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Harmonic measures and unique ergodicity of foliations

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Abstract.

In this article, we provide a short review of general results on harmonic measures on foliated manifolds and discuss briefly about uniquely ergodic foliations.

§1. Introduction

Harmonic measures for foliations of Riemannian manifolds have been introduced by Lucy Garnett [8], see also [6], [7] and [13]. If $(M, \mathcal{F})$ is a foliated Riemannian manifold, $\Delta_{\mathcal{F}}$ is the *foliated Laplace operator* on M (that is, $\Delta_{\mathcal{F}}$ acts on the space of C^2-functions on M and is defined by the formula

$$(1) \qquad \Delta_{\mathcal{F}} f(x) = \Delta_{L_x}(f|L_x)(x), \quad x \in M,$$

Δ_{L_x} being the Laplace operator on the leaf L_x of $\mathcal{F}$ through x equipped with the Riemannian metric induced from M) and μ is a Borel probability measure on M, then μ is said to be *harmonic* whenever

$$(2) \qquad \int_M \Delta_{\mathcal{F}} f \, d\mu = 0$$

for all $f \in \mathrm{C}^2(M)$.

Assume for all the article that M is compact. Then, harmonic measures exist, can be characterized as measures which are invariant under

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the operators $D_t^{\mathcal{F}}$ of *leaf-wise heat diffusion* and locally (in a foliated chart U) expressed as

$$\mu = \int_T h_t \cdot \mathrm{vol}_{P_t}\, d\sigma, \tag{3}$$

where T is a transversal, P_t ($t \in T$) is a plaque of the chart under consideration (see Figure 1) and h_t is a harmonic function on P_t.

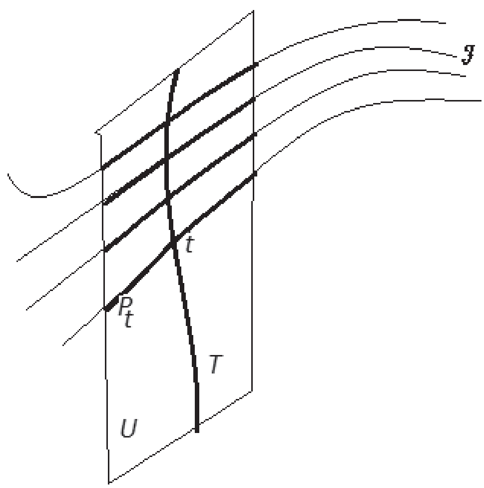

Fig. 1. A foliated chart.

All the harmonic measures on $(M, \mathcal{F})$ form a convex subset $\mathcal{H}(M, \mathcal{F})$ of the set $\mathcal{M}(M)$ of all the Borel probability measures on M. All the extreme points of $\mathcal{H}(M, \mathcal{F})$ form the set $\mathcal{E}(M, \mathcal{F})$ of *ergodic* measures. A foliation $\mathcal{F}$ is said to be *uniquely ergodic* whenever the set $\mathcal{H}(M, \mathcal{F})$ (equivalently, $\mathcal{E}(M, \mathcal{F})$) consists of a single element. Both, simple and interesting, examples of uniquely ergodic (in this sense) foliations do exist (see, for instance, [2] and Section 4 below).

In this article, we provide a short review of general results on harmonic measures on foliated manifolds and discuss briefly about uniquely ergodic foliations. With regard to the last topic, we are grateful to the referee for his critical remarks on the first version of the paper and bringing to our attention results of [10] mentioned here in the last section (Example 4.6).

§2. Existence

The simplest example of a harmonic measure is provided by so called *transverse invariant measures*, that is measures on transversals which are invariant under holonomy maps. If σ is such a transverse invariant measure on $(M, \mathcal{F})$, then one can define a harmonic measure μ by (3) takimg $h_t = 1$ for all $t \in T$ and normalizing the result to become a probablility meaure. There exist foliations which admit no non-trivial invariant measures: indeed, the existence of such measures has been shown to have some influence on the topology of $(M, \mathcal{F})$, see [11], [9] and [13], pp. 101–105. Contrary to that, harmonic measures on compact foliated manifolds do exist always.

The original Garnett's proof of existence of harmonic measures on compact foliated manifolds followed the lines of the classical proof of existence of invariant measures for continuous transformations of compact topological spaces (see [14], pp. 150–152). Roughly speaking, if $(D_t^{\mathcal{F}})$ is a family of leaf-wise diffusion operators on $(M, \mathcal{F})$ and (σ_n) is an arbitrary sequence of Borel probability measures on M, then the accumulation points of the sequence (μ_n) given by

$$\mu_n = \frac{1}{n} \cdot \sum_{n=0}^{n-1} D_n^{\mathcal{F}} \sigma_n, \tag{4}$$

where $D_t^{\mathcal{F}} \sigma$ ($\sigma \in \mathcal{M}(M)$, $t \geq 0$) denotes the measure given by

$$\int_M f \, dD_t^{\mathcal{F}} \sigma = \int_M D_t^{\mathcal{F}} f \, d\sigma \quad f \in C(M), \tag{5}$$

are $D_t^{\mathcal{F}}$-invariant, therefore harmonic.

Here, we shall present and apply another proof, the one which can be found in [13].

First, let us consider a compact Hausdorff space X, the linear space $V = C(X)$ of all the continuous real functions on X and a linear subspace W of V. Denote by $\mathcal{M}(X)$ the space of all Borel probability measures on X and by $\mathcal{M}_W(X)$ the subspace of $\mathcal{M}(X)$ consisting of all the measures μ such that

$$\int_X f \, d\mu = 0 \tag{6}$$

for all the elements f of W.

Then, the following holds.

Proposition 2.1. *The following conditions are equivalent*:

(i) *any function $f \in W$ is non-positive at some points of X*,

(ii) $\mathcal{M}_W(X) \neq \emptyset$.

Proof. Implication (ii) $\Longrightarrow$ (i) is obvious: if $W \ni f > 0$ everywhere on X, then $\int_X f d\mu > 0$ for all the Borel probability measures μ on X.

Assume now that W satisfies (i) and let $l(f) = 0$ for all $f \in W$ and $q(h) = \max(-h)$ for all $h \in V$. The functional $q\colon V \to \mathbb{R}$ is semi-linear in the following sense:

$$q(h_1 + h_2) \leq q(h_1) + q(h_2)$$

and

$$q(ah) = aq(h)$$

for all $h, h_1, h_2 \in V$ and $a \in \mathbb{R}$, $a \geq 0$. Obviously, $l(f) \leq q(f)$ for all $f \in W$. Therefore, by the classical Hahn–Banach theorem, there exists a linear extension $L\colon V \to \mathbb{R}$ of l such that $L(h) \leq q(h)$ for all $h \in V$. The functional $-L$ is positive (i.e. $-L(h) > 0$ whenever $h > 0$) and therefore, by the Riesz representation theorem ([12], Theorem 2.14), corresponds to a Borel measure ν on X. The normalized measure $\mu = -\nu/L(1)$ satisfies (6). Q.E.D.

Now, if $(M, \mathcal{F})$ is a compact foliated Riemannian manifold and f is a smooth function on M, then f attains its maximum at some point x and $\Delta_{\mathcal{F}} f(x) \leq 0$. Therefore, the space $W = \{\Delta_{\mathcal{F}} f, f \in C^2(M)\}$ satisfies condition (i), therefore also (ii), of Proposition 2.1 which yields the following.

Theorem 2.2. *On an arbitrary compact foliated Riemannian manifold $(M, \mathcal{F})$ the spaces $\mathcal{H}(M, \mathcal{F})$ and $\mathcal{E}(M, \mathcal{F})$ are nonempty.*

The same argument applies to minimal subsets of foliated manifolds.

Let us recall that a subset A of a foliated manifold $(M, \mathcal{F})$ is (1) *saturated* whenever A is a union of leaves, that is whenever any leaf of $\mathcal{F}$ which intersects A is entirely contained in A, and (2) *minimal* whenever it is nonempty, closed and saturated, and contains no proper subsets enjoying all these properties. The Zorn Lemma implies immediately that the closure of any leaf of any foliation of a closed manifold M contains a minimal set. Closed leaves themselves always constitute minimal sets.

Let us recall also that the *support* of a measure μ on a topological space X is the collection $\operatorname{supp}\mu$ of all the points $x \in X$ such that $\mu(U) > 0$ for all open neighbourhoods U of x.

Since any leaf-wise smooth and continuous function f defined on a minimal set A attains its maximum at some points of A and has non-positive leaf Laplacian there, our Proposition 2.1 implies also the following.

Corollary 2.3. *For any minimal subset set A of a closed foliated manifold $(M, \mathcal{F})$, there exists a harmonic measure on M supported in A.*

Remark 2.4. (1) Proposition 2.1 implies also existence of invariant measures for continuous transformations F of compact topological spaces X. Indeed, the space $W = \{f - f \circ F; f \in C(X)\}$ satisfies condition (i) therein: $f(x_0) - f(F(x_0)) \leq 0$ at points x_0 where f attains its minimum.

(2) Since finitely generated abelian groups of continuous transformations of a compact space X admit invariant measures (see, for instance, [13], p. 98), Proposition 2.1 implies also the following, surprising the author a bit, fact: *if X is compact, $F_i\colon X \to X$, $i = 1, \ldots, n$, are continuous and pairwise commuting, and $f_i\colon X \to \mathbb{R}$, $i = 1, \ldots, n$, are continuous as well, then there exist points x and y of X such that*

$$\sum_{i=1}^{n}(f_i - f_i \circ F_i)(x) \leq 0 \quad \text{and} \quad \sum_{i=1}^{n}(f_i - f_i \circ F_i)(y) \geq 0. \tag{7}$$

Problem 2.5. In the situation described in Remark 2.4 (2), prove the existence of points x and y satisfying (7) using just the methods of set-theoretic topology.

Finally, let us recall that if μ is a probability measure on a group G acting on a compact space X, then a probability measure ν on X is said to be *μ-stationary* whenever $\nu = \int_G g^*\nu \, d\mu$, that is whenever

$$\int_X \left(f - \int_G (f \circ g) \, d\mu \right) d\nu = 0$$

for all $f \in C(X)$. Again, at a point x_0 of X where f attains its minimum one has

$$f(x_0) - \int_G f(g(x_0)) \, d\mu \leq 0,$$

that is the subspace W of $C(X)$ generated by all the functions of the form $f - \int_G (f \circ g) \, d\mu$ satisfies condition (i) of Proposition 2.1. This proves the following.

Corollary 2.6. *For any G, μ and X as above, μ-stationary measures on X do exist.*

§3. Selected properties

As mentioned in Introduction, harmonic measures μ are invariant with respect to the operators $D_t^{\mathcal{F}}$ of heat diffusion along the leaves. This means that for any $f\colon M \to \mathbb{R}$ one has

$$\int_M D_t^{\mathcal{F}} f \, d\mu = \int_M f \, d\mu. \tag{8}$$

This implies the following.

Proposition 3.1. *The support* $\operatorname{supp}\mu$ *of any harmonic measure* μ *on a closed foliated manifold* $(M, \mathcal{F})$ *is saturated.*

Proof. Let x be a point of $\operatorname{supp}\mu$ and y another point of the leaf L_x through x. Let U be an open neighbourhood of y and f a smooth non-negative real function on M such that $f(y) > 0$ and $\operatorname{supp} f \subset U$. Then, for any $t > 0$,

$$\int_U f d\mu = \int_M f d\mu = \int_M D_t^{\mathcal{F}} f d\mu > 0$$

since $D_t^{\mathcal{F}} f \geq 0$ everywhere on M and $D_t^{\mathcal{F}} f(x) > 0$. This implies that $\mu(U) > 0$ and $y \in \operatorname{supp}\mu$. Q.E.D.

One has the following Lebesgue decomposition for harmonic measures.

Theorem 3.2. *If* μ *and* ν *are two harmonic probability measures on* $(M, \mathcal{F})$, *then* ν *can be decomposed as* $\nu = \nu_1 + \nu_2$, *where* ν_1 *and* ν_2 *are harmonic,* ν_1 *is absolutely continuous with respect to* μ *while* μ *and* ν_2 *are mutually singular. Moreover,* ν_1 *and* ν_2 *are concentrated on disjoint saturated sets.*

Proof. Roughly speaking, $\nu_2 = \nu|\tilde{A}$ and $\nu_1 = \nu|M \setminus \tilde{A}$ where $\tilde{A}$ is the *essential saturation* (that is, the union of all the leaves L such that the leaf volume of $L \cap A$ is strictly positive) of a subset A of M such that $\mu(A) = 0$ and $\nu|M \setminus A$ is absolutely continuous with respect to μ. The existence of such an A follows from the standard Lebesgue decomposition theorem ([12], Theorem 6.9) while more details of this proof can be found in [13]. Q.E.D.

Moreover, if we call a leaf L of $\mathcal{F}$ *eventually wandering* if it is non-compact and proper (that is, the manifold topology of L coincides with that induced from M) and we denote by $EW(\mathcal{F})$ the union of all eventually wandering leaves of $\mathcal{F}$. Certainly, the set $EW(\mathcal{F})$ and its complement are saturated. With this notation, we have the following.

Theorem 3.3. *For any harmonic measure μ on $(M, \mathcal{F})$ one has*

$$\mu(EW(\mathcal{F})) = 0. \tag{9}$$

In other words, this means that *any harmonic measure is supported in the union of all eventually non-wandering leaves.*

The proof of Theorem 3.3 is rather technical and can be found, for example, in [13].

§4. Unique ergodicity

First, let us observe that our Corollary 2.3 implies directly the following.

Proposition 4.1. *Any uniquely ergodic foliation $\mathcal{F}$ on a closed manifold M contains exactly one minimal set, in particular, at most one compact leaf.*

Proof. Indeed, if not, M would contain two minimal sets A_i, $i = 1, 2$, each of them supporting a harmonic measure μ_i. Certainly $\mu_1 \neq \mu_2$. Q.E.D.

Example 4.2. If a closed foliated manifold $(M, \mathcal{F})$ admits a unique minimal set, this set consists of a single closed leaf L and all the other leaves are proper (and non-compact, therefore eventually wandering), then $\mathcal{F}$ is uniquely ergodic (with respect to any Riemannian structure on M). Indeed, by Theorem 3.3, any ergodic harmonic measure on $(M, \mathcal{F})$ has to be supported in L, by Theorem 3.2 it has to be absolutely continuous with respect to the leaf volume on L and—since the only harmonic functions on closed Riemannian manifolds are constant—any harmonic measure on any closed Riemannian manifold coincides with the normalized volume. In particular, the standard Reeb foliation of S^3 is uniquely ergodic and its unique harmonic measure coincides with the normalized volume of the toral leaf; indeed, the toral leaf is the only eventually non-wandering leaf of the Reeb foliation.

Example 4.3. Assume that B is a closed Riemannian manifold of negative curvature and $\rho \colon \Gamma = \pi_1(B) \to \mathrm{PSL}(d, \mathbb{C})$ is a group homomorphism. Assume also that ρ is (i) *strongly irreducible*, that is no finite family of projective subspaces of $\mathbb{C}P^{d-1}$ is ρ-invariant, and (ii) *contracting*, that is for any probability measure μ on $\mathbb{C}P^{d-1}$ there exists a sequence (g_n) of elements of Γ for which the measures $\rho(\gamma_n) * \mu$ converge to a Dirac mass. Then the foliation $\mathcal{F}$ obtained by suspension of ρ (that is, $\mathcal{F}$ is the foliation of $M = (\tilde{B} \times \mathbb{C}P^{d-1})/\Gamma$—where Γ acts on $\tilde{B}$ *via* covering transformation and on $\mathbb{C}P^{d-1}$ *via* ρ—obtained by the natural projection of the product foliation $\tilde{\mathcal{F}} = \{\tilde{B} \times \{*\}\}$ of the product of the universal cover $\tilde{B}$ of B by $\mathbb{C}P^{d-1}$) is uniquely ergodic ([2], Theorem 3.2).

Finally, we shall discuss briefly about a relation between the unique ergodicity of a foliation $\mathcal{F}$ and the image of the Laplace operator $\Delta_{\mathcal{F}}$. To this end, let us consider first a compact space X and a subspace W of $C(X)$ which satisfies condition (i) of Proposition 2.1. Such a subspace W will be called *uniquely ergodic* whenever the set $\mathcal{M}_W(X)$ consists of a unique element. With this terminology we shall prove the following.

Proposition 4.4. *If $C(X) = W \oplus \mathbb{R}$, then the subspace W is uniquely ergodic.*

Proof. This is rather obvious: if $C(X) = W \oplus \mathbb{R}$, then (according to the Riesz representation theorem again) the conditions $\int f \, d\mu = 0$ for $f \in W$ and $\int c \, d\mu = c$ for $c \in \mathbb{R}$ define a unique element μ of $\mathcal{M}_W(X)$. Q.E.D.

Applying Proposition 4.4 to the space $W = \operatorname{im} \Delta_{\mathcal{F}}$ we obtain directly the following.

Corollary 4.5. *If $C(M) = \operatorname{im} \Delta_{\mathcal{F}} \oplus \mathbb{R}$, then a foliation $\mathcal{F}$ of a compact Riemannian manifold M is uniquely ergodic.*

One can ask how about the converse implications in Proposition 4.4 and Corollary 4.5.

First, observe that our Proposition 4.4 can be applied also to classical dynamical systems to show that a continuous transformation T of a compact space X is uniquely ergodic if only $C(X) = W_T \oplus \mathbb{R}$, where W_T is a linear subspace of $C(X)$ generated by all the functions $f - f \circ T$ with $f \in C(X)$.

Example 4.6. Given $\alpha \in \mathbb{R}$, denote by $R_\alpha \colon S^1 \to S^1$ the rotation by angle $\alpha \cdot \pi$. By Proposition 12.6.3 in [10], there exists an irrational number α and a smooth (even, analytic) function $\phi \colon S^1 \to \mathbb{R}$ such that $\phi = \Phi \circ R_\alpha - \Phi$ for some "highly discontinuous" function $\Phi \colon S^1 \to \mathbb{R}$. By "highly discontinuous" we mean here "Borel measurable and such that $\lambda(U \cap \Phi^{-1}(V)) > 0$ for all non-empty open sets $U \subset S^1$ and $V \subset \mathbb{R}$", λ being the Lebesgue measure on S^1.

Now, if $\phi = \Psi \circ R_\alpha - \Psi + c$ for some $c \in \mathbb{R}$ and a continuous function $\Psi \colon S^1 \to \mathbb{R}$, then the difference $F = \Phi - \Psi$ would satisfy $F \circ R_\alpha - F = c$ everywhere on S^1. Since R_α is ergodic and Ψ is continuous, $c \neq 0$, say $c > 0$. Since F is bounded on a set A of positive measure and $F(R_\alpha^n(x)) = F(x) + nc \to \infty$ as $n \to \infty$ for all $x \in S^1$, we arrive at a contradiction with the classical Poincaré Recurrence Theorem applied to A and R_α.

Therefore, $C(S^1) \neq W_{R_\alpha} \oplus \mathbb{R}$ for this particular value of α and the converse to Proposition 4.4 does not hold in full generality.

Let us also observe that any single closed Riemannian manifold M (which can be considered as a foliated one in codimension zero) is uniquely ergodic: the only harmonic probability measure is the normalized volume. Also, since the equation $\Delta_M u = f$ has a solution u if and only if $\int_M f = 0$ (see, for instance, [3], Theorem 4.7), $C(M) = \operatorname{im} \Delta_M \oplus \mathbb{R}$.

The above suggests that one can ask whether the converse to Corollary 4.5 holds either in general or under some conditions (and try to find them). One can also ask whether the converse to Proposition 4.4 holds for all (or those satisfying suitable—which ones?—conditions) closed subspaces W of $C(X)$, X being, as before, compact. At the moment, we do not know any reasonable answer to these questions. It seems that working towards the affirmative answer one could follow the lines of the classical proofs of the Hahn-Banach Theorem, see—for example—[1].

Finally, let us mention that recently an interesting definition of the Laplace operator on Finsler manifolds appeared in [4] (see also [5]). With this definition one should be able to develop a theory of harmonic measures, diffusion operators and Brownian motion, and discuss the problem of unique ergodicity for foliations of manifolds eqipped with (leafwise) Finsler structures.

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