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# Deformation Theory of CR-Structures and Its Application to Deformations of Isolated Singularities I

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## Introduction

Let $(V, o)$ be a normal isolated singularity in $C^N$ of complex dimension $n$. We would like to study a deformation theory of complex structures of $(V, o)$. This problem is studied in several ways. For example, (1) Grauert's method (cf. [Gr1]), (2) Douady's method (cf. [Dou]), (3) Kuranishi's approach (cf. [Ku1], [Ku2]), etc. In this paper, we recall Kuranishi's approach and give a review of some contribution, done by T. Akahori and K. Miyajima (cf. [Ku1], [Ku2], [Ak1]-[Ak5], [Ak-My1], [My1]).

Now we set the intersection of $V$ with the real hypersphere centered at $o$ of radius $\epsilon$, namely

$$M = V \cap S_{\epsilon}^{2N-1}.$$

This $M$ is a non-singular real $2n-1$ dimensional $C^\infty$ manifold, and over this $M$, a CR structure is induced from $V$. Namely, ${}^0T'' = C \otimes TM \cap T''N \mid_M$, where $N = V - o$. Conversely, this CR structure $(M, {}^0T'')$ determines the normal Stein space $V$, uniquely. Noting this result, in order to give a versal family of deformations of singularities, Kuranishi initiated his deformation theory of CR strucutres for a normal isolated singularity. To see Kuranishi's approach and to see our contribution, we recall Kodaira-Spencer's theory for deformation theory of complex structures of compact complex manifolds.

Let $X$ be a complex manifold, and let $(X, T''X)$ denote the complex structure. Then, the deformation theory of complex structures proceeds as follows.

1) *Formulation.* Any deformation of the given complex structure $T''X$, can be parametrized by an element $\phi$ of $\Gamma(X, T'X \otimes (T''X)^*)$,

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which satisfies the deformation equation

$$\overline{\partial}^{(1)}_{T'X}\phi + R_2(\phi) = 0.$$

Here $\overline{\partial}_{T'X}$ means the Cauchy-Riemann operator associated with the holomorphic vector bundle $T'X$. And we have the deformation complex

$$0 \to \Gamma(X, T'X) \overset{\overline{\partial}_{T'X}}{\to} \Gamma(X, T'X \otimes (T''X)^*) \overset{\overline{\partial}_{T'X}}{\to} \cdots$$
$$\overset{\overline{\partial}_{T'X}}{\to} \Gamma(X, T'X \otimes \wedge^p(T''X)^*) \overset{\overline{\partial}_{T'X}}{\to} \Gamma(X, T'X \otimes \wedge^{p+1}(T''X)^*) \overset{\overline{\partial}_{T'X}}{\to} \cdots$$

on $X$ (note that this is an elliptic complex).

Therefore our geometrical problem becomes a problem of a non-linear partial differential equations. To solve this, that is to say, to construct our solutions for this non-linear partial differential equation, there are two methods, namely, Kuranishi's method (see [Ku4]) and Kodaira-Spencer's method (see [Kod]). Kuranishi's method is to give a particular solution space by adding a new equation (Kuranishi's ingenious method; it is $\overline{\partial}^*\phi = 0$ in the compact complex manifold case). This method is applicable in many fields (for example, recent work of Donaldson's (see [Don])). For deformation theory of CR structures, by his method (adding some new equations), Kuranishi gave a special solution space, which is parametrized by $H^1(X, T'X)$ in [Ku1], [Ku2]. Actually, in order to make this special solution space clear, I started my research. On the other hand, Kodaira-Spencer's method is "so-called" power series method and obviously quite elementary. This method is divided into two parts.

2) *Formal Construction.* We construct the formal power series $\phi(t) = \sum \phi_\mu(t)t^\mu$ satisfying;

$$\phi_1(t) = \sum_{\lambda}^{q} \beta_\lambda t_\lambda,$$

where $\beta_\lambda$ is a base of $H^1(X, T'X)$ and $q = \dim_C H^1(X, T'X)$, and

$$\overline{\partial}^{(1)}_{T'X}\phi_{\mu+1} + (\overline{\partial}^{(1)}_{T'X}\phi^\mu(t) + R_2(\phi^\mu)) \equiv 0 \bmod \ t^{\mu+2}.$$

by using the Kodaira-Hodge decomposition theorem for the standard $\overline{\partial}_{T'X}$(this is elliptic).

3) *Convergence.* By using the ellipticity of the standard $\overline{\partial}_{T'X}$, we prove that our $\phi(t)$ converges on $\{t : t \in C^q, \ | t |< \epsilon\}$ where $\epsilon$ is chosen to be a sufficiently small positive number.

There are several similarities between our case (the case of CR structures over the link $M = V \cap S_\epsilon^{2N-1}$) and the above case (the complex structure case). For example, over the link, we have $\overline{\partial}_b$-operator and $T'$ bundle (correspond to the standard $\overline{\partial}$-operator and the holomorphic tangent bundle). Therefore it is quite natural to try to construct deformation theory of CR structures over the link just like the compact complex manifold case. If we adopt Kodaira-Spencer's method in the CR case, the deformation complex should be

$$0 \to \Gamma(M,T') \overset{\overline{\partial}_{T'}}{\to} \Gamma(M,T' \otimes ({}^0T'')^*) \overset{\overline{\partial}_{T'}}{\to} \cdots$$
$$\overset{\overline{\partial}_{T'}}{\to} \Gamma(M,T' \otimes \wedge^p({}^0T'')^*) \overset{\overline{\partial}_{T'}}{\to} \Gamma(M,T' \otimes \wedge^{p+1}({}^0T'')^*) \overset{\overline{\partial}_{T'}}{\to} \cdots$$

(note that this complex is subellptic), where we denote

$$T' = \overline{{}^0T''} + C\zeta,$$

and $\zeta$ is a supplementary vector field of $\overline{{}^0T''} + {}^0T''$ (note that its choice is not canonical), and $\overline{\partial}_{T'}$ is the tangential Cauchy-Riemann operator associated with the holomorphic vector bundle $T'$. For this $\overline{\partial}_{T'}$, if $\dim_R M = 2n-1 \geq 5$, we have the Neumann operator $N$ satisfying for $\mu \in \Gamma(M,T' \otimes ({}^0T'')^*)$

$$\mu = \Box_{T'} N_{T'} \mu + H_{T'} \mu,$$

just like the Green operator for a compact complex manifold. However, there is one major difference between them. Even in the strongly pseudo-convex case, only 1/2 estimate holds for the $\overline{\partial}_b$-Neumann problem. Hence the Neumann operator gains only 1 derivative in the strongly pseudo-convex CR manifolds case, in contrast to the compact complex manifolds case where the Green operator gains 2 derivatives. Therefore in proving the convergence of the formal solution, we encounter severe difficulty.

To avoid this difficulty, Kuranishi [Ku4] proceeded as follows: He added a new equation (it resembled $\overline{\partial}^* \phi = 0$, but a complicated one) to the defomation equation, and fortunately this system of partial differential equations can be solved by using the Nash method (it is impossible to be solved by the Banach-inverse mapping theorem) (see [Ku1], [Ku2])). By this method, he obtained a versal family of CR structures. However, because of using Nash-Moser's inverse mapping theorem, he could not put a complex structure over the parameter space of this versal family. In order to improve this point, we proposed a new technique.

Our approach is to follow Kodaira-Spencer's method. Of course, we have to overcome the above analytical difficulty. Here is our approach. Even though the Neumann operator gains only 1 derivative, it still gains 2 derivatives in the direction ${}^0T'' + \overline{{}^0T''}$. So noting this fact, we follow the following line.

(Step 1) We establish the deformation theory of CR structures which vary in the direction ${}^0T'' + \overline{{}^0T''}$.

(Step 2) We obtain a new Neumann type operator which corresponds to Step 1 (obviously, we have to show a new a priori estimate).

This project was succesfully done in the case of $\dim_R M \geq 7$. Our result is that we find a suitable solution for the $\overline{\partial}_{T'}$-equation and fortunately it works well in the deformation theory of CR structures.

This work leads us to a study of the relation between Hodge theory of isolated singularities and deformations of CR structures (cf. [Ak-My2]). This will be discussed in Part II in this book. And there, Miyajima will give an idea about the application of the deformation theory of CR structures to deformations of normal isolated singularities, in the case of $\dim_R M \geq 5$.

## §1. Kuranishi's original approach

We start with recalling Kuranishi's original approach to deformation theory of isolated singularities, and discuss several problems, which arose from his work. In the Introduction, we wrote that we improved his result, but from the beginning of our work, it seems that the point of view of Kuranishi is different from ours. Even though in his paper he wrote that he initiated his work in order to construct the versal family of CR structures, his main interest seems to be a geometry of real hypersurfaces (it reminds readers of Cartan's work). With this in mind, we briefly sketch his approach.

### 1.1. Deformation equation

Let $N$ be a complex $n$-dimensional manifold. Let $M$ be a real hypersurface of $N$. Then, a CR-structure ${}^0T''$ on $M$ is induced from the complex structure of $N$. That is to say,

$${}^0T'' = C \otimes TM \cap T''N \mid_M .$$

By using a local coordinate of $N$, this is explicitly written as follows. We assume that, for a reference point $p$ of $M$, we take a coordinate neighborhood $U$ of $p$ in $N$, and a system of complex coordinates $(z^1, \ldots, z^n)$.

Let $r = 0$ be a $C^\infty$ defining equation of $M$ (we assume $dr \neq 0$ on $M$). We use the notation

$$
\begin{aligned}
r_j &= \partial r / \partial z^j, \\
r_{\overline{j}} &= \partial r / \partial \overline{z}^j.
\end{aligned}
\tag{1.1.1}
$$

Then our ${}^0T''$ is written as follows.

$$
{}^0T'' = \{ \sum_{j=1}^n a^{\overline{j}} \partial / \partial \overline{z}^j \mid \sum_{j=1}^n a^{\overline{j}} r_{\overline{j}} = 0 \}. \tag{1.1.2}
$$

We put a hermitian metric on $N$. With respect to this metric, we consider the dual vector field $P'$ of $\partial r$ (resp. the dual vector field $P''$ of $\overline{\partial} r$). We set a real supplementary vector field $P$ to ${}^0T'' + \overline{{}^0T''}$ by

$$
\sqrt{-1} P = P' - P''. \tag{1.1.3}
$$

Now we set for $j = 1, \ldots, n$,

$$
Z_{\overline{j}} = \partial / \partial \overline{z}^j - r_{\overline{j}} P''. \tag{1.1.4}
$$

Then our ${}^0T''_{|U \cap M}$ is generated by $Z_{\overline{j}}, j = 1, \ldots, n$. If we set

$$
P' = \sum_{j=1}^n p^j \frac{\partial}{\partial z^j}, \; P'' = \sum_{j=1}^n p^{\overline{j}} \frac{\partial}{\partial \overline{z}^j},
$$

then we have

$$
\sum_j p^j h_j = \sum_j p^{\overline{j}} h_{\overline{j}} = 1, \; p^{\overline{j}} = \overline{p^j},
$$

and there is one relation among the $Z_{\overline{j}}$'s:

$$
\sum_{j=1}^n r_{\overline{j}} Z_{\overline{j}} = 0.
$$

Next let

$$
Z^{\overline{k}} = i^* d\overline{z}^k - p^{\overline{k}} i^* d'' h
$$

where $i : U \cap M \hookrightarrow U$ is the injection and $d'' h = \sum_j \frac{\partial}{\partial \overline{z}^j} d\overline{z}^k$. Then $Z^{\overline{1}}, \ldots, Z^{\overline{n}}$ generate $({}^0T'')^*_{|U \cap M}$ and satisfy

$$
\sum_{j=1}^n r_{\overline{k}} Z_{\overline{k}} = 0.
$$

We set

$$T' = \overline{{}^0T''} + P,$$

and consider the natural isomorphism from $T'$ to $T'N\,|_M$, induced from the inclusion map $T' \hookrightarrow CTM$ and the projection map $CTN\,|_M \rightarrow T'N\,|_M$. We use the notation $\tau$ for the inverse map of this isomorphism:

$$\tau : T'N\,|_M \rightarrow T'.$$

Then an element $\phi \in \Gamma(M, \mathrm{Hom}({}^0T'', (T'N)_{|M}))$ defines a subbundle ${}^\phi T''$ of $CTM$ by

$${}^\phi T'' = \{X - \tau \circ \phi(X) \mid X \in {}^0 T''\}.$$

${}^\phi T''$ is an almost CR structure on $M$ (cf. §2.1). The condition that ${}^\phi T''$ is a CR structure was described by Kuranishi as follows:

**Theorem 1.1.1.** *(see Theorem 3.1 in [Ku1]) Let $\phi \in \Gamma(M, (T'N)\,|_M \otimes ({}^0T'')^*)$ be sufficiently small so that ${}^\phi T''$ is defined. Let $z = (z_1, \ldots, z_n)$ be a chart of $N$. Write*

$$\phi = \sum_{k=1}^{k=n} \phi^k \partial/\partial z^k, \;\; \phi^k = \sum_{l=1}^{l=n} \phi^k_{\bar{l}} Z^{\bar{l}},$$

*where $\sum_{l=1}^{k=n} p^{\bar{l}} \phi^k_{\bar{l}} = 0$. Then*

$$P(\phi) = \bar{\partial}_b \phi - \sum_{j,k,l} (\partial^\tau \phi^k_{\bar{l}} / \partial z^j) \phi^j \wedge Z^{\bar{l}} (\partial/\partial z^k)$$
$$+ \sum_i h_i \phi^i \wedge \sum_{k,l} (\bar{\partial}_b p^{\bar{l}} - \sum_j (\partial^\tau p^{\bar{l}} / \partial z^j) \phi^j) \phi^k_{\bar{l}} (\partial/\partial z^k)$$

*is independent of the choice of the chart $z$. ${}^\phi T''$ is integrable if and only if*

$$P(\phi) = 0$$

*(cf. §2.1 for $\bar{\partial}_b$).*

## 1.2. Kuranishi's construction of the versal family of CR structures

As mentioned in the Introduction, our contribution is that we can apply Kodaira-Spencer's method to the deformation theory of CR structures. For the local structure of the "moduli space of CR structures", our contribution would be enough. However, in order to study a global "moduli", we surely have to adopt Kuranishi's line. But (1.2.1) below is not suitable. "A modified new construction" will be necessary, and this would lead to a kind of invariants as in Seiberg-Witten invariants ([Don]) for non-singular compact manifolds. If this is introduced, surely, this invariant must be an invariant of the isolated singularity $(V, x)$. So in order to understand an isolated singularity, our CR geometrical method would give a very important device for isolated singularities. In any case, we recall the family which Kuranishi constructed. Note that an almost CR structure ${}^{\phi}T''$ induces the operator $\overline{\partial}_{{}^{\phi}T''} : \Gamma(M, C) \to \Gamma(M, ({}^{\phi}T'')^*)$. Kuranshi considered the operator $\overline{\partial}^{\phi}{}_b : \Gamma(M, C) \to \Gamma(M, ({}^{0}T'')^*)$ corresponding to $\overline{\partial}_{{}^{\phi}T''}$ under the natural isomorphism $\lambda^{\phi} :{}^{0} T'' \to^{\phi} T''$. Then we consider the set of $\psi \in \Gamma(M, T' \otimes ({}^{0}T'')^*)$ satisfying

$$\text{(1.2.1)} \qquad \begin{aligned} &\rho^{\psi}\psi = \rho^{\psi}t \\ &N^{\psi}(\overline{\partial}_b^{*\psi} P(\psi) + \overline{\partial}_b^{\psi}\overline{\partial}_b^{\sharp\psi}\psi) = 0. \end{aligned}$$

Here $\rho^{\psi}$ means the harmonic projection operator with respect to the $\overline{\partial}^{\psi}{}_b$-harmonic theory. We do not explain the notation in detail. See [Ku1] for the precise definitions. If $\psi$ is so small, this set coincides with the set of $\psi$ satisfying

$$\begin{aligned} P(\psi) &=0 \\ \overline{\partial}_b^{\sharp\psi}\psi &=0, \end{aligned}$$

where $\overline{\partial}_b^{\sharp\phi}$ means the "modified " adjoint operator of $\overline{\partial}^{\phi}$ with respect to the Levi metric.

Therefore the family constructed by Kuranishi seems to be a natural extension of Kuranishi's family in the compact complex manifolds case. We note that, in the compact complex manifold case, Kuranishi gave a complex analytic structure on the set of small $\phi \in A_M^{0,1}(T'M)$ satisfying

$$\begin{aligned} P(\phi) &= 0 \\ \overline{\partial}_b^{*}\phi &= 0. \end{aligned}$$

And as mentioned in the Introduction, this method (adding the new equation $\overline{\partial}_b^{*}\phi = 0$) is not available in the CR case (because Kohn's Neumann operator gains only 1 derivative, not like the Green operator). For this reason, even by the Nash-technique, we cannot solve the equation without modifying $\overline{\partial}_b^{*\psi}$. Here, "to solve " means that there is a finite dimensional Euclidean space $\mathcal{H}$ such that a small neighborhood of the origin parametrizes a neighborhood of the solution space of (1.2.1).

## 1.3. $\mathbf{C^\infty}$ parametrization of real hypersurfaces

We have to explain that the above family of solutions of (1.2.1) has a special geometrical meaning. For this, we must describe the "moduli space" of real hypersurfaces in a complex manifold, which are "very close" to the original real hypersurface $M$. If $M$ and $M'$ are both real hypersurfaces of the same complex manifold $N$ and close in the $C^\infty$ sense, then $M'$ is called "very close" to $M$. Kuranishi showed that if a real hypersurface in $N$ is close enough to $M$, then this real hypersurface corresponds to an element $\zeta \in \Gamma(M, T')$, a $T'$-valued global vector field. We can obtain this real hypersurface by wiggling the original real hypersurface $M$ in $N$ under a diffeomorphism generated by $\zeta \in \Gamma(M, T')$. In this way, we have a map from a small neighborhood of the origin of $\Gamma(M, T')$ into $\Gamma(M, T' \otimes ({}^0T'')^*)$ such that its linearization is $\overline{\partial}_{T'}$.

## 1.4. Versality (Equivalence problem)

Now we see the geometrical aspect of (1.2.1). Kuranishi proved the following property ([Ku1]): For any given family of deformations of complex manifold $N$, denoted $N_\omega$, there is an embedding $f$ of $M$ into $N_\omega$ and an element $t$ of $\mathcal{H}$ such that ${}^{\psi(t)}T''$ is "very close" to the structure induced by $f$. Namely, there is a complex manifold $\mathcal{N}$ with boundary such that there is a smooth map $\rho$ from $\mathcal{N}$ to the interval [0,1] and the boundary $= \rho^{-1}(0) \cup \rho^{-1}(1)$, and

$$\begin{aligned}
\rho^{-1}(0) &= (M, {}^{\psi(t)}T'') \\
\rho^{-1}(1) &= (M, \text{ the CR structure induced from} \\
&\qquad\qquad \text{the complex structure } N_\omega \text{ by } f).
\end{aligned}$$

There are several problems which should be considered in the spirit of the Kuranishi deformation theory of CR structures.

**Problem 1**. To determine holomorphic convex hulls.

For a subset $M$ of $N$, in general, it is difficult to determine the holomorphic convex hull $\widetilde{M}$ of $M$ in $N$. In fact, $\widetilde{M}$ is defined by

$$\widetilde{M} = \{p \,:\, p \in N, \mid f(p) \mid \leq \sup_{q \in M} \mid f(q) \mid, \quad \text{for any holomorphic function on } N\}.$$

Similarly, the local holomorphic convex hull is defined as follows. For a reference point $p$ of $M$, and for a neighborhood $U$ of $p$ in $N$,

$$\widetilde{M \cap U} = \{p \ : \ p \ \in \ U, \ | \ f(p) \ | \leq \sup_{q \in U \cap M} \ | \ f(q) \ |,$$
$$\text{for any holomorphic function on } U\}.$$

So $\widetilde{M}, \widetilde{M \cap U}$ are defined by highly transcendental method. The problem is to construct holomorphic convex hulls $\widetilde{M}, \widetilde{M \cap U}$ by using deformation theory of $M$ or $M \cap U$ (rather, it is better to say a replacement of $M$, $M \cap U$ in $N$ respectively) and a geometry of ${}^0T''$ on $M$. Obviously, this problem is closely related to Kuranishi's problem "to prove Rossi's filling holes theorem by a geometrical method".

**Problem 2**. Equivalence problem.

The standard equivalence problem of real hypersurfaces were solved by Cartan, Tanaka, Chern-Moser. However, this equivalence is very strong. Namely, let $M$ be a real hypersurface in a complex manifold $N$, and let $M'$ be a real hypersurface in a complex manifold $N'$. In Chern-Moser's sense, in the real analytic category, the CR structure on $M$, induced from $N$ is equivalent to the CR structure on $M'$, induced from $N'$ if and only if there is a biholomorphic map from a neighborhood of $M$ in $N$ to a neighborhood of $M'$ in $N'$. Our equivalence differs from this. We assume that $M, M'$ are both strongly pseudo-convex. Then, by Rossi's theorem with Stein factorization theorem, we have two Stein spaces, $V, V'$. The problem is that if $V$ and $V'$ are isomorphic to each other as germs of isolated singularities, is it possible to express this situation in the CR geometrical way? Furthermore, the Stein factorization procedure is highly transcendental. Is it possible to construct a Stein space $V$ from CR geometry on $M$? So, Problem 2 is somewhat related to Problem 1.

**Problem 3**. Seiberg-Witten type invariants for isolated singularities.

The reason we posed this problem is that our approach is the so-called coordinate-free approach, so in this sense, our approach seems to be accessible to the introduction of a kind of "Seiberg-Witten invariants" for CR-structures just as in the differential geometric and topological way (though the Levi metric works in the construction). However, what we really need is an invariant for isolated singularities, which characterizes the "global moduli space of isolated singularities". At present, we cannot overcome this difficulty.

From the next section on, we recall our improvement. We hope that our setup could be of any help for the solution of the above mathematical problems.

## §2. CR structures and $\overline{\partial}_b$

We begin with recalling the definition of $\overline{\partial}_b$. Sometimes, we use notation different from Kuranishi's one.

### 2.1. $\overline{\partial}_b$-operator

Let $N$ be a complex manifold of complex dimension n. Let $M$ be a smooth real hypersurface in $N$. This means that for every point $p$ of $M$, there is a local defining function $\rho$ of $M$ over a neighborhood of $p$, satisfying $(d\rho)(p) \neq 0$. Then as is well known, over this $M$, we can introduce the tangential Cauchy-Riemann structure ${}^0T''$. Namely let

$$ {}^0T'' = C \otimes TM \cap T''N \,|_M \,. \tag{2.1.1} $$

Then, this ${}^0T''$ satisfies

$$ {}^0T'' \cap \overline{{}^0T''} = 0 \,, \ \dim_C \frac{C \otimes TM}{{}^0T'' + \overline{{}^0T''}} = 1 \tag{2.1.2} $$

$$ [\Gamma(M, {}^0T''), \Gamma(M, {}^0T'')] \subset \Gamma(M, {}^0T''). \tag{2.1.3} $$

And we can define the tangential Cauchy-Riemann operator $\overline{\partial}_b$. Namely, for any $C^\infty$ function $f$ in $M$, we set an element of $\Gamma(M, ({}^0T'')^*)$ by

$$ \overline{\partial}_b f(X) \ = \ Xf, \ X \in {}^0T''. $$

And we have a differential complex

$$ \begin{aligned} 0 \to \Gamma(M, C) \to \Gamma(M, ({}^0T'')^*) \to \Gamma(M, \wedge^2({}^0T'')^*) \to \cdots \\ \to \Gamma(M, \wedge^p({}^0T'')^*) \to \Gamma(M, \wedge^{p+1}({}^0T'')^*) \to \cdots . \end{aligned} $$

The explicit form of $\overline{\partial}_b$ is given by

$$ \overline{\partial}_b f = \sum_k \frac{\partial f}{\partial \overline{z}_k} Z^{\overline{k}} $$

in terms of the notation in Sect. 1.1.

This notion of ${}^0T''$ is generalized to an intrinsic structure on $M$ as follows. Let $M$ be a $C^\infty$ manifold with real dimension $2n - 1$. We assume that $M$ is orientable. Let $E$ be a subbundle of the complexified tangent bundle $C \otimes TM$ satisfying

$$E \cap \overline{E} = 0, \ \dim_C \frac{C \otimes TM}{E + \overline{E}} = 1, \tag{2.1.4}$$

$$[\Gamma(M, E), \Gamma(M, E)] \subset \Gamma(M, E), \tag{2.1.5}$$

where $\Gamma(M, E)$ denotes the space consisting of $E$-valued $C^\infty$ sections. $E$ is called a CR structure on $M$ and the pair $(M, E)$ a CR manifold. For a CR manifold $(M, E)$, we can introduce a natural $\overline{\partial}_b$-operator in the same manner as above:

$$\overline{\partial}_b^{(p)} \ : \ \Gamma(M, \wedge^p E^*) \to \Gamma(M, \wedge^{p+1} E^*).$$

If there is no confusion, we abbreviate $\overline{\partial}_b$ for $\overline{\partial}_b^{(p)}$. And we have a differential complex

$$\begin{aligned} 0 \to \Gamma(M, C) \to \Gamma(M, E^*) \to \Gamma(M, \wedge^2 E^*) \to \cdots \\ \to \Gamma(M, \wedge^p E^*) \to \Gamma(M, \wedge^{p+1} E^*) \to \cdots. \end{aligned}$$

For an orientable CR manifold $(M, {}^0T'')$, we set a $C^\infty$ vector bundle decomposition

$$C \otimes TM = {}^0T'' + {}^0T' + C \otimes F, \tag{2.1.6}$$

where ${}^0T' = \overline{{}^0T''}$ and $F$ is a non-vanishing real $C^\infty$ vector field on $M$ satisfying for every point $p$ of $M$,

$$F_p \notin {}^0T'_p + {}^0T''_p,$$

and $C \otimes F$ means the line bundle generated by $F$. For each point $p$ of $M$, we define a hermitian form $L_p$ on ${}^0T''_p$ by

$$L_p(X, Y)F_p = -\sqrt{-1}[X', \overline{Y'}]_{C\otimes F}(p) \qquad \text{for } X, \ Y \in {}^0T''_p \tag{2.1.7}$$

where $X'$, $Y'$ are in $\Gamma(M, {}^0T'')$ such that $X'_p = X$ and $Y'_p = Y$ hold, and $[X', \overline{Y'}]_{C\otimes F}$ denotes the projection of $[X', \overline{Y'}]$ to $C \otimes F$ according to the splitting (2.1.6). $L_p$ is called the Levi-form at $p$ and a CR manifold $(M, {}^0T'')$ is called strongly pseudo- convex if $L_p$ has definite sign at every point $p$ of $M$.

In the case that $(M, {}^0T'')$ is a CR manifold as in Sect.1.1, we can choose a local coordinate $(z_1, \ldots, z_n)$ of $N$ such that

$$r \ = \ 2\mathrm{Im}\ z_n \ - \ h(z_1, \ldots, z_{n-1}, \overline{z}_1, \ldots, \overline{z}_{n-1}, \mathrm{Re}\ z_n)$$

where $h$ is a real valued $C^\infty$ function satisfying grad $h(p) = 0$. Then $(M, {}^0T'')$ is strongly psudoconvex if and only if the complex Hessian $(\partial^2 h/\partial z_i \partial \overline{z}_j\ (p))_{1 \leq i,j \leq n-1}$ is positive or negative definite.

### 2.2. $T'$-bundle and $\overline{\partial}_{T'}$-operator

Let $(M, {}^0T'')$ be an orientable CR manifold and fix the splitting (2.1.6). We set

$$T' = {}^0T' + C \otimes F.$$

Then, this $T'$-bundle admits a CR structure in the following sense. For $u$ in $\Gamma(M, T')$, we set a first order differential operator

$$\overline{\partial}_{T'} \ :\ \Gamma(M, T') \ \rightarrow\ \Gamma(M, T' \otimes ({}^0T'')^*)$$

by $\overline{\partial}_{T'} u(X) = [X, u]_{T'}$ for $X \in {}^0T''$. We have to explain this definition more precisely. For each point $p$ of $M$, for $X \in {}^0T''_p$, we take $X' \in \Gamma(M, {}^0T'')$ satisfying

$$X'_p = X.$$

$\overline{\partial}_{T'} u(X)$ is determined by

$$\overline{\partial}_{T'} u(X) = [X', u]_{T'},$$

where $[X', u]_{T'}$ means the $T'$-part of $[X', u]$ according to the splitting (2.1.6). Obviously, this definition makes sense. Because for any $C^\infty$ function, for any $Z \in \Gamma(M, {}^0T'')$, and for any $u \in \Gamma(M, T')$,

$$\begin{aligned}[fZ, u]_{T'} &= (-u(f)Z + f[Z, u])_{T'} \\ &= f[Z, u]_{T'}.\end{aligned}$$

This means that our definition does not depend on the $C^\infty$ extension of $X$. As for scalar valued differential forms, we can define $\overline{\partial}^{(p)}_{T'}$-operator. For example, for $\phi \in \Gamma(M, T' \otimes ({}^0T'')^*)$,

$$\overline{\partial}^{(1)}_{T'} \phi(X, Y) = [X, \phi(Y)]_{T'} - [Y, \phi(X)]_{T'} - \phi([X, Y]) \ for\ X, Y \in {}^0T''\ .$$

Then it satisfies $\overline{\partial}^{(1)}_{T'} \overline{\partial}_{T'} = 0$ (hence $T'$ is a holomorphic vector bundle over a CR manifold in N. Tanaka's sense ([Ta])). And we have a differential complex

$$\begin{aligned}0 \rightarrow \Gamma(M, T') \rightarrow \Gamma(M, T' \otimes ({}^0T'')^*) \rightarrow \Gamma(M, T' \otimes \wedge^2 ({}^0T'')^*) \rightarrow \cdots \\ \rightarrow \Gamma(M, T' \otimes \wedge^p ({}^0T'')^*) \rightarrow \Gamma(M, T' \otimes \wedge^{p+1} ({}^0T'')^*) \rightarrow \cdots.\end{aligned}$$

We note that the $T'$-bundle is a generalization of the holomorphic tangent bundle. In fact, if M is a real hypersurface in a complex manifold $N$, we consider

$$T'N\,|_M,$$

the restriction of the holomorphic tangent bundle $T'N$ to the real hypersurface $M$. Then the composite of the inclusion map of $T'$ to $C\otimes TM \subset C\otimes TN$ and the projection of $C\otimes TN$ to $T''$, induces an isomorphism $i$ from $T'$ to $T'N\,|_M$ and preserves

$$\overline{\partial}_{T'N} i(u(X)) = i((\overline{\partial}_{T'}u)(X)), \qquad \text{for} \quad X \in {}^0T'',$$

if $u$ satisfies some conditions (this will be discussed in §4.1), where $\overline{\partial}_{T'N}$ means the standard $\overline{\partial}$- operator on $N$, and for $X \in {}^0T''$, the left hand side makes sense.

## §3. Geometry on deformations of CR manifolds

In this section, we briefly recall the deformation theory of strongly pseudo-convex CR-structures. Throughout this section, $(M,{}^0T'')$ is a strongly pseudo-convex compact CR manifold and we fix the splitting (2.1.6). For the detailed discussion, see [Ak1], [Ak2], [Ak3].

### 3.1. Almost CR manifolds

Let $E$ be an almost CR structure on $M$. Then, by using the $C^\infty$ vector bundle decomposition (2.1.6), we have a homomorphism from $E$ to ${}^0T''$, the composite of the inclusion of $E$ to $C\otimes TM$ and the projection of $C\otimes TM$ to ${}^0T''$.

**Definition 3.1.1.** *Let $(M,{}^0T'')$ be a CR manifold. An almost CR structure $E$ is of finite distance from $(M,{}^0T'')$ if the above homomorphism is an isomorphism.*

**Proposition 3.1.2.** *Let $(M,{}^0T'')$ be a CR manifold and $E$ an almost CR structure of finite distance from ${}^0T''$. Then there exists a $\phi \in \Gamma(M, T' \otimes ({}^0T'')^*)$ satisfying*

$$\begin{aligned} E &= {}^\phi T'' \\ &= \{X'; X' = X + \phi(X), X \in {}^0T''\}. \end{aligned}$$

Namely, $\phi$ defines a bundle homomorphism ${}^0T'' \to T'$ whose graph coincides with $E$. For the proof, see [Ak1].

### 3.2. Deformation equation

By Proposition 3.1,2, we see that for a given CR manifold $(M,{}^0T'')$, an almost CR manifold of finite distance from $(M,{}^0T'')$ is parametrized by $\Gamma(M, T'\otimes({}^0T'')^*)$. Now, in this section, we see under what condition this ${}^\phi T''$ is actually a CR structure. For this, we have to introduce notation. Let $\phi$ be an element of $\Gamma(M, T'\otimes({}^0T'')^*)$. We set elements,$R_2(\phi)$ and $R_3(\phi)$ of $\Gamma(M, T'\otimes\wedge^2({}^0T'')^*)$ by

(3.2.1)

$$R_2(\phi)(X,Y) =[\phi(X),\phi(Y)]_{T'} - \phi([X,\phi(Y)]_{{}^0T''} + [\phi(X),Y]_{{}^0T''})$$

(3.2.2)

$$R_3(\phi)(X,Y) =\phi([\phi(X),\phi(Y)]_{{}^0T''})$$

for $X$,$Y$ in $\Gamma(M,{}^0T'')$.

We see that these $R_2(\phi)$,$R_3(\phi)$ make sense as elements of $\Gamma(M, T'\otimes\wedge^2({}^0T'')^*)$. In fact, for any $C^\infty$ functions $f$ and $g$, and $X', Y' \in \Gamma(M, {}^0T'')$, by a simple direct computation of brackets, we have

(3.2.3)

$$\begin{aligned} &R_2(\phi)(fX',gY')\\ &\quad=[\phi(fX'),\phi(gY')]_{T'} - \phi([fX',\phi(gY')]_{{}^0T''} + [\phi(fX'),gY']_{{}^0T''}\\ &\quad=fg\{[\phi(X'),\phi(Y')]_{T'} - \phi([X',\phi(Y')]_{{}^0T''}) + [\phi(X'),Y']_{{}^0T''})\}. \end{aligned}$$

By (3.2.3), $R_2(\phi)(X'_p, Y'_p)$ depends only on $X_p$ and $Y_p$. Hence $R_2(\phi)$ is an element of $\Gamma(M, T'\otimes\wedge^2({}^0T'')^*)$. Obviously $R_3(\phi)\in\Gamma(M, T'\otimes\wedge^2({}^0T'')^*)$ holds for the same reason.

In this notation, we have

**Proposition 3.2.1.** *(see Theorem 2.1 in [Ak1]) Let $\phi$ be an element of $\Gamma(M, T'\otimes({}^0T'')^*)$. Then an almost CR structure $(M,{}^\phi T'')$ is a CR structure if and only if $\phi$ satisfies the following non-linear equation:*

$$\begin{aligned} P(\phi) &=\overline{\partial}^{(1)}_{T'}\phi + R_2(\phi) + R_3(\phi)\\ &=0. \end{aligned} \tag{3.2.4}$$

### 3.3. $E_j$-structures

Now we recall the subbundles $E_j$, which played quite successful roles in deformation theory of CR-structures. We set a subspace $\Gamma_i$ of $\Gamma(M, T'\otimes\wedge^i({}^0T'')^*)$ by

$$\Gamma_i = \{\ u\ ; u\in\Gamma(M, {}^0T'\otimes\wedge^i({}^0T'')^*), (\overline{\partial}^{(i)}_{T'}u)_{C\otimes F} = 0\},$$

where $(\overline{\partial}^{(i)}_{T'}u)_{C\otimes F}$ denotes the projection of $\overline{\partial}^{(i)}_{T'}u$ to $C\otimes F\otimes\wedge^{i+1}({}^0T'')$ according to (2.2.1). Then we have

**Theorem 3.3.1.** *(see Proposition 2.1 in [Ak3]) There is a subbundle $E_i$ of $T'\otimes\wedge^i({}^0T'')^*$ satisfying*

$$\Gamma_i = \Gamma(M, E_i).$$

*And there is a differential subcomplex*

$$0\to\Gamma(M,E)_1)\xrightarrow{\overline{\partial}_1}\Gamma(M,E_2)\xrightarrow{\overline{\partial}_2}\Gamma(M,E_3)\xrightarrow{\overline{\partial}_3}\cdots$$
$$\xrightarrow{\overline{\partial}_{i-1}}\Gamma(M,E_i)\xrightarrow{\overline{\partial}_i}\Gamma(M,E_{i+1})\xrightarrow{\overline{\partial}_{i+1}}\cdots$$

*where $\overline{\partial}_i$ means the restriction of $\overline{\partial}^{(i)}_{T'}$ to $\Gamma(M,E_i)$.*

By Theorem 3.3.1 we have an injection $i:\mathrm{Ker}\ \overline{\partial}_i\hookrightarrow\mathrm{Ker}\ \overline{\partial}^{(i)}_{T'}$.

**Theorem 3.3.2.** *(see Theorems 2.3 and 2.4 in [Ak3]) The injection induces an isomorphism*

$$i:\ \mathrm{Ker}\ \overline{\partial}_i/\,\mathrm{Im}\ \overline{\partial}_{i-1}\to\mathrm{Ker}\ \overline{\partial}^{(i)}_{T'}/\,\mathrm{Im}\ \overline{\partial}^{(i-1)}_{T'},$$

*where $2\le i\le n-1$, and the surjective map*

$$i:\mathrm{Ker}\ \overline{\partial}_1\to\mathrm{Ker}\ \overline{\partial}^{(1)}_{T'}/\,\mathrm{Im}\ \overline{\partial}^{(0)}_{T'}.$$

### 3.4. Local expression for $\mathbf{E_j}$.

The explicit expression for the differential complex $(\Gamma(M,E_i),\overline{\partial}_i)$ is as follows. We briefly recall only the results. For the proof see [Ak2], [Ak3].

Let $\{U_k,h_k\}_{k\in K}$ be a local coordinate covering of $M$ such that $K$ is a finite set and $U_k$ is homeomorphic to $R^{2n-1}$. And let $\{\rho_k\}_{k\in K}$ be a partition of unity subordinate to the coordinate covering of $M$. Since $(M,{}^0T'')$ is strongly pseudo-convex, there exists a moving frame $\{e^k_1,.,,e^k_{n-1}\}$ of ${}^0T''\,|_{U_k}$ such that

$$[e^k_i,\overline{e}^k_j]_{C\otimes F}=\sqrt{-1}\delta_{i,j}F.$$

By using these frames, we have the following lemmas. For the proof, see Lemmas 3.1–3.4 in [Ak3] respectively.

**Lemma 3.4.1.** *Let $\phi$ be an element of $\Gamma(M, {}^0T' \otimes ({}^0T'')^*)$. Then $\phi$ belongs to $\Gamma(M, E_1)$ if and only if*

$$\phi^k_{i,j} - \phi^k_{j,i} = 0 \ for\ 1 \leq i, j \leq n-1,$$

*where by $\phi^k_{i,j}$ we denote the $C^\infty$-functions defined by*

$$\phi(e^k_j) = \sum_i \phi^k_{i,j}\bar{e}^k_i.$$

As for the vector bundle $E_2$, we have the following lemma.

**Lemma 3.4.2.** *Let $\phi$ be an element of $\Gamma(M, {}^0T' \otimes \wedge^2({}^0T'')^*)$. Then $\phi$ belongs to $\Gamma(M, E_2)$ if and only if*

$$\phi^k_{i,(j,\alpha)} - \phi^k_{j,(i,\alpha)} + \phi^k_{\alpha,(i,j)} = 0$$

*for all i,j,$\alpha$ satisfying $1 \leq i, j, \alpha \leq n-1$, where $\phi^k_{\alpha,(i,j)}$ denotes the $C^\infty$-function defined by*

$$\phi(e^k_i, e^k_j) = \sum_\alpha \phi^k_{\alpha,(i,j)}\bar{e}^k_\alpha.$$

**Lemma 3.4.3.** *For $\phi \in \Gamma(M, E_1)$, we have*

$$(\bar{\partial}_1\phi)^k_{\alpha,(i,j)} = e^k_i\phi^k_{\alpha,j} - e^k_j\phi^k_{\alpha,i} + the\ terms\ of\ order\ zero\ of\ \phi$$

*and for $\mu \in (M, E_2)$,*

$$(\bar{\partial}_2\mu)^k_{\alpha,(i,j,l)} = e^k_i\mu^k_{\alpha,(j,l)} - e^k_j\mu^k_{\alpha,(i,l)} + e^k_l\mu^k_{\alpha,(i,j)} + the\ terms\ of\ order\ zero\ of\ \mu.$$

We put the inner product on $\Gamma(M, E_1)$, induced by the Levi metric. Let $\bar{\partial}^*_1$ denotes the adjoint operator of $\bar{\partial}_1$. Then the following lemma follows from these lemmas. Here we remark that $\{e^k_1, e^k_2, \ldots, e^k_{n-1}\}$ are orthonormal with respect to this inner product.

**Lemma 3.4.4.** *For all $\phi$ in $\Gamma(M, E_1)$, $\bar{\partial}^*_1$ can be expressed by*

$$(\bar{\partial}^*_1\phi)^k_{\alpha,i} = -\frac{1}{2}\sum_j \bar{e}^k_j\phi^k_{\alpha,(i,j)} - \frac{1}{2}\sum_j \bar{e}^k_j\phi^k_{i,(j,\alpha)} + the\ terms\ of\ order\ 0\ of\ \phi\ .$$

**3.5. An a priori estimate.**

First, we introduce new norms $\|\phi\|'_{(m)}$ and $\|\phi\|''_{(m)}$ on $\Gamma(M, E_p)$. Let $\{U_k, h_k\}_{k\in K}$ be a local coordinate covering of $M$, $\{\rho_k\}_{k\in K}$ be a partition of unity subordinate to this coordinate covering, and $\{e_1^k, .., e_{n-1}^k\}$ the moving frame of ${}^0T''\ |_{U_k}$ as in §3.4. For $\phi \in \Gamma(M, E_p)$ and $I = (i_1, \ldots, i_p)$, a $C^\infty$-funciton $\phi_{\alpha,I}^k$ on $U_k$ is defined by

$$\phi(e_{i_1}^k, \ldots, e_{i_p}^k) = \sum_{\alpha} \phi_{\alpha,I}^k \bar{e}_\alpha^k.$$

Then we define the norms $\|\phi\|'_{(m)}$ and $\|\phi\|''_{(m)}$ respectively by

$$\begin{aligned}
\|\phi\|'^2_{(m)} = &\sum_{k\in K', i,\alpha,I} \|(\rho_k e_i^k \phi_{\alpha,I}^k) h_k^{-1}\|^2_{(m)} \\
&+ \sum_{k\in K', i,\alpha,I} \|(\rho_k \overline{e_i^k} \phi_{\alpha,I}^k) h_k^{-1}\|^2_{(m)} \\
&+ \|\phi\|^2_{(m)}, \\
\|\phi\|''^2_{(m)} = &\sum_{k\in K', i,j,\alpha,I} \|(\rho_k e_i^k e_j^k \phi_{\alpha,I}^k) h_k^{-1}\|^2_{(m)} \\
&+ \sum_{k\in K', i,j,\alpha,I} \|(\rho_k e_i^k \overline{e_j^k} \phi_{\alpha,I}^k) h_k^{-1}\|^2_{(m)} \\
&+ \sum_{k\in K', i,j,\alpha,I} \|(\rho_k \overline{e_i^k} e_j^k \phi_{\alpha,I}^k) h_k^{-1}\|^2_{(m)} \\
&+ \sum_{k\in K', i,j,\alpha,I} \|(\rho_k \overline{e_i^k e_j^k} \phi_{\alpha,I}^k) h_k^{-1}\|^2_{(m)} \\
&+ \|\phi\|^2_{(m+1)}.
\end{aligned}$$

By direct computation (using integration by parts ), we can prove the following theorem. For the notation, for example, $\|\ \|'_{(m)}$-norms, and $\|\ \|''_{(m)}$-norms, see [Ak3].

**Theorem 3.5.1.** *(see Theorem 4.1(new estimate) in [Ak3]). Suppose that $(M, {}^0T'')$ is strongly pseudo convex and $\dim_R M = 2n-1 \geq 7$. Then the following estimate holds.*

$$\|\bar{\partial}_1^* \phi\|^2_{(0)} + \|\bar{\partial}_2 \phi\|^2_{(0)} + \|\phi\|^2_{(0)} \geq C\|\phi\|'^2_{(0)}$$

*for all $\phi \in \Gamma(M, E_2)$, where $C$ is a positive constant.*

Following the standard functional analysis method with Theorem 3.5.1, we have Theorem 3.5.2.

**Theorem 3.5.2.** *(see Theorem 4.1 in [Ak3]). Under the assumption of Theorem 3.5.1, we have a Neumann operator*

$$N : \Gamma_2(M, E_2) \rightarrow \Gamma_2(M, E_2)$$

*such that*

*a) $N$ is bounded,*

*b) if $\phi \in \Gamma(M, E_2)$, $N\phi$ is also in $\Gamma(M, E_2)$,*

*c) if $N\phi = 0$, $\phi \in \Gamma(M, E_2)$, then $\overline{\partial}\phi = 0$, and $\overline{\partial}^*\phi = 0$, and*

*d) if $\phi \in \Gamma(M, E_2)$, then $\phi = \overline{\partial}\overline{\partial}^* N\phi + \overline{\partial}^*\overline{\partial}N\phi + \alpha$, $\alpha \in \mathbf{H}$ where $\mathbf{H}$ is the null space of $N$.*

*Here we use the notation $\Gamma_2(M, E_2)$ for the Hilbert space obtained as the completion of $\Gamma(M, E_2)$ with respect to the $L_2$-norm.*

### 3.6. Some estimates.

In this section, we recall some a priori estimates for the Neumann operator obtained in §3.5. By the standard argument, we have the following estimate from Theorem 3.5.1.

$$\|\phi\|'_{(m-1/2)} \leq C_m\{\|\overline{\partial}_2\phi\|_{(m)} + \|\overline{\partial}_1^*\phi\|_{(m)} + \|\phi\|_{(m)}\} \tag{3.6.1}$$

for all $\phi \in \Gamma(M, E_2)$, and

$$\|\phi\|'_{(m+1/2)} \leq C'_m\{\|\Box\phi\|_{(m-1/2)} + \|\phi\|_{(m-1/2)}\}, \tag{3.6.2}$$

for all $\phi \in \Gamma(M, E_2)$, where $\Box = \overline{\partial}_1\overline{\partial}_1^* + \overline{\partial}_2^*\overline{\partial}_2$.

More precisely, we have the following theorem.

**Theorem 3.6.1.** *(see Theorem 5.1 in [Ak3]). The following estimate holds:*

$$\|\phi\|''_{(m-1/2)} \leq C''_m\{\|\Box\phi\|_{(m-1/2)} + \|\phi\|_{(m-1/2)}\}$$

*for all $\phi \in \Gamma(M, E_2)$, where $m$ is a non-negative integer.*

Thus we have

**Corollary 3.6.2.** *(see Corollary 5.2 in [Ak3]).*

$$\|N\mu\|''_{(m-1/2)} \leq C'_m \|\mu\|_{(m-1/2)}$$

*for all* $\mu \in \Gamma(M, E_2)$, *where* $m$ *is a non-negative integer.*

**Lemma 3.6.3.** *(see Lemma 5.3 in [Ak3]). The following estimate holds.*

$$\|R_2(\phi)\|_{(m-1/2)} \leq C_m \|\phi\|'^2_{(m-1/2)},$$

*for all* $\phi \in \Gamma(M, E_2)$, *where we assume* $m \geq n+1$.

For the definition of $R_2(\phi)$, see (3.2.1) in this paper.

**Proposition 3.6.4.** *(see Proposition 5.4 in [Ak3]). Suppose that* $R_2(\phi)$ *is in* $\Gamma(M, E_2)$. *Then,*

$$\|\overline{\partial}_1^* N R_2(\phi)\|'_{(m-1/2)} \leq C_m \|\phi\|'^2_{(m-1/2)} \text{ for all } \phi \in \Gamma(M, E_2)$$

*holds.*

### 3.7. An application to the deformation theory of CR structures.

Let $(M, {}^0T'')$ be a compact strongly pseudo-convex CR manifold. By using the differential complex in Theorem 3.3.1, and a new Hogde type decomposition theorem in Theorem 3.5.2, we can discuss the deformation theory of CR structures.

Then, we have

**Theorem 3.7.1.** *(see Theorem 6.2 in [Ak3]). Under the assumption* $\dim_R M = 2n - 1 \geq 7$ *and* $H^2(M, T') = 0$, *there is an* $E_1$*-valued* $C^2$*-class section* $\phi(t)$, *parametrized complex analytically by a neighborhood* $U$ *of the origin in the Euclidean space* $\mathcal{H}$, *satisfying*

(1) $\phi(o) = 0$

(2) $P(\phi(t)) = \overline{\partial}_{T'}^{(1)} \phi(t) + R_2(\phi(t)) = 0$, *and*

(3) *the linear term of* $\phi(t)$ *is equal to* $\sum_{\lambda=1}^{q} \beta_\lambda t_\lambda$, *where* $\{\beta_\lambda\}_{1 \leq \lambda \leq q}$ *is a basic system of* $\mathcal{H}$, $q = \dim_C \mathcal{H}$ *and* $\{t_i\}_{1 \leq i \leq q}$ *are local coordinates of* $U$.

*Here* $\mathcal{H}$ *is a subspace of* $\Gamma(M, E_1)$ *such that* $\mathcal{H} \simeq \mathrm{Ker}\ \overline{\partial}_{T'}^{(1)} / \mathrm{Im}\ \overline{\partial}_{T'}^{(0)}$ *holds (cf. Theorem 3.3.2) and* $m$ *is a sufficiently large integer such that* $m \geq n+2$ *holds. (Note that* $R_3(\phi) = 0$ *holds for* $\phi \in \Gamma(M, E_1)$.*)*

This theorem is proved by the standard Kodaira-Spencer deformation theory using Lemma 3.7.3 and Proposition 3.7.4 below. Since its argument is a prototype of the ones in Section 4 and in Part II, we will give a brief sketch of the proof.

*A sketch of the proof.* Let $\phi(t)$ be a $\Gamma(M, E_1)$-valued holomorphic function and

$$\phi(t) = \sum \phi_{k_1 k_2 \dots k_q} t_1^{k_1} \dots t_q^{k_q}$$

be the power series expansion of $\phi(t)$ with $\phi(0) = 0$. For simplicity, we abbreviate

$$\phi(t) = \sum_{\lambda=1}^{\infty} \phi_\lambda(t),$$

where $\phi_\lambda(t)$ is a homogeneous polynomial of degree $\lambda$ in $(t_1, .., t_q)$. Let

$$\phi^\mu(t) = \sum_{\lambda=1}^{\mu} \phi_\lambda(t).$$

For any $\Gamma(M, E_1)$-valued holomorphic functions $\phi(t)$ and $\psi(t)$, we indicate by $\phi(t) \equiv_\mu \psi(t)$ that the power series expansion of $\phi(t) - \psi(t)$ in $(t_1, .., t_q)$ contains no term of degree $\lambda < \mu$.

Clearly the conditions (1) and (2) are equivalent to the system of congruence

$$(3.7.1)_\mu \qquad \overline{\partial}^{(1)}_{T'} \phi(t) + R_2(\phi(t)) \equiv_{\mu+1} 0 \ (\mu = 1, 2, \dots).$$

Since $R_2(\phi(t)$ is of second order with respect to $\phi(t)$, we obtain

$$(3.7.2) \qquad R_2(\phi(t)) \equiv_{\mu+1} R_2(\phi^{\mu-1}(t)).$$

Hence we can rewrite $(3.7.1)_\mu$ as follows:

$$(3.7.3)_\mu \qquad \overline{\partial}^{(1)}_{T'} \phi_\mu(t) + R_2(\phi^{\mu-1}(t)) \equiv_{\mu+1} 0 \ (\mu = 1, 2, \dots).$$

Further, these are equivalent to the following:

$$(3.7.4)_\mu \qquad \overline{\partial}^{(1)}_{T'} \phi_\mu(t) + P(\phi^{\mu-1}(t)) \equiv_{\mu+1} 0 \ (\mu = 1, 2, \dots)$$

because of $\phi^\mu(t) = \phi_\mu(t) + \phi^{\mu-1}(t)$ and $P(\phi^{\mu-1}(t)) = \overline{\partial}^{(1)}_{T'} \phi^{\mu-1}(t) + R_2(\phi^{\mu-1}(t))$.

Now we shall construct $\phi(t)$ by induction on $\mu$. We set $\phi_0 = 0$ and $\phi_1(t) = \sum_{\lambda=1}^{q} \beta_\lambda t_\lambda$. Then clearly $(3.7.5)_1$ holds.

Suppose that $\phi^{\mu-1}(t)$ is already determined and satisfies $(3.7.4)_{\mu-1}$. Then we will study the differential equation $(3.7.4)_\mu$.

$$(3.7.5)_\mu \qquad \overline{\partial}^{(1)}_{T'} \phi_\mu(t) + P(\phi^{\mu-1}(t)) \equiv_{\mu+1} 0.$$

We recall Theorem 4.10 in [Ak1]. (We note that this lemma holds for any twice continuously differentiable $\phi$.)

**Lemma 3.7.2.** *(see Theorem 4.10 in [Ak1]). For any element* $\phi \in \Gamma(M, T')$,

$$\overline{\partial}^{\phi}_{T'} P(\phi) = 0.$$

From the assumption $P(\phi^{\mu-1}(t)) \equiv_\mu 0$ and Lemma 3.7.2, we obtain

$$\overline{\partial}^{(2)}_{T'}(P(\phi^{\mu-1}(t)) \equiv_{\mu+1} \overline{\partial}^{\phi^{\mu-1}(t)}_{T'}(P(\phi^{\mu-1}(t))) = 0. \tag{3.7.5}$$

Hence, under the assumption $H^2(M, T') = 0$, the partial differential equation $(3.7.4)_\mu$ has a solution in $\Gamma(M, T' \otimes ({}^0T'')^*)$. And the following proposition enables us to choose the solution relying on the Hodge decomposition in Theorem 3.5.2, that assures $\phi_\mu(t) \in \Gamma(M, E_1)$.

**Proposition 3.7.3.** *Given a $\Gamma(M, E_1)$-valued polynomial $\phi^{\mu-1}(t)$ in $(t_1, .., t_q)$ satisfying $P(\phi^{\mu-1}(t)) \equiv_\mu 0$, the homogeneous part of degree $\mu$ in $(t_1, \dots, t_q)$ of $P(\phi^{\mu-1}(t))$ takes its value in $\Gamma(M, E_2)$.*

Hence, if we set

$$\phi_\mu(t) = -\overline{\partial}^* N\{\text{the } \mu\text{-th homogeous polynomial term of } P(\phi^{\mu-1}(t))\},$$

$\phi^{\mu-1}(t) + \phi_\mu(t)$ satisfies $(3.7.1)_\mu$, where $N$ denotes the new Neumann operator obtained in Theorem 3.5.2.

The convergence of

$$\phi(t) = \phi_1(t) + \phi_2(t) + \dots$$

is proved by the standard Kodaira-Spencer argument: For two powerseries

$$A(t) = \sum_{\nu=(\nu_1,\dots,\nu_q)} a_\nu t_1^{\nu_1} \dots t_q^{\nu_q}$$

and

$$B(t) = \sum_{\nu=(\nu_1,\dots,\nu_q)} b_\nu t_1^{\nu_1} \dots t_q^{\nu_q},$$

we denote

$$A(t) << B(t)$$

if $|a_\nu| \le |b_\nu|$ holds for all $\nu$. Let

$$A(t) := \frac{b}{16c} \sum_{\mu \ge 1} \frac{c^\mu}{\mu^2} (t_1 + \dots + t_q)^\mu$$

be a convergent power series where $b$ and $c$ are positive constants. Then we have

$$||\phi_\mu(t)||'_{(m-\frac{1}{2})} << ||\overline{\partial}^* NR_2(\phi^{(\mu-1)}(t))||'_{(m-\frac{1}{2})} << C||\phi^{(\mu-1)}(t)||'^2_{(m-\frac{1}{2})}$$

by Proposition 3.6.4. Hence

$$||\phi_\mu(t)||'_{(m-\frac{1}{2})} << A(t) \qquad \text{follows from} \qquad ||\phi^{(\mu-1)}(t)||'_{(m-\frac{1}{2})} << A(t)$$

if we choose $b$ and $c$ sufficiently large at the beginning, because

$$||\phi_\mu(t)||'_{(m-\frac{1}{2})} << C||\phi^{(\mu-1)}(t)||'^2_{(m-\frac{1}{2})} << CA(t)^2 << \frac{b}{c}CA(t)$$

holds (remark that $A(t)^2 << \frac{b}{c}A(t)$ holds (cf. (5.116) in [Ko])).

We note that the assumption $H^2(M,T') = 0$ is not essential. We will discuss in Part II the case of $H^2(M,T') \neq 0$ and the completeness of $\phi(t)$ (which is called the Kuranishi versality). In any case, we have

**Corollary 3.7.4.** *The deformation $\phi(t)$ constructed in Theorem 3.7.1 is versal.*

## §4. Geometry of the deformations of strongly pseudo-convex domains

### 4.1. $T'N$-valued complex.

Let $N$ be a complex manifold and $\Omega$ be a relatively compact strongly pseudo-convex subdomain of $N$. We assume $\dim_C N \geq 4$. Let $T'N$ be the holomorphic tangent bundle on $N$. Then, there is a first order differential operator $\overline{\partial}_{T'N}$ from $\Gamma(\overline{\Omega}, T'N)$ to $\Gamma(\overline{\Omega}, T'N \otimes (T''N)^*)$, where $\Gamma(\overline{\Omega}, T'N)$ denotes the space of $T'N$-valued sections smooth up to the boundary $b\Omega$. Namely, for $u$ in $\Gamma(\overline{\Omega}, T'N)$,

$$\overline{\partial}_{T'N} u(X) = [X, u]_{T'N},$$

where $X \in \Gamma(\overline{\Omega}, T''N)$, and $[X,u]_{T'N}$ denotes the projection to $T'N$ according to the decomposition of the vector bundle $C \otimes TN = T'N + T''N$. Then, as is well known, we can define a first order differential operator $\overline{\partial}^{(p)}_{T'N}$ from $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p (T''N)^*)$ to $\Gamma(\overline{\Omega}, T'N \otimes \wedge^{p+1}(T''N)^*)$

in the same way as for scalar valued forms, and we have a differential complex.

$$0 \to \Gamma(\overline{\Omega}, T'N) \to \Gamma(\overline{\Omega}, T'N \otimes (T''N)^*) \to \Gamma(\overline{\Omega}, T'N \otimes \wedge^2(T''N)^*) \to$$
$$\to \Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*) \to \Gamma(\overline{\Omega}, T'N \otimes \wedge^{p+1}(T''N)^*) \to,$$

while we have the restriction map $\tau_p$

$$\tau_p : \Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*) \ \to \Gamma(b\Omega, T' \otimes \wedge^p({}^0T'')^*),$$

given by

$$\tau_p\phi(X_1, .., X_p) = (i)^{-1}(\phi(X_1, .., X_p)) \quad \text{for} \quad X_j \in^0 T'',$$

where $i : T' \to T'N_{|M}$ denotes the isomorphism in §3.2. Henceforth, we abbreviate $\tau$ for $\tau_p$. Then, we have

**Lemma 4.1.1.** *(see Lemma 1.1 in [Ak4]) Let $\phi$ be an element of $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$ satisfying*

$$\tau\phi \in \Gamma(b\Omega, {}^0T' \otimes \wedge^p({}^0T'')^*) \ \textit{and} \ \overline{\partial}^{(p)}_{T'}\tau\phi \in \Gamma(b\Omega, {}^0T' \otimes \wedge^{p+1}({}^0T'')^*).$$

*Then,*

$$\tau(\overline{\partial}^{(p)}_{T'}\phi) = \overline{\partial}^{(p)}_{T'}(\tau\phi).$$

Similarly, we have the following lemma.

**Lemma 4.1.2.** *(see Lemma 1.2 in [Ak4]) Let $\phi$ be an element of $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$ satisfying $\tau\phi \in \Gamma(b\Omega, {}^0T' \otimes \wedge^p({}^0T'')^*)$ and $\tau(\overline{\partial}^{(p)}_{T'N}\phi) \in \Gamma(b\Omega, {}^0T' \otimes \wedge^{p+1}({}^0T'')^*)$. Then,*

$$\tau(\overline{\partial}^{(p)}_{T'N}\phi) = \overline{\partial}^{(p)}_{T'}(\tau\phi).$$

## 4.2. Almost complex manifolds and deformation equation

In this section, we recall the deformation theory of complex structures and the deformation equation.

Let $N$ be a $C^\infty$ differentiable manifold of real dimension $2n$. Let $E$ be a $C^\infty$ subvector bundle of the complexified tangent bundle $C \otimes TN$ satisfying

$$C \otimes TN = E \oplus \overline{E}.$$

$E$ is called an almost complex structure and the pair (N,E) an almost complex manifold. Now let $(N, T''N)$ be a complex manifold. Then, by using the canonical decomposition

$$C \otimes TN = T'N + T''N,$$

we have a homomorphism from $E$ to $T''N$, the composite of the inclusion of $E$ to $C \otimes TN$ and the projection of $C \otimes TN$ to $T''N$.

**Definition 4.2.1..** *Let $(N, T''N)$ be a complex manifold and $E$ an almost complex structure. $E$ is of finite distance from $T''N$ if the above homomorhism is an isomorphism.*

Then, we have

**Proposition 4.2.2.** *If $E$ is an almost complex manifold of finite distance from $T''N$, then there is a $\phi \in \Gamma(N, T'N \otimes (T''N)^*)$ satisfying*

$$\begin{aligned} E =& {}^{\phi}T''N \\ =& \{X'; X' = X + \phi(X), X \in T''N\}. \end{aligned}$$

By Proposition 4.2.2, we see that for a given CR manifold $(M, {}^0T'')$, almost CR manifolds of finite distance from $(M, {}^0T'')$ are parametrized by $\Gamma(N, T'N \otimes ({}^0T''N)^*)$. Now, in this section, we see when this ${}^{\phi}T''N$ is actually a complex manifold. For this, we must introduce notation. Let $\phi$ be an element of $\Gamma(N, T'N \otimes (T''N)^*)$. For this $\phi$, we set an element $R_2(\phi)$ of $\Gamma(N, TN' \otimes \wedge^2(T''N)^*)$ by

$$(4.2.1) \quad R_2(\phi)(X, Y) = [\phi(X), \phi(Y)]_{T'N} - \phi([X, \phi(Y)]_{T''N} + [\phi(X), Y]_{T''N})$$

for $X, Y$ in $\Gamma(N, T''N)$.

We remark that $R_2(\phi)$ makes sense as an element of $\Gamma(N, T'N \otimes \wedge^2(T''N)^*)$ for the same reason as in §3.2. In this notation, we have

**Proposition 4.2.3.** *Let $\phi$ be an element of $\Gamma(N, T'N \otimes (T''N)^*)$. Then an almost complex structure $(N, {}^{\phi}T''N)$ is a complex structure if and only if $\phi$ satisfies the following non-linear equation.*

$$(4.2.2) \qquad \begin{aligned} P(\phi) =& \overline{\partial}_{T'}^{(1)}\phi + R_2(\phi) \\ =& 0. \end{aligned}$$

**4.3. $\mathcal{E}_j$-structures**

As in the CR structure case, we introduce a subcomplex which satisfies a certain boundary condition.

We introduce a subspace $\mathcal{E}_p$ of $(\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$ by

$$\mathcal{E}_p = \{\ \phi\ :\ \phi \in \Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*),\ \tau\phi \in \Gamma(b\Omega, E_p)\}.$$

For $\mathcal{E}_p$, we show the following theorems.

**Theorem 4.3.1.** *(see Theorem 3.3 in [Ak4]). There is a differential subcomplex of* $(\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*), \overline{\partial}^{(p)}_{T'N})$.

$$0 \to \mathcal{E}_0 \xrightarrow{\overline{\partial}} \mathcal{E}_1 \xrightarrow{\overline{\partial}_1} \mathcal{E}_2 \xrightarrow{\overline{\partial}_2} \cdots$$
$$\xrightarrow{\overline{\partial}_{p-1}} \mathcal{E}_p \xrightarrow{\overline{\partial}_p} \mathcal{E}_{p+1} \xrightarrow{\overline{\partial}_{p+1}} \cdots$$

*where $\overline{\partial}_p$ means the restriction of $\overline{\partial}^{(p)}_{T'N}$ to $\mathcal{E}_p$.*

For the proof, it is enough to show

$$\overline{\partial}^{(p)}_{T'N}\mathcal{E}_p \subset \mathcal{E}_{p+1}.$$

For $\phi$ in $\mathcal{E}_p$,

$$\tau\overline{\partial}^{(p)}_{T'N}\phi = \overline{\partial}^{(p)}_{T'}(\tau\phi) \text{ (by Lemma 4.1.1).}$$

By Theorem 3.3.1, we have

$$\overline{\partial}^{(p)}_{T'}(\tau\phi) \in \Gamma(b\Omega, E_{p+1}) \ (\tau\phi \text{ being in } \Gamma(b\Omega, E_p)).$$

So this completes the proof of Theorem 4.3.1.

Henceforth we write this complex by

$$(\mathcal{E}_p, \overline{\partial}_p).$$

For this complex, we have the following theorem.

**Theorem 4.3.2.** *(see Theorem 3.4 in [Ak4]). The injection* $\operatorname{Ker}\overline{\partial}_{(p)} \hookrightarrow \operatorname{Ker}\overline{\partial}^{(p)}_{T'N}$ *induces an isomorphism*

$$i:\ \operatorname{Ker}\overline{\partial}_p/\operatorname{Im}\overline{\partial}_{p-1} \to \operatorname{Ker}\overline{\partial}^{(p)}_{T'N}/\operatorname{Im}\overline{\partial}^{(p-1)}_{T'N} \ \textit{if } p \geq 2$$

*and in the case of $p = 1$, the injection induces a surjective map*

$$\operatorname{Ker}\overline{\partial}_1 \to \operatorname{Ker}\overline{\partial}^{(1)}_{T'N}/\operatorname{Im}\overline{\partial}_{T'N} \to 0.$$

### 4.4. Estimates

In this section, we recall the new a priori estimate in [Ak4], [Ak5] for the subcomplex $(\mathcal{E}_p, \overline{\partial}_p)$. For this purpose, we make preparations.

Let $\{U_k, h_k\}_{k \in K}$ be a coordinate covering of $N$ such that

$$\bigcup_{k \in K} U_k \supset \overline{\Omega} \qquad \text{and} \qquad K \text{ is finite.}$$

Let $K'$ be a subset of $K$ satisfying, for $k \in K'$,

$$U_k \cap b\Omega \neq 0.$$

Let $\{\rho_k\}_{k \in K}$ be a partition of unity subordinate to the above covering. In this paper, we use the Levi metric defined by Greiner and Stein (cf. Chapter 4 in [Gr-St]). Then, for a point $p \in M$, there are a coordinate open set $U_k$ and an orthonormal basis $(e_1^k, .., e_{n-1}^k, e_n^k)$ of $T''N\,|_{U_k}$ satisfying

$$(e_j^k)_q \in^0 T_q'',$$

where $q \in b\Omega \cap U_k$ and $j = 1, \ldots, n-1$,

$$\begin{aligned}[e_i^k, \overline{e}_j^k] = \sqrt{-1}(\delta_{i,j} + O(\rho))(e_n^k - \overline{e}_n^k) \\ + \sum_{r=1}^{n-1} a_{i,j}^{k,r} e_r^k + \sum_{r=1}^{n-1} b_{i,j}^{k,r} \overline{e}_r^k,\end{aligned}$$

on $U_k$, and $e_n^k$ is globally defined in a neighborhood of the $b\Omega$, where $a_{i,j}^{k,r}$ and $b_{i,j}^{k,r}$ are $C^\infty$-functions on $U_k$, $\rho$ is the defining function of $b\Omega$ and $O(\rho)$ stands for a $C^\infty$-function which vanishes on $b\Omega$ (therefore by using integration by parts, we can neglect the $O(\rho)$-term).

Now we put an $L^2$-norm and the $\| \ \|'$-norm on $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T'')^*)$. $I^p$ denotes the family of all ordered set $(i_1, .., i_p)$ of integers with $1 \leq i_1 < i_2 < ... < i_p \leq n$. For any $\phi$ in $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T'')^*)$, $I \in I^p$ and $l (1 \leq l \leq n)$, we define $C^\infty$-functions $\phi_{l,I}^k$ on $U_k$ by

$$\phi(e_{i_1}^k, .., e_{i_p}^k) = \sum_l \phi_{l,I}^k \overline{e}_l^k,$$

where $I = (i_1, .., i_p)$. Using these functions $(\rho_k \phi_{l,I}^k) h_k^{-1}$ in $C_o^\infty(R^{2n})$, we define the $L^2$ -norm on $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$ by

$$\|\phi\|^2 = \sum_{l,I,k} \|(\rho_k \phi_{l,I}^k) h_k^{-1}\|^2$$

where $\| \ \|^2$ means the $L^2$-norm on $C_o^\infty(R^{2n})$ and $C_o^\infty(R^{2n})$ means the space of $C^\infty$-functions on $R^{2n}$ with compact support. Next we introduce a $\| \ \|'$-norm on $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$ by

$$\begin{aligned}\|\phi\|'^2 = \textstyle\sum_{k\in K', i\neq n, l, I} &\|(\rho_k e_i^k \phi_{l,I}^k)h_k^{-1}\|^2 \\ + \sum_{k\in K', i\neq n, l, I} &\|(\rho_k \overline{e}_i^k \phi_{l,I}^k)h_k^{-1}\|^2 \\ + \sum_{k\in K', n\in I, l} &\|(\rho_k \overline{e}_n^k \phi_{l,I}^k)h_k^{-1}\|^2 \\ + \sum_{k\in K', n\notin I, l} &\|(\rho_k e_n^k \phi_{l,I}^k)h_k^{-1}\|^2 \\ + \sum_{k\neq K', I, l} &\|(\rho_k \phi_{l,I}^k)h_k^{-1}\|_{(1)}^2\end{aligned}$$

where $\| \ \|_{(1)}$ means the Sobolev 1-norm on $C_o^\infty(R^{2n})$.

Henceforth we omit $h_k^{-1}$,$\rho_k$ and the index $k$ for brevity.

We set a vector field $\xi$ on $b\Omega$ by

$$\tau(-\sqrt{-1}\overline{e}_n)$$

and fix the decomposition of the vector bundle $C\otimes T(b\Omega) = {}^0T'' + {}^0T' + C\otimes F$, where $F = \xi$.

With these preparations, we consider the following space $B^2$ of $\mathcal{E}_2$.

$$\begin{aligned}B^2 = \{\phi; \phi \in \Gamma(\overline{\Omega}, T'N \otimes \wedge^2(T''N)^*), \langle\sigma(\vartheta, d\rho)\phi, y\rangle = 0 \\ \text{on } b\Omega \text{ for any } y \text{ in } E_1 \text{ and } \tau\phi \in \Gamma(b\Omega, E_2)\},\end{aligned}$$

where $\langle , \rangle$ denotes the inner product defined by the Levi-metric, and $\vartheta$ denotes the formal adjoint operator of $\overline{\partial}_{T'}^{(1)}$.

On $B^2$, we have the following a priori estimate (the key estimate).

**Theorem 4.4.1.** *(see Theorem 4.3 in [Ak4] and Corollary 6.2 in [Ak5]). Assume that $\Omega$ is strongly pseudo-convex and* $\dim_C \Omega \geq 4$. *Then* *the following estimate holds.*

$$\begin{aligned}\|\vartheta\phi\|^2+&\|\overline{\partial}\phi\|^2 + \|\phi\|^2\\ \geq& c\{\sum_{i<j}\sum_{\alpha=1}^{n-1}\{\sum_{l=1}^{n-1}\|e_l\phi_{\alpha,(i,j)}\|^2 + \sum_{l=1}^{n-1}\|\overline{e}_l\phi_{\alpha,(i,j)}\|^2\}\}\\ &+\sum_{i<j}\{(\sum_{l\neq i,j}\|e_l\phi_{n,(i,j)}\|^2) + \|\overline{e}_i\phi_{n,(i,j)}\|^2 + \|\overline{e}_j\phi_{n,(i,j)}\|\}\\ =& c\|\phi\|'^2\end{aligned}$$

*for all $\phi$ in $B^2$, where $c$ is a positive constant independent of $\phi$ and for brevity, we write $\overline{\partial}$ for $\overline{\partial}^{(1)}_{T'N}$.*

This was first proved by direct computation (see [Ak4]). Later, it was proved in a fairly wide framework in [Ak5]. The proof of this theorem in [Ak5] relied on estimates established in Theorem 3.5.1 and the following Proposition 4.4.2. In order to see Proposition 4.4.2, we have to recall some notation. We recall $T'$-bundle on $M$, and $\overline{\partial}_{T'}$ -operator. Let $\vartheta_{T'}$ be the formal adjoint operator of $\overline{\partial}_{T'}$ with respect to the Levi-metric. And we set a $C^\infty$ vector bundle decomposition of $T' \otimes \wedge^p({}^0T'')^*$,

$$T' \otimes \wedge^p({}^0T'')^* = E_p + E_p^\perp. \tag{4.4.1}$$

Here $E_p^\perp$ is the complement of $E_p$ with respect to the Levi metric. Then, our proposition is stated as follows.

**Proposition 4.4.2.** *(see Theorem 6.1 in [Ak5]). Suppose that* $\dim_R M = 2n-1 \geq 7$. *Then we have*

$$\|\vartheta_{T'}\psi\| + \|(\overline{\partial}_{T'}\psi)_{E_2^\perp}\| + \|\psi\| \geq c\|\psi\|'$$

*for $\psi \in \Gamma(M, E_1^\perp)$, where $c$ is a positive constant, and $(\overline{\partial}_{T'}\psi)_{E_2^\perp}$ means the projection of $\overline{\partial}_{T'}\psi$ to $E_2^\perp$ according to (4.4.1).*

### 4.5. The new Hodge decomposition theorem

Based on our estimate (Theorem 4.4.1), we discuss a new Hodge decomposition theorem, which differs from the standard one (see [Kohn]), and apply it in solving the Cauchy-Riemann equation in the subcomplex $(\mathcal{E}_p, \overline{\partial}_p)$. We note that our new Neumann operator preserves the boundary condition. Let $G$ be a holomorphic vector bundle on $N$. Let $D^p$ be the Cauchy-Riemann operator for $G$-valued $p$ forms and $D_b^p$ the induced operator over the boundary $b\Omega$. Let $F_p$ be a subbundle of $G \otimes \wedge^p({}^0T'')^*$ over the boundary $b\Omega$. We set

$$\mathcal{F}^p = \{\phi; \phi \in \Gamma(\overline{\Omega}, G \otimes \wedge^p(T''N)^*), \ \tau\phi \in \Gamma(b\Omega, F_p)\},$$

where $\tau$ means the restriction map $\phi$ to the element of $\Gamma(b\Omega, G \otimes \wedge^p({}^0T'')^*)$. In the same way as in §4.4, we put the $L^2$-norm, the inner product and also $\| \ \|'$-norm on $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T''N)^*)$. We set

$$\begin{aligned} B^p = \{\phi; \phi \in \Gamma(\overline{\Omega}, G \otimes \wedge^p(T''N)^*), \ \tau\phi \in \Gamma(b\Omega, F_p) \\ \text{and} \quad \langle \sigma(D^{*p-1}, d\rho)\phi, y\rangle = 0 \quad \text{for any} \quad y \in F_{p-1} \text{ on } b\Omega\}, \end{aligned}$$

where $D^{*p-1}$ denotes the formal adjoint operator of $D^{p-1}$, $\sigma(D^{*p-1}, d\rho)$ means the symbol at $d\rho$, and $\rho$ is the defining function for $b\Omega$ in $N$. For brevity, we write $D^*$ for $D^{*p}$ and $D$ for $D^p$.

In this notation, our theorem is stated as follows.

**Theorem 4.5.1.** *(see Theorem 5.1 in [Ak4]) Suppose that*

$$D_b^{p-1}\Gamma(b\Omega, F_{p-1}) \subset \Gamma(b\Omega, F_p) \tag{A.1}$$

*and*

$$\|D^*\phi\|^2 + \|D\phi\|^2 + \|\phi\|^2 \geq c\|\phi\|'^2 \tag{A.2}$$

*for all $\phi$ in $B^p$, where $c$ is a positive constant independent of $\phi$. Then, there are the new Neumann operator $N$; $\mathcal{L}_2^p \to \mathcal{L}_2^p$ and the new harmonic operator $H$; $\mathcal{L}_2^p \to \mathbf{H}$ satisfying*

(1) *$H$ and $N$ are bounded,*
(2) *if $\phi$ is in $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T'')^*)$, then $\mathbf{H}\phi$ and $N\phi$ are in $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T'')^*)$,*
(3) *if $\phi$ is in $\Gamma(\overline{\Omega} \otimes \wedge^p(T'')^*)$, then $\phi = (DD^* + D^*D)N\phi + H\phi$*
(4) *$HN = NH$*
(5) *if $\phi$ is in $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T'')^*)$, then $D^*N\phi$ is in $\mathcal{F}^{p-1}$, and in addition, if $\phi$ is in $\mathcal{F}^p$, $D\phi = 0$ and $H\phi = 0$, then $DD^*N\phi = \phi$ where $L_2^p$ denotes the $L^2$-completion of $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T'')^*)$ and $\mathbf{H} = \{\phi; \phi \in B^p, D, phi = 0$ and $D^*\phi = 0\}$.*

*[Brief sketch of the proof].* We note that Kohn's standard Neumann operator relies on Morrey's estimate (in Kohn's case, the (A.1)

part is trivial). We briefly recall the proof, which is carried out in the standard functional analysis. First, we set a map $T$ from $B^p$ to

$$\Gamma(\overline{\Omega}, G \otimes \wedge^{p-1}(T'')^* \times \Gamma(\overline{\Omega}, G \otimes \wedge^{p+1}(T'')^*)$$

by $T\phi = (D^*\phi, D\phi)$. We complete $T$ and use the same notation $T$ for this. $D(T)$ denotes the domain on which this operator is defined. For $D(T)$, just by the standard argument, we have

**Lemma 4.5.2.** *$D(T)$ is dense in $\mathcal{L}_2^p$ and*

$$D(T) \cap \Gamma(\overline{\Omega}, G \otimes \wedge^p(T'')^*) = B^p.$$

And also we have the following lemma and proposition.

**Lemma 4.5.3.** *Let $\mu$ be in $\Gamma(\overline{\Omega}, G \otimes \wedge^{P+1}(T'')^*)$, and suppose that*

$$\langle \phi, \sigma(D^*, d\rho)\mu \rangle = 0 \textit{ for all } \phi \textit{ in } B^p \textit{ on } b\Omega.$$

*Then $\tau\psi$ in $\Gamma(b\Omega, F_{p-1})$.*

**Proposition. 4.5.4.**

$$\begin{aligned} D(T^*) \cap \{&\Gamma(\overline{\Omega}, G \otimes \wedge^{p-1}(''N)^*) \times \Gamma(\overline{\Omega}, G \otimes \wedge^{p+1}(T''N)^*)\} \\ =&\{(\psi, \mu) : \psi \in \Gamma(\overline{\Omega}, G \otimes \wedge^{p-1}(T''N)^*), \tau\psi \in \Gamma(b\Omega, F_{p-1}), \\ &\mu \in \Gamma(\overline{\Omega}, G \otimes \wedge^{p+1}(T''N)^*) \textit{ and } \langle \sigma(D^*, d\rho)\mu, y \rangle = 0 \\ &\textit{for any } y \textit{ in } F_p \textit{ on } b\Omega\}, \end{aligned}$$

*where $D(T^*)$ means the domain of $T^*$.*

So, we obtain that for $\phi$ in $B^p$ satisfying $T\phi$ in $D(T^*)$,

$$T^*T\phi = \Box\phi. \tag{4.5.1}$$

With these preparations, we prove Theorem 4.5.1. We follow Kohn-Nirenberg's approach in [K-N]. Namely, we first set

$$\mathbf{H} = \{\phi : \phi \in D(T), T\phi = 0\}.$$

Obviously, $\mathbf{H}$ is finite dimensional and so closed in $\mathcal{L}_2^p$. Next we study $B^p \cap H'$, where $H'$ is the complement of $\mathbf{H}$ in $\mathcal{L}_2^p$. We consider the problem of finding a solution $\psi \in B^p \cap H'$

$$(T\psi, T\phi) = (\alpha, \phi) \quad \text{for} \quad \phi \in B^p \cap H',$$

where $\alpha$ is in $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T''N)^*)$.

By the assumption (A.2), there is a unique element $\psi$ in $B^p \cap H'$ (Theorem 4.1 in [Ak4]), And $\psi$ satisfies the boundary condition

$$T\psi \in D(T^*). \tag{4.5.2}$$

That is to say, for any $\alpha$ in $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T''N)^*)$, there is a $\psi$ in $B^p$ satisfying

$$T^*T\psi = \alpha - H\alpha,$$

where $H$ is the projection of $\mathcal{L}_2^p$ to $\mathbf{H}$.

We set

$$N\alpha = \psi$$

and call $N$ the new Neumann operator. We see that our new Neumann operator $N$ satisfies the relation (5). We recall (4.5.2). Namely, for $\alpha$ in $\Gamma(\overline{\Omega}, G \otimes \wedge^p(T''N)^*)$,

$$TN\alpha \in D(T^*).$$

By the definition of $T$, $TN\alpha = (D^*N\alpha, DN\alpha)$. And by Proposition 4.5.4, we get

$$D^*N\alpha \in \mathcal{F}^{p-1}$$

and

$$\langle \sigma(D^*, d\rho)DN\alpha, y\rangle = 0 \quad \text{for all} \quad y \in F_p \quad \text{on} \quad b\Omega.$$

It remains to prove that under the assumptions $h\phi = 0$, $D\phi = 0$ and $\phi \in \mathcal{F}^p$, we obtain

$$DD^*N\phi = 0.$$

For this, we set

$$\mu = DD^*N\phi - \phi.$$

Then, form $D\phi = 0$, $D\mu = 0$ follows. In addition, we have

$$\begin{aligned} D^* &= D^*DD^*N\phi - D^*\phi \\ &= D^*(\phi - H\phi - D^*DN\phi) - D^*\phi \\ &= D^*\phi - D^*\phi \\ &= 0. \end{aligned}$$

Next we see that $\mu$ is in $B^p$, i.e., in $D(T)$. For this, we compute the following by integration by parts. For $\psi$ in $\mathcal{F}^{p-1}$,

$$\begin{aligned} (\mu, D\psi) &= (DD^*N\phi - \phi, D\psi) \\ &= (DD^*N\phi, D\psi) - (\phi, D\psi) \\ &= (\phi - H\phi - D^*DN\phi, D\psi) - (\phi, D\psi) \\ &= -(H\phi, D\psi) - (D^*DN\phi, D\psi). \end{aligned}$$

We note that the boundary term vanishes, so we have

$$(H\phi, D\psi) = (D^* H\phi, \psi) = 0.$$

Similarly,

$$(D^* DN\phi, D\psi) = 0.$$

Because

$$\tau D\psi = D_b\psi \quad \text{is in} \quad \Gamma(b\Omega, F_p),$$

we have

$$\langle \sigma(D^*, d\rho) DN\phi, D\psi \rangle = 0 \quad \text{on} \quad b\Omega.$$

Thus,

$$(\mu, D\psi) = -(DN\phi, DD\psi) = 0.$$

On the other hand,

$$\begin{aligned}(\mu, D\psi) =& (D^*\mu, \psi) - \int_{b\Omega} \langle \sigma(D^*, d\rho)\mu, \psi \rangle d(b\Omega) \\ =& -\int_{b\Omega} \langle \sigma(D^*, d\rho)\mu, \psi \rangle d(b\Omega).\end{aligned}$$

Hence

$$\langle \sigma(D^*, d\rho)\mu, \psi \rangle = 0 \quad \text{on} \quad b\Omega \quad \text{for} \quad \psi \in \mathcal{F}^{p-1}.$$

By the definition of $B^p$ and Lemma 4.5.2, we obtain

$$\mu \in D(T).$$

Combined with $D\mu = 0$ and $D^*\mu = 0$, we have

$$T\mu = 0.$$

Therefore

$$\mu \in \mathbf{H}.$$

Furthermore, for any $\alpha \in \mathbf{H}$,

$$(\mu, \alpha) = (DD^* N\phi - \phi, \alpha) = (DD^* N\phi, \alpha) \quad (\text{by} \quad H\phi = 0).$$

The boundary term vanishes, so we have

$$(DD^* N\phi, \alpha) = (D^* N\phi, D^*\alpha) = 0 \quad (\text{by} \quad \alpha \in \mathbf{H}).$$

So $\mu = 0$. This is the outline of the proof.

From this theorem with Theorem 4.4.1, we immediately obtain Corollary 4.5.5.

**Corollary 4.5.5.** *In the case of $G = T'N$, $F_p = E_p$, and $p = 2$, the new Neumann operator exists.*

### 4.6. Some estimates

In order to try to construct a versal family, we will review some estimates for the Neumann operator $N$ obtained in §4.5.

First, we put the tangential Sobolev $(0, m)$-norm and $\|\ \|'$-norm on $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$. $I^p$ denotes the family of all ordered sets $(i_1, .., i_p)$ of integers with $1 \leq i_1 < i_2 < ... < i_p \leq n$. For $\phi$ in $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$, $I \in I^p$ and $l(1 \leq l \leq n)$, we define $C^\infty$ functions $\phi^k_{l,I}$ on $U_k$ by

$$\phi(e^k_{i_1}, .., e^k_{i_p}) = \sum_l \phi^k_{l,I} \bar{e}^k_l, \text{ where } I = (i_1, .., i_p).$$

Using these functions, we define the tangential Sobolev $(0, m)$-norm on $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$ by

$$\|\phi\|^2_{(0,m)} = \sum_{l,I,k} \|(\rho_k \phi^k_{l,I}) h_k^{-1}\|^2_{(0,m)}$$

where for $k \in K'$, $\|\ \|^2_{(0,m)}$ means the tangential Sobolev $(0, m)$-norm on $C^\infty_0(R^{2n}_+)$ (here $C^\infty_0(R^{2n}_+)$ means the space of $C^\infty$-functions on the upper half plane $R^{2n}_+$ with compact support), and for $k \notin K'$, $\|\ \|^2_{(0,m)}$ means the tangential Sobolev $(0, m)$-norm on $C^\infty_0(R^{2n})$ with compact support (for the definition of the tangential Sobolev norm, see Definition 2.5.1 in [Hö]). Next we introduce $\|\ \|'_{(0,m)}$-norm on $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$ by

$$\begin{aligned}
\|\phi\|'^2_{(0,m)} = & \sum_{k\in K', i\neq, l, I} \|(\rho_k e^k_i \phi^k_{l,I}) h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', i\neq, l, I} \|(\rho_k \bar{e}^k_i \phi^k_{l,I}) h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', n\in I, l} \|(\rho_k \bar{e}^k_n \phi^k_{l,I}) h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\in K', n\notin I, l} \|(\rho_k e^k_n \phi^k_{l,I}) h_k^{-1}\|^2_{(0,m)} \\
& + \sum_{k\notin K', l, I} \|(\rho_k \phi^k_{l,I}) h_k^{-1}\|^2_{(1,m)}.
\end{aligned}$$

From now on, we assume $m > 2n$ unless we note otherwise. With these preparations, we show the following more precise estimate (cf. Theorem 4.4.1 in this paper).

**Proposition 4.6.1.** *(see Theorem 6.1 in [Ak4]) Assume that $\Omega$ is strongly pseudo-convex and $\dim_C \Omega \geq 4$. Then the following estimate holds.*

$$\|\vartheta\phi\|^2_{(0,m)} + \|\overline{\partial}\phi\|^2_{(0,m)} + \|\phi\|^2_{(0,m)} \geq c\|\phi\|'^2_{(0,m)}$$

*for all $\phi$ in $B^2$, where $c$ is a positive constant independent of $\phi$ and $m$ is a non-negative integer.*

For $\Box = \overline{\partial}\vartheta + \vartheta\overline{\partial}$, we show some estimates by using this proposition. To do so, we must introduce a new norm. For $\mu$ in $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N)^*)$, we set

$$\begin{aligned}
\|\mu\|''^2_{(0,m)} = & \sum_{k\in K', i<j<n, l, I} \|(\rho_k e^k_i e^k_j \mu^k_{l,I})h^{-1}_k\|^2_{(0,m)} \\
& + \sum_{k\in K', i<j<n, l, I} \|(\rho_k e^k_i \overline{e}^k_j \mu^k_{l,I})h^{-1}_k\|^2_{(0,m)} \\
& + \sum_{k\in K', i<j<n, l, I} \|(\rho_k \overline{e}^k_i e^k_j \mu^k_{l,I})h^{-1}_k\|^2_{(0,m)} \\
& + \sum_{k\in K', i<j<n, l, I} \|(\rho_k \overline{e}^k_i \overline{e}^k_j \mu^k_{l,I})h^{-1}_k\|^2_{(0,m)} \\
& + \sum_{k\in K', i<n, l, n\in I} \|(\rho_k \overline{e}^k_n \overline{e}^k_i \mu^k_{l,I})h^{-1}_k\|^2_{(0,m)} \\
& + \sum_{k\in K', i<n, l, n\in I} \|(\rho_k \overline{e}^k_n e^k_i \mu^k_{l,I})h^{-1}_k\|^2_{(0,m)} \\
& + \sum_{k\in K', i<n, l, n\notin I} \|(\rho_k e^k_n \overline{e}^k_i \mu^k_{l,I})h^{-1}_k\|^2_{(0,m)} \\
& + \sum_{k\in K', i<n, l, n\notin I} \|(\rho_k e^k_n e^k_i \mu^k_{l,I})h^{-1}_k\|^2_{(0,m)} \\
& + \sum_{k\in K', l, I} \|(\rho_k e^k_n \overline{e}^k_n \mu^k_{l,I})h^{-1}_k\|^2_{(0,m)} \\
& + \sum_{k\in K', l, I} \|(\rho_k \overline{e}^k_n e^k_n \mu^k_{l,I} h^{-1}_k\|^2_{(0,m)} + \|\mu\|^2_{(1,m)}.
\end{aligned}$$

After this, we omit the suffix $k$ and the functions $\rho_k, h_k$. In this notation, our theorem is stated as follows.

**Theorem 4.6.2.** *(see Theorem 6.2 in [Ak4]) For $\mu$ in $B^2$,*

$$\|\mu\|''^2_{(0,m)} \leq c_m\{\|\Box\mu\|^2_{(0,m)} + \|\mu\|^2_{(0,m)}\}.$$

From this proposition, we immediately have

**Proposition 4.6.3.** *(see Theorem 6.3 in [Ak4]) For $\mu$ in $\Gamma(\overline{\Omega}$, $T'N \otimes \wedge^2(T''N)^*)$, we have*

$$\|\vartheta N\mu\|'_{(0,m)} \leq c_m\|\mu\|_{(0,m)},$$

*where $c_m$ is a positive constant.*

Using this proposition, we can discuss an a priori estimate for $R_2(\phi)$.

Combining this with the standard argument of functional analysis, we obtain the following main theorem.

**Theorem 4.6.4.** *(see Theorem 6.6 in [Ak4]) For $\phi$ in $\mathcal{E}_1$ satisfying $\phi_{n,n} = 0$ on $b\Omega$,*

$$\|\vartheta NR_2(\phi)\|'_{(0,m)} \leq c_m\|\phi\|'^2_{(0,m)}.$$

## 4.7. Construction

We construct a versal family of deformations of $\Omega$, consisting of $T'N \otimes (T''N)^*$-valued $\mathcal{A}^m$-class elements. For this purpose, we must introduce a new subspace $\mathcal{E}'_1$ of $\mathcal{E}_1$.

First we set a linear map $t$ from $\Gamma(\overline{\Omega}, T'N \otimes \wedge^p(T''N^*)$ to $\Gamma(b\Omega, T' \otimes \wedge^{p-1}({}^0T^{P''})^*)$ by

$$t\phi(X_1, .., X_{p-1}) = \tau(\phi(e_n, X_1, .., X_{p-1})),$$

where $X_j$ is an element of ${}^0T''$ and $e_n$ is as introduced in §3.4. By using this $t$, we introduce

$$\mathcal{E}'_p = \{\phi : \phi \in \mathcal{E}_p,\ t\phi = 0\}.$$

Then for $\mathcal{E}'_p$, the following theorem holds.

**Theorem 4.7.1.** *(see Theorem 7.1 in [Ak4]). The injection $\mathcal{E}'_p \cap \operatorname{Ker}\overline{\partial}_p \hookrightarrow \operatorname{Ker}\overline{\partial}_p$ induces an isomorphism*

$$\{\mathcal{E}'_p \cap \operatorname{Ker}\overline{\partial}_p\}/\{\mathcal{E}'_p \cap \operatorname{Im}\overline{\partial}_{p-1}\} \backsimeq \operatorname{Ker}\overline{\partial}_p/\operatorname{Im}\overline{\partial}_{p-1} \ \textit{if}\ p \geq 2,$$

*and a surjective map*

$$\mathcal{E}_1' \cap \operatorname{Ker} \overline{\partial}_1 \to \operatorname{Ker} \overline{\partial}^{(1)}_{T'N} / \operatorname{Im} \overline{\partial}_{T'N} \to 0.$$

Then from Theorem 4.7.1, we immediately have the following corollary.

**Corollary 4.7.2.** *(see Corollary 7.2 in [Ak4]). There is a finite dimensional sub-vector space $\mathcal{H}$ of $\mathcal{E}_1'$ such that the map in Theorem 4.3.2 induces an isomorphism*

$$\mathcal{H} \simeq \operatorname{Ker} \overline{\partial}^{(1)}_{T'N} / \operatorname{Im} \overline{\partial}_{T'N}.$$

With this corollary in mind, we will construct a versal family of $\mathcal{A}^m$ class. Our main theorem in this section is as follows.

**Theorem 4.7.3.** *(see Theorem 7.4 in [Ak4]). Under the assumptions $\dim_C \Omega = n \geq 4$ and $H^2(\Omega, T'N) = 0$, there is an $\mathcal{E}_1'$ valued $\mathcal{A}^m$ class element $\phi(t)$, parametrized complex analytically by a neighborhood $U$ of the origin in the Euclidean space $\mathcal{H}$ satisfying*

(1) $\phi(0) = 0$,
(2) $\overline{\partial}^{(1)}_{T'N}\phi(t) + R_2(\phi(t)) = 0$, *and*
(3) *the linear term of $\phi(t)$ is equal to $\sum_{\lambda}^{q} \beta_\lambda t_\lambda$, where $\{\beta_\lambda\}_{1\leq\lambda\leq q}$ is a system of bases of $\mathcal{H}$ and $\{t_i\}_{1\leq i\leq q}$ are local coordinates of $U$. Here $m$ is a sufficiently large integer satisfying $m \geq n+2$.*

The construciton of $\phi(t)$ is the same as in the CR case (cf.Sect.3.7). However, in order to assumre the convergence, we need to construct $\phi(t)$ so that it is $\mathcal{E}_1'$-valued (cf. Theorem 4.6.4). Hence we introduce an operator $\mathbf{A} : \mathcal{E}_p \to \mathcal{E}_p'$ having the following properties:

$$\overline{\partial}^{(p)}_{T'N}\mathbf{A} = \overline{\partial}^{(p)}_{T'N},$$
$$\|\mathbf{A}\phi\|'_{(0,m)} \leq c_m \|\phi\|'_{(0,m)}.$$

See [Ak4] for the construction of $\mathbf{A}$.

By an argument parallel to that in the proof of Theorem 3.7.1, replacing $\Gamma(M, E_1)$ and the Hodge decomposition in Theorem 3.5.2 by $\mathcal{E}_1'$ and the Hodge decomposition in Theorem 4.5.1, we can trace the construciton in §3.7. The following lemma and proposition correspond to Lemma 3.7.2 and Proposition 3.7.3 in §3.7 respectively.

**Lemma 4.7.4.** *For any element $\phi$ in $\Gamma(\overline{\Omega}, T'N \otimes (T''N)^*)$,*

$$\overline{\partial}^{\phi}_{T'N} P(\phi) = 0.$$

**Proposition 4.7.5.** *Given $\mathcal{E}_1'$ valued holomorhic function $\phi^{\mu-1}(t)$ in $(t_1, .., t_q)$ satisfying $P(\phi^{\mu-1}(t)) \equiv_\mu 0$, the homogeneous part of degree $\mu$ in $(t_1, .., t_q)$ of $P(\phi^{\mu-1}(t))$ takes its value in $\mathcal{E}_2$.*

Only difference is that $\phi_\mu(t)$ is given by

$$\phi_\mu(t) = \mathbf{A}\{-\vartheta N\{\text{the } \mu \text{ th homogeous polynomial term of } P(\phi^{\mu-1}(t))\}\},$$

not by

$$\phi_\mu(t) = -\vartheta N\{\text{the } \mu \text{ th homogeous polynomial term of } P(\phi^{\mu-1}(t))\}.$$

This is required because $\vartheta N\{$the $\mu$ th homogeous polynomial term of $P(\phi^{\mu-1}(t))\}$ is not necessarily $\mathcal{E}_1'$-valued though it is certainly $\mathcal{E}_1$-valued. This adjustment is necessary for the convergence procedure. In order to carry out the convergence process in §3.2, using Theorem 4.6.4 instead of Proposition 3.6.4, we need the property that $\phi(t)$ is $\mathcal{E}_1'$-valued. Thus Theorem 4.7.4 is proved.

We note that the assumption $H^2(\Omega, T'N) = 0$ is not essential either in this case. The same modification as in the CR case is possible. And the proof of the Kuranishi versality is also the same as in the CR case.

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