Advanced Studies in Pure Mathematics 8, 1986
Complex Analytic Singularities
pp. 95–99

Topological Restrictions on the Links of Isolated Complex Singularities

Alan H. Durfee[1)]

This note is a preliminary report on joint work with Richard Hain; the full details will appear shortly [4]. We will sketch a proof of the following result:

Theorem 1. *Let L be the link of an isolated singular point of a complex algebraic variety X of dimension n. If s, $t<n$ and $s+t\geq n$, then* the *cup product*

$$H^s(L, Q)\otimes H^t(L, Q)\longrightarrow H^{s+t}(L, Q)$$

vanishes.

Recall that the *link* L of a singular point is by definition the intersection of the variety with a small sphere about that point. If the variety is n-dimensional and the singularity is isolated, then L is a real $(2n-1)$-manifold.

This theorem shows that there are restrictions on the topology of isolated singular points of complex varieties. For example, any manifold of the form $K\times M\times N$ where $\dim K<n$, $\dim M<n$, $\dim K+\dim M\geq n$ and $\dim K+\dim M+\dim N=2n-1$ cannot occur as such a link. The case $n=2$ of this theorem was shown in [14], where the Grauert contraction theorem was used. Apparently this is the first known such restriction in higher dimensions. In fact, this result gives a necessary condition for contraction. For example, $S^2\times S^2\times S^3$ cannot occur as the link of an isolated singular point, but it occurs as the link of $P^1\times P^1$ in $P^1\times P^1\times C^2$. Hence this set cannot be contracted to a point. The techniques used in this paper are similar to those used by Morgan to find an example of a homotopy type which cannot occur as a smooth algebraic variety. On the other hand, for any finite simplicial complex there is a quasi-projective variety (non-compact, with singularities) of the same homotopy type [1, Section 9].

Received October 20, 1984.
Revised December 10, 1985.
1) Research partially supported by the Max Planck Institut für Mathematik, Bonn, Germany.

Theorem 1 will follow immediately from the next three results.

Proposition 1. *Let L be as above. Then $H^m(L)$, for all m, has a weight filtration*

$$0=W_{-1}\subset W_0\subset\cdots\subset W_{2m}=H^m(L).$$

Cohomology with complex coefficients is all that is needed here, although the filtration is actually defined on cohomology with rational coefficients.

We shall see that this proposition, as well as the following two propositions, each have two proofs, one using mixed Hodge theory and the other by reducing modulo p and using the results of the Weil conjectures. (This is the usual situation for proofs of this type.) For example, this proposition follows from the existence of a mixed Hodge structure on the local cohomology of a point in a variety [2, Hodge III, 8.3.8]. A component of this mixed Hodge structure is a functorial weight filtration W on the cohomology groups $H^m(L; Q)$, for all m. The fact that the weights go from 0 to $2m$ is easily derived from the construction. More generally, a mixed Hodge structure on a punctured neighborhood of one variety in another has been constructed by several authors [3, 6, 8].

Proposition 1 can also be proved by reducing modulo p; the weight filtration is then defined by eigenspaces of the Frobenius action. This type of argument is well known, and is sketched for the case of quasiprojective varieties in [1]. A proof of Proposition 1 using these techniques is given in [5].

Proposition 2. *Let L be as above. Then*

$$W_mH^m(L)=H^m(L) \qquad \text{for } m<n, \quad \text{and}$$
$$W_mH^m(L)=0 \qquad \text{for } m\geq n.$$

This says that there are restrictions on the weight filtration: for $m<n$, the group $H^m(L)$ can only have classes of weight 0 through m, and for $m\geq n$, the group $H^m(L)$ can only have classes of weight $m+1$ through $2m$. This result is only true for links of isolated singular points, unlike Proposition 1 which is true more generally for the punctured neighborhood of one variety in another.

Again there are two ways of proving Proposition 2. The first (and original way) uses the decomposition theorem in intersection homology, which at this moment only has a Weil-conjecture-type proof [3, 12, 7]. The second proof uses only Hodge theory, and specifically, the Strong Lefschetz Theorem [13, 11].

Proposition 3. *The weight filtration on $H^{\cdot}(L)$ in Proposition* 1 *is preserved by cup products: for all s, t,*

$$W_s \cup W_t \subset W_{s+t}.$$

Once again this proposition has two proofs. The Weil-conjecture proof is simply to notice that the weight filtration, which is defined by the eigenspaces of the Frobenius action, is multiplicative. The Hodge-theoretic proof is one of the principal results in [4]. We sketch this next.

A mixed Hodge structure on the cohomology of a space X is simply two filtrations, the weight filtration W and the Hodge filtration F, satisfying various properties. (For the rest of this discussion we will work over the complex numbers to simplify matters. The weight filtration should be defined over the real or rational numbers.) This mixed Hodge structure is usually the cohomology of a *mixed Hodge complex*, that is, a chain complex K with two filtrations W and F. For example, if X is a smooth quasiprojective variety with compactification $\bar{X}$ such that $\bar{X}-X=D$ is a divisor with normal crossings, then K is the complex $E(\bar{X}\log D)$ of smooth forms on $\bar{X}$ with logarithmic singularities along D, with its weight and Hodge filtrations defined as usual.

To prove Proposition 3, over needs (roughly speaking) to find a mixed Hodge complex with a product which preserves these filtrations and induces the cup product in cohomology. Such a complex is called a *multiplicative mixed Hodge complex*. For example, the complex $E(\bar{X}\log D)$ with its wedge product is a multiplicative mixed Hodge complex for X.

Unfortunately, all the mixed Hodge complexes used to prove Proposition 1 do not have a suitable wedge product. Hence it is necessary to find another mixed Hodge complex which does. The following construction works in the more general situation of the link of one projective variety in another, and even for the complement of one link in another. Without loss of generality, we may replace the variety with its isolated singular point by a pair (X, D), where X is a smooth projective variety and D is a divisor with normal crossings. Thus L is the boundary of a neighborhood of D in X. If D has only one irreducible component, the complex $E(X\log D)$ localized along D is a multiplicative mixed Hodge complex for L; "localizing" here means dividing by the ideal of smooth forms vanishing on D. If E is another divisor, the complex $E(X\log D\cup E)$ localized along D is similarily a mixed Hodge complex for the complement $L-(L\cap E)$.

When $D=D_1\cup\cdots\cup D_r$ has more than one irreducible component, one proceeds as follows: Let T_i be a neighborhood of D_i, and let $T_i^*=T_i-(D\cap T_i)$. If the neighborhoods are chosen correctly, $T=\bigcup T_i$ is a neighborhood of D and $L=\bigcup T_i^*$. The complex $E(X\log D\cup E)$ localized along D has cohomology isomorphic to $H^*(T_i^*)$. One then combines these complexes by taking "compatible forms" in the sense of Thom or Sullivan. This then gives a multiplicative mixed Hodge complex whose cohomology is isomorphic to the cohomology of L. A similar procedure is used in [9] for quasiprojective varieties. (This construction was also discovered independently by V. Navarro.)

In addition, more than just the property of the cup product (as in Proposition 3) is true:

If L is a link as above, then the real homotopy type of L has a mixed Hodge structure. In particular, the Malcev Lie algebra of the fundamental group of L has a mixed Hodge structure, and when L is simply-connected, the higher homotopy groups of L have a mixed Hodge structure.

Typical corollaries are some of the following:

—Let L be as above. Then Poincaré duality

$$H_c^k(L) \longrightarrow H_{2n-1-k}(L)$$

is an isomorphism of mixed Hodge structures of type $(-n, -n)$. Similarly, Alexander duality is also an isomorphism of mixed Hodge structures.

—Let L be the complement of an "algebraic" link in the three-sphere. Then L is a formal space (in the sense of Sullivan).

—There is a smooth simply-connected 11-manifold L with the rational cohomology ring of $2(S^2 \times S^9)\#(S^5 \times S^6)\#2(S^4 \times S^7)$ which cannot occur as the link of an isolated singular point of a 6-dimensional variety. (Even though the cohomology ring of L could occur, the rational homotopy type cannot.)

Bibliography

[1] P. Deligne, Poids dans la cohomologie des variétés algébriques, Actes ICM, Vanouver 1974, 79–85.

[2] ——, Théorie de Hodge II and III, Publ. Math. IHES, **40** (1971), 5–57, and **44** (1975), 5–77.

[3] A. Durfee, Mixed Hodge structures on punctured neighborhoods, Duke Math. J., **50** (1983), 1017–1040.

[4] A. Durfee, and R. Hain, Mixed Hodge structures on the homotopy of links preprint, October 1985.

[5] A. Durfee, The l-adic cohomology of links, preprint, Barcelona conference on Hodge theory, 1985.

[6] F. Elzein, Mixed Hodge structures, Trans. Amer. Math. Soc., **275** (1983), 71–106.

[7] ——, Mixed Hodge structures, Proc. Sym. Pure Math., **40** Part I (1983), 345–352.

[8] F. Guillen, V. Aznar and F. Puerta, Théorie de Hodge via schemas cubiques, preprint, June 1982, Universidad Politecnica de Barcelona.

[9] R. Hain, The de Rham homotopy theory of complex algebraic varieties, preprint, The Institute for Advanced study, 1986.
[10] J. Morgan, The algebraic topology of smooth algebraic varieties, Publ. Math. IHES, **48** (1978), 137–204.
[11] V. Navarro, Asterisque 130, to appear.
[12] J. Steenbrink, Mixed Hodge structures associated with isolated singularities, Proc. Sym. Pure Math., **40** Part 2 (1983), 513–536.
[13] ——, Mixed Hodge structures and singularities, Asterisque, to appear.
[14] D. Sullivan, On the intersection ring of compact three manifolds, Topology, **14** (1975), 275–277.

Department of Mathematics
Mount Holyoke College
South Hadley, Massachusetts 01075
U.S.A.