

The Fusion Matrix and the Verlinde Loop Operators in Conformal Field Theory

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*to Professor Masatake Kuranishi
on the occasion of his seventieth birthday*

§1. INTRODUCTION.

Two-dimensional conformal field theories are at the center of some of the most promising recent developments in high-energy and statistical physics [1]. At the same time, their mathematical structure has revealed itself to be quite rich [2-4]. In particular, the Verlinde dimension formulae for Wess-Zumino-Witten models have been studied extensively in the mathematical literature, especially from the algebraic-geometric [5], as well as the differential-geometric viewpoint [6].

The Verlinde dimension formulae are only one consequence of a deeper phenomenon, namely the Verlinde fusion formula, which links the fusion rules of a conformal field theory to the matrix S realizing the fundamental modular transformation $\tau \rightarrow -1/\tau$ on the characters $\chi^i(e^{2\pi i\tau})$ of the theory. The Verlinde fusion formula was originally conjectured by Verlinde [7], and subsequently proven by Moore and Seiberg [3]. Other arguments were also given by Dijkgraaf and Verlinde [8], Witten [9], and Cardy [10]. All these proofs have however remained largely inaccessible to a broader audience, partly due to their reliance on diagrammatic manipulations, and partly due to the subtleties inherent to the fusion procedure in a conformal field theory.

Given the importance of the subject, it may be useful to have a simple and precise exposition of the mathematical structure inherent to the Verlinde fusion formula. This is the goal of this paper. The key

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tool is still a set of identities for the fusion matrix, among which is the fundamental pentagon identity due to Moore and Seiberg. The nature of some of these identities was however sometimes obscure, since they may or may not be gauge-dependent, and can require delicate limiting processes in complicated compositions of chiral vertex operators. We clarify these issues, and also take the opportunity to provide a gauge-independent formula for the Verlinde loop operators.

§2. THE FUSION AND THE BRAIDING MATRICES

In this section, we introduce the algebraic structure relevant to the fusion matrix and to the Verlinde holonomy operators in a rational conformal field theory.

1. Chiral algebras

In a two-dimensional conformal field theory, conformal invariance means that the Hilbert space of states carries a representation of the Virasoro algebra. The Virasoro algebra is the central extension by \mathbb{C} of the Lie algebra of formal meromorphic vector fields on \mathbb{C}^* with poles at the origin, and Lie bracket given by

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}. \quad (1)$$

Here the generators $L_n = -z^{n+1}\partial_z$, $n \in \mathbb{Z}$, form a basis for the space of formal meromorphic vector fields, and the constant c is one of the defining parameters of the theory, called its *central charge*. More generally, let \mathcal{A} be a possibly larger *chiral algebra*, i.e., an algebra whose enveloping algebra contains the Virasoro algebra. We say that the left sector of the theory is rational and invariant under \mathcal{A} if

(i) *Its Hilbert space of states decomposes as a finite sum of tensor products of Hilbert spaces*

$$H = \bigoplus_{i \in I, \tilde{i} \in \tilde{I}} H_i \otimes \tilde{H}_{\tilde{i}} \quad (2)$$

with the following properties:

- Each H_i is an irreducible representation of \mathcal{A} ;
- Each H_i decomposes in turn as

$$H_i = \bigoplus_{d=0}^{\infty} H_i^d \quad (3)$$

where the H_i^d 's are finite-dimensional orthogonal eigenspaces of the operator L_0 , with eigenvalue $h_i + d$, $d \in \mathbb{N}$, and $h_i \geq 0$. The lowest eigenvalue h_i is often also called the conformal dimension of the representation H_i ;

- There is a unique space H_0 , with $h_0 = 0$. The eigenspace H_0^0 is one-dimensional and generated by a unit vector Ω , which we shall refer to as the (chiral) vacuum. The chiral vacuum Ω is $SL(2, \mathbb{C})$ invariant, in the sense that it is annihilated by the canonical $SL(2, \mathbb{C})$ subalgebra of the Virasoro algebra

$$L_n \Omega = 0, \quad n = 0, \pm 1. \tag{4}$$

It is useful to keep in mind some of the simplest examples of rational conformal field theories, namely the unitary minimal models and the $SU_K(2)$ Wess-Zumino-Witten models. In the first case, the central charge c is given by $c = 1 - 6[m(m + 1)]^{-1}$, the chiral algebra is just the Virasoro algebra, and I consists of $m(m - 1)/2$ chiral fields, indexed by pairs (p, q) of positive integers with $1 \leq q \leq p \leq m - 1$. The corresponding L_0 eigenvalues $h_{p,q}$ are given by $h_{p,q} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}$. In the second case, the central charge is $c = \frac{3K}{K+2}$, and the chiral algebra \mathcal{A} is the Kac-Moody algebra $SU_K(2)$ with canonical generators J_m^a satisfying the relation

$$[J_m^a, J_m^b] = i\epsilon^{abc} J_{m+n}^c + Km\delta_{m+n,0}\delta^{ab}.$$

The primary chiral fields in the theory correspond now to the integrable representations of the loop group. They are indexed by their spins $\frac{j}{2}$, which are required to satisfy $0 \leq j \leq K$. Their conformal dimensions are $h_j = \frac{j(j+2)}{4(K+2)}$. The enveloping algebra of $SU_K(2)$ contains the Virasoro algebra, in view of the classical Sugawara construction $L_n = \frac{1}{2(K+2)} \sum_{m,a} : J_{n-m}^a J_m^a :$. Here $::$ denotes normal ordering, that is, the generators J_n^a are indexed so that the annihilators correspond to n strictly positive, and are set to the right in the product.

2. Chiral vertices

The fundamental notion in the operator formalism for conformal field theory is that of *chiral vertex operators*, or simply *chiral vertices*. A chiral vertex operator φ_{jk}^i is a holomorphic function $\varphi_{jk}^i(z)$ on \mathbb{C}^* , valued in the space of complex homomorphisms

$$\varphi_{jk}^i(z) : H_j \otimes H_k \rightarrow H_i$$

which satisfies the *conformal Ward identities* in the sense that

$$\begin{aligned} L_n \varphi_{jk}^i(z)(\xi_j \otimes \xi_k) &= \varphi_{jk}^i(z)(\Delta_z(L_n)(\xi_j \otimes \xi_k)) \\ \frac{\partial}{\partial z} \varphi_{jk}^i(z)(\xi_j \otimes \xi_k) &= \varphi_{jk}^i(z)(L_{-1}\xi_j \otimes \xi_k) \end{aligned} \tag{5}$$

where Δ_z is the imbedding of the Virasoro algebra into its tensor product with itself given by

$$\Delta_z(L_n) = \sum_{k=-1}^{\infty} \binom{n+1}{k+1} z^{n-k} L_k \otimes 1 + 1 \otimes L_n. \tag{6}$$

The Ward identities with respect to a larger chiral algebra \mathcal{A} can be formulated in the same way. In fact, the Virasoro generators L_n can be viewed as the Fourier-Laurent expansion of a holomorphic tensor $T(z) = \sum_{n=-\infty}^{\infty} L_n z^{-n-2}$, and the Virasoro commutation relations can be shown to be equivalent to the fact that $T(z)$ transforms as a projective connection, i.e., as a rank two tensor with an additional Schwarzian term. Similarly, if we assume that the chiral algebra \mathcal{A} is generated by the Fourier-Laurent coefficients J_n^a

$$J_n^a = \frac{1}{2\pi i} \oint_{|z|=1} dz z^{h_a+n-1} J^a(z)$$

of a finite number of *local currents* $J^a(z)$, $a = 1, \dots, N$, which transform as tensors of rank h_a under local conformal transformations, then the Ward identities with respect to \mathcal{A} are given simply by (5-6), and the conformal dimension 2 of $T(z)$ replaced now by h_a

$$\Delta_z(J_n^a) = \sum_{k=1-h_a}^{\infty} \binom{n+h_a-1}{k+h_a-1} z^{n-k} J_k^a \otimes 1 + 1 \otimes J_n^a.$$

As usual, we assume that $L_n^\dagger = L_{-n}$ and, in the case of a chiral algebra, $(J_n^a)^\dagger = J_{-n}^a$.

Strictly speaking, we require $\varphi_{jk}^i(z)$ to be defined only on the subspaces $H_i^m \otimes H_k^n$ for all m, n (and in particular over all *finite* linear combinations of such states). It is well-known in quantum field theory that this restriction cannot be removed, since it is at the root of the fundamental *operator product expansions* which encode the singularities and other short-distance properties of the composition of vertex operators. In our context, these issues underlie the properties of braiding and fusion to be discussed later.

It is often convenient, in order to avoid a proliferation of parentheses, to use also the notations $\varphi_{jk}^i(z; \xi_j)\xi_k$ and $\varphi_{jk}^i(z; \xi_j \otimes \xi_k)$ for $\varphi_{jk}^i(z)(\xi_j \otimes \xi_k)$.

In practice, we shall rely mostly on the following consequences of the above Ward identities. For any $\xi \in H_j$, the chiral vertex operator $\varphi_{jk}^i(z)$ leads to an operator $\varphi_{jk}^i(z; \xi)$ from H_k to H_i given by

$$\varphi_{jk}^i(z; \xi)\xi_k = \varphi_{jk}^i(z)(\xi \otimes \xi_k).$$

Consider the case when ξ is a *Virasoro primary state*, i.e., a state satisfying

$$L_n \xi = 0, \quad n \geq 1, \tag{7}$$

(observe that any state in H_j^0 is a Virasoro primary state, since otherwise $L_n \xi$, $n > 0$, would be a state in H_j with lower L_0 eigenvalue than h_j). The conformal Ward identities imply then the following commutation relation

$$[L_n, \varphi_{jk}^i(z; \xi)] = (z^{n+1} \frac{\partial}{\partial z} + z^n(n+1)h_j)\varphi_{jk}^i(z; \xi). \tag{8}$$

By setting $n = 0$, we can determine this way the complete dependence of $\varphi_{jk}^i(z)$ with respect to z

$$\varphi_{jk}^i(z) = z^{L_0} \varphi_{jk}^i(1) z^{-L_0} \otimes z^{-L_0}. \tag{9}$$

In presence of a larger chiral algebra \mathcal{A} , we can define analogously *chiral primary states* as states annihilated by J_n^α for all $n \geq 1$. For $\xi \in H_j$ chiral primary, the Ward identities become

$$[J_n^\alpha, \varphi_{jk}^i(z; \xi)] = z^n \varphi_{jk}^i(z; J_0^\alpha \xi). \tag{10}$$

The space of chiral vertex operators of type $\varphi_{jk}^i(z)$ will be denoted V_{jk}^i . We assume that it is finite-dimensional and set

$$\dim V_{jk}^i = N_{jk}^i.$$

We can now state our assumptions on N_{jk}^i in a rational conformal field theory:

(ii) *there exists an involution $j \rightarrow j^*$ such that 0 is self-conjugate, and $N_{jk}^0 = \delta_{j^*k}$. Furthermore, for any chiral vertex $\varphi_{jj^*}^0(z)$ in $V_{jj^*}^0$, the bilinear form on $H_j^0 \otimes H_{j^*}^0$ given by*

$$\xi_j \otimes \xi_{j^*} \rightarrow \langle \Omega | \varphi_{jj^*}^0(z)(\xi_j \otimes \xi_{j^*}) \rangle_{|z=1} \tag{11}$$

is non-degenerate. It is symmetric when j is self-conjugate;

We note that it follows from the Ward identities (5) and the irreducibility of H_j with respect to the chiral algebra that $\lim_{z \rightarrow 0} \varphi_{j0}^j(z; \xi \otimes \Omega)$ must be proportional to ξ (c.f. [4]). Henceforth, we adopt the following standard normalization for $\varphi_{j0}^j(z)$

$$\lim_{z \rightarrow 0} \varphi_{j0}^j(z; \xi \otimes \Omega) = \xi \tag{12}$$

4. Fusion and braiding of chiral vertices

We begin by discussing the composition of chiral vertex operators. Already formally, a chiral vertex operator $\varphi_{pq}^l(w)$ can be composed (on the left) with a chiral vertex operator $\varphi_{jk}^i(z)$ only if l is either k or j , in which case we have the two possibilities

$$\begin{aligned} \varphi_{jk}^i(z)\varphi_{pq}^k(w) &: H_j \otimes H_p \otimes H_q \rightarrow H_i \\ \varphi_{jk}^i(z)\varphi_{pq}^j(w) &: H_k \otimes H_p \otimes H_q \rightarrow H_i \end{aligned}$$

given more explicitly by

$$\begin{aligned} \varphi_{jk}^i(z)\varphi_{pq}^k(w)(\xi_j \otimes \xi_p \otimes \xi_q) &= \varphi_{jk}^i(z; \xi_j)\varphi_{pq}^k(w; \xi_p \otimes \xi_q) \\ \varphi_{jk}^i(z)\varphi_{pq}^j(w)(\xi_k \otimes \xi_p \otimes \xi_q) &= \varphi_{jk}^i(z; \varphi_{pq}^j(w; \xi_p \otimes \xi_q))\xi_k. \end{aligned} \tag{13}$$

As we noted before however, the chiral vertex operator $\varphi_{jk}^i(z)$ is only densely defined on $H_j \otimes H_k$, and for an arbitrary state $\xi_k = \sum_{d=0}^\infty \xi_k^{(d)}$, the series $\sum_{d=0}^\infty \varphi_{jk}^i(z; \xi_j \otimes \xi_k^{(d)})$ may not converge. Thus the compositions (13) cannot always be carried out, and we impose the following condition:

(iii) *The compositions (13) are convergent and lead to holomorphic functions of z and w in*

$$|z| > |w| > 0.$$

They can be analytically continued to the whole cut plane

$$z \notin \mathbb{R}_-, w \notin \mathbb{R}_-, w - z \notin \mathbb{R}_-. \tag{14}$$

Furthermore, they are injective as maps from $V_{j_1 k_1}^{k_0} \otimes V_{j_2 k_2}^{k_1} \otimes \dots \otimes V_{j_N k_N}^{k_{N-1}}$ and from $V_{j_1 k_1}^{j_0} \otimes V_{j_2 k_2}^{j_1} \otimes \dots \otimes V_{j_N k_N}^{j_{N-1}}$ to the space of operator-valued functions;

The fundamental properties of the composition of chiral vertex operators are encoded in the existence of the following *fusion* and the *braiding* matrices F and R .

(iv) For every $i, j, k, l \in I$, there exists an isomorphism $F \begin{bmatrix} i & j \\ k & l \end{bmatrix}$

$$F \begin{bmatrix} i & l \\ j & k \end{bmatrix} : \bigoplus_{p \in I} V_{jp}^i \otimes V_{kl}^p \rightarrow \bigoplus_{q \in I} V_{ql}^i \otimes V_{jk}^q \quad (15)$$

which allows to interchange the two types of composition of chiral vertices possible respectively on $\bigoplus_{p \in I} V_{jp}^i \otimes V_{kl}^p$ and on $\bigoplus_{q \in I} V_{ql}^i \otimes V_{jk}^q$. More precisely, if the image by $F \begin{bmatrix} i & l \\ j & k \end{bmatrix}$ of a basis $\varphi_{jp;\alpha}^i \otimes \varphi_{kl;\beta}^p$ of $\bigoplus_{p \in I} V_{jp}^i \otimes V_{kl}^p$ is given by

$$F \begin{bmatrix} i & l \\ j & k \end{bmatrix} (\varphi_{jp;\alpha}^i \otimes \varphi_{kl;\beta}^p) = \sum_{q \in I} \sum_{\gamma \in N_{ql}^i, \delta \in N_{jk}^q} \varphi_{ql;\gamma}^i \otimes \varphi_{jk;\delta}^q F_{qp} \begin{bmatrix} i & l \\ j & k \end{bmatrix}_{\alpha \otimes \beta}^{\gamma \otimes \delta} \quad (16)$$

with $\varphi_{ql;\gamma}^i \otimes \varphi_{jk;\delta}^q$ a basis for $\bigoplus_{q \in I} V_{ql}^i \otimes V_{jk}^q$, then we have

$$\varphi_{jp;\alpha}^i(z) \varphi_{kl;\beta}^p(w) = \sum_{q \in I} \sum_{\gamma \in N_{ql}^i, \delta \in N_{jk}^q} \varphi_{ql;\gamma}^i(w) \varphi_{jk;\delta}^q(z-w) F_{qp} \begin{bmatrix} i & l \\ j & k \end{bmatrix}_{\alpha \otimes \beta}^{\gamma \otimes \delta} \quad (17)$$

on the overlap of the domains of definition of both sides of the equation. In particular, for each fixed z (respectively w), one side of the equation can be viewed as the analytic continuation in w (respectively in z) of the other side to its domain of definition.

We shall often drop the indices α, \dots for the bases of chiral vertex operators and for the entries of the matrices $F_{qp} \begin{bmatrix} i & j \\ k & l \end{bmatrix}$ and abbreviate the above equations by their matrix forms

$$\begin{aligned} \varphi_{jp}^i \varphi_{kl}^p &= \sum_{q \in I} \varphi_{ql}^i \varphi_{jk}^q F_{qp} \begin{bmatrix} i & l \\ j & k \end{bmatrix} \\ \varphi_{jp}^i(z) \varphi_{kl}^p(w) &= \sum_{q \in I} \varphi_{ql}^i(w) \varphi_{jk}^q(z-w) F_{qp} \begin{bmatrix} i & l \\ j & k \end{bmatrix}. \end{aligned} \quad (18)$$

We would like to stress that although the operator $F_{qp} \begin{bmatrix} i & l \\ j & k \end{bmatrix}$ is an intrinsic object, the entries $F_{qp} \begin{bmatrix} i & l \\ j & k \end{bmatrix}_{\alpha \otimes \beta}^{\gamma \otimes \delta}$ are basis dependent. In particular,

under a change of gauge

$$\begin{aligned}
 \varphi_{jp;\alpha}^i &\rightarrow \lambda_{jp;\alpha}^i \varphi_{jp;\alpha}^i \\
 \varphi_{kl;\beta}^p &\rightarrow \lambda_{kl;\beta}^p \varphi_{kl;\beta}^p \\
 \varphi_{ql;\gamma}^i &\rightarrow \lambda_{ql;\gamma}^i \varphi_{ql;\gamma}^i \\
 \varphi_{jk;\delta}^q &\rightarrow \lambda_{jk;\delta}^q \varphi_{jk;\delta}^q
 \end{aligned}
 \tag{19}$$

the fusion matrix elements scale as

$$F_{qp} \begin{bmatrix} i & l \\ j & k \end{bmatrix}_{\alpha \otimes \beta}^{\gamma \otimes \delta} \rightarrow \frac{\lambda_{jp;\alpha}^i \lambda_{kl;\beta}^p}{\lambda_{ql;\gamma}^i \lambda_{jk;\delta}^q} F_{qp} \begin{bmatrix} i & l \\ j & k \end{bmatrix}_{\alpha \otimes \beta}^{\gamma \otimes \delta}
 \tag{20}$$

We note that the right hand side of (17) describes in particular the singularities of the composition of $\varphi_{jp}^i(z)$ and $\varphi_{kl}^p(w)$ as z approaches w . In fact, by expanding $\varphi_{jk}^q(z-w)$ in an orthonormal basis $\xi_q^{(d)}$ of H_q^d , we find

$$\begin{aligned}
 &\varphi_{ql}^i(w) \varphi_{jk}^q(z-w) (\xi_l \otimes \xi_j \otimes \xi_k) \\
 &= \varphi_{ql}^i(w) ((z-w)^{L_0} \varphi_{jk}^q(1; \xi_j \otimes \xi_k) \otimes \xi_l) (z-w)^{-(h_j+h_k)} \\
 &= \sum_{d=0}^{\infty} (z-w)^{h_q-h_j-h_k+d} \sum_{\xi_q^{(d)}} \varphi_{ql}^i(w; \xi_q^{(d)} \otimes \xi_l) \langle \xi_q^{(d)} | \varphi_{jk}^q(1; \xi_j \otimes \xi_k) \rangle.
 \end{aligned}$$

In particular, the leading singularities are of the form $(z-w)^{h_q-h_j-h_k}$, and the above expression has *regular singular behavior*.

(v) *There exists an isomorphism* $R \begin{bmatrix} i & l \\ j & k \end{bmatrix}$

$$R \begin{bmatrix} i & l \\ j & k \end{bmatrix} : \oplus_{p \in I} V_{jp}^i \otimes V_{kl}^p \rightarrow \oplus_{q \in I} V_{kq}^i \otimes V_{jl}^q
 \tag{21}$$

which interchanges the composition of chiral vertices allowable on the two spaces, i.e., in a basis with matrix $R_{qp} \begin{bmatrix} i & l \\ j & k \end{bmatrix}_{\alpha \otimes \beta}^{\gamma \otimes \delta}$ and

$$R \begin{bmatrix} i & l \\ j & k \end{bmatrix} (\varphi_{jp;\alpha}^i \otimes \varphi_{kl;\beta}^p) = \sum_{q \in I} \sum_{\gamma \in N_{kq}^i, \delta \in N_{jl}^q} \varphi_{kq;\gamma}^i \otimes \varphi_{jl;\delta}^q R_{qp} \begin{bmatrix} i & l \\ j & k \end{bmatrix}_{\alpha \otimes \beta}^{\gamma \otimes \delta}
 \tag{22}$$

we have

$$\varphi_{jp;\alpha}^i(z) \varphi_{kl;\beta}^p(w) = \sum_{q \in I} \sum_{\gamma \in N_{kq}^i, \delta \in N_{jl}^q} \varphi_{kq;\gamma}^i(w) \varphi_{jl;\delta}^q(z) R_{qp} \begin{bmatrix} i & l \\ j & k \end{bmatrix}_{\alpha \otimes \beta}^{\gamma \otimes \delta}
 \tag{23}$$

in the sense of analytic continuation.

This last property involving analytic continuation requires clarification. Indeed, the region where both sides of (23) are analytic, say as functions of w , is $\{w - z \notin \mathbb{R}_-\} \cap \{z - w \notin \mathbb{R}_-\}$, and thus splits into two disconnected components, where the imaginary part of $w - z$ can be either strictly positive or strictly negative. To each of these components corresponds an identity (23), with a distinct R matrix. We shall denote by $R^{(\pm)}$ the R matrix according to the sign of the imaginary part of $w - z$, and for simplicity, often just by R the $R^{(+)}$ matrix. It is useful to visualize the two corresponding analytic continuations as follows .

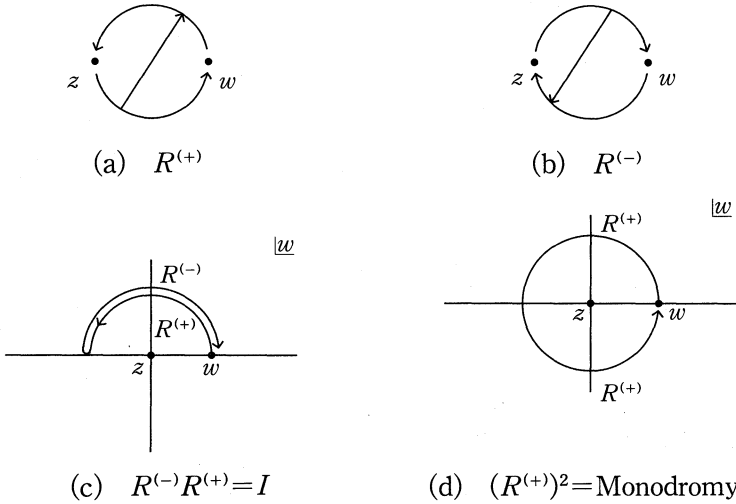


Figure 1. The braiding matrices $R^{(+)}$ and $R^{(-)}$

In the case of $R^{(+)}$, we interchange the positions of z and w , keeping the imaginary part of $w - z$ positive. The argument of the vector $w - z$ goes from 0 to π in the process. In the case of $R^{(-)}$, the imaginary part of $w - z$ remains negative, and the argument of the vector $w - z$ goes from 0 to $-\pi$. This implies

$$R^{(+)}R^{(-)} = R^{(-)}R^{(+)} = I \tag{24}$$

since $R^{(-)}R^{(+)}$ corresponds to the vector $w - z$ first rotating from 0 to π , and the vector $z - w$ then rotating from 0 to $-\pi$. But this means that $w - z$ goes from 0 to π , and then back from π to 0, and nothing

is changed. On the other hand, $(R^{(+)})^2$ corresponds to $w - z$ increasing from 0 to π , and then $z - w$ also increasing from 0 to π . This means that $w - z$ increases all the way from 0 to 2π , and w describes a full circle around z , going counterclockwise. Thus $(R^{(+)})^2$ is just the monodromy of the $\varphi_{kq}^i(w)\varphi_{jl}^q(z)$ as w describes a counterclockwise loop around z . Similarly $(R^{(-)})^2$ is the monodromy as w runs clockwise around the same loop.

As before, it is often convenient to abbreviate (23) by its matrix version

$$\varphi_{jp}^i(z)\varphi_{kl}^p(w) = \sum_{q \in I} \varphi_{kq}^i(w)\varphi_{jl}^q(z)R_{qp} \begin{bmatrix} i & l \\ j & k \end{bmatrix}.$$

The existence of the braiding and fusion matrices already imply the following fundamental properties:

- (a) $N_{jk}^i = N_{i^*j}^{k^*}$. Fusion realizes an isomorphism between $\oplus_{p \in I} V_{i^*p}^0 \otimes V_{jk}^p = V_{i^*i}^0 \otimes V_{jk}^i$ and $\oplus_{q \in I} V_{qk}^0 \otimes V_{i^*j}^q = V_{k^*k}^0 \otimes V_{i^*j}^{k^*}$. Since both spaces $V_{i^*i}^0$ and $V_{k^*k}^0$ are one-dimensional, our claim follows. Note in particular that $N_{j0}^i = N_{i^*j}^0 = \delta_{ij}$;
- (b) V_{jk}^i and V_{kj}^i are isomorphic. Indeed, R realizes an isomorphism between the spaces $\oplus_{p \in I} V_{jp}^i \otimes V_{k0}^p$ and $\oplus_{q \in I} V_{kq}^i \otimes V_{j0}^q$. In view of (a), these spaces reduce respectively to $V_{jk}^i \otimes V_{k0}^k$ and $V_{kj}^i \otimes V_{j0}^j$. If we adopt the standard basis (12) for φ_{j0}^j and φ_{k0}^k , the spaces V_{j0}^j and V_{k0}^k reduce to \mathbb{C} , and R provides then the desired isomorphism.

5. $SL(2, \mathbb{C})$ invariance of the two and three-point functions

We shall now show that the $SL(2, \mathbb{C})$ invariance of the vacuum determines completely the z -dependence of the two and three point functions in a conformal field theory. Only the z -independent coefficient depends on the theory proper, and it is completely determined by the leading terms of the corresponding chiral vertex operators. In fact, iterating the conformal Ward identity (8) gives

$$\begin{aligned} & [\varphi_{p_N i_N}^{l_N}(z_N; \xi_N) \varphi_{p_{N-1} i_{N-1}}^{i_N} (z_{N-1}; \xi_{N-1}) \cdots \varphi_{p_0 i_0}^{i_1} (z_0; \xi_0), L_m] \\ &= - \sum_{k=0}^N (z_k^{m+1} \frac{\partial}{\partial z_k} + h_k(m+1)) \varphi_{p_N i_N}^{l_N}(z_N; \xi_N) \times \\ & \qquad \qquad \qquad \varphi_{p_{N-1} i_{N-1}}^{i_N} (z_{N-1}; \xi_{N-1}) \cdots \varphi_{p_0 i_0}^{i_1} (z_0; \xi_0) \end{aligned}$$

when all the ξ_k are Virasoro primary states in H_k^0 . If we take $l_N = i_0 = 0$, apply the above operator to the vacuum Ω , and use the fact that $L_m^\dagger = L_{-m}$, together with the $SL(2, \mathbb{C})$ invariance (4) of the vacuum we find that for $m = 0, \pm 1$

$$\sum_{k=0}^N \left(z_k^{m+1} \frac{\partial}{\partial z_k} + h_k(m+1) \right) \times \langle \Omega | \varphi_{p_N i_N}^{l_N}(z_N; \xi_N) \times \varphi_{p_{N-1} i_{N-1}}^{i_N}(z_{N-1}; \xi_{N-1}) \cdots \varphi_{p_0 i_0}^{i_1}(z_0; \xi_0) \Omega \rangle = 0. \quad (25)$$

These differential equations can be solved completely for $N = 1$ and $N = 2$. In fact, when $N = 1$, the equation with $m = -1$ implies that the function $\langle \Omega | \varphi_{i^* l}^0(z_1; \xi_{l^*}) \varphi_{l_0}^l(z_2; \xi_l) \Omega \rangle$ depends only on $z_1 - z_0$, and the equation with $m = 0$ implies that the dependence is of the form $(z_1 - z_2)^{-h_{l^*} - h_l}$ up to a z -independent coefficient, which must be a bilinear form in ξ_{l^*} and ξ_l . This bilinear form can then be determined by letting z_2 and z_1 tend successively to 0. We find

$$\lim_{z_1 \rightarrow 0} z_1^{h_{l^*} + h_l} \lim_{z_2 \rightarrow 0} \langle \Omega | \varphi_{i^* l}^0(z_1; \xi_{l^*}) \varphi_{l_0}^l(z_2; \xi_l) \Omega \rangle = \langle \Omega | \widehat{\varphi}_{i^* l}^0(\xi_{l^*} \otimes \xi_l) \rangle \quad (26)$$

where the leading part $\widehat{\varphi}_{jk}^i$ of a chiral vertex operator

$\varphi_{jk}^i(z)$ is the operator from $H_j^0 \otimes H_k^0$ to H_i^0 defined by

$$\begin{aligned} \widehat{\varphi}_{jk}^i(\xi_j \otimes \xi_k) &= \lim_{z \rightarrow 0} z^{-h_i + h_j + h_k} \varphi_{jk}^i(z; \xi_j \otimes \xi_k) \\ &= \text{projection on } H_i^0 \text{ of } \varphi_{jk}^i(1; \xi_j \otimes \xi_k). \end{aligned} \quad (27)$$

We note that this is the same form we had introduced in (11). Altogether, we have then

$$\langle \Omega | \varphi_{i^* l}^0(z_1; \xi_{l^*}) \varphi_{l_0}^l(z_2; \xi_l) \Omega \rangle = \langle \Omega | \widehat{\varphi}_{i^* l}^0(\xi_{l^*} \otimes \xi_l) \rangle (z_1 - z_2)^{-h_{l^*} - h_l} \quad (28)$$

Similarly, for the three-point function, the equation (25) with $m = -1$ implies that the correlation function

$$\langle \varphi_{i^* i}^0(z_1; \xi_{i^*}) \varphi_{jk}^i(z_2; \xi_j) \varphi_{k_0}^k(z_3; \xi_k) \Omega | \Omega \rangle$$

is of the form $f(z_1 - z_3, z_2 - z_3)$, for some function $f(u, v)$. If we substitute this in the equations with $m = 0$ and $m = 1$, we can solve algebraically for $\partial_u f$ and $\partial_v f$. Integrating back in z_1 and z_2 , we find

$$(z_1 - z_2)^{h_k - h_{i^*} - h_j} (z_1 - z_3)^{h_j - h_{i^*} - h_k} (z_2 - z_3)^{h_{i^*} - h_j - h_k}$$

again up to a z -independent, trilinear coefficient in ξ_{i^*} , ξ_j and ξ_k . As before, we determine this coefficient by taking the limits $z_3 \rightarrow 0$, $z_2 \rightarrow 0$, and $z_1 \rightarrow 0$, in that order

$$\lim_{z_1 \rightarrow 0} z_1^{2h_{i^*}} \langle \Omega | \varphi_{i^* i}^0(z_1; \xi_{i^*}) \lim_{z_2 \rightarrow 0} z_2^{h_{i^*} - h_j - h_k} \varphi_{jk}^i(z_2; \xi_j \otimes \xi_k) \rangle$$

Now the limit $\lim_{z_2 \rightarrow 0} z_2^{-h_{i^*} + h_j + h_k} \varphi_{jk}^i(z_2; \xi_j \otimes \xi_k)$ must exist and produce a pure L_0 eigenstate η , belonging to, say H_i^d for some d . In particular $h_i + d = h_{i^*}$ and $\varphi_{i^* i}^0(z_1; \xi_{i^*})\eta = z_1^{-h_{i^*} - h_i - d} \varphi_{i^* i}^0(1; \xi_{i^*})\eta$ in view of the scaling properties of $\varphi_{i^* i}^0(z_1; \xi_{i^*})$. The right hand side in the above equation reduces then to $\langle \Omega | \widehat{\varphi}_{i^* i}^0(\xi_{i^*} \otimes \eta) \rangle$, and we arrive at

$$\begin{aligned} & \langle \varphi_{i^* i}^0(z_1; \xi_{i^*}) \varphi_{jk}^i(z_2; \xi_j) \varphi_{k0}^k(z_3; \xi_k) \Omega | \Omega \rangle \\ &= \langle \Omega | \widehat{\varphi}_{i^* i}^0(\xi_{i^*} \otimes \widehat{\varphi}_{jk}^i(\xi_j \otimes \xi_k)) \rangle \\ & \times (z_1 - z_2)^{h_k - h_{i^*} - h_j} (z_1 - z_3)^{h_j - h_{i^*} - h_k} (z_2 - z_3)^{h_{i^*} - h_j - h_k}. \end{aligned} \tag{29}$$

Incidentally, we have shown that $h_i \leq h_{i^*}$ and, since $i \rightarrow i^*$ is an involution,

$$h_i = h_{i^*}. \tag{30}$$

Next, we consider correlation functions of the form

$$\langle \Omega | \varphi_{k^* k}^0(z_2; \varphi_{i^* j}^{k^*}(z_1 - z_2; \xi_{i^*} \otimes \xi_j)) \varphi_{k0}^k(z_3; \xi_k) \Omega \rangle$$

By the fusion identity (17)

$$\varphi_{i^* i}^0(z_1) \varphi_{jk}^i(z_2) = \varphi_{k^* k}^0(z_2) \varphi_{i^* j}^{k^*}(z_1 - z_2) F_{ik} \begin{bmatrix} 0 & k \\ i^* & j \end{bmatrix}$$

it follows that this correlation function is also proportional to (29). We may as before determine the coefficient of proportionality by letting the insertion points go successively to 0, beginning with z_3 , then $z_1 - z_2$, and then z_2 . We find

$$\begin{aligned} & \langle \Omega | \varphi_{k^* k}^0(z_2; \varphi_{i^* j}^{k^*}(z_1 - z_2; \xi_{i^*} \otimes \xi_j)) \varphi_{k0}^k(z_3; \xi_k) \Omega \rangle \\ &= \langle \Omega | \widehat{\varphi}_{k^* k}^0(\varphi_{i^* j}^{k^*}(\xi_{i^*} \otimes \xi_j) \otimes \xi_k) \rangle \\ & \times (z_1 - z_2)^{h_k - h_{i^*} - h_j} (z_1 - z_3)^{h_j - h_{i^*} - h_k} (z_2 - z_3)^{h_{i^*} - h_j - h_k}. \end{aligned} \tag{31}$$

We note that there are in particular N_{jk}^i independent correlation functions of the form (29), corresponding to a choice of basis $\varphi_{jk;\alpha}^i(z)$, $\alpha = 1, \dots, N_{jk}^i$, for the space of chiral vertex operators in V_{jk}^i .

6. Special values of the fusion and braiding matrices

We had stressed earlier that the fusion and braiding matrix entries are basis dependent in general. However, certain entries are intrinsic (up to the canonical normalization (12)), and we shall show that

$$R_{kj}^{(\pm)} \begin{bmatrix} i & 0 \\ k & j \end{bmatrix} R_{jk}^{(\pm)} \begin{bmatrix} i & 0 \\ j & k \end{bmatrix} = e^{\pm 2\pi i(h_i - h_j - h_k)}, \tag{32}$$

$$F_{ik} \begin{bmatrix} i & 0 \\ k & j \end{bmatrix} = I. \tag{33}$$

To see (32), we apply, say, $R^{(+)}$, twice to $\varphi_{kj}^i(w; \xi_k)\varphi_{j0}^j(z; \xi_j)\Omega$. Then, as we noted before, w describes a counterclockwise loop around z . If we let z tend to 0, we recognize $(R^{(+)})^2$ as the monodromy of $\varphi_{kj}^i(w; \xi_k \otimes \xi_j)$ as w describes a counterclockwise loop around 0. In view of the w dependence of $\varphi_{kj}^i(w)$ given in (9), this equals $e^{2\pi i(h_i - h_j - h_k)}$. The case of $(R^{(-)})^2$ is identical, this time producing the monodromy clockwise, and thus an additional $-$ sign in the exponent. This establishes (32).

To see (33), we consider the fusing identity

$$\varphi_{jk}^i(z)\varphi_{k0}^k(w) = \varphi_{i0}^i(w)\varphi_{jk}^i(z-w)F_{ik} \begin{bmatrix} i & 0 \\ j & k \end{bmatrix}. \tag{34}$$

This implies the identity between three-point functions

$$\begin{aligned} &\langle \Omega | \varphi_{i^*i}^0(y; \xi_{i^*})\varphi_{jk}^i(z; \xi_j)\varphi_{k0}^k(w; \xi_k)\Omega \rangle \\ &= \langle \Omega | \varphi_{i^*i}^0(y; \xi_{i^*})\varphi_{i0}^i(w; \xi_i)\varphi_{jk}^i(z-w; \xi_j)\Omega \rangle F_{ik} \begin{bmatrix} i & 0 \\ j & k \end{bmatrix}. \end{aligned} \tag{35}$$

In view of the expressions (29) and (30) for the two sides of the equation, we can equate the coefficients

$$\begin{aligned} &\langle \Omega | \widehat{\varphi}_{i^*i}^0(\xi_{i^*} \otimes \widehat{\varphi}_{jk}^i(\xi_j \otimes \widehat{\varphi}_{k0}^k(\xi_k))) \rangle \\ &= \langle \Omega | \widehat{\varphi}_{i^*i}^0(\xi_{i^*} \otimes \widehat{\varphi}_{i0}^i(\widehat{\varphi}_{jk}^i(\xi_j \otimes \xi_k) \otimes \Omega) F_{ik} \begin{bmatrix} i & 0 \\ j & k \end{bmatrix}. \end{aligned}$$

Since $\widehat{\varphi}_{i^*i}^0$ is non-degenerate, this implies

$$\widehat{\varphi}_{jk}^i\widehat{\varphi}_{k0}^k = \widehat{\varphi}_{i0}^i\widehat{\varphi}_{jk}^i F_{ik} \begin{bmatrix} i & 0 \\ j & k \end{bmatrix}. \tag{36}$$

The desired identity for F follows now from the normalization (12).

We digress briefly here to note the differences between an abstract fusion identity of the form $F(\varphi_{jk}^i \otimes \varphi_{i0}^i) = \varphi_{i0}^i \otimes \varphi_{jk}^i F_{ik} \begin{bmatrix} i & 0 \\ j & k \end{bmatrix}$, and identities of the form (34) and (36). The abstract fusion identity is just a relation between chiral vertex operators as abstract elements in the vector spaces $V_{jk}^i \otimes V_{i0}^i$ and $V_{i0}^i \otimes V_{jk}^i$. An identity of the form (34) contains a lot more information, since it is an identity between the chiral vertex operators as *operator-valued functions*. It depends on the insertion points z and w , and incorporates descendant states in the H_i representation propagating between the vertices. An identity of the form (36) is an intermediate version, since it still relates the *leading terms of the chiral vertices as operators*, but no longer depends on either insertion points or descendants. Heuristically speaking, the insertion points and descendant states are necessary to insure the correct singularities in the correlation functions, but the fusing/braiding identities are already dictated by relations between primary states. A useful and reliable diagrammatic formalism for fusing and braiding in conformal field theory should preferably be based on identities of the intermediate type (36). There is at present no systematic way for reducing an arbitrary z and descendant-dependent identity to this simpler type of counterpart. In the few simple cases where it can be done, it already yields useful information. The preceding is one example, and we shall also need the one below, obtained by an analogous argument.

Consider the following braiding identity

$$\varphi_{jk}^i(z)\varphi_{k0}^k(w) = \varphi_{kj}^i(w)\varphi_{j0}^j(z)R_{jk} \begin{bmatrix} i & 0 \\ j & k \end{bmatrix}.$$

By going to the 3-point functions $\langle \Omega | \varphi_{i^*i}^0(x)\varphi_{jk}^i(z)\varphi_{k0}^k(w)\Omega \rangle$ and $\langle \Omega | \varphi_{i^*i}^0(x)\varphi_{kj}^i(w)\varphi_{j0}^j(z)\Omega \rangle$, and comparing their coefficients (29), we arrive at (using counterclockwise braiding)

$$\begin{aligned} &\langle \Omega | \widehat{\varphi}_{i^*i}^0(\xi_{i^*})\widehat{\varphi}_{jk}^i(\xi_j \otimes \xi_k) \rangle \\ &= e^{-\pi i(h_{i^*} - h_j - h_k)} \langle \Omega | \widehat{\varphi}_{i^*i}^0(\xi_{i^*})\widehat{\varphi}_{kj}^i(\xi_k \otimes \xi_j) \rangle R_{jk} \begin{bmatrix} i & 0 \\ j & k \end{bmatrix}. \end{aligned}$$

Since $\widehat{\varphi}_{i^*i}^0$ is non-degenerate, and ξ_{i^*} is arbitrary, it follows that

$$\widehat{\varphi}_{jk}^i(\xi_j \otimes \xi_k) = e^{-\pi i(h_{i^*} - h_j - h_k)} \widehat{\varphi}_{kj}^i(\xi_k \otimes \xi_j) R_{jk} \begin{bmatrix} i & 0 \\ j & k \end{bmatrix} \quad (37)$$

We note that the relation (37) between the leading terms $\widehat{\varphi}_{jk}^i$ and $\widehat{\varphi}_{kj}^i$ vertices is very simple, although it is manifestly false for the full chiral

vertices, since $\varphi_{jk}^i(z; \xi_j \otimes \xi_k)$ behaves very differently with respect to descendants in the first factor ξ_j or in the second factor ξ_k . We shall often abbreviate the operator $R_{jk} \begin{bmatrix} i & 0 \\ l & k \end{bmatrix} : V_{jk}^i \otimes \mathbb{C} \rightarrow V_{kj}^i \otimes \mathbb{C}$ simply by $R_{jk}^i : V_{jk}^i \rightarrow V_{kj}^i$.

§3. GAUGE-INVARIANT IDENTITIES OF THE FUSION MATRIX

In this section, we describe some gauge-invariant (i.e. independent of the choice of bases in the spaces of chiral vertex operators) identities relevant to the Verlinde holonomy operators. Of particular importance is the pentagon identity due to Moore and Seiberg [3].

1. The pentagon identity

This is the identity resulting from two different ways of rewriting the expression

$$\varphi_{jk}^i(x) \varphi_{lm}^k(y) \varphi_{np}^m(z).$$

On one hand, we can successively braid the last two factors, then the first two, and then fuse the last two. The result is

$$\begin{aligned} & \sum_{s \in I} \varphi_{jk}^i(x) \varphi_{ns}^k(z) \varphi_{lp}^s(y) R_{sm}^{(23)} \begin{bmatrix} k & p \\ l & n \end{bmatrix} \\ &= \sum_{s,r \in I} \varphi_{nr}^i(z) \varphi_{js}^r(x) \varphi_{lp}^s(y) R_{rk}^{(12)} \begin{bmatrix} i & s \\ j & n \end{bmatrix} R_{sm}^{(23)} \begin{bmatrix} k & p \\ l & n \end{bmatrix} \\ &= \sum_{s,r,q \in I} \varphi_{nr}^i(z) \varphi_{qp}^r(y) \varphi_{jl}^q(x-y) F_{qs}^{(23)} \begin{bmatrix} r & p \\ j & l \end{bmatrix} R_{rk}^{(12)} \begin{bmatrix} i & s \\ j & n \end{bmatrix} R_{sm}^{(23)} \begin{bmatrix} k & p \\ l & n \end{bmatrix} \end{aligned} \quad (38)$$

where the upper indices for the matrices F and R indicate on which pair of factors they act. On the other hand, we can begin by fusing the first two factors

$$\sum_{q \in I} \varphi_{qm}^i(y) \varphi_{jl}^q(x-y) \varphi_{np}^m(z) F_{qk}^{(12)} \begin{bmatrix} i & m \\ j & l \end{bmatrix}.$$

Next, we braid the first and last factors. To write the result in the correct order, we reintroduce briefly indices α, β , and γ for the bases of

chiral vertex operators. Then

$$\begin{aligned}
 \varphi_{qm;\alpha}^i(y)\varphi_{jl;\beta}^q(x-y)\varphi_{np;\gamma}^m(z) &= \varphi_{qm;\alpha}^i(y;\varphi_{jl;\beta}^q(x-y))\varphi_{np;\gamma}^m(z) \\
 &= \sum_r \varphi_{nr;\lambda}^i(z)\varphi_{qp;\mu}^r(y;\varphi_{jl;\beta}^q(x-y))R_{rm} \begin{bmatrix} i & p \\ q & n \end{bmatrix}_{\alpha\otimes\gamma}^{\lambda\otimes\mu} \\
 &= \sum_r \varphi_{nr;\lambda}^i(z)\varphi_{qp;\mu}^r(y)\varphi_{jl;\beta}^q(x-y)R_{rm} \begin{bmatrix} i & p \\ q & n \end{bmatrix}_{\alpha\otimes\gamma}^{\lambda\otimes\mu}. \quad (39)
 \end{aligned}$$

This means that in this case, the R matrix acts as an operator from the pair of first and third indices to the pair of the first and second indices. It is convenient to introduce the permutation operator $P : a \otimes b \rightarrow b \otimes a$. The above transformation can then be represented by $P^{(23)}R^{(13)}$, where this time $R^{(13)}$ acts as an operator from the pair of first and third indices to itself.

We can now equate the coefficients of the two expressions (38) and (39), obtaining the pentagon identity

$$P^{(23)}R_{rm}^{(13)} \begin{bmatrix} i & p \\ q & n \end{bmatrix} F_{qk}^{(12)} \begin{bmatrix} i & m \\ j & l \end{bmatrix} = \sum_{s \in I} F_{qs}^{(23)} \begin{bmatrix} r & p \\ j & l \end{bmatrix} R_{rk}^{(12)} \begin{bmatrix} i & s \\ j & n \end{bmatrix} R_{sm}^{(23)} \begin{bmatrix} k & p \\ l & n \end{bmatrix} \quad (40)$$

as operators of the type

$$V_{jk}^i \otimes V_{lm}^k \otimes V_{np}^m \rightarrow V_{nr}^i \otimes V_{qp}^r \otimes V_{jl}^q. \quad (41)$$

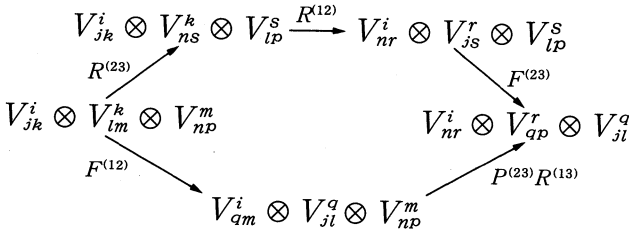


Figure 2. The Pentagon Identity

Sometime, it is convenient to abbreviate (40) further as

$$P^{(23)}R^{(13)}F^{(12)} = F^{(23)}R^{(12)}R^{(23)} \quad (42)$$

when all the indices as well as the summation over intermediate indices s is implicit from the context.

2. Relation between the fusion and the braiding matrices

This relation is an easy consequence of the pentagon identity (40) with the choice $p = 0$. In this case, we must have $m = n, q = r$, and the pentagon is an identity of isomorphisms

$$V_{jk}^i \otimes V_{lm}^k \otimes V_{m0}^m \rightarrow V_{mq}^i \otimes V_{q0}^q \otimes V_{jl}^q$$

With the standard normalization (12), V_{m0}^m and V_{q0}^q reduce to \mathbb{C} . Thus the permutation P on the tensor products $V_{jl}^q \otimes \mathbb{C} \rightarrow \mathbb{C} \otimes V_{jl}^q$ is just the identity. Also, in view of (33), the fusion matrices on the right hand side also reduce to the identity in this case. The pentagon identity reduces then to

$$(R_{qm}^i \otimes 1)F_{qk} \begin{bmatrix} i & m \\ j & l \end{bmatrix} = R_{qk} \begin{bmatrix} i & l \\ j & m \end{bmatrix} (1 \otimes R_{lm}^k) \tag{43}$$

as mappings from $V_{jk}^i \otimes V_{lm}^k$ to $V_{mq}^i \otimes V_{jl}^q$. Here $R_{qm}^i = R_{qm} \begin{bmatrix} i & 0 \\ q & m \end{bmatrix}$ is the isomorphism $V_{qm}^i \rightarrow V_{mq}^i$ from (37). Recalling that $R^{(+)}R^{(-)} = 1$, we can also express this relation as

$$F_{qk} \begin{bmatrix} i & m \\ j & l \end{bmatrix} = (R_{mq}^{(-)i} \otimes 1)R_{qk} \begin{bmatrix} i & l \\ j & m \end{bmatrix} (1 \otimes R_{lm}^{(+)k}). \tag{44}$$

3. The inverse of the fusion matrix entries

The fusion condition (iii) requires that the fusion matrix $F \begin{bmatrix} i & m \\ j & l \end{bmatrix}$ be invertible as an operator from $\oplus_k V_{jk}^i \otimes V_{lm}^k$ to $\oplus_q V_{qm}^i \otimes V_{jl}^q$. We shall however show now that each matrix entry $F_{qk} \begin{bmatrix} i & m \\ j & l \end{bmatrix}$ is invertible in itself. Denote this entry by F for simplicity. Since $R^{(+)}R^{(-)} = 1$, and since we can interchange $R^{(+)}$ and $R^{(-)}$ in (44)

$$F = (R^{(+)} \otimes 1)R^{(-)}(1 \otimes R^{(-)})$$

we obtain as a consequence of the above

$$1 = (R^{(+)} \otimes 1)F(1 \otimes R^{(-)})(R^{(-)} \otimes 1)F(1 \otimes R^{(+)}).$$

Multiplying both sides on the left with $(1 \otimes R^{(+)})$ and on the right with $(1 \otimes R^{(-)})$ gives

$$1 = (R^{(+)} \otimes R^{(+)})F(R^{(-)} \otimes R^{(-)})F.$$

In particular we obtain

$$F^{-1} = (R^{(+)} \otimes R^{(+)})F(R^{(-)} \otimes R^{(-)}).$$

The expression F^{-1} here of course does not refer to the inverse of the whole fusion matrix $F \begin{bmatrix} i & m \\ j & l \end{bmatrix}$, as would customarily be the case. Rather it is the inverse of the entry $F_{qp} \begin{bmatrix} i & m \\ j & l \end{bmatrix}$, and the preceding identity can be written more precisely as

$$\left(F_{qp} \begin{bmatrix} i & m \\ j & l \end{bmatrix} \right)^{-1} = (R_{pj}^{(+)i} \otimes R_{ml}^{(+)p})F_{pq} \begin{bmatrix} i & j \\ m & l \end{bmatrix} (R_{qm}^{(-)i} \otimes R_{jl}^{(-)q}). \quad (45)$$

4. The pentagon identity in terms of the fusion matrix

To obtain this relation, we substitute R in terms of F in the pentagon identity

$$\begin{aligned} & P^{(23)}(R^{(+)(1)} \otimes 1)F^{(13)}(1 \otimes R^{(-)(3)})F^{(12)} \\ &= F^{(23)}(R^{(+)(1)} \otimes 1)F^{(12)}(1 \otimes R^{(-)(2)})(R^{(+)(2)} \otimes 1)F^{(23)}(1 \otimes R^{(-)(3)}. \end{aligned}$$

Since $R^{(+)}R^{(-)} = 1$, and the factors $R^{(\pm)}$ commute with F and P when they act on distinct factors in the tensor product, we find

$$P^{(23)}F^{(13)}F^{(12)} = F^{(23)}F^{(12)}F^{(23)} \quad (46)$$

as an identity of operators of the type

$$V_{jk}^i \otimes V_{lm}^k \otimes V_{pn}^m \rightarrow V_{rn}^i \otimes V_{qp}^r \otimes V_{jl}^q. \quad (47)$$

Explicitly,

$$P^{(23)}F_{rm}^{(13)} \begin{bmatrix} i & n \\ q & p \end{bmatrix} F_{qk}^{(12)} \begin{bmatrix} i & m \\ j & l \end{bmatrix} = \sum_{s \in I} F_{qs}^{(23)} \begin{bmatrix} r & p \\ j & l \end{bmatrix} F_{rk}^{(12)} \begin{bmatrix} i & n \\ j & s \end{bmatrix} F_{sm}^{(23)} \begin{bmatrix} k & n \\ l & p \end{bmatrix}. \quad (48)$$

5. The Verlinde loop operators and the fusion matrix

In his ground-breaking paper [7], E. Verlinde introduced for each field j and each homology cycle C on the torus an operator $V_j(C)$ on characters, defined intuitively by inserting the identity operator as the leading term in the fusion of the fields j and j^* , transporting the field j^* around the C cycle, and letting it fuse again with the j field. This

sequence of manipulations is represented by the transformations on the two point function indicated in Figure 3. Our present purpose is to realize $V_j(C)$ explicitly in terms of the fusion matrix.

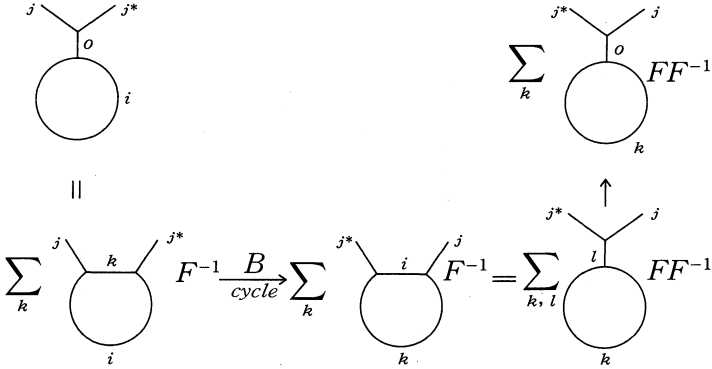


Figure 3. The Verlinde loop operator $V_j(B)$

To do so, we represent the diagram on the left hand-side by

$$\lim_{w \rightarrow 0} \lim_{z \rightarrow w} (z - w)^{2h_j} \text{Tr}_{H_i} (q^{L_0} \varphi_{0i}^i(w) \varphi_{jj^*}^0(z - w)) \tag{49}$$

which corresponds to inserting the identity as the fusion of the fields j and j^* . Here $q = e^{2\pi i \tau}$ is the complex modulus of the torus. Next, the two-point function in (49) can be rewritten as

$$\begin{aligned} &\text{Tr}_{H_i} (q^{L_0} \varphi_{0i}^i(w) \varphi_{jj^*}^0(z - w)) \\ &= \sum_{k \in I} \text{Tr}_{H_i} (q^{L_0} \varphi_{jk;\alpha}^i(z) \varphi_{j^*i;\beta}^k(w)) F_{k0}^{-1} \begin{bmatrix} i & i \\ j & j^* \end{bmatrix}^{\alpha \otimes \beta} \end{aligned}$$

The precise interpretation of the trace on the right hand side is that of a functional-valued tensor of type $(dz)^{L_0} \otimes (dw)^{L_0}$, acting on $H_j \otimes H_{j^*}$. In particular, if we choose for the sake of simplicity incoming states ξ_j and ξ_{j^*} which are chiral primary states in H_j and H_{j^*} respectively, the trace becomes a tensor of type $(dz)^{h_j} \otimes (dw)^{h_j}$. The transport of j^* around homology cycles can now be realized as

$$\begin{aligned} A : w &\rightarrow e^{2\pi i} w \\ B : w &\rightarrow qw. \end{aligned} \tag{50}$$

Here we are viewing the complex torus as the quotient space $\mathbb{C}^* / \{w \sim qw\}$, and A, B are the usual canonical homology basis. Since we shall

be interested later in a modular transformation taking B to $A^{-1} : w \rightarrow e^{-2\pi i}w$, we consider rather transport around these loops. Explicitly, we find

$$\begin{aligned} & \text{Tr}_{H_i}(q^{L_0}\varphi_{jk;\alpha}^i(z)\varphi_{j^*i;\beta}^k(e^{-2\pi i}w))(dz)^{h_j} \otimes (d(e^{-2\pi i}w))^{h_j} \\ &= e^{-2\pi i(h_i-h_k)}\text{Tr}_{H_i}(q^{L_0}\varphi_{jk;\alpha}^i(z)\varphi_{j^*i;\beta}^k(w))(dz)^{h_j} \otimes (dw)^{h_j} \\ & \text{Tr}_{H_i}(q^{L_0}\varphi_{jk;\alpha}^i(z)\varphi_{j^*i;\beta}^k(qw))(dz)^{h_j} \otimes (d(qw))^{h_j} \\ &= \text{Tr}_{H_k}(q^{L_0}\varphi_{j^*i;\beta}^k(w)\varphi_{jk;\alpha}^i(z))(dz)^{h_j} \otimes (dw)^{h_j}. \end{aligned} \quad (51)$$

The first identity in (51) is an easy consequence of the scaling property (9) of chiral vertex operators. To establish the second one, we also exploit the cyclicity of the trace to rewrite the left hand side as

$$\begin{aligned} & \text{Tr}_{H_i}(\varphi_{jk;\alpha}^i(z)\varphi_{j^*i;\beta}^k(qw)q^{L_0})(dz)^{h_j} \otimes (d(qw))^{h_j} \\ &= \text{Tr}_{H_i}(\varphi_{jk;\alpha}^i(z)q^{L_0}\varphi_{j^*i;\beta}^k(w))(dz)^{h_j} \otimes (dw)^{h_j} \\ &= \sum_{d=0}^{\infty} \langle \xi_i^d | \varphi_{jk;\alpha}^i(z) q^{L_0} \varphi_{j^*i;\beta}^k(w) \xi_i^d \rangle (dz)^{h_j} \otimes (dw)^{h_j} \\ &= \sum_{d,c=0}^{\infty} \langle \xi_i^d | \varphi_{jk;\alpha}^i(z) | \xi_k^c \rangle \langle \xi_k^c | q^{L_0} \varphi_{j^*i;\beta}^k(w) \xi_i^d \rangle (dz)^{h_j} \otimes (dw)^{h_j} \\ &= \sum_{c,d=0}^{\infty} \langle \xi_k^c | q^{L_0} \varphi_{j^*i;\beta}^k(w) \xi_i^d \rangle \langle \xi_i^d | \varphi_{jk;\alpha}^i(z) | \xi_k^c \rangle (dz)^{h_j} \otimes (dw)^{h_j} \\ &= \text{Tr}_{H_k}(q^{L_0}\varphi_{j^*i;\beta}^k(w)\varphi_{jk;\alpha}^i(z))(dz)^{h_j} \otimes (dw)^{h_j} \end{aligned}$$

as was to be shown. It is now easy to fuse again, select the identity component, and to obtain respectively for the cycle A^{-1}

$$\begin{aligned} & \text{Tr}_{H_i}(q^{L_0}\varphi_{0i}^i(w)\varphi_{jj^*}^0(z-w)) \\ & \rightarrow \text{Tr}_{H_i}(q^{L_0}\varphi_{0i}^i(w)\varphi_{jj^*}^0(z-w)) \times \\ & \sum_{k \in I} F_{0k} \begin{bmatrix} i & i \\ j & j^* \end{bmatrix}_{\alpha \otimes \beta} e^{-2\pi i(h_k-h_i)} F_{k0}^{-1} \begin{bmatrix} i & i \\ j & j^* \end{bmatrix}^{\alpha \otimes \beta} \end{aligned} \quad (52)$$

and for the cycle B

$$\begin{aligned} & \text{Tr}_{H_i}(q^{L_0}\varphi_{0i}^i(w)\varphi_{jj^*}^0(z-w)) \\ & \rightarrow \sum_{k \in I} \text{Tr}_{H_k}(q^{L_0}\varphi_{0k}^k(z)\varphi_{j^*j}^0(w-z)) F_{0i} \begin{bmatrix} k & k \\ j^* & j \end{bmatrix}_{\beta \otimes \alpha}^{\lambda \otimes \mu} F_{k0}^{-1} \begin{bmatrix} i & i \\ j & j^* \end{bmatrix}^{\alpha \otimes \beta} \end{aligned} \quad (53)$$

To pass to characters, we take the limits indicated in (52-53) in the beginning and final expressions

$$\begin{aligned} \lim_{w \rightarrow 0} \lim_{z \rightarrow w} (z-w)^{2h_j} \text{Tr}_{H_i} (q^{L_0} \varphi_{0i}^i(w) \varphi_{jj^*}^0(z-w)) \\ = \lim_{w \rightarrow 0} \text{Tr}_{H_i} ((q^{L_0} \varphi_{0i}^i(w; \widehat{\varphi}_{jj^*}^0(\xi_j \otimes \xi_{j^*}))). \end{aligned} \tag{54}$$

Consider the following particular braiding identity

$$\varphi_{0i}^i(x) \varphi_{i0}^i(y) = \varphi_{i0}^i(y) \varphi_{00}^0(x) R_{0i} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}.$$

By going to the two-point function, we see easily that

$$\widehat{\varphi}_{0i}^i(\Omega \otimes \xi) = \widehat{\varphi}_{i0}^i(\xi \otimes \Omega) R_{0i} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}. \tag{55}$$

In the standard normalization, $\widehat{\phi}_{i0}^i(\xi \otimes \Omega) = \xi$. Thus, the right hand side of (54) becomes

$$\begin{aligned} \lim_{w \rightarrow 0} \sum_{\eta \in H_i} \langle \eta | q^{L_0} \varphi_{0i}^i(w; \langle \Omega | \widehat{\varphi}_{jj^*}^0(\xi_j \otimes \xi_{j^*}) \Omega) \eta \rangle \\ = \langle \widehat{\varphi}_{jj^*}^0(\xi_j \otimes \xi_{j^*}) | \Omega \rangle \chi^i(q) R_{0i} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \end{aligned}$$

where $\chi^i(q)$ is the character of the chiral algebra for the H_i representation

$$\chi^i(q) = \text{Tr}_{H_i}(q^{L_0}).$$

Similarly, we can evaluate the limit on the right hand side of (53)

$$\begin{aligned} \lim_{w \rightarrow 0} \lim_{z \rightarrow w} (z-w)^{2h_j} \text{Tr}_{H_k} (q^{L_0} \varphi_{0k}^k(z) \varphi_{j^*j}^0(w-z)) \\ = e^{-2\pi h_j} \langle \widehat{\varphi}_{j^*j}^0(\xi_{j^*} \otimes \xi_j) | \Omega \rangle \chi_k(q) R_{0k} \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}. \end{aligned}$$

The ratio $\widehat{\varphi}_{j^*j}^0/\widehat{\varphi}_{jj^*}^0$ can be expressed in terms of the fusion matrix, since

$$\varphi_{j^*0}^{j^*}(x) \varphi_{jj^*}^0(y) = \varphi_{0j^*}^{j^*}(y) \varphi_{j^*j}^0(x-y) F_{00} \begin{bmatrix} j^* & j^* \\ j^* & j \end{bmatrix}$$

so that, after passing to limits in the 3-point functions as before

$$\frac{\widehat{\varphi}_{j^*j}^0}{\widehat{\varphi}_{jj^*}^0} = \frac{1}{F_{00} \begin{bmatrix} j^* & j^* \\ j^* & j \end{bmatrix} R_{0j^*} \begin{bmatrix} j^* & 0 \\ 0 & j^* \end{bmatrix}}. \tag{56}$$

Altogether, we obtain the following realization of the Verlinde loop operators, where we have added a common factor $e^{2\pi i h_j}$ to simplify later formulas:

Definition. For each chiral field j , the Verlinde loop operators $V_j(C)$ are defined by

$$V_j(C)\chi^i(q) = \sum_{k \in I} \chi^k(q)(V_j(C))_k^i$$

with

$$\begin{aligned} (V_j(A^{-1}))_k^i &= \delta_i^k \sum_{l \in I} F_{0l} \begin{bmatrix} i & i \\ j & j^* \end{bmatrix}_{\alpha \otimes \beta} e^{-2\pi i(h_l - h_i - h_j)} F_{l0}^{-1} \begin{bmatrix} i & i \\ j & j^* \end{bmatrix}^{\alpha \otimes \beta} \\ (V_j(B))_k^i &= \sum_{\alpha, \beta} \frac{F_{0i} \begin{bmatrix} k & k \\ j^* & j^* \end{bmatrix}_{\beta \otimes \alpha} F_{k0}^{-1} \begin{bmatrix} i & i \\ j & j^* \end{bmatrix}^{\alpha \otimes \beta}}{F_{00} \begin{bmatrix} j^* & j^* \\ j^* & j \end{bmatrix}} \times \frac{R_{0k} \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}}{R_{0i} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} R_{0j^*} \begin{bmatrix} j^* & 0 \\ 0 & j^* \end{bmatrix}} \end{aligned} \tag{57}$$

It is easy to check that the right hand side of (57) is gauge-invariant under the scale transformations (19), as it should be.

§4. THE GAUGE-FIXED FUSIONMATRIX

Our primary goal in this section is to establish the famous Verlinde formulae, one version of which says that the holonomy operators $\varphi_j(B)$ coincide precisely with the matrices N_{jk}^i in the basis $\chi^i(q)$ of chiral characters. For this, we need to select more carefully the bases $\varphi_{jk;\alpha}^i(z)$ of the spaces of chiral vertex operators V_{jk}^i , so that the fusion matrix entries assume their simplest form, and actually develop symmetries that would not hold in arbitrary bases.

1. Partial gauge-fixing

Henceforth we assume the standard normalization (12). Next, recall that the identity (55) implies that $\widehat{\varphi}_{0i}^i$ and $\widehat{\varphi}_{i0}^i$ are proportional as operators on primary states. Thus, we may choose the normalization of $\widehat{\varphi}_{0i}^i$ so that

$$\begin{aligned} \widehat{\varphi}_{0i}^i(\Omega \otimes \xi) &= \xi, \quad \xi \in H_i^0 \\ R_{0i} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} &= 1. \end{aligned} \tag{58}$$

This gauge already implies the following special values for F and for R

$$\begin{aligned} F_{kj} \begin{bmatrix} i & j \\ k & 0 \end{bmatrix} &= F_{ki} \begin{bmatrix} i & j \\ 0 & k \end{bmatrix} = I \\ R_{ij} \begin{bmatrix} i & j \\ k & 0 \end{bmatrix} &= e^{\pi i(h_i - h_k)} \\ R_{0i} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} &= I. \end{aligned} \tag{59}$$

Indeed, as in the argument leading to (36), the fusing identity

$$\varphi_{kj}^i(z)\varphi_{0j}^j(w) = \varphi_{kj}^i\varphi_{k0}^k(z-w)F_{kj} \begin{bmatrix} i & j \\ k & 0 \end{bmatrix}$$

implies the following algebraic counterpart

$$\widehat{\varphi}_{kj}^i(\xi_k \otimes \widehat{\varphi}_{0j}^j(\Omega \otimes \xi_j)) = \widehat{\varphi}_{kj}^i(\xi_k \otimes \xi_j)F_{kj} \begin{bmatrix} i & j \\ k & 0 \end{bmatrix}$$

from which the first identity in F follows at once. The second is established in the same way, from the fusing identity

$$\varphi_{0i}^i(z)\varphi_{kj}^i(w) = \varphi_{kj}^i(w)\varphi_{0k}^k(z-w)F_{ki} \begin{bmatrix} i & j \\ 0 & k \end{bmatrix}.$$

As for the identities in R , we consider first the braiding identity

$$\varphi_{kj}^i(z)\varphi_{0j}^j(w) = \varphi_{0i}^i(w)\varphi_{kj}^i(z)R_{ij} \begin{bmatrix} i & j \\ k & 0 \end{bmatrix}$$

which implies the algebraic identity

$$\begin{aligned} &\langle \Omega | \widehat{\varphi}_{i^*i}^0(\xi_{i^*} \otimes \varphi_{kj}^i(\xi_k \otimes \xi_j)) \rangle \\ &= e^{-\pi i(h_i - h_k)} \langle \Omega | \widehat{\varphi}_{i^*i}^0(\xi_{i^*} \otimes \widehat{\varphi}_{kj}^i(\xi_k \otimes \xi_j)) \rangle R_{ij} \begin{bmatrix} i & j \\ k & 0 \end{bmatrix} \end{aligned}$$

and hence the first identity on R in (59). As for the other identity, we note that the one we just established implies that $R_{i0} \begin{bmatrix} i & 0 \\ i & 0 \end{bmatrix} = 1$. Since (32) implies

$$R_{i0}^{(+)} \begin{bmatrix} i & 0 \\ i & 0 \end{bmatrix} R_{0i}^{(-)} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = I \tag{60}$$

it follows that both factors in (60) are the identity. Reversing the roles of $R^{(+)}$ and $R^{(-)}$, we obtain the desired statement.

2. Rigid gauges

Except for V_{i0}^i and V_{0i}^i , the bases for the various spaces V_{jk}^i of chiral vertices can be chosen independently of each other. However, we saw earlier that the existence of fusion implies that the spaces V_{jk}^i and V_{ki}^{j*} are (projectively) isomorphic. We can make use of that fact to coordinate the choices of bases in these two spaces, arriving in this way at a simpler fusion matrix. In order for the choices to be consistent, however, we have to proceed with some care.

A first step is to fix in the space of chiral vertices of the form V_{ii}^0 an arbitrary representative chiral vertex $\varphi_{ii^*}^0(z)$, which so far is only unique up to a multiplicative constant. To be consistent with our earlier normalization (12), we can choose $\widehat{\varphi}_{00}^0(\cdot \otimes \Omega)$ to be the identity. This choice of $\varphi_{ii^*}^0$ actually dictates a choice of representative $\varphi_{i^*i}^0$ in $V_{ii^*}^0$. This is evident for $i = i^*$. More generally, we note that the fusing identity

$$\varphi_{i0}^i(z)\varphi_{i^*i}^0(w) = \varphi_{0i}(w)^i\varphi_{ii^*}^0(z-w)F_{00}\begin{bmatrix} i & i \\ i & i^* \end{bmatrix} \tag{61}$$

implies, after passing to the 3-point function and taking limits, that the algebraic vertices $\widehat{\varphi}_{i^*i}^0$ and $\widehat{\varphi}_{ii^*}^0$ are proportional as bilinear forms on $H_i^0 \otimes H_{i^*}^0$. After normalizing φ_{i0}^i and φ_{0i}^i as in (12) and (58), and after selecting $\varphi_{ii^*}^0$, we can normalize $\varphi_{i^*i}^0$ when $i \neq i^*$ so that

$$\frac{\widehat{\varphi}_{i^*i}^0}{\widehat{\varphi}_{ii^*}^0} = F_{00}\begin{bmatrix} i & i \\ i & i^* \end{bmatrix} = 1. \tag{62}$$

When $i = i^*$, the identity (62) holds automatically by our assumption that $\widehat{\varphi}_{ii}^0$ is then a symmetric bilinear form on $H_i^0 \otimes H_i^0$.

Next, we fix i, j, k , and recall that fusion also implied that V_{ki}^{j*} and V_{jk}^i have the same dimension. We can make this relation more explicit, using the above bases for $V_{jj^*}^0$ and $V_{ii^*}^0$. Now fusion provides an isomorphism between

$$V_{jj^*}^0 \otimes V_{ki}^{j*} \rightarrow V_{ii^*}^0 \otimes V_{jk}^i \tag{63}$$

and hence between

$$V_{ki}^{j*} \rightarrow V_{jk}^i \tag{64}$$

since, in presence of a basis, both $V_{jj^*}^0$ and $V_{ii^*}^0$ can be identified with \mathbb{C} . More concretely, let $\varphi_{ki}^{j*}(z)$ be any chiral vertex operator in V_{ki}^{j*} . Then

$F(\varphi_{jj^*}^0 \otimes \varphi_{k i^*}^{j^*})$ can be written in a unique way as $\varphi_{i i^*}^0 \otimes \varphi_{jk}^i$, for some well-defined φ_{jk}^i in V_{jk}^i . Starting with a basis $\varphi_{k i^*}^{j^*}$ for $V_{k i^*}^{j^*}$, we obtain this way another basis $\varphi_{jk; \alpha}^i$, with respect to which the fusion matrix F reduces by construction to the identity

$$F_{ij^*} \begin{bmatrix} 0 & i^* \\ j & k \end{bmatrix} = I. \quad (65)$$

The leading terms of the chiral vertices $\varphi_{k i^*}^{j^*}$ and $\varphi_{jk; \alpha}^i$ can be related as operators. In fact, in view of (65), the fusion relation in its analytic form can be written as

$$\varphi_{jj^*}^0(z) \varphi_{k i^*}^{j^*}(w) = \varphi_{i i^*}^0(w) \varphi_{jk}^i(z-w). \quad (66)$$

If we pass to the 3-point function and take suitable limits, then we obtain as before the corresponding relation between the leading chiral vertices

$$\widehat{\varphi}_{jj^*}^0(\xi_j \otimes \widehat{\varphi}_{k i^*}^{j^*}(\xi_k \otimes \xi_{i^*})) = \widehat{\varphi}_{i i^*}^0(\widehat{\varphi}_{jk}^i(\xi_j \otimes \xi_k) \otimes \xi_{i^*}). \quad (67)$$

This suggests a slightly different interpretation of the above discussion. A (leading) chiral vertex operator $\widehat{\varphi}_{jk}^i$ is a bilinear form

$$H_j^0 \otimes H_k^0 \rightarrow H_i^0,$$

and can thus be also viewed as a bilinear form $H_k^0 \otimes (H_i^0)^\dagger \rightarrow (H_j^0)^\dagger$, where $(H_j^0)^\dagger, (H_i^0)^\dagger$ are respectively the dual of H_j^0 and $(H_i^0)^\dagger$, that is, the spaces of linear functionals on H_j^0 and H_k^0 . The basic observation underlying the above construction is that a choice of the non-degenerate forms $\widehat{\varphi}_{jj^*}^0$ and $\widehat{\varphi}_{i i^*}^0$ allows us to identify the duals $(H_i^0)^\dagger$ and $(H_j^0)^\dagger$ with the Hilbert spaces $H_{i^*}^0$ and $H_{j^*}^0$ of the conjugate fields i^* and j^* . With this identification, the leading term of the chiral vertex $\widehat{\varphi}_{jk}^i$ corresponds then to a bilinear form

$$H_k^0 \otimes H_{i^*}^0 \rightarrow H_{j^*}^0.$$

Note that duality holds *at the operator level* only between the leading parts $\widehat{\varphi}_{jk}^i$ and $\widehat{\varphi}_{k i^*}^{j^*}$ of the chiral vertices and that there is no simple and direct duality statement between the full vertices themselves. Note also that without a choice of representatives $\varphi_{jj^*}^0$ and $\varphi_{i i^*}^0$, we still have a projective correspondence between $(H_j)^\dagger, (H_i)^\dagger$ and $H_{j^*}^0, H_{i^*}^0$, correspondence which is quite different from the usual identification of a

Hilbert space with its adjoint. Avoiding notions of adjoints and complex conjugation is desirable in the chiral (i.e. holomorphic) sector of a conformal field theory.

We have seen that given a basis $\varphi_{jk;\alpha}^i$ for V_{jk}^i , there is a unique basis $\varphi_{ki^*}^{j^*}$ for $V_{ki^*}^{j^*}$ with respect to which the identity (65) holds. This process leads to a closed cycle

$$V_{jk}^i \rightarrow V_{ki^*}^{j^*} \rightarrow V_{i^*j}^{k^*} \rightarrow V_{jk}^i. \tag{68}$$

Note that if we start with the standard normalized basis $\varphi_{i0}^i(z)$ for V_{i0}^i , we get back this way our earlier choices of $\varphi_{0i^*}^{i^*}(z)$ and $\varphi_{i^*i}^0(z)$ for $V_{0i^*}^i$ and $V_{i^*i}^0$. More generally, the set of all spaces of chiral vertices can be divided into such disjoint cycles. Within each cycle, we select a basis, say, for V_{jk}^i , and deduce the bases for $V_{ki^*}^{j^*}$ and $V_{i^*j}^{k^*}$ accordingly. It should be noted that from $V_{i^*j}^{k^*}$, we do come back to the original basis for V_{jk}^i . In fact, this last step leads to the basis $\tilde{\varphi}_{jk;\alpha}^i$ given by

$$\tilde{\varphi}_{jk;\alpha}^i = \varphi_{jk;\alpha}^i \frac{\hat{\varphi}_{k^*k}^0 \hat{\varphi}_{j^*j}^0 \hat{\varphi}_{ii^*}^0}{\hat{\varphi}_{kk^*}^0 \hat{\varphi}_{j^*j^*}^0 \hat{\varphi}_{i^*i}^0}. \tag{69}$$

With our earlier choice (62), the new basis coincides with the basis $\varphi_{jk;\alpha}^i$ we started with. Thus we can now assume that the identity holds for all fields i, j, k in these bases, which we shall refer henceforth to as a rigid gauge.

3. The inverse of the fusion matrix in a rigid gauge

The starting point is the pentagon identity (40), in which we set $i = 0$. Since the only non-zero component in $F_{\star\star}^{[0\star]}$ is $F_{m^*j^*} \begin{bmatrix} 0 & m \\ j & l \end{bmatrix}$, we can also set $k = j^*, r = n^*$, and arrive at

$$\begin{aligned} P^{(23)} F_{rm}^{(13)} \begin{bmatrix} 0 & r^* \\ q & p \end{bmatrix} F_{qj^*}^{(12)} \begin{bmatrix} 0 & m \\ j & l \end{bmatrix} \delta_{q^*m} \\ = \sum_{s \in I} F_{qs}^{(23)} \begin{bmatrix} r & p \\ j & l \end{bmatrix} F_{rj^*}^{(12)} \begin{bmatrix} 0 & r^* \\ j & s \end{bmatrix} F_{sm}^{(23)} \begin{bmatrix} j^* & r^* \\ l & p \end{bmatrix}. \end{aligned} \tag{70}$$

So far our equations have been gauge-invariant, but now we select a rigid gauge, so that the identity (65) applies. Acting on a basis

$$\varphi_{jj^*}^0 \otimes \varphi_{lm;\lambda}^{j^*} \otimes \varphi_{pr^*;\mu}^m \in V_{jj^*}^0 \otimes V_{lm}^{j^*} \otimes V_{pr^*}^m$$

and expressed in terms of a basis

$$\varphi_{rr^*}^0 \otimes \varphi_{qp;\alpha}^r \otimes \varphi_{jl;\beta}^q \in V_{rr^*}^0 \otimes V_{qp}^r \otimes V_{jl}^q$$

the matrix elements of the left hand side of (70) are

$$(\delta_\mu^\kappa \delta_\lambda^\rho) \delta_\kappa^\alpha \delta_\rho^\beta \delta_{q^*m} = \delta_{q^*m} \delta_\mu^\alpha \delta_\lambda^\beta$$

with the matrix elements of $P^{(23)}$ contributing $\delta_\mu^\kappa \delta_\lambda^\rho$. On the other hand, the matrix elements of the right hand side are

$$\sum_{s \in I} F_{qs} \begin{bmatrix} r & p \\ j & l \end{bmatrix}_{\gamma \otimes \delta}^{\alpha \otimes \beta} \delta_\rho^\gamma F_{sm} \begin{bmatrix} j^* & r^* \\ l & p \end{bmatrix}_{\lambda \otimes \mu}^{\rho \otimes \delta}$$

Equating the two expressions gives

$$\sum_{i \in I} F_{qs} \begin{bmatrix} r & p \\ j & l \end{bmatrix}_{\gamma \otimes \delta}^{\alpha \otimes \beta} F_{sm} \begin{bmatrix} j^* & r^* \\ l & p \end{bmatrix}_{\lambda \otimes \mu}^{\gamma \otimes \delta} = \delta_{q^*m} \delta_\mu^\alpha \delta_\lambda^\beta. \tag{71}$$

This gives two ways of computing F^{-1} . First, by comparing with

$$\sum_{s \in I} F_{qs} \begin{bmatrix} r & p \\ j & l \end{bmatrix}_{\gamma \otimes \delta}^{\alpha \otimes \beta} F_{sm^*}^{-1} \begin{bmatrix} r & p \\ j & l \end{bmatrix}_{\mu \otimes \lambda}^{\gamma \otimes \delta} = \delta_{q^*m} \delta_\mu^\alpha \delta_\lambda^\beta$$

we obtain

$$F_{sm^*}^{-1} \begin{bmatrix} r & p \\ j & l \end{bmatrix}_{\mu \otimes \lambda}^{\gamma \otimes \delta} = F_{sm} \begin{bmatrix} j^* & r^* \\ l & p \end{bmatrix}_{\lambda \otimes \mu}^{\gamma \otimes \delta}. \tag{72}$$

Next, by comparing with

$$\sum_{s \in I} F_{q^*s}^{-1} \begin{bmatrix} j^* & r^* \\ l & p \end{bmatrix}_{\gamma \otimes \delta}^{\beta \otimes \alpha} F_{sm} \begin{bmatrix} j^* & r^* \\ l & p \end{bmatrix}_{\lambda \otimes \mu}^{\gamma \otimes \delta} = \delta_{q^*m} \delta_\mu^\alpha \delta_\lambda^\beta$$

we obtain a second expression for F^{-1}

$$F_{q^*s}^{-1} \begin{bmatrix} j^* & r^* \\ l & p \end{bmatrix}_{\gamma \otimes \delta}^{\beta \otimes \alpha} = F_{qs} \begin{bmatrix} r & p \\ j & l \end{bmatrix}_{\gamma \otimes \delta}^{\alpha \otimes \beta}. \tag{73}$$

4. Symmetry of the gauge-fixed fusion matrix

Comparing the above two expressions for F^{-1} leads to a useful symmetry symmetries of the gauge-fixed fusion matrix:

$$F_{sm} \begin{bmatrix} j^* & r^* \\ l & p \end{bmatrix}_{\lambda \otimes \mu}^{\gamma \otimes \delta} = F_{s^* m^*} \begin{bmatrix} p^* & l \\ r^* & j \end{bmatrix}_{\mu \otimes \lambda}^{\delta \otimes \gamma} \tag{74}$$

We could have anticipated this symmetry, since $F_{sm} \begin{bmatrix} j^* & r^* \\ l & p \end{bmatrix}$ sends $V_{lm}^{j^*} \otimes V_{pr^*}^m$ to $V_{sr^*}^{j^*} \otimes V_{lp}^s$, while $F_{s^* m^*} \begin{bmatrix} p^* & l \\ r^* & j \end{bmatrix}$ sends $V_{r^* m^*}^{p^*} \otimes V_{jl}^{m^*}$ to $V_{s^* l}^{p^*} \otimes V_{r^* j}^{s^*}$. In a rigid gauge, these spaces are respectively isomorphic under the correspondences $V_{lm}^{j^*} \rightarrow V_{mj}^{m^*}$, $V_{pr^*}^m \rightarrow V_{r^* m^*}^{p^*}$, $V_{sr^*}^{j^*} \rightarrow V_{r^* j}^{s^*}$, and $V_{lp}^s \rightarrow V_{ps^*}^{l^*} \rightarrow V_{s^* l}^{p^*}$ which we discussed in (68). We also note that symmetries of the form (74) can only hold in a chosen gauge, since the entries on the two sides of the equation scale differently under an arbitrary change of scales (19).

5. Proof of the Verlinde fusion formula

Consider the pentagon identity (40), with the choice of fields

$$\begin{aligned} P^{(23)} F_{rm}^{(13)} \begin{bmatrix} j^* & n \\ q & p \end{bmatrix} F_{ql}^{(12)} \begin{bmatrix} j^* & m \\ k & i^* \end{bmatrix} \\ = \sum_{s \in I} F_{qs}^{(23)} \begin{bmatrix} r & p \\ k & i^* \end{bmatrix} F_{rl}^{(12)} \begin{bmatrix} j^* & n \\ k & s \end{bmatrix} F_{sm}^{(23)} \begin{bmatrix} l & n \\ i^* & p \end{bmatrix}. \end{aligned}$$

Next, we set $m = r = 0$, $l = i^*$. In view of (33) and (65), we are led to $p^* = q = j^*$, $s = k^*$. Finally, we set $n = j^*$, and arrive at

$$\begin{aligned} P^{(23)} F_{00}^{(13)} \begin{bmatrix} j^* & j^* \\ j^* & j \end{bmatrix} F_{j^* i^*}^{(12)} \begin{bmatrix} j^* & 0 \\ k & i^* \end{bmatrix} \\ = F_{j^* k^*}^{(23)} \begin{bmatrix} 0 & j \\ k & i^* \end{bmatrix} F_{0i^*}^{(12)} \begin{bmatrix} j^* & j^* \\ k & k^* \end{bmatrix} F_{k^* 0}^{(23)} \begin{bmatrix} i^* & j^* \\ i^* & j \end{bmatrix}. \end{aligned} \tag{75}$$

This is an identity of operators of the type

$$V_{ki^*}^{j^*} \otimes V_{i^* 0}^{i^*} \otimes V_{jj^*}^0 \rightarrow V_{0j^*}^{j^*} \otimes V_{jj^*}^0 \otimes V_{ki^*}^{j^*} \tag{76}$$

We now express it in terms of the respective bases $\varphi_{ki^*, \beta}^{j^*} \otimes \varphi_{i^* 0}^{i^*} \otimes \varphi_{jj^*}^0$ and $\varphi_{0j^*}^{j^*} \otimes \varphi_{jj^*}^0 \otimes V_{ki^*, \alpha}^{j^*}$, chosen in a rigid gauge.

On the left hand side, $F \begin{bmatrix} * & 0 \\ * & * \end{bmatrix}$ acts as the identity matrix, $F_{00} \begin{bmatrix} j^* & j^* \\ j^* & j \end{bmatrix}$ is just a scalar, and $P^{(23)}$ also reduces to the identity, since the space $V_{i^*0}^{i^*} \otimes V_{jj^*}^0$ on which it acts is one-dimensional. Thus we get as the matrix element

$$(\text{Left Hand Side})_{\beta}^{\alpha} = \delta_{\beta}^{\alpha} F_{00} \begin{bmatrix} j^* & j^* \\ j^* & j \end{bmatrix}.$$

On the right hand side, again in a rigid gauge, $F \begin{bmatrix} 0 & j \\ k & i^* \end{bmatrix}$ is the identity, and we are left with

$$\begin{aligned} (\text{Right Hand Side})_{\beta}^{\alpha} &= \sum_{\gamma, \delta} \delta_{\gamma}^{\alpha} F_{0i^*} \begin{bmatrix} j^* & j^* \\ k & k^* \end{bmatrix}_{\beta \otimes \delta} F_{k^*0} \begin{bmatrix} i^* & j^* \\ i^* & j \end{bmatrix}^{\delta \otimes \gamma} \\ &= \sum_{\delta} F_{0i^*} \begin{bmatrix} j^* & j^* \\ k & k^* \end{bmatrix}_{\beta \otimes \delta} F_{k^*0} \begin{bmatrix} i^* & j^* \\ i^* & j \end{bmatrix}^{\delta \otimes \alpha}. \end{aligned}$$

Setting these two expressions to be equal, α to be equal to β , and summing with respect to α yields the following identity in a rigid gauge

$$N_{jk}^i = \sum_{\alpha, \delta} \frac{F_{0i^*} \begin{bmatrix} j^* & j^* \\ k & k^* \end{bmatrix}_{\alpha \otimes \delta} F_{k^*0} \begin{bmatrix} i^* & j^* \\ i^* & j \end{bmatrix}^{\delta \otimes \alpha}}{F_{00} \begin{bmatrix} j^* & j^* \\ j^* & j \end{bmatrix}}. \tag{77}$$

Our final task is to relate the right hand side of (77) to the Verlinde operators. We can do this using the symmetries of the fusion matrix in a rigid gauge. First, the formula (72) for F^{-1} implies

$$F_{k0}^{-1} \begin{bmatrix} i & i \\ j & j^* \end{bmatrix}^{\alpha \otimes \beta} = F_{k0} \begin{bmatrix} j^* & i^* \\ j^* & i \end{bmatrix}^{\alpha \otimes \beta}.$$

Next, the symmetry identity (74) implies

$$F_{k0} \begin{bmatrix} j^* & i^* \\ j^* & i \end{bmatrix}^{\alpha \otimes \beta} = F_{k^*0} \begin{bmatrix} i^* & j^* \\ i^* & j \end{bmatrix}^{\beta \otimes \alpha}$$

while the other factor in the identity (57) for $(V_j(B))_k^i$ becomes

$$F_{0i} \begin{bmatrix} k & k \\ j^* & j \end{bmatrix}_{\beta \otimes \alpha} = F_{0i^*} \begin{bmatrix} j^* & j^* \\ k & k^* \end{bmatrix}_{\alpha \otimes \beta}.$$

Since all the relevant braiding matrix elements in (57) are the identity in this gauge, these equations imply the following statement for the operators $V_j(B)$:

Theorem (Verlinde Fusion Formula): *Under the assumptions (i-v), the Verlinde loop operators $V_j(C)$ defined by the expression (57), can also be rewritten as*

$$\begin{aligned} V_j(A^{-1})\chi^i &= \lambda_{(j)i}\chi^i \\ V_j(B)\chi^i &= \sum_j N_{jk}^i \chi^k. \end{aligned} \tag{78}$$

The statement about the Verlinde loop operators $V_j(A^{-1})$, namely that they are diagonal on the basis $\chi^i(q)$ of characters, is already clear from their very definition (57), which expresses also the eigenvalues $\lambda_{(j)i}$ explicitly in terms of the fusion matrix.

So far our discussion has not required any assumption about the theory on the torus. If however, we assume now that the theory is modular invariant, in the sense that the modular transformation $S : \tau \rightarrow -1/\tau$ which acts on the chiral characters $S\chi^i(q) = \sum_k S_k^i \chi^k(q)$ and interchanges the cycles A^{-1} and B , also interchanges the Verlinde loop operators $V_j(A^{-1})$ and $V_j(B)$, then we may conclude that

$$\sum_p N_{jp}^i S_k^p = S_k^i \lambda_{(j)k} \iff N_j S = D_j S^{-1} \tag{79}$$

where we have introduced the following matrices $(N_j)_k^i = N_{jk}^i$, $(D_j)_k^i = \lambda_{(j)i} \delta_k^i$. This gives in particular the eigenvalues $\lambda_{(j)k}$ in terms of the matrix S . In fact, set $i = 0$. Since $N_{jp}^0 = \delta_{j^*p}$, we get

$$\lambda_{(j)k} = \frac{S_k^{j^*}}{S_k^0}.$$

6. The Verlinde dimension formula

For the sake of completeness, we include a derivation of the well-known Verlinde dimension formula from the Verlinde fusion formula. We consider here the space $V_{j_2 \dots j_n}^{j_1}$ ($n \geq 3$) of n -point conformal blocks on the sphere, which is isomorphic to $\oplus_{p_1, \dots, p_{n-2}} V_{j_2 p_1}^{j_1} \otimes V_{j_3 p_2}^{p_1} \otimes \dots \otimes V_{j_{n-1} j_n}^{p_{n-2}}$. Note that the product of a number of matrices N_j can be reduced to a product of diagonal matrices: $N_{j_2} N_{j_3} \dots N_{j_{n-1}} = S D_{j_2} D_{j_3} \dots D_{j_{n-1}} S^{-1}$

Thus

$$\begin{aligned} \dim_{\mathbb{C}} V_{j_2 \dots j_n}^{j_1} &= (N_{j_2} N_{j_3} \dots N_{j_{n-1}})_{j_n}^{j_1} \\ &= \sum_i S_i^{j_1} \left(\frac{S_i^{j_2} S_i^{j_3} \dots S_i^{j_{n-1}}}{(S_i^0)^{n-2}} \right) (S^{-1})_{j_n}^i. \end{aligned} \tag{80}$$

The Verlinde dimension formula for the space B_g of 0-point conformal blocks on a Riemann surface of genus g follows from (80) together with the sewing property of conformal field theory. The sewing property implies that B_g is isomorphic to the space obtained from the spaces of $2g$ -point conformal blocks on the sphere upon identifying (via conjugation) the representation labels j_1, \dots, j_{2g} in pairs and summing over the remaining free g labels, i.e.,

$$\begin{aligned} B_g &= \bigoplus_{j_1, \dots, j_g \in I} \bigoplus_{p_1, \dots, p_{2g-2} \in I} \\ &\quad V_{j_2 p_1}^{j_1} \otimes V_{j_2^* p_2}^{p_1} \otimes V_{j_3 p_3}^{p_2} \otimes V_{j_3^* p_4}^{p_3} \otimes \dots \otimes V_{j_g p_{2g-2}}^{p_{2g-3}} \otimes V_{j_g^* j_1}^{p_{2g-2}}. \end{aligned}$$

Therefore

$$\begin{aligned} \dim_{\mathbb{C}} B_g &= \sum_{j_1, \dots, j_g} \left(N_{j_2} N_{j_2^*} \dots N_{j_g} N_{j_g^*} \right)_{j_1}^{j_1} \\ &= \sum_{j_2, \dots, j_g} \text{Tr} \left(N_{j_2} N_{j_2^*} \dots N_{j_g} N_{j_g^*} \right) \\ &= \sum_{i, j_2, \dots, j_g} \frac{1}{(S_i^0)^{2g-2}} S_i^{j_2} S_i^{j_2^*} \dots S_i^{j_g} S_i^{j_g^*}. \end{aligned} \tag{81}$$

This formula simplifies further to

$$\dim_{\mathbb{C}} B_g = \sum_{i \in I} \frac{1}{(S_i^0)^{2g-2}}. \tag{82}$$

if S is assumed to be symmetric, i.e., $S_i^j = S_j^{i^*}$. This is the case for the Wess–Zumino–Witten models with group $SU(2)$ and integral level $K > 0$. The index set is in this case the weight set $I = \{0, 1, \dots, K\}$ corresponding to the level K integrable representations of the loop algebra of $sl(2)$, and all indices are self-conjugate. The S -matrix is given by

$$S_j^i = \sqrt{\frac{2}{K+2}} \sin \frac{(i+1)(j+1)\pi}{K+2},$$

and (82) gives the now familiar dimension formula

$$\dim_{\mathbb{C}} B_g = \sum_{i=0, \dots, K} \left(\frac{K+2}{2} \right)^{g-1} \left(\sin \frac{(i+1)\pi}{K+2} \right)^{2-2g}. \quad (83)$$

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