$\frac{\lambda_j}{\kappa} \notin \mathbf{Z},\ 1 \leq j \leq n, \quad \frac{1}{\kappa}\sum_{j=1}^{n} \lambda_j \notin \mathbf{Z}. \tag{30}$$

Proposition 5.2. *Under the condition* (30) *there is an isomorphism*

$$H_1(X_{n,1}, \mathcal{L}^*) \cong H_1^{lf}(X_{n,1}, \mathcal{L}^*)$$

and the above homology group is spanned by $[\gamma_j]$, $1 \leq j \leq n-1$.

Proof. By the hypothesis we can apply Theorem 3.1 for $X_{n,1}$ and obtain the isomorphism

$$H_1(X_{n,1}, \mathcal{L}^*) \cong H_1^{lf}(X_{n,1}, \mathcal{L}^*).$$

Moreover, we can also apply Theorem 3.2 and conclude that the homology group $H_1^{lf}(X_{n,1}, \mathcal{L}^*)$ is spanned by the bounded chambers $[\gamma_j]$, $1 \leq j \leq n-1$. Q.E.D.

In the following we consider $[\gamma_j]$ as an element of $H_1(X_{n,1}, \mathcal{L}^*)$ by means of the above isomorphism and the integrals

$$\int_{\gamma_i} \eta_j, \quad 1 \leq i \leq n-1,\, 1 \leq j \leq n$$

are convergent. We assume the condition (30). Then, we have the following theorem.

Theorem 5.3. *The period map ϕ induces an isomorphism*

$$H_1(X_{n,1}, \mathcal{L}^*) \cong N[|\Lambda| - 2].$$

The above isomorphism is equivariant with respect to the action of the pure braid group P_n.

Proof. First, we show that $\phi([\gamma_i])$, $1 \leq i \leq n-1$, are linearly independent. As is shown in [31] the determinant

$$\det \left(\int_{\gamma_i} \eta_j \right)_{1 \leq i \leq n-1, 1 \leq j \leq n-1}$$

is given by gamma functions as

$$\frac{\Gamma\left(-\frac{\lambda_1}{\kappa} + 1\right) \cdots \Gamma\left(-\frac{\lambda_n}{\kappa} + 1\right)}{\Gamma\left(-\frac{\lambda_1}{\kappa} - \cdots - \frac{\lambda_n}{\kappa} + 1\right)}$$

up to a multiplication of a non-zero constant. The above determinant does not vanish under the condition (30) and we obtain that ϕ is injective. Since

$$\dim H_1(X_{n,1}, \mathcal{L}^*) = \dim N[|\Lambda| - 2] = n - 1$$

we obtain the isomorphism. The action of the pure braid group P_n on the homology group $H_1(X_{n,1}, \mathcal{L}^*)$ is given by the horizontal sections of the local system $\mathcal{H}_{n,1}$ on X_n. On the other hand, we have shown that $\phi([\gamma_i])$, $1 \leq i \leq n-1$, form a basis of the space of solutions of the KZ equation with values in $N[|\Lambda| - 2]$. Since the solutions are given by hypergeometric integrals with respect to the cycles $[\gamma_i]$, $1 \leq i \leq n-1$, the monodromy representation of the KZ equation is identified with the above action of P_n on $H_1(X_{n,1}, \mathcal{L}^*)$. Q.E.D.

Now we describe a relation between the monodromy representation

$$\overline{\theta}_\kappa \colon P_n \longrightarrow \mathrm{GL}(N[|\Lambda| - 2])$$

and the Gassner representation. First, we recall the Gassner representation. The fundamental group of $X_{n,1}$ is isomorphic to the free group with n generators. Let $\pi \colon \widetilde{X}_{n,1} \to X_{n,1}$ be the covering corresponding to the kernel of the abelianization map

$$\alpha \colon \pi_1(X_{n,1}, x_0) \longrightarrow \mathbf{Z}^{\oplus n}.$$

The covering transformation group $\mathbf{Z}^{\oplus n}$ acts on the homology group $H_1(\widetilde{X}_{n,1}; \mathbf{Z})$ and it turns out that $H_1(\widetilde{X}_{n,1}; \mathbf{Z})$ is a free module of rank $n-1$ over the group ring $\mathbf{Z}[\mathbf{Z}^{\oplus n}]$, which is isomorphic to the ring of Laurent polynomials

$$\Lambda_n = \mathbf{Z}[q_1^{\pm 1}, \dots, q_n^{\pm 1}]$$

with indeterminates $q_1, \dots, q_n$. The pure braid group P_n acts as diffeomorphisms of $X_{n,1}$, which induces an action of P_n on $H_1(\widetilde{X}_{n,1}; \mathbf{Z})$ commuting with the action of Λ_n. Thus we obtain a linear representation

$$\rho_n \colon P_n \longrightarrow \mathrm{GL}(n-1, \Lambda_n)$$

which is called the Gassner representation. By comparing the action of P_n on the homology of local systems we have the following proposition.

Proposition 5.4. *The representation*

$$\overline{\theta}_\kappa \colon P_n \longrightarrow \mathrm{GL}(N[|\Lambda| - 2])$$

is equivalent to the Gassner representation specialized at

$$q_j = e^{-2\pi\sqrt{-1}\lambda_j/\kappa}, \quad 1 \le j \le n.$$

The Burau representation of the braid group B_n corresponds to the case $q_1 = \cdots = q_n = q$. We consider the homomorphism $\beta \colon \mathbf{Z}^{\oplus n} \to \mathbf{Z}$ defined by $\beta(x_1, \dots, x_n) = x_1 + \cdots + x_n$ and denote by $\widetilde{X}'_{n,1}$ the abelian covering of $X_{n,1}$ corresponding the the kernel of $\beta \circ \alpha$. The homology group $H_1(\widetilde{X}'_{n,1}; \mathbf{Z})$ is a free module over $\Lambda = \mathbf{Z}[q, q^{-1}]$ by means of deck transformations and the braid group B_n acts on $H_1(\widetilde{X}'_{n,1}; \mathbf{Z})$ commuting with the action of Λ. Thus we have a linear representation

$$\rho_n \colon B_n \longrightarrow \mathrm{GL}(n-1, \Lambda),$$

which is called the Burau representation. Considering the case $\lambda_1 = \cdots = \lambda_n = \lambda$, we obtain the monodromy representation

$$\overline{\theta}_\kappa \colon B_n \longrightarrow \mathrm{GL}(N[n\lambda - 2])$$

as the Burau representation specialized at $q = e^{-2\pi\sqrt{-1}\lambda/\kappa}$.

5.3. Homological representations

In this section we describe a relation between the monodromy representations of the KZ equations and homological representations of the braid groups in the case $m \geq 2$.

As in the previous sections we consider the projection $\pi_{n,m} \colon X_{m+n} \to X_n$, the fiber $X_{n,m}$ over $p \in X_n$ and the local system $\mathcal{L}$ associated with the multi-valued function $\Phi_{n,m}$. Let us recall that $Y_{n,m} = X_{n,m}/\mathfrak{S}_m$ and that $\overline{\mathcal{L}}$ is the induced local system on $Y_{n,m}$. We see that $Y_{n,m}$ is expressed as the complement of a complex hypersurface in $\mathbf{C}^m$. We take the base point $p = (1,\ldots,n)$. In the following we suppose that the highest weights $\lambda_1,\ldots,\lambda_n$ and the parameter κ satisfy the condition in Theorem 3.2 for $Y_{n,m}$.

For the purpose of describing the homology group $H^{lf}_m(X_{n,m}, \mathcal{L}^*)$ and $H^{lf}_m(Y_{n,m}, \overline{\mathcal{L}}^*)$ we introduce the following notation. For non-negative integers $m_1, \ldots, m_{n-1}$ satisfying

$$m_1 + \cdots + m_{n-1} = m$$

we define a bounded chamber $\Delta_{m_1,\ldots,m_{n-1}}$ in $\mathbf{R}^m$ by

$$\begin{aligned}
&1 < t_1 < \cdots < t_{m_1} < 2,\\
&2 < t_{m_1+1} < \cdots < t_{m_1+m_2} < 3,\\
&\cdots\\
&n-1 < t_{m_1+\cdots+m_{n-2}+1} + \cdots + t_m < n.
\end{aligned}$$

We put $M = (m_1, \ldots, m_{n-1})$ and we write Δ_M for $\Delta_{m_1,\ldots,m_{n-1}}$. We denote by $\overline{\Delta}_M$ the image of Δ_M by the projection map from $X_{n,m}$ to $Y_{n,m}$. The bounded chamber Δ_M defines a homology class $[\Delta_M] \in H^{lf}_m(X_{n,m}, \mathcal{L}^*)$ and its image $\overline{\Delta}_M$ defines a homology class

$$[\overline{\Delta}_M] \in H^{lf}_m(Y_{n,m}, \overline{\mathcal{L}}^*).$$

Under the condition in Theorem 3.2 we have the following proposition.

Proposition 5.5. *The above bounded chambers $[\overline{\Delta}_M]$ for non-negative integers $m_1,\ldots,m_{n-1}$ with $m_1 + \cdots + m_{n-1} = m$ form a basis of $H^{lf}_m(Y_{n,m},\overline{\mathcal{L}})$.*

In particular, we have

$$\dim H_m(Y_{n,m}, \overline{\mathcal{L}}^*) = \dim H^{lf}_m(Y_{n,m}, \overline{\mathcal{L}}^*) = \binom{m+n-2}{m}. \tag{31}$$

We denote the above dimension by $d_{n,m}$.

The period map ϕ in (28) induces

$$\phi\colon H_m(Y_{n,m}, \overline{\mathcal{L}}^*) \longrightarrow N[|\Lambda| - 2m].$$

Let us notice that we have $\dim N[|\Lambda| - 2m] = d_{n,m}$. We have the following theorem for $m \geq 2$.

Theorem 5.6. *For generic $\lambda_1, \ldots, \lambda_n$ and κ the period map ϕ gives an isomorphism*

$$H_m(Y_{n,m}, \overline{\mathcal{L}}^*) \cong N[|\Lambda| - 2m]$$

which is equivariant with respect to the action of the pure braid group P_n. In particular, the linear representation

$$\rho_{n,m}\colon P_n \longrightarrow \mathrm{Aut} H_m(X_{n,m}, \mathcal{L}^*)$$

and the monodromy representation of the KZ equation

$$\overline{\theta}_{\kappa,m}\colon P_n \longrightarrow \mathrm{GL}(N[|\Lambda| - 2m])$$

are equivalent.

Proof. We denote by I_m the set of $(n-1)$-tuple of non-negative integers $(m_1, \ldots, m_{n-1})$ with $m_1 + \cdots + m_{n-1} = m$. We suppose the condition in Theorem 3.2 for $\lambda_1, \ldots, \lambda_n$ and κ. By Proposition 5.5 we have a basis $\{c_M\}$ for $H_m(Y_{n,m}, \overline{\mathcal{L}}^*)$ which is in one-to-one correspondence with the set I_m. Since $N[|\Lambda| - 2m]$ has also a basis which is in one-to-one correspondence with the set I_m the solutions of the KZ equation by hypergeometric integrals in Theorem 5.1 can be written in the matrix form

$$\left(\int_{c_M} \omega_{M'} \right)_{M, M' \in I_m}$$

where $\omega_{M'}$ is a multi-valued m form on $X_{n,m}$. As is shown in [31] the determinant of the above matrix is expressed by gamma functions in $\lambda_1, \ldots, \lambda_n$ and κ (see [18]) and it turns out that the determinant does not vanish for generic $\lambda_1, \ldots, \lambda_n, \kappa$. This show that $\phi(c_M)$, $M \in I_m$, are linearly independent and that ϕ is an isomorphism. By means of the above expression of the basis of solutions of the KZ equation by hypergeometric integrals the monodromy representations of the KZ equation can be described by the action of P_n on the homology group $H_m(Y_{n,m}, \mathcal{L}^*)$. This shows that the isomorphism ϕ is equivariant with respect to the action of P_n. Q.E.D.

Let us describe a relation to the homological representations of the braid groups. We have

$$H_1(Y_{n,m};\mathbf{Z}) \cong \mathbf{Z}^{\oplus n} \oplus \mathbf{Z} \tag{32}$$

where the first n components correspond to normal loops of the images of hyperplanes $w_i = z_\ell$, $\ell = 1, \dots, n$, and the last component corresponds to the normal loop of the image of the diagonal hyperplanes $w_i = w_j$, $1 \le i < j \le m$, namely, the discriminant set. Let $\pi\colon \tilde{Y}_{n,m} \to Y_{n,m}$ be the covering corresponding to the kernel of the abelianization map $\alpha\colon \pi_1(Y_{n,m}, x_0) \to H_1(Y_{n,m};\mathbf{Z})$. The homology group $H_m(\tilde{Y}_{n,m};\mathbf{Z})$ has a structure of the module over

$$\Lambda_{n,1} = \mathbf{Z}[q_1^{\pm 1}, \dots q_n^{\pm 1}, t^{\pm 1}]$$

and the pure braid group P_n acts on $H_m(\tilde{Y}_{n,m};\mathbf{Z})$ commuting with the action of $\Lambda_{n,1}$. This is called the homological representation of P_n. The representation $\rho_{n,m}$ in Theorem 5.6 is obtained by the specialization

$$q_j = e^{-2\pi\sqrt{-1}\lambda_j/\kappa},\ 1 \le j \le n, \quad t = e^{2\pi\sqrt{-1}/\kappa}.$$

If we set $q_1 = \cdots = q_n = q$, then we obtain linear representations of the braid group B_n with two parameters q and t. This is geometrically described in the following way. We consider the homomorphism

$$\beta\colon H_1(Y_{n,m};\mathbf{Z}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z} \tag{33}$$

defined by $\beta(x_1, \dots, x_n, y) = (x_1 + \cdots + x_n, y)$. Let us denote by $\tilde{Y'}_{n,m}$ the covering over $Y_{n,m}$ corresponding to the kernel of $\beta \circ \alpha$. Then the homology group $H_m(\tilde{Y'}_{n,m};\mathbf{Z})$ is a free module of rank $d_{n,m}$ over $\Lambda_{1,1} = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$. The braid group B_n acts on $H_m(\tilde{Y'}_{n,m};\mathbf{Z})$ commuting with the action of $\Lambda_{1,1}$. This homological representation of the braid group B_n was extensively studied by Bigelow [1] and Krammer [21] in the case $m = 2$ and they showed that the representation is faithful. In terms of local systems these representations appear as

$$\overline{\theta}_{\kappa,m}\colon B_n \longrightarrow \mathrm{GL}(N[n\lambda - 2m])$$

in the case $\lambda_1 = \cdots = \lambda_n = \lambda$ by means of the specialization

$$q = e^{-2\pi\sqrt{-1}\lambda/\kappa}, \quad t = e^{2\pi\sqrt{-1}/\kappa}.$$

Remark. (1) It turns out that the representation $r_{\kappa,m}$ defined in (11) is irreducible. It follows from the fact the the corresponding homological representation $\rho_{n,m}$ is irreducible as is shown in [6]. There is a general treatment of correspondence with the irreducibility of a representation of the holonomy Lie algebra and the associated monodromy representation (see [22], [23]).

(2) In the present article we treated the case where the parameters $\lambda_1, \dots, \lambda_n$ and κ are generic. In the case of conformal field theory this hypothesis of genericity is not always satisfied. The period map ϕ might not be an isomorphism because of the resonance at infinity. It turns out that the space of conformal blocks is isomorphic to a quotient of the homology group $H_m(Y_{n,m}, \overline{\mathcal{L}}^*)$. We refer the reader to [29], [10], [19] for this subject.

References

[1] S. J. Bigelow, Braid groups are linear, *J. Amer. Math. Soc.* **14** (2001), 471–486.

[2] J. S. Birman, *Braids, links, and mapping class groups*, Princeton Univ. Press, Princeton, NJ, 1974.

[3] E. Brieskorn, Sur les groupes de tresses [d'après V. I. Arnol'd], in *Séminaire Bourbaki, 24ème année* (1971/1972), Exp. 401, Lecture Notes in Math. **317**, Springer, Berlin, 1973, 21–44.

[4] K.-T. Chen, Iterated integrals of differential forms and loop space homology, *Ann. of Math.* (2) **97** (1973), 217–246.

[5] K.-T. Chen, Iterated path integrals, *Bull. Amer. Math. Soc.* **83** (1977), 831–879.

[6] Z. Chen, *Stratifications of complementary spaces of hyperplane arrangements and homology groups with local coefficients*, Thesis, the University of Tokyo, 2005.

[7] E. Date, M. Jimbo, A. Matsuo and T. Miwa, Hypergeometric-type integrals and the $\mathfrak{sl}(2, \mathbf{C})$ Knizhnik–Zamolodchikov equation, *Internat. J. Modern Phys. B* **4** (1990), 1049–1057.

[8] P. Deligne, P. Griffiths, J. Morgan and D. Sullivan, Real homotopy theory of Kähler manifolds, *Invent. Math.* **29** (1975), 245–274.

[9] V. G. Drinfel'd, Quasi-Hopf algebras, *Leningrad Math. J.* **1** (1990), 1419–1457.

[10] B. Feigin, V. Schechtman and A. Varchenko, On algebraic equations satisfied by hypergeometric correlators in WZW models. I, *Comm. Math. Phys.* **163** (1994), 173–184.

[11] V. G. Knizhnik and A. B. Zamolodchikov, Current algebra and Wess–Zumino model in two dimensions, *Nuclear Phys. B* **247** (1984), 83–103.

[12] T. Kohno, On the holonomy Lie algebra and the nilpotent completion of the fundamental group of the complement of hypersurfaces, *Nagoya Math. J.* **92** (1983), 21–37.

[13] T. Kohno, Série de Poincaré–Koszul associée aux groupes de tresses pures, *Invent. Math.* **82** (1985), 57–75.
[14] T. Kohno, Homology of a local system on the complement of hyperplanes, *Proc. Japan Acad. Ser. A Math. Sci.* **62** (1986), 144–147.
[15] T. Kohno, Monodromy representations of braid groups and Yang–Baxter equations, *Ann. Inst. Fourier* (*Grenoble*) **37** (1987), 139–160.
[16] T. Kohno, Bar complex of the Orlik-Solomon algebra, *Topology Appl.* **118** (2002), 147–157.
[17] T. Kohno, *Conformal field theory and topology*, Translations of Mathematical Monographs **210**, Amer. Math. Soc., Providence, RI, 2002.
[18] T. Kohno, Homological representations of braid groups and KZ connections, *J. Singul.* **5** (2012), 94–108.
[19] T. Kohno, Local systems on configuration spaces, KZ connections and conformal blocks, *Acta Math. Vietnam.* **39** (2014), 575–598.
[20] M. Kontsevich, Vassiliev's knot invariants, in *I. M. Gel'fand Seminar*, Adv. Soviet Math. **16**, Amer. Math. Soc., Providence, RI, 1993, 137–150.
[21] D. Krammer, Braid groups are linear, *Ann. of Math.* (2) **155** (2002), 131–156.
[22] I. Marin, *Représentations linéaires des tresses infinitésimales*, Thesis, Univ. Paris XI-Orsay, 2001.
[23] I. Marin, Sur les représentations de Krammer génériques, *Ann. Inst. Fourier* (*Grenoble*) **57** (2007), 1883–1925.
[24] J. Milnor, *Morse theory*, Ann. of Math. Studies **51**, Princeton Univ. Press, Princeton, NJ, 1963.
[25] J. W. Morgan, The algebraic topology of smooth algebraic varieties, *Inst. Hautes Études Sci. Publ. Math.* **48** (1978), 137–204.
[26] P. Orlik and H. Terao, *Arrangements of hyperplanes*, Grundlehren der Mathematischen Wissenschaften **300**, Springer, Berlin, 1992.
[27] P. Orlik and H. Terao, *Arrangements and hypergeometric integrals*, MSJ Memoirs **9**, Math. Soc. Japan, Tokyo, 2001.
[28] V. V. Schechtman and A. N. Varchenko, Hypergeometric solutions of Knizhnik–Zamolodchikov equations, *Lett. Math. Phys.* **20** (1990), 279–283.
[29] R. Silvotti, Local systems on the complement of hyperplanes and fusion rules in conformal field theory, *Internat. Math. Res. Notices* **1994**, 111 ff., approx. 17 pp. (electronic).
[30] D. Sullivan, Infinitesimal computations in topology, *Inst. Hautes Études Sci. Publ. Math.* **47** (1977), 269–331.
[31] A. Varchenko, *Multidimensional hypergeometric functions and representation theory of Lie algebras and quantum groups*, Advanced Series in Mathematical Physics **21**, World Sci. Publishing, River Edge, NJ, 1995.

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