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Selberg-Ihara's Zeta function for p-adic Discrete Groups

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Dedicated to Prof. Friedrich Hirzebruch and Prof. Ichiro Satake on their sixtieth birthdays

Contents

Introduction

0-1. Let G be $\mathrm{PSL}(2, \boldsymbol{R})$ and let Γ ($\subset G$) be a Fuchsian group of the first kind. In [Sel], a zetafunction $Z_\Gamma(s)$ was introduced and proved to have many important properties which resemble those of usual L-functions, such as Euler product, functional equation, and analogue of Riemann Hypothesis. This function, now called with the name of Selberg, is generalized to any discrete subgroup Γ of a semi-simple Lie group of $\boldsymbol{R}$-rank one, when G/Γ is compact by Gangolli [Gan], and later by Gangolli-Warner [G-W] to the case when G/Γ has a finite volume. Meanwhile, an analogue of $Z_\Gamma(s)$ was introduced by Ihara [I-1], for a cocompact torsion-free discrete subgroup Γ of $\mathrm{PSL}(2, K)$ or $\mathrm{PL}(2, K)$, where K is a p-adic field. Especially it was shown that Ihara's zeta function $Z_\Gamma(u)$ is a

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rational function of the indeterminate u. This result is based on a remarkable structure theorem of such Γ, which in particular asserts that it is a free group with explicitly constructible basis (i.e. minimal set of generators).

0-2. In this paper, we shall extend Ihara's results to the case when G is a semi-simple algebraic group over a $\boldsymbol{p}$-adic field K and Γ is a discrete subgroup of G such that G/Γ is compact, under the assumption that G has K-rank one, and that it has an affine Tits system (G, B, N, S). In this case, S consists of two elements, say, s_1, s_2, and Bruhat-Tits' building X attached to (G, B, N, S) is a tree with two distinct kinds of vertices ([T-1, 2]). It is known that Γ acts on X freely and that Γ is again a free group ([Ser-1]). As in Ihara [I-1], our zetafunction $Z_\Gamma(u;\rho)$ is defined, for an n-dimensional unitary representation ρ of Γ, by the infinite (formal) product

$$Z_\Gamma(u;\rho):=\prod_{\{\gamma\}_\Gamma}\det\{I_n-\rho(\gamma)u^{\deg\{\gamma\}_\Gamma}\}^{-1} \tag{0.1}$$

where the product is taken over the set of "primitive hyperbolic" Γ-conjugacy classes $\{\gamma\}_\Gamma$, and its degree is defined as follows. Recall that $U_1=B\cup Bs_1B$, $U_2=B\cup Bs_2B$ are the subgroups of G which contain B properly and which represent the G-conjugacy classes of the maximal (open) compact subgroups. We define the length function $l: G\to\boldsymbol{N}\cup\{0\}$ by

$$l(g)=m\Longleftrightarrow g\in G_m:=U_1(s_1s_2)^m U_1, \tag{0.2}$$

and put

$$\deg\{\gamma\}_\Gamma:=\operatorname*{Min}_{x\in G} l(x^{-1}\gamma x). \tag{0.3}$$

0-3. To state our main result, put

$$q_i:=\#(B\backslash Bs_iB) \quad\text{and}\quad h_i:=\#(U_i\backslash G/\Gamma) \quad (i=1,2).$$

It follows from our assumption on the compactness of G/Γ that h_i (the class number of Γ with respect to U_i) is finite. Let $\mathscr{H}(G, U_1)$ be the Hecke algebra of the pair (G, U_1), and let $A_{1,\rho}$ be the following matrix given by the Brandt representation of $\mathscr{H}(G, U_1)$ attached to ρ:

$$A_{1,\rho}:=(a_{ij})\in M(nh_1, \boldsymbol{C}),\ \text{with}\ a_{ij}=\sum_{\gamma}\rho(\gamma), \tag{0.4}$$

where the last sum is taken over the set $\Gamma\cap x_i^{-1}G_1x_j$, $\{x_i\ (1\leq i\leq h_1)\}$ being a complete set of representatives of $U_1\backslash G/\Gamma$. See the text (§ 7) for another interpretation of $A_{1,\rho}$. The purpose of this paper is to prove the following:

Theorem (0.5). *Let the notation and assumptions be as above. Then* Γ *is a free group of rank* $r=q_1h_1-h_2+1$ $(=q_2h_2-h_1+1)$, *and we have*

$$(0.6) \qquad Z_\Gamma(u;\rho)^{-1}=(1-u)^{n(r-1)}(1+q_2u)^{n(h_2-h_1)} \times \det\{I_{nh_1}-(A_{1,\rho}-q_2+1)u+q_1q_2u^2\}.$$

Let $\rho=\mathbf{1}$ be the trivial representation, and put $Z_\Gamma(u)=Z_\Gamma(u;\mathbf{1})$. Combining the above result and that of Garland [Gar] (see also [Cas]), we get an information on the spectral decomposition of $L^2(G/\Gamma)$, which was first found by Ihara [I-1] for $G=\mathrm{PSL}_2(K)$:

Corollary (0.7). *We have the following equality*

$$-\operatorname*{Res}_{u=1}\frac{d}{du}\log Z_\Gamma(u)=r=\textit{the multiplicity of the Steinberg representation in } L^2(G/\Gamma).$$

There are at least three different ways to prove the above theorem. The first, which is based on the combinatorial argument on the structure of Γ, is a reformulation of the ideas of Ihara [I-1] in terms of the Tits system. We shall describe this proof in some detail. The second proof, which uses the graph-theoretic interpretation of the Γ-conjugacy classes, will be sketched briefly. The third method, which uses the Selberg's original idea of trace formula, is not complete yet. It would be an interesting problem to give an interpretation of our results and some questions arising therefrom, in terms of the spectral decomposition of $L^2(G/\Gamma)$. This will be treated in the subsequent paper [Ha].

0-4. As a matter of fact, our zetafunction $Z_\Gamma(u;\rho)$ can be defined for subgropus Γ of a much wider class of groups G; and the above theorem holds in such general setting. In the Appendix, we shall study the class of groups for which the whole procedure of the evaluation of $Z_\Gamma(u;\rho)$ developed in this paper can be applied without any change of notation.

Throughout the following, for any set S, $\sharp(S)$ will denote its cardinal number. For any ring A and a positive integer n, $M(n, A)$ will denote the ring of all n by n matrices whose entries are elements of A. The symbol $[x]$ ($x\in \boldsymbol{R}$) will be the largest integer n such that $n\leq x$.

§ 1. Groups with axiom (G, l, I), (G, l, II)

Let G be an abstract group. Assume that we are given a map

$$l: G \longrightarrow \boldsymbol{N}\cup\{0\}$$

called *the length*, satisfying the following conditions (G, l, I), (G, l, II), where G_l denotes the set of all elements of G with length l ($l=0, 1, 2, \cdots$) and $U:=G_0$.

(G, l, I) *For any l, G_l is non-empty, U is a subgroup of G, and*

$$G_l^{-1}=G_l, \quad UG_lU=G_l, \quad \#(U\backslash G_l)<\infty.$$

According to this condition we can define the Hecke algebra $\mathscr{H}(G, U)$ with respect to the pair (G, U) as in Shimura [Sh]. Since each G_l is a union of finite number of U-double cosets, it can be considered as an element of $\mathscr{H}(G, U)$ (by taking formal sum instead of disjoint union).

$(\mathrm{G}, l, \mathrm{II})$ *There exist two natural numbers q_1, q_2 such that*

$$G_1^2=G_2+(q_2-1)G_1+q_2(q_1+1)U \tag{1.1}$$

$$G_1G_l=G_{l+1}+(q_2-1)G_l+q_1q_2G_{l-1} \quad (2\leq l), \tag{1.2}$$

where the products G_1^2, G_1G_l are taken in $\mathscr{H}(G, U)$.

From this it follows immediately that

$$\#(U\backslash G_l)=(q_1q_2)^{l-1}q_2(q_1+1) \qquad \text{for } 1\leq l. \tag{1.3}$$

The class of groups satisfying our axiom (G, l, I), (G, l, II) has been studied in Appendix. Here we collect some of the results. The proofs are given in Appendix. Let $\Omega:=\{\omega_j;\ 1\leq j\leq t:=q_2(1+q_1)\}$ be a complete set of representatives of $U\backslash G_1$. Then one has the following

Lemma (1.4).
(i) *For each $\omega\in\Omega$, there is a unique $\rho\in\Omega$ such that $\rho\omega\in U$.*
(ii) *For each $\omega\in\Omega$, $\#\{\rho\in\Omega;\ \rho\omega\in G_1\}=q_2-1$.*
(iii) *For all other $\rho\in\Omega$, one has $\rho\omega\in G_2$.*

Lemma (1.5). *Any element $g\in G$ is expressed as a product*

$$g=u\omega_{j_1}\omega_{j_2}\cdots\omega_{j_l} \ (u\in U,\ \omega_{j_1},\ \cdots,\ \omega_{j_l}\in\Omega),$$

for which one has $l(g)\leq l$, with equality if and only if $g\in G_l$. Moreover, the last condition is equivalent to

$$\omega_{j_n}\omega_{j_{n+1}}\in G_2 \qquad \text{for } n=1, 2, \cdots, l-1. \tag{$*$}$$

Lemma (1.6). *Suppose that $x, y, z\in G_1$.*
(i) *If $xy\in G_2$ and $yz\in G_i$, then $xyz\in G_{i+1}$ $(i=0, 1, 2)$.*
(ii) *If $xy\in G_i$ and $yz\in G_2$, then $xyz\in G_{i+1}$ $(i=0, 1, 2)$.*

Corollary (1.7). *Suppose that $x=x_1x_2\cdots x_m\in G_m$, and $y=y_1y_2\cdots y_n\in G_n$ are (free) products of $x_1, \cdots, x_m\in G_1$, $y_1, \cdots, y_n\in G_1$ respectively. If $x_my_1\in G_1$, then one has $l(xy)=m+n-1$, and vice versa.*

These results have simple interpretations in terms of trees (cf. Appendix). We now describe it briefly. Recall first the following:

Definition (1.8). Let q_1, q_2 be two positive integers. By a semi-regular bipartite tree of valency (q_1+1, q_2+1), we mean a connected tree $X(q_1, q_2)=(V_1, V_2; E)$ whose set of vertices is a disjoint union of two subsets V_1, V_2, and each vertex $P \in V_1$ (resp. $Q \in V_2$) is adjacent exactly to q_1+1 (resp. q_2+1) vertices of V_2 (resp. V_1) (see Fig. 1 in § 2).

Let V_1 be the coset space $U\backslash G$, on which G acts by the right multiplication. For each $P=Ux \in V_1$, the set $V(P; 2)$ consisting of the points $Q=Uy$ such that $l(xy^{-1})=1$ has the cardinality $t=q_2(1+q_1)$(=independent of P), as one sees immediately from (G, l, I) and (1.3).

Lemma (1.9). *The set $V(P; 2)$ is divided into a disjoint union of q_1+1 subset $E_j(P)$ $(0\leq j\leq q_1)$; and for any two distinct points $Q=Uy$, $R=Uz$ of $E_j(P)$, one has $l(yz^{-1})=1$.*

Proof. See Appendix Lemma (10.9).

Now we consider the (disjoint) union E^* of all pairs $(P, E_j(P))$, $P \in V_1$:

$$(1.10) \qquad E^* := \bigcup_{P\in V_1} \bigcup_{j=0}^{q_1} (P, E_j(P))$$

One can regard E^* as a fibre space over V_1, whose fibre over the point P is a set of q_1+1 subsets $E_j(P)$. This fibre space structure is compatible with the action of G. On the other hand one can introduce an equivalence relation on E^*:

$$(1.11) \qquad (P, E_j(P)) \simeq (Q, E_k(Q)) \Longleftrightarrow E_j(P)\cup\{P\}=E_k(Q)\cup\{Q\}.$$

One can denote the equivalence class of $(P, E_j(P))$ by the q_2+1 points set $E_j(P)\cup\{P\}$. Note that each set $E_j(P)\cup\{P\}$ also has the same property as in the Lemma (1.9). Call V_2 the set of equivalence classes: $V_2:=E^*/\simeq$. Now we define a bipartite graph X, whose set of vertices consists of the disjoint union of V_1 and V_2; and we require that two vertices $P \in V_1$ and $E_j \in V_2$ are ajacent (i.e., joined by an edge $e=[P, E_j]$) iff $P \in E_j$. It is not difficult to see that X is a semi-regular bipartite tree of valency (q_1+1, q_2+1). Moreover, the action of G is naturally extended to X, in such a way that is transitive on V_1. This gives the first half of the following: (cf. Theorem (11.10))

Theorem (1.12). *Let q_1, q_2 be the given positive integers. There exists a bijection between the following objects:*

(i) *The group G with length function which satisfy the axiom* (G, l, I), (G, l, II).

(ii) *The groups G which acts on the semiregular bipartite tree $X(q_1, q_2)$, of valency (q_1+1, q_2+1), whose action is transitive on the first set V_1 of its vertices.*

Moreover, one can identify the coset space $U\backslash G$ and the set V_1; the length function l on $U\backslash G$ is then equal to the half of the distance d_X defined by the tree $X(q_1, q_2)$: $l(Ux, Uy)=d_X(Ux, Uy)/2$.

Note that we are assuming that G acts on $X(q_1, q_2)$ from the right, so that on $V_1 \simeq U\backslash G$, the action of G is that of the right multiplication. Let $P_0 \in V_1$ be the distinguished point which is corresponding to U. Then one has

$$g \in G_l \Longleftrightarrow d_X(P_0, P_0 g)=2l. \tag{1.13}$$

§2. Tits system and building

The most important class of groups G and the length function l, which satisfy the conditions (G, l, I), (G, l, II) in the preceding paragraph, is supplied by the socalled Tits system, or B-N pair. We shall review the definition and some of the basic facts on Tits system that we need for our study. Recall that a Tits system is a quadruplet (G, B, N, S) consisting of a group G, subgroups B, N of G, and a subset S of $W:=N/(B\cap N)$ which satisfy the following conditions:

(T. 1) $B\cup N$ generates G and $B\cap N$ is a normal subgroup of N.

(T. 2) S consists of a finite number of elements of order 2, and W is generated by them.

(T. 3) $sBw\subset BwB\cup BswB$ $(s \in S,\ w \in W)$.

(T. 4) For any $s \in S$, we have $sBs \not\subset B$.

In (T. 3) and (T. 4), expressions like sBw, BwB make sense since s, w are defined modulo $T:=N\cap B\subset B$. The group W is called the Weyl group of the Tits system.

Fact (2.1). (W, S) is a Coxeter system.

Fact (2.2). $G=BWB$, and the mapping $w \to BwB$ is a bijection of W onto $B\backslash G/B$ (Bruhat decomposition).

For $X\subset S$, let W_X be the subgroup of W generated by X. Put $G_X:=BW_XB$.

Fact (2.3). G_X is a subgroup of G and is generated by $\bigcup_{s\in X} BsB$.

Moreover, the assignment $X \to G_X$ is a bijection between the power set of S and the set of subgroups of G containing B.

Fact (2.4). Let $(X_i)_{i \in I}$ is a family of subset of X; if $X = \bigcap_i X_i$ then $G_X = \bigcap_i G_{X_i}$. And we have $G_{X_i} \subset G_{X_j}$ if and only if $X_i \subset X_j$.

We call G_X the standard parabolic subgroup of type X. It is known that any subgroup containing B is of this type.

Now we assume, throughout this paper, that

(2.5) (i) S consists of two elements, say, s_1, s_2.
(ii) $s_1 s_2$ is of infinite order.

Then $W = \langle s_1, s_2; s_1^2 = s_2^2 = 1 \rangle$ is isomorphic to the infinite dihedral group. Let U_1, U_2 be the standard parabolic subgroups corresponding to the subsets $\{s_1\}$, $\{s_2\}$ of S respectively:

$$U_1 := B \cup Bs_1 B, \qquad U_2 := B \cup Bs_2 B. \tag{2.6}$$

Moreover, we assume that the cardinalities

$$q_i := \sharp(B \backslash Bs_i B) \quad (i = 1, 2) \tag{2.7}$$

are finite. Then it follows immediately that

$$\sharp(B \backslash U_i) = 1 + q_i \quad (i = 1, 2). \tag{2.8}$$

Let $\mathscr{H}(G, B)$ be the Hecke algebra of (G, B). By a result of Iwahori-Matsumoto [I-M], we have

$$\mathscr{H}(G, B) \simeq \boldsymbol{Z}[T_1, T_2]_{nc} \tag{2.9}$$

(non-commutative ring generated by T_1, T_2)

with $Bs_1 B \rightarrow T_1$, $Bs_2 B \rightarrow T_2$, and with the fundamental relations

$$T_i^2 = (q_i - 1)T_i + q_i \quad (i = 1, 2). \tag{2.10}$$

We have also the Hecke algebras $\mathscr{H}(G, U_i)$ for which we have

$$\mathscr{H}(G, U_i) \simeq \boldsymbol{Z}[T] \ (i = 1, 2), \quad \text{with } U_i s_1 s_2 U_i \longrightarrow T. \tag{2.11}$$

Now we recover the length function $l: G \rightarrow \boldsymbol{N} \cup \{0\}$ as follows. First we define $l(w)$ $(w \in W)$ to be the number of s_2 in the reduced expression of w. Then we put

$$l(g) = l(w), \qquad \text{if } g \in BwB. \tag{2.12}$$

It is immediate to see that the condition (G, l, I) is satisfied. Notice that $U = l^{-1}(0) = U_1$. We shall show that it also satisfy (G, l, II).

Lemma (2.13). *Let* $w=w_1w_2\cdots w_r$ $(w_i \in S)$ *be the reduced expression of* $w \in W$. *Then one has*

$$\sharp(B\backslash BwB)=\prod_{i=1}^{r} \sharp(B\backslash Bw_iB).$$

Proof. See [I-M], and [Mac], Proposition (3.1.7). Q.E.D.

Lemma (2.14). *As an element of* $\mathscr{H}(G, B)$, *we have*

$$G_l=(1+T_1)T_2(T_1T_2)^{l-1}(1+T_1) \quad (l\geq 1).$$

The relations (1.1), (1.2) now follow easily from this and (2.10). Thus we see that a Tits system of affine type of rank two, together with a choice of one of its two generators in S, gives a group G satisfying (G, l, I), (G, l, II).

Next we review the Bruhat-Tits building X attached to the Tits system (G, B, N, S) of affine type of rank two. By this assumption on the rank, it is a tree described as follows. The set V of its vertices is a disjoint union of V_1 and V_2, each of which is in one-to-one correspondence with $U_1\backslash G$ and $U_2\backslash G$ respectively. The set E of its edges is in one-to-one correspondence with $B\backslash G$. Let $\varphi_i : B\backslash G \to U_iG$ be the natural projection $(i=1, 2)$. Then two vertices P, Q are joined by a (non-oriented) edge e if and only if $P=\varphi_1(e)$, $Q=\varphi_2(e)$, where we regard P, Q, e as elements of $U_1\backslash G$, $U_2\backslash G$, and $B\backslash G$ respectively.

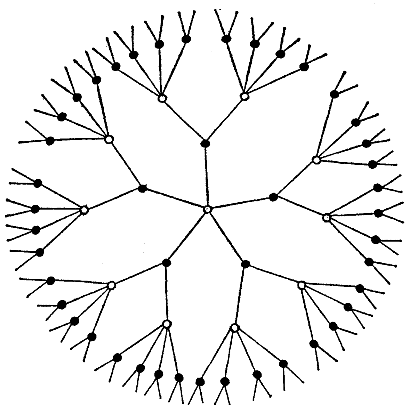
One can describe it also as follows. First notice that the normalizer of any parabolic subgroup of G coincides with itself. So one can identify the cosets U_ig in $U_i\backslash G$ (resp. Bg in $B\backslash G$) with the set of conjugates $g^{-1}U_ig$ (resp. $g^{-1}Bg$) of the parabolic subgroups of each type. Then the two vertices are adjacent (i.e., joined by an edge) if and only if the intersection of the corresponding parabolic subgroups is a conjugate of B.

(2.15) **Example.** $P_0 \longleftrightarrow U_1$, $Q_0 \longleftrightarrow U_2$, $e_0 \longleftrightarrow B$

$q_1=4$, $q_2=2$

In the figure below, black vertices $\{\bullet\}$ and white ones $\{\circ\}$ correspond to the conjugates $\{gU_1g^{-1}\}$ and $\{gU_2g^{-1}\}$ respectively. Each black vertex has $1+q_1$ adjacent (white) vertices, and each white one has $1+q_2$ adjacent (black) vertices.

Let P_0 and Q_0 be, as in Fig-1, the adjacent pair of vertices corresponding to U_1, U_2 respectively. For the sake of simplicity, we identify V_i with $U_i\backslash G$, E with $B\backslash G$, and denote them simply by V_i, E respectively $(i=1, 2)$.



(Figure-1)

§ 3. *P*-adic algebraic groups

Let G be a semi-simple algebraic group defined over a local field K, and assume that G has K-rank one and has an affine Tits system (G, B, N, S). Note that the last condition is satisfied if G is simply connected. In this case B is called the Iwahori subgroup. Let W be the (affine) Weyl group of our Tits system. The assumption on the K-rank implies that our Tits system has rank two, so that, writing $S=\{s_1, s_2\}$, all the conditions of the preceding paragraph are satisfied.

#	Type	simply connected groups	(d_1, d_2)
(1)	A_1	$SL(2, K)$	(1, 1)
(2)	$C\text{-}BC_1$	$SU(3, L/K)$; $[L:K]=2$, $L/K=$ramified	(1, 1)
(3)	${}^dA_{2d-1}$ $(d\geq 2)$	$SL(2, D)$; $D=K$-simple div. algebra $[D:K]=d^2$	(d, d)
(4)	${}^2A_3''$	$SU(4, L/K)$; $[L:K]=2$, $L/K=$unramified	(3, 3)
(5)	${}^2A_2'$	$SU(3, L/K)$; $[L:K]=2$, $L/K=$unramified	(3, 1)
(6)	${}^2C_3$	$Sp(3)=U(3, D)$; $D=$div. quaternion$/K$	(3, 2)
(7)	${}^2C_2$	$Sp(2)=U(2, D)$; $D=$div. quaternion$/K$	(1, 2)
(8)	${}^2C\text{-}B_3$	$SU_{\text{skew}}(4, D/K)$; $[L:K]=2$, $L/K=$ramified	(3, 2)
(9)	${}^2C\text{-}B_2$	$SU_{\text{skew}}(3, D/K)$; $[L:K]=2$, $L/K=$ramified	(1, 2)
(10)	${}^4D_4$	$SU_{\text{skew}}(4, D/K)$; $[L:K]=2$, $L/K=$unramified	(1, 4)
(11)	${}^4D_5$	$SU_{\text{skew}}(5, D/K)$; $L=K\oplus K$	(3, 4)

In (8)-(11), $SU_{\text{skew}}(r, D/K)$ denotes the special unitary group of a quaternion skew-hermitian form $q(x)$ of rank r, with Witt index$=1$, and L denotes the center of the even Clifford algebra of $q(x)$.

Fact (3.1). The two subgroups $U_i := B \cup Bs_iB$ ($i=1, 2$) form a representatives of the G-conjugacy classes of the maximal open compact subgroups of G.

Let q denote the cardinal number of the residue field of K, i.e. $O/\boldsymbol{p} \simeq \boldsymbol{F}_p$.

Fact (3.2). There are two positive integers d_1, d_2 such that $\sharp(B\backslash Bs_iB) = q^{d_i}$ ($=q_i$, say, for $i=1, 2$).

According to the classification of Tits [T-2], there are 11 types of simple and semi-simple groups over a local field K, up to the central isogeny, which have K-rank one. They are listed in the above table.

§ 4 Structure of the discrete subgroups Γ

Let G be a group with a length function $l: G \to \boldsymbol{N} \cup \{0\}$ satisfying the axiom (G, l, I), (G, l, II) in § 1. And let Γ be a subgroup of G which satisfies:

(Γ, I) Γ is torsion free, and $\Gamma \cap x^{-1}Ux = \{1\}$ for any $x \in G$.

(Γ, II) $\sharp(U\backslash G/\Gamma) < \infty$.

We put $h := \sharp(U\backslash G/\Gamma)$, and denote by $\{x_1, \cdots, x_h\}$ a complete set of representatives in $U\backslash G/\Gamma$, which is fixed once and for all. Thus we have

(4.1) $$G = \bigcup_{j=1}^{h} Ux_j\Gamma \quad \text{(disjoint)}.$$

Also put, for each $l=1, 2, \cdots$:

(4.2) $$S_{ij}^{(l)} := \Gamma \cap x_i^{-1} G_l x_j \quad (1 \leq i, j \leq h).$$

Note that (Γ, I), (Γ, II) imply that $S_{ij}^{(l)}$ are all finite set. One can easily show, as in [I-1], that the mapping $G_l \to (S_{ij}^{(l)})$ defines a ring homomorphism φ: $\boldsymbol{Z}[G_l; l=0, 1, 2, \cdots] \to M(h, \boldsymbol{Z}[\Gamma])$, where we identify the set $S_{ij}^{(l)}$ and the formal sum of its elements in the group ring $\boldsymbol{Z}[\Gamma]$. And one can show

(4.3) $$\sum_{i=1}^{h} \sharp(S_{ij}^{(l)}) = \sum_{j=1}^{h} \sharp(S_{ij}^{(l)}) = \sharp(U\backslash G_l) = (q_1q_2)^{l-1}q_2(1+q_1) \quad (l \geq 1).$$

We shall denote $S_{ij}^{(1)}$ simply by S_{ij}. From the assumption (G, l, II) and the above remark, it follows that Γ is generated by $\bigcup_{i,j} S_{ij}$. More precisely, one has

Lemma (4.4). *Let i, j ($1 \leq i, j \leq h$) be given. Then*

(i) $\Gamma=\bigcup_{l=0}^{\infty} S_{ij}^{(l)}$ (*disjoint*).

(ii) *Each element* $\gamma \in S_{ij}^{(l)}$ *has a following expression*

$$(*) \qquad \gamma=\sigma_{ii_1}\sigma_{i_1i_2}\cdots\sigma_{i_{l-1}j}\ (\sigma_{i_ki_{k+1}} \in S_{i_ki_{k+1}};\ i_0=i,\ i_l=j).$$

with

$$(**) \qquad \sigma_{i_{k-1}i_k}\sigma_{i_ki_{k+1}} \in S^{(2)}_{i_{k-1}i_{k+1}} \quad (k=1, 2, \cdots, l-1).$$

(iii) *In the above expression, the indices* $(i_1, i_2, \cdots, i_{l-1})$ *and the elements* $\sigma_{i_ki_{k+1}}$ $(0\leq k\leq l-1)$ *are uniquely determined by* γ *and* (i, j).

Proof. (i) This follows trivially from $G=x_i^{-1}Gx_j=\bigcup_{l=0}^{\infty} x_i^{-1}G_lx_j$ (disjoint). (ii) Let γ be any element of $S_{ij}^{(l)}$. Then one sees that $x_i\gamma x_j^{-1} \in G_l$, so that from Lemma (1.5), there exists a unique expression of $x_i\gamma x_j^{-1}$ of the following form:

$$x_i\gamma x_j^{-1}=u\omega_{j_1}\omega_{j_2}\cdots\omega_{j_l}\ (u \in U,\ \omega_{j_1}, \cdots, \omega_{j_l} \in \Omega),$$

with $\qquad \omega_{j_n}\omega_{j_{n+1}} \in G_2 \qquad$ *for* $n=1, 2, \cdots, l-1$.

Put

$$g_0:=u\omega_{j_1}\omega_{j_2}\cdots\omega_{j_l}x_j,$$
$$g_1:=\omega_{j_2}\omega_{j_3}\cdots\omega_{j_l}x_j,$$
$$\cdots,$$
$$g_l:=x_j.$$

There exists a unique index i_{l-1} such that $g_{l-1} \in Ux_{i_{l-1}}\Gamma$; writing $g_{l-1}=\omega_{j_l}x_j=v_{l-1}x_{i_{l-1}}\sigma_{i_{l-1}j}$ $(v_{l-1} \in U, \sigma_{i_{l-1}j} \in \Gamma)$, one sees that $\sigma_{i_{l-1}j} \in S_{i_{l-1}j}$. Next let i_{l-2} be the unique index such that $g_{l-2}\sigma_{i_{l-1}j} \in Ux_{i_{l-2}}\Gamma$; and write

$$g_{l-2}=\omega_{j_{l-1}}\omega_{j_l}x_j=v_{l-2}x_{i_{l-2}}\sigma_{i_{l-2}i_{l-1}}\sigma_{i_{l-1}j}\ (v_{l-2} \in U,\ \sigma_{i_{l-2}i_{l-1}} \in \Gamma),$$

One sees immediately that $\sigma_{i_{l-2}i_{l-1}} \in S_{i_{l-2}i_{l-1}}$. Repeating this procedure, one gets an expression (*) of γ as asserted. The condition (**) follows from the corresponding condition $\omega_{j_n}\omega_{j_{n+1}} \in G_2$, since one has

$$\sigma_{i_{k-1}i_k}\sigma_{i_ki_{k+1}}=x_{i_{k-1}}^{-1}(v_{k-1}^{-1}\omega_{j_k}\omega_{j_{k+1}}v_{k+1})x_{i_{k+1}} \in \Gamma\cap x_{i_{k-1}}^{-1}G_2x_{i_{k+1}}.$$

(iii) The uniqueness follows easily from the above arguments.

Q.E.D.

Note that $S_{ij}=S_{ji}^{-1}$. Let S be a subset of $\cup S_{ij}-\{1\}$ such that $\cup S_{ij}=S\cup S^{-1}\cup\{1\}$.

Corollary (4.5). *The subgroup* Γ *of* G *satisfying* (Γ, I), (Γ, II) *is a free group with free generators* S.

Proof. The fact that Γ is a free group is proved in Appendix, Corollary (10.12). Therefore our assertion follows immediately from Lemm (4.4). Q.E.D.

However, in general, it is difficult to determine the set S explicitly, as has been done in [I-1]. This is partly because the same element σ can belong to many of the subsets S_{ij}.

Remark (4.6). Using the action of G on the tree $X(q_1, q_2)$, the above Lemma (4.4) can be interpreted as follows. We first note that

$$(4.7) \qquad S_{ij}^{(l)}=\{\gamma \in \Gamma;\, d_X((P_0x_j, P_0x_i\gamma)=2l\} \quad (P_0=U \in U\backslash G),$$

and that, if we put $P_j:=P_0x_j$ $(1\leq j\leq h)$, then we have

$$(4.8) \qquad V_1=\bigcup_{\gamma\in\Gamma}\{P_j;\, 1\leq j\leq h\}\gamma \quad \text{(disjoint)}.$$

In particular, each point $P \in V_1$ determines uniquely the index i $(1\leq i\leq h)$ and $\gamma \in \Gamma$ such that $P=P_0x_i\gamma$.

Now for each $\gamma \in S_{ij}^{(l)}=\Gamma\cap x_i^{-1}G_lx_j$, one has $d_X(P_0x_j, P_0x_i\gamma)=2l$. Put $R_0:=P_0x_i\gamma$, $R_l:=P_0x_j$, and let

$$C(R_0, R_l)=[R_0, Q_1, R_1, Q_2, R_2, \cdots, Q_l, R_l]$$

be the geodesic path from R_0 to R_l in $X(q_1, q_2)$. Then each $R_k \in V_1$ determines an index i_k $(1\leq i_k\leq h)$ and an element $\gamma_k \in \Gamma$. Now it is easy to see that

$$\begin{aligned}
&\gamma_l=1,\\
&\gamma_{l-1}=\sigma_{i_{l-1}j},\\
&\gamma_{l-2}=\sigma_{i_{l-2}i_{l-1}}\sigma_{i_{l-1}j},\\
&\quad\cdots,\\
&\gamma_0=\sigma_{ii_1}\sigma_{i_1i_2}\cdots\sigma_{i_{l-1}j}.
\end{aligned}$$

Lemma (4.9). *For any element σ of S_{ij} $(1\leq i, j\leq h)$, one has*

$$\sum_{k=1}^{h}\#\{\tau \in S_{ki};\, \tau\sigma \in S_{kj}^{(l)}\}=\begin{cases}1 & \cdots\ l=0\\ q_2-1 & \cdots\ l=1\\ q_1q_2 & \cdots\ l=2.\end{cases}$$

Proof. Note that $\sigma \in S_{ij}$ (resp. $\tau \in S_{ki}$) is equivalent to $d_X(P_0x_i\sigma, P_0x_j)=2$ (resp. $d_X(P_0x_i, P_0x_k\tau)=2$). Now the set $B:=\{R \in V_1;\, d_X(P_0x_i, R)=2\}$ is divided, according to its distance from $P_0x_j\sigma^{-1}$, into three disjoint subsets: $B=B_0\cup B_1\cup B_2$, with

$$B_l=\{R \in V_1;\ d_X(R, P_0x_j\sigma^{-1})=2l\} \quad (l=0, 1, 2).$$

From the above remark, it follows that if $R \in B_l$, then R is expressed as $R=P_0x_k\tau$ with $1 \leq k \leq h$, $\tau \in S_{ki}^{(l)}$. Now the assertion follows from Lemma (1.4). Q.E.D.

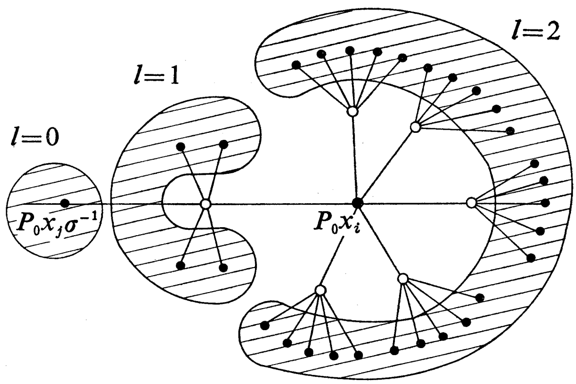


Figure-2

Finally we shall make the following observation.

Lemma (4.10). *Let Γ be a subgroup of G satisfying the conditions* (Γ, I), (Γ, II). *Then one has:*

(i) $h_2:=(1+q_1)h/(1+q_2)$ *is an integer.*

(ii) *The free rank r of Γ is given by $r=q_1h-h_2+1$.*

Proof. Consider the quotient graph $Y=X(q_1, q_2)/\Gamma$, for which one has $\sharp(V_1Y)=\sharp(V_1/\Gamma)=h$. Putting $h_2=\sharp(V_2Y)=\sharp(V_2/\Gamma)$, one sees that the number of edges in Y is $(q_1+1)h=(q_2+1)h_2$, hence (i). To prove (ii), one notes that the Euler characteristic of Y is

$$\begin{aligned}\chi(Y)&=(h+h_2)-(q_1+1)h\\&=\dim H^0(\Gamma, \boldsymbol{R})-\dim H^1(\Gamma, \boldsymbol{R})\\&=1-r.\end{aligned}\tag{4.11}$$

Q.E.D.

Remark (4.12). It would be interesting to interprete h_2 and prove the equality (i) combinatorially, without using the graph Y. In fact, using the notation of (1.10), h_2 is seen to be the number of the equivalence classes $E_j(P) \cup \{P\}$ modulo Γ; and (i) can be proved by counting the cardinality of E^*/Γ in two different ways.

§ 5. Γ-conjugacy classes of given degree

In this and the next sections, the notation and the assumptions will be the same as in the previous sections. We begin with the same definition as in [I-1]:

Definition (5.1) For any Γ-conjugacy class $\{\gamma\}_\Gamma$ ($\gamma \in \Gamma$), we put

$$\deg\{\gamma\}_\Gamma := \operatorname*{Min}_{x \in G} l(x^{-1}\gamma x),$$

and call it the *degree* of $\{\gamma\}_\Gamma$, or of γ.

Note that, from $G = \bigcup_{i=1}^h Ux_i\Gamma$, one has

$$(5.2) \qquad \deg\{\gamma\}_\Gamma := \operatorname*{Min}_{1 \le i \le h} \{l;\ \delta \in S_{ii}^{(l)},\ \delta \text{ is } \Gamma\text{-conjugate to } \gamma\}.$$

Now let γ be an element of $S_{ii}^{(l)}$, and let

$$\gamma = \sigma_{ii_1}\sigma_{i_1i_2}\cdots\sigma_{i_{l-1}i} \ (\sigma_{i_ki_{k+1}} \in S_{i_ki_{k+1}};\ i_0 = i_l = i).$$

be the expression as in Lemma (4.4). For the sake of simplicity, put $\sigma_m = \sigma_{i_{m-1}i_m}$ ($m = 1, 2, \cdots, l$). Let k ($1 \le k \le [l/2]$) be such that

$$(5.3) \qquad \begin{aligned} &i_1 = i_{l-1} \quad \text{and} \quad \sigma_l\sigma_1 = 1, \\ &i_2 = i_{l-2} \quad \text{and} \quad \sigma_{l-1}\sigma_2 = 1, \\ &\qquad \cdots, \\ &i_k = i_{l-k} \quad \text{and} \quad \sigma_{l-k+1}\sigma_k = 1, \\ &i_{k+1} \ne i_{l-k-1} \quad \text{or} \quad \sigma_{l-k}\sigma_{k+1} \ne 1. \end{aligned}$$

Lemma (5.4). *Let $\gamma \in S_{ii}^{(l)}$ satisfy the above condition. Then one has*
(i) *If $\sigma_{l-k}\sigma_{k+1} \in S_{i_{l-k-1}i_{k+1}}^{(2)}$ then* $\deg\{\gamma\}_\Gamma = l - 2k$.
(ii) *If $\sigma_{l-k}\sigma_{k+1} \in S_{i_{l-k-1}i_{k+1}}^{(1)}$ then* $\deg\{\gamma\}_\Gamma = l - 2k - 1$.

Proof. By the condition (5.3) one sees that

$$\gamma = \sigma_1\sigma_2\cdots\sigma_k(\sigma_{k+1}\cdots\sigma_{l-k})\sigma_k^{-1}\cdots\sigma_1^{-1},$$

so that γ is Γ-conjugate to $\gamma' := \sigma_{k+1}\cdots\sigma_{l-k}$. Again by (5.3) one sees that $\gamma' \in S_{i_ki_k}^{(l-2k)}$. Also from $i_{l-k} = i_k$, one has $\sigma_{l-k}\sigma_{k+1} \in S_{i_{l-k-1}i_{k+1}}^{(m)}$ with $m = 1$ or 2. Consider the element $\gamma'' := \sigma_{k+1}^{-1}\gamma'\sigma_{k+1} = \sigma_{k+2}\cdots\sigma_{l-k-1}\sigma_{l-k}\sigma_{k+1}$, which is Γ-conjugate to γ. Since the condition (**) of Lemma (4.4) is satisfied with only possible exception for $m = 1$ in the last factor, we see from (1.7) that $\gamma'' \in S_{i_{k+1}i_{k+1}}^{(l-2k+2-m)}$. Thus we have $\deg\{\gamma\}_\Gamma \ge l - 2k + 2 - m$. Now let $\delta \in \Gamma$ be such that $\delta^{-1}\gamma''\delta \in S_{ff}^{(n)}$, and suppose that $n < l - 2k + 2 - m$. Using Lemma (4.4), one finds $s \ge 1$ such that $\delta \in S_{i_{k+1}f}^{(s)}$, so that δ is expressed as

$$\delta=\tau_{i_{k+1}f_1}\tau_{f_1f_2}\cdots\tau_{f_{s-1}f}\ (\tau_{f_tf_{t+1}}\in S_{f_tf_{t+1}}),$$

with the similar condition as $(**)$ of Lemma (4.4). We may assume that δ has been so chosen that $s(\geq 1)$ is minimal among such elements. Suppose first that $m=2$. If we have

$$\tau^{-1}_{i_{k+1}f_1}\sigma_{i_{k+1}i_{k+2}}\in S^{(2)}_{f_1i_{k+1}}\cup S^{(1)}_{f_1i_{k+2}},\quad \text{or}$$
$$\sigma_{i_ki_{k+1}}\tau_{i_{k+1}f_1}\in S^{(2)}_{i_kf_1}\cup S^{(1)}_{i_kf_1},$$

then using Lemma (1.4) it follows easily that $n=l(\delta^{-1}\gamma''\delta)\geq l-2k+2s-2\geq l-2k$, which contradicts to our assumption. Thus we have either $\tau_{i_{k+1}f_1}=\sigma_{i_{k+1}i_{k+2}}$ and $f_1=i_{k+2}$, or $\tau_{i_{k+1}f_1}=\sigma^{-1}_{i_ki_{k+1}}$ and $f_1=i_{k+1}$. Replacing γ'' by γ' or by $\sigma_{l-k}\sigma_{k+1}\cdots\sigma_{l-k-1}$, this reduces the problem to the case of δ with $s-1$, which contradicts to the minimality of s. The case $m=1$ can be treated similarly, if we regard the product $\sigma_{l-k}\sigma_{k+1}$ as a single element of $S^{(1)}_{i_{l-k-1}i_{k+1}}$. Q.E.D.

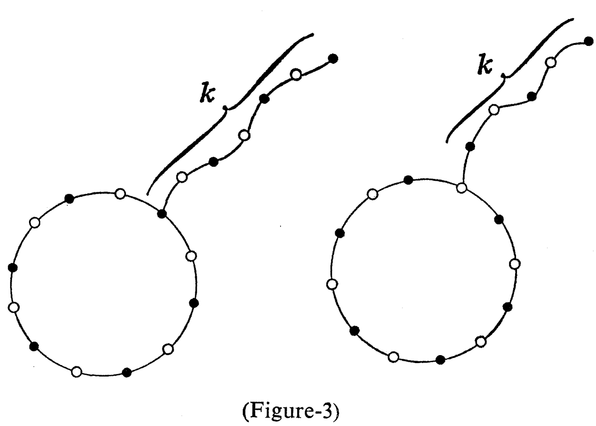
The geometric interpretation of this result is quite simple and helpful to understand the situation. We first note that for $\gamma\in S^{(l)}_{ii}$, the expression

$$(*)\qquad \gamma=\sigma_{ii_1}\sigma_{i_1i_2}\cdots\sigma_{i_{l-1}i}\ (\sigma_{i_ki_{k+1}}\in S_{i_ki_{k+1}};\ i_0=i_l=i),$$

with

$$(**)\qquad \sigma_{i_{k-1}i_k}\sigma_{i_ki_{k+1}}\in S^{(2)}_{i_{k-1}i_{k+1}}\ (k=1,2,\cdots,l-1)$$

corresponds to the geodesic path C (i.e., the path without backtracking) in the tree $X(q_1,q_2)$, joining P_0x_i and $P_0x_i\gamma$. Now, if we project C to the quotient graph $X(q_1,q_2)/\Gamma$, we get a closed path $\bar{C}$ in $X(q_1,q_2)/\Gamma$. In



(Figure-3)

general, however, $\bar{C}$ has a backtracking at the origin$=$terminal, as is illustrated in the above figures.

From the proof of Lemma (5.4) and the above interpretation, we immediately have the following

Lemma (5.5). *Let $\gamma \in S_{ii}^{(l)}$ and let*

$$\gamma=\sigma_1\sigma_2\cdots\sigma_l \ (\sigma_k=\sigma_{i_k i_{k+1}} \in S_{i_k i_{k+1}};\ i_0=i_l=i)$$

be the expression as in Lemma (4.4). *Moreover, suppose that $\sigma_l\sigma_1 \in S_{i_{l-1}i_1}^{(2)}$. Then*

(i) *The elements $\gamma_j:=\sigma_{j+1}\sigma_{j+2}\cdots\sigma_l\sigma_1\cdots\sigma_j$ $(0\leq j\leq l-1)$ are exactly those which are Γ-conjugate to γ and which are contained in some $S_{mm}^{(l)}$ $(1\leq m\leq h$; l is fixed).*

(ii) *The above l elements γ_j $(0\leq j\leq l-1)$ are not necessarily distinct. If exactly d of them are mutually distinct, then each one belongs to exactly d different $S_{mm}^{(l)}$'s; i.e., $\sharp\{m;\ \gamma_j \in S_{mm}^{(l)}\}=d$.*

§ 6. Zeta function $Z_\Gamma(u;\rho)$

Let γ $(\neq 1)$ be an element of Γ. Since Γ is a free group of finite rank, the centralizer $C_\Gamma(\gamma)$ of γ in Γ is an infinite cyclic group. We call γ or, the conjugacy class $\{\gamma\}_\Gamma$, to be primitive, if $C_\Gamma(\gamma)$ is generated by γ.

Definition. Let $\rho:\Gamma\rightarrow U(n)$ be an n-dimensional unitary representation of Γ. Then the zeta function $Z_\Gamma(u;\rho)$ of Γ attached to ρ is defined by an infinite product

$$Z_\Gamma(u;\rho):=\prod_{\{\gamma\}_\Gamma}\det(I_n-\rho(\gamma)u^{\deg\{\gamma\}_\Gamma})^{-1}. \tag{6.1}$$

Let $G=\bigcup_{i=1}^{h} Ux_i\Gamma$ be a decomposition of G into disjoint union whose set of representatives is fixed throughout the following. Consider the Brandt representation of $\mathscr{H}(G,U)$ attached to ρ:

$$\begin{aligned}\varphi_\rho:\mathscr{H}(G,U)&\longrightarrow M(nh,\boldsymbol{C})=M(h,\boldsymbol{C})\otimes M(n,\boldsymbol{C})\\ UyU&\longrightarrow\Big(\sum_{\gamma\in\Gamma\cap x_i^{-1}UyUx_j}\rho(\gamma)\Big)_{1\leq i,j\leq h}.\end{aligned} \tag{6.2}$$

We put

$$A_{l,\rho}:=\varphi_\rho(G_l) \qquad \text{for } l=0,1,2,\cdots \tag{6.3}$$

Also put $h_1:=h$, $h_2:=(q_1+1)h_1/(q_2+1)$; recall that h_2 is an integer, and that the rank of Γ is equal to $h_1q_1-h_2+1=h_2q_2-h_1+1$ (cf. Lemma (4.10)).

Now our main result of this paper is the following

Theorem (6.4). *The zeta function $Z_\Gamma(u;\rho)$ is a rational function of u which has the following expression:*

$$Z_\Gamma(u;\rho)^{-1}=(1-u)^{n(r-1)}(1+q_2u)^{n(h_2-h_1)}$$
$$\times\det\{I_{nh_1}-(A_{1,\rho}-q_2+1)u+q_1q_2u^2\}.$$

We shall prove this theorem. Taking the logarithmic derivative of (6.1), one gets

$$u\frac{d}{du}\log Z_\Gamma(u;\rho)=\sum_{l=0}^{\infty}N_{l,\rho}u^l, \tag{6.5}$$

with

$$N_{l,\rho}:=\sum_{d|l}d\sum_{P=\{\delta\}_\Gamma}\operatorname{tr}\rho(\delta^{l/d}) \tag{6.6}$$

where the second sum in (6.6) is extended over the set of primitive Γ-conjugacy classes P such that $\deg P=d$. Suppose that $\delta\in S_{ii}^{(d)}$ is a representative of P. Then by Lemma (5.4) we see that $\gamma=\delta^{l/d}\in S_{ii}^{(l)}$, and that, if we express γ as in Lemma (5.5) then by the primitiveness of $P=\{\delta\}_\Gamma$, there are exactly d mutually different ones among the l expressions γ_j $(0\le j\le l-1)$. It follows that each of them is contained in exactly d distinct $S_{mm}^{(l)}$'s $(1\le m\le h)$. Thus we have

$$N_{l,\rho}=\sum_{j=1}^{h}\sum_{\substack{\gamma\in S_{jj}^{(l)}\\ \deg\{\gamma\}_\Gamma=l}}\operatorname{tr}\rho(\gamma). \tag{6.7}$$

For each $l\ge 1$, k $(0\le k\le[(l-1)/2])$, and i $(1\le i\le h)$, put

$$S_{ii}^{(l,k,+)}:=\{\gamma\in S_{ii}^{(l)};\ (5.3)\text{ with }\sigma_{l-k}\sigma_{k+1}\in S_{i_{l-k-1}i_{k+1}}^{(2)}\}$$
$$S_{ii}^{(l,k,-)}:=\{\gamma\in S_{ii}^{(l)},\ (5.3)\text{ with }\sigma_{l-k}\sigma_{k+1}\in S_{i_{l-k-1}i_{k+1}}^{(1)}\},$$

where we use the abbreviated notation as in Lemma (5.4), and k is determined by (5.3). Then one has for each i $(1\le i\le h)$ and $l\ge 1$,

$$S_{ii}^{(l)}=\bigcup_{k=0}^{[(l-1)/2]}S_{ii}^{(l,k,+)}\cup\bigcup_{k=0}^{[(l-2)/2]}S_{ii}^{(l,k,-)}\ \text{(disjoint)}. \tag{6.8}$$

Lemma (6.9). (i) *For $k\ge 1$, the mapping*

$$\phi^+:\sum_{i=1}^{h}S_{ii}^{(l,k,+)}\longrightarrow\sum_{j=1}^{h}S_{jj}^{(l-2k,0,+)} \tag{6.10}$$

$$\gamma=\sigma_1\sigma_2\cdots\sigma_k(\sigma_{k+1}\cdots\sigma_{l-k})\sigma_k^{-1}\cdots\sigma_1^{-1}\longrightarrow\sigma_{k+1}\sigma_{k+2}\cdots\sigma_{l-k}$$

is $(q_1-1)q_2(q_1q_2)^{k-1}$*-to-one.*

(ii) *For* $k\geqq 0$, *the mapping*

$$\phi^-:\sum_{i=1}^{h}S_{ii}^{(l,k,-)}\longrightarrow\sum_{j=1}^{h}S_{jj}^{(l-2k-1,0,+)} \tag{6.11}$$

$$\gamma=\sigma_1\sigma_2\cdots\sigma_k(\sigma_{k+1}\cdots\sigma_{l-k})\sigma_k^{-1}\cdots\sigma_1^{-1}\longrightarrow\sigma_{k+2}\cdots\sigma_{l-k}\sigma_{k+1}$$

is $(q_2-1)(q_1q_2)^k$*-to-one.*

Proof. We first prove (i). Suppose we are given $\gamma'=\sigma_{k+1}\cdots\sigma_{l-k}\in S_{i_ki_k}^{(l-2k)}$, and consider the number of choices for $\sigma_k,\cdots,\sigma_1$. We first show that there are $(q_1-1)q_2$ choices for σ_k. Indeed, σ_k should be chosen under the following conditions:

$$\begin{aligned}
&\sigma_k\sigma_{k+1}\in S_{i_ki_{k+1}}^{(2)}\ \text{i.e.,}\ d_X(P_0x_{i_{k+1}}\sigma_{k+1}^{-1},P_0x_{i_{k-1}}\sigma_k)=2\\
&\sigma_{l-k}\sigma_k^{-1}\in S_{i_{l-k-1}i_{k-1}}^{(2)}\ \text{i.e.,}\ d_X(P_0x_{i_{k-1}}\sigma_k,P_0x_{i_{l-k-1}}\sigma_{l-k})=2\\
&\sigma_{l-k}\sigma_{k+1}\in S_{i_{l-k-1}i_{k+1}}^{(2)}\ \text{i.e.,}\ d_X(P_0x_{1_{k+1}}\sigma_{k+1}^{-1},P_0x_{i_{l-k-1}}\sigma_{l-k})=2.
\end{aligned}$$

Noting that the index i_k has been given, we see that to choose σ_k is equivalent to choose the vertex $P_0x_{i_{k-1}}\sigma_k\in V_1$, which is at the distance 4 from the (given) vertices $P_0x_{i_{k+1}}\sigma_{k+1}^{-1}$, $P_0x_{i_{l-k-1}}\sigma_{l-k}$. Now it is easy to see from Lemma (4.9) that there exactly $(q_1-1)q_2$ such vertices (see also Figure 2). Next for each choice of σ_k, one can show by Lemma (4.9) that there are q_1q_2 choices for σ_{k-1}. Similarly for each choice of $\sigma_k,\sigma_{k-1},\cdots,\sigma_{j+1}$, there are q_1q_2 choices for σ_j. This proves (i). To prove (ii), it suffices to note that, for given $\sigma_{k+2},\cdots,\sigma_{l-k}$, there are exactly q_2-1 choices of σ_{k+1} such that $\gamma'=\sigma_{k+2}\cdots\sigma_{l-k}\sigma_{k+1}\in S_{i_{k+1}i_{k+1}}^{(l-2k-1)}$, which follows easily from Lemma (4.9). Then one can proceed as in the case (i). Q.E.D.

Let f be a class function on Γ. For a finite subset S of Γ, we denote by $f(S)$ the sum of $f(\gamma)$, $\gamma\in S$.

Corollary (6.12). *For any class function f on Γ, one has*

$$\sum_{i=1}^{h}f(S_{ii}^{(l,k,+)})=(q_1-1)q_2(q_1q_2)^{k-1}\sum_{i=1}^{h}f(S_{ii}^{(l-2k,0,+)}),$$

$$\sum_{i=1}^{h}f(S_{ii}^{(l,k,-)})=(q_2-1)(q_1q_2)^{k}\sum_{i=1}^{h}f(S_{ii}^{(l-2k-1,0,+)}).$$

Proof. Let χ be the character of ρ, i.e., $\chi(\gamma)=\operatorname{tr}\rho(\gamma)$ $(\gamma\in\Gamma)$. Then we have

$$\begin{aligned}\operatorname{tr}(A_{l,\rho})&=\sum_{i=1}^{h}\chi(S_{ii}^{(l)})\\&=\sum_{k=0}^{[(l-1)/2]}\sum_{i=1}^{h}\chi(S_{ii}^{(l,k,+)})+\sum_{k=0}^{[(l-2)/2]}\sum_{i=1}^{h}\chi(S_{ii}^{(l,k,-)})\\&=\sum_{i=1}^{h}\chi(S_{ii}^{(l,0,+)})+(q_1-1)q_2\sum_{k=1}^{[(l-1)/2]}(q_1q_2)^{k-1}\sum_{i=1}^{h}\chi(S_{ii}^{(l-2k,0,+)})\\&\quad+(q_2-1)\sum_{k=0}^{[(l-2)/2]}(q_1q_2)^{k}\sum_{i=1}^{h}\chi(S_{ii}^{(l-2k-1,0,+)})\end{aligned}$$

Thus from (6.7), we obtain

$$\begin{aligned}\operatorname{tr}(A_{l,\rho})&=N_{l,\rho}+(q_1-1)q_2\sum_{k=1}^{[(l-1)/2]}(q_1q_2)^{k-1}N_{l-2k,\rho}\\&\quad+(q_2-1)\sum_{k=0}^{[(l-2)/2]}(q_1q_2)^{k}N_{l-2k-1,\rho}.\end{aligned}\tag{6.13}$$

Q.E.D.

Lemma (6.14).

$$\begin{aligned}N_{l,\rho}&=\operatorname{tr}(A_{l,\rho})-(q_2-1)\operatorname{tr}(A_{l-1,\rho})\\&\quad-\sum_{k=1}^{l-2}\left\{(q_2-1)+(q_1-q_2)q_2\sum_{m=0}^{l-2-k}(-q_2)^m\right\}\operatorname{tr}(A_{k,\rho}).\end{aligned}$$

Proof. We prove this by induction on l. Note first that the assertion for $l=1$ follows from (6.13). Suppose that it is true for $l=1, 2, \cdots, p-1$. Using (6.13) for $l=p$, and the induction hypothesis, we have an equality:

$$\begin{aligned}N_{p,\rho}&=\operatorname{tr}(A_{p,\rho})-(q_2-1)\operatorname{tr}(A_{p-1,\rho})\\&\quad-(q_1-1)q_2\sum_{k=1}^{[(p-1)/2]}(q_1q_2)^{k-1}\operatorname{tr}(A_{p-2k,\rho})\\&\quad+(q_1-1)(q_2-1)q_2\sum_{k=1}^{[(p-1)/2]}(q_1q_2)^{k-1}\sum_{j=1}^{p-2k-1}\operatorname{tr}(A_{j,\rho})\\&\quad+(q_1-1)(q_1-q_2)q_2^2\sum_{k=1}^{[(p-1)/2]}(q_1q_2)^{k-1}\sum_{j=1}^{p-2k-2}\sum_{m=0}^{p-2k-2-j}(-q_2)^m\operatorname{tr}(A_{j,\rho})\\&\quad-(q_2-1)\sum_{k=1}^{[(p-2)/2]}(q_1q_2)^{k}\operatorname{tr}(A_{p-2k-1,\rho})\\&\quad+(q_2-1)^2\sum_{k=0}^{[(p-2)/2]}(q_1q_2)^{k}\sum_{j=1}^{p-2k-2}\operatorname{tr}(A_{j,\rho})\\&\quad+(q_2-1)(q_1-q_2)q_2\sum_{k=0}^{[(p-2)/2]}(q_1q_2)^{k}\sum_{j=1}^{p-2k-3}\sum_{m=0}^{p-2k-3-j}(-q_2)^m\operatorname{tr}(A_{j,\rho}).\end{aligned}\tag{6.15}$$

Now a direct calculation shows that the total sum of the right hand side of (6.15) is equal to that of (6.14) for $l=p$. Q.E.D.

Using (6.5) and (6.14), we have

$$(6.16)\quad u\frac{d}{du}\log Z_{\Gamma}(u;\rho)=\sum_{l=1}^{\infty}\operatorname{tr}(A_{l,\rho})u^l$$
$$-(q_2-1)\sum_{l=2}^{\infty}\operatorname{tr}(A_{l-1,\rho})u^l-(q_2-1)\sum_{l=3}^{\infty}\sum_{k=1}^{l-2}\operatorname{tr}(A_{k,\rho})u^l$$
$$-(q_1-q_2)q_2\sum_{l=3}^{\infty}\sum_{k=1}^{l-2}\sum_{m=0}^{l-2-k}(-q_2)^m\operatorname{tr}(A_{k,\rho})u^l$$
$$=\left\{1-(q_2-1)\sum_{k=1}^{\infty}u^k-(q_1-q_2)q_2\sum_{k=2}^{\infty}\sum_{m=0}^{k-2}(-q_2)^m u^k\right\}\sum_{l=1}^{\infty}\operatorname{tr}(A_{l,\rho})u^l$$
$$=\left\{1-\frac{(q_2-1)u}{1-u}-\frac{(q_1-q_2)q_2u^2}{(1-u)(1+q_2u)}\right\}\sum_{l=1}^{\infty}\operatorname{tr}(A_{l,\rho})u^l$$
$$=\frac{1-q_1q_2u^2}{(1-u)(1+q_2u)}\cdot\sum_{l=1}^{\infty}\operatorname{tr}(A_{l,\rho})u^l.$$

Applying the Brandt representation φ_ρ to the both sides of (1.1), (1.2), we have

$$(6.17)\qquad A_{1,\rho}^2=A_{2,\rho}+(q_2-1)A_{1,\rho}+q_2(q_1+1)A_{0,\rho},$$
$$(6.18)\qquad A_{1,\rho}A_{l,\rho}=A_{l+1,\rho}+(q_2-1)A_{l,\rho}+q_1q_2A_{l-1,\rho}\quad (l\geq 2).$$

It is easy to see that these equalities are equivalent to the following one in the ring $\boldsymbol{Z}[A_{l,\rho};l\geq 0]\,[[u]]$ of formal power series.

$$(6.19)\qquad \sum_{l=0}^{\infty}A_{l,\rho}u^l=\frac{(1-u)(1+q_2u)}{1-(A_{1,\rho}-q_2+1)u+q_1q_2u^2}.$$

Taking the trace of both sides of (6.19), we have

$$(6.20)\qquad u\frac{d}{du}\log Z_{\Gamma}(u;\rho)=(1-q_1q_2u^2)\operatorname{tr}\{1-(A_{1,\rho}-q_2+1)u+q_1q_2u^2\}^{-1}$$
$$-nh\cdot\frac{1-q_1q_2u^2}{(1-u)(1+q_2u)}.$$

Now the following equality is easily proved:

$$(6.21)\qquad \operatorname{tr}\{1-(A_{1,\rho}-q_2+1)u+q_1q_2u^2\}^{-1}=\frac{u}{1-q_1q_2u^2}$$
$$\times\frac{d}{du}\log\det\{1-(A_{1,\rho}-q_2+1)u+q_1q_2u^2\}^{-1}+\frac{nh}{1-q_1q_2u^2}.$$

Combining (6.20) and (6.21) together with $Z_\Gamma(0; \rho)=1$, we get the following expression of $Z_\Gamma(u; \rho)$:

$$Z_\Gamma(u; \rho)^{-1}=(1-u)^{(q_1q_2-1)nh/(q_2+1)}(1+q_2u)^{(q_1-q_2)nh/(q_2+1)} \times \det\{I_{nh_1}-(A_{1,\rho}-q_2+1)u+q_1q_2u^2\}.$$

Note that, by Lemma (4.10), we have

$$\frac{(q_1q_2-1)h}{q_2+1}=\frac{q_2(q_1+1)h_1}{q_2+1}-\frac{(q_2+1)h_1}{q_2+1}=q_2h_2-h_1=r-1,$$
$$\frac{(q_1-q_2)h}{q_2+1}=\frac{(q_1+1)h_1}{q_2+1}-\frac{(q_2+1)h_1}{q_2+1}=h_2-h_1.$$

This completes the proof of the theorem (6.4).

§ 7. Remarks

Finally we shall give some remarks on our results.

7-1. First, we shall give a graph-theoretic interpretation of our results. The basic ideas have been suggested by [Ser-1] (see also [Su]). Let Y denote the finite quotient graph $X(q_1, q_2)/\Gamma$. Then the group Γ can be identified with the fundamental group of Y, or its realization as a CW-complex, with respect to a (fixed) point. And there is a bijective correspondence between the set of Γ-conjugacy classes and the set of geodesic cycles. Note that, by definition, a geodesic cycle is a closed path in Y without backtracking, modulo the equivalence induced by shifting the origin. A cycle is called primitive, if it is not a power of another cycle. This corresponds to the primitive Γ-conjugacy classes. Thus, denoting by $|P|$ the length of the geodesic cycle P in Y, we get

$$Z_\Gamma(u; \rho)=\prod_{P:\text{primitive}} \det(I_n-\rho(\langle P\rangle)u^{|P|/2})^{-1}, \tag{7.1}$$

where the product is extended over all primitive cycles P of Y, and $\langle P\rangle$ denotes the corresponding conjugacy class of Γ. Taking the logarithmic derivative of (7.1), we get

$$N_{l,\rho}=\sum_C \chi(\langle C\rangle), \tag{7.2}$$

where C runs through the set of geodesic path in Y of length $2l$, with origin a vertex belonging to V_1Y (=image of V_1). In this context, the matrix $A_{l,\rho}$ has the following interpretation. Let $S_\rho(\Gamma)$ be the $\boldsymbol{C}$-vector space

$$(7.3) \qquad S_\rho(\Gamma):=\{f: V_1\to \boldsymbol{C}^n; f(P\gamma)=f(P)\rho(\gamma) \text{ for any } \gamma\in\Gamma\}.$$

Since Γ acts on $X(q_1, q_2)$ without fixed point, it follows that $S_\rho(\Gamma)$ is of finite dimension with $\dim S_\rho(\Gamma)=n\sharp(V_1Y)=nh$. Note that $S_\rho(\Gamma)$ is regarded as a space of $\boldsymbol{C}^n$-valued functions on $U\backslash G$, on which one can define the representation of the Hecke algebra $\mathscr{H}(G, U)$ by the convolution. It is immediate to see that the resulting representation is equivalent to the Brandt representation φ_ρ defined by (6.2); and $A_{l,\rho}$ corresponds to the following linear operator on $S_\rho(\Gamma)$:

$$(7.4) \qquad (A_{l,\rho}f)(P):=\sum_{\substack{Q\in V_1\\ d_X(P,Q)=2l}} f(Q).$$

Let $V_1^*=\{P_k; 1\leq k\leq h\}$ be a complete set of representatives of $V_1/\Gamma=V_1Y$, and let $e_j:=(0,\cdots,0,1,0,\cdots,0)$ $(1\leq j\leq n)$ be the standard unit vectors of $\boldsymbol{C}^n$. Then the functions $f_{k,j}$ $(1\leq k\leq h,\ 1\leq j\leq n)$ of $S_\rho(\Gamma)$ which are determined by the condition $f_{k,j}(P_k)=e_j$, $f_{k,j}(P_m)=0$ $(m\neq k)$ from a basis of $S_\rho(\Gamma)$. Using this basis, it is easy to show

$$(7.5) \qquad \mathrm{tr}(A_{l,\rho})=\sum_{\substack{\gamma\in\Gamma, P\in V_1^*\\ d_X(P,P\gamma)=2l}} \chi(\gamma).$$

Using these interpretations, one can simplify the proofs of the results in § 6.

7-2. Suppose that (G, I) is obtained from a Tits system (G, B, N, S). Then one can proceed the same calculation of $Z_\Gamma(u;\rho)$, with U_2 instead of U_1. Thus one gets a second formula for $Z_\Gamma(u;\rho)$:

$$(7.6) \qquad \begin{aligned} Z_\Gamma(u;\rho)^{-1}&=(1-u)^{n(r-1)}(1+q_1u)^{n(h_1-h_2)}\\ &\quad\times\det\{I_{nh_2}-(A'_{1,\rho}-q_1+1)u+q_1q_2u^2\}. \end{aligned}$$

where $A'_{1,\rho}$ is defined similarly as $A_{1,\rho}$. Note that, while the zeta function $Z_\Gamma(u;\rho)$ defined by (7.1) is independent of the choice between U_1 and U_2, the final results (6.4) and (7.6) are not symmetric in q_1, q_2 and h_1, h_2. It is, therefore, an interesting problem to explain this difference in the two expressions. It is also an important problem to ask the possible relation between our results on $Z_\Gamma(u;\rho)$ and the spectral decomposition of $L^2(G/\Gamma)$. We shall study these problems in the subsequent paper [Ha]. Here we content ourselves with the following observation.

Suppose that G is an algebraic group over a local field K satisfying the conditions of § 3. Suppose moreover, that $\rho=\mathbf{1}$, the trivial representation, and put $A_1=A_{1,1}$. Then, by the same argument as in [I-1], we see that A_1 has the eigenvalue $q_2(1+q_1)$ with multiplicity one. It follows that

$Z_\Gamma(u; \mathbf{1})^{-1}$ has the factor $(1-u)$ with multiplicity $r=\dim H^1(\Gamma, \boldsymbol{R})$. This, combined with a result of [Car], [Gas], implies the following result which generalizes that of Ihara [I-1]:

Proposition (7.7). *We have the following equality*

$$-\operatorname*{Res}_{u=1} \frac{d}{du} \log Z_\Gamma(u) = r = \text{the multiplicity of the Steinberg representation in } L^2(G/\Gamma).$$

We shall give a different proof of this in [Ha], which is independent of [Gar], [Cas].

Appendix. Bipartite trees, Hecke algebras, and flowers of groups (by Ki-ichiro Hashimoto)

Contents

§ 8. Introduction

In [I-1], Ihara studied the discrete subgroups of $SL(2, K)$ over a $\boldsymbol{p}$-adic field K. There he established, among others, a remarkable structure theorem which states that any discrete torsion free subgroup Γ of $SL(2, K)$ is a free group, whose free basis can be constructed in an explicit way. In fact what he did is more; such structure theorem was proved for subgroups of more general groups G satisfying certain axioms, and a zeta function for such Γ has been introduced and studied with a number of applications. The proofs are based on somewhat mysterious combinatorial arguments, as was written in the introduction of [Ser-1]. Later Serre [Ser-1] gave a graph-theoretic interpretation of the first result, generalizing it to much more general class of groups G. In one of the main theorems (Th. 13 in § I.5.4), he established a structure theorem of groups acting on a tree X. G is then recovered as the fundamental group $\pi_1(\boldsymbol{G}, Y, T)$ of the graph of groups $(\boldsymbol{G}, Y)$ at a maximal tree $T\,(\subset Y)$, where Y is the quotient graph X/G. Ihara's structure theorem is generalized to subgroups Γ of such G.

Here we note that, according to a philosophy of Tits, the tree attached to $G=SL(2, K)$ is regarded as the analogue of the upper half-plane for $SL(2, \boldsymbol{R})$. Based on this idea, a theory of harmonic analysis on trees has been constructed (cf. [Car]). Ihara's main result in [I-1] on the zeta function of Γ ($\subset SL(2, K)$) is particulary interesting if viewed from this point.

However, in such full generality, it seems difficult to extend the results of [I-1] on zeta functions, or to get a deep result on the arithmetic of a subgroup Γ of G.

We note here that in [I-1], an essential role has been played by the relations (G, l, II) (cf. (11.12)) of Hecke operators (a $\boldsymbol{p}$-adic analogue of the Laplacian), which have been given in [Ser-1] a simple interpretation in terms of graph theory, or trees. It is, therefore, natural to study the class of groups acting on trees, *which are not necessarily homogeneous*, for which one can expect a nice relation for elements of its Hecke algebra, that will lead us to the evaluation of zeta functions for its (discrete) subgroups Γ, as well as its application to the spectral decomposition of $L^2(G/\Gamma)$. We require that the Hecke operators satisfy, instead of the relations (G, l, II) of [I-1], one of those which appear in the theory of Iwahori-Matsumoto [I-M] (see (1.1), (1.2)).

The purpose of this note is to prove the equivalence between the following classes of objects consisting of groups and some extra data:

(8.1) *Groups G satisfying similar axioms* (G, l, I), (G, l, II) *as in* [I-1], *the latter describing a structure of the Hecke algebra $\mathscr{H}(G, U)$ of G with respect to a subgroup U* (cf. (1.1), (1.2)).

(8.2) *Groups G with an action on a semi-regular bipartite tree $X(q_1, q_2)=(V_1, V_2; E)$ of valency (q_1+1, q_2+1), which is transitive on V_1, one of its two kinds of vertices.*

(8.3) *Fundamental groups $\pi_1(G, F, T)$ of the "flowers F", of groups with certain regularity condition* (cf. (11.4) (11.5)); here *a flower* is a finite graph described as in the following:

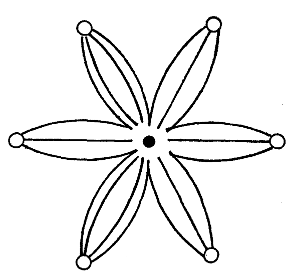


Fig. (8.4)

The equivalence of (8.2) and (8.3) follows as a special case of Serre's description of the groups acting on a tree, mentioned above. We included it in our main result, not only because of its beautiful characterization of our class, but also because it shows very well the range of it. As a simplest case, our class contains the groups which have the Tits system (G, B, N, S), where S consists of two elements generating the infinite dihedral group, and the flower reduces to $\bullet\!\!-\!\!\!-\!\!\circ$. This is the case for any simply connected groups over the local fields K with K-rank one. In fact our motivation has been to extend the results of [I-1] to such groups. However, (8.3) shows that our class covers much more wide class of groups which in general fail to have a Tits system.

Notation. For a finite set S, $\sharp(S)$ denotes its cardinality. By a graph, we mean, unless otherwise stated, a *non-oriented* one. If X is a graph, we denote VX (resp. EX) the set of its vertices (resp. edges). If X is a tree, d_X: $VX\times VX \to N\cup\{0\}$ denotes the distance on X. For a vertex $P\in VX$ and $l\in N\cup\{0\}$, we put $VX(P; l):=\{Q\in VX;\ d_X(P, Q)=l\}$.

§ 9. Groups with axioms (G, l, I), (G, l, II)

We shall prove the results described in § 1. Let (G, l) be as in § 1. We begin with the following remark. Our axioms $(G, l, \mathrm{I}, \mathrm{II})$ can be viewed as a generalization of those of Ihara [I-1], in two ways. First, if we put $q_1=q_2=q$, then we see that our axioms can be derived from those of [I-1], by considering only G_{2l}'s (see Remark (11.14)). This is what we obtain by replacing $PL_2(K)$ by $PSL_2(K)$ in [I-1]. On the other hand, axioms of [I-1] is recovered also by putting $q_2=1$ in our axioms. In terms of trees, this corresponds to considering the barycentric subdivision of the homogeneous tree attached to $PL_2(K)$ (see § 10).

Proof of Lemma (1.4). Since (i) is trivial, and (iii) follows from (i), (ii) and (1.1), we only need to show (ii). From the definition of the product $G_1\cdot G_1$ in $\mathscr{H}(G, U)$, we have

$$G_1=\sum_i U\omega_i \Longrightarrow G_1^2=\sum_{i,j} U\omega_i\omega_j,$$

where the formal sum is taken with multiplicities (cf. [Sh], Chap. 3). Comparing this with (1.1), we see that, for any $\xi\in G_1$,

$$q_2-1=\sharp\{(i,j);\ U\omega_i\omega_j=U\xi\}=\sharp\{j;\ \xi\omega_j^{-1}\in \textstyle\sum_i U\omega_i=G_1\}.$$

The assertion (ii) follows from this by taking $\xi=\omega^{-1}\in G_1^{-1}=G_1$. Q.E.D.

It follows from (1.1) and Lemma (1.4), (i) that we have $t=\sharp(U\backslash G_1)$

$=q_2(q_1+1)$; and the induction using (1.2) shows

(1.3) $$\sharp(U\backslash G_l)=(q_1q_2)^{l-1}q_2(q_1+1) \qquad \text{for } l\geq 1.$$

Proof of Lemma (1.5). Repeated application of the argument as above shows that

(9.6) $$G_1^l=\sum_{j_1,j_2\cdots,j_l} U\omega_{j_1}\omega_{j_2}\cdots\omega_{j_l},$$

with multiplicity taking into account. On the other hand, we see from (G, l, I), (G, l, II) and the induction on l that

$$G_1^l=G_l+c_1G_{l-1}+\cdots+c_lU \quad (c_i\in N\cup\{0\}).$$

Comparing these two expressions for G_1^l, we get the assertions except for the equivalence; $l(g)=l\Leftrightarrow(*)$. To prove this, we note that this assertion is equivalent to the identity

(9.7) $$G_l=\sum_{j_1,j_2,\cdots,j_l}^{(*)} U\omega_{j_1}\omega_{j_2}\cdots\omega_{j_l},$$

where $\Sigma^{(*)}$ is the partial sum of (9.6) of those which satisfy the condition $(*)$. From what we have seen, it follows that G_l is contained in the sum $\Sigma^{(*)}$. On the other hand, from Lemma (1.4) we see that the number of U-cosets in this sum is $(q_1q_2)^{l-1}q_2(q_1+1)$, which is equal to $\sharp(U\backslash G_l)$ (cf. (1.3)). This proves our asssertion. Q.E.D.

We remark that, under the condition (G, l, I), (G, l, II) is equivalent to the statements in Lemmas (1.4) and (1.5).

Following [I-1], we call a product xy $(x, y\in G)$ *free*, if $l(xy)=l(x)+l(y)$. The free product of n elements $x_1\cdots x_n$ is defined similarly. It is easy to see from the above lemma that, if xy, yz are free products and $y\notin U$, then xyz is free. In particular for $x_1, x_2, \cdots, x_l\in G_1$, we have

(9.8) $$l(x_1x_2\cdots x_l)=l \Longleftrightarrow l(x_ix_{i+1})=2 \quad (i=1, 2, \cdots, l-1).$$

The following is also an easy consequence of Lemma (1.5).

Lemma (9.9). *If the product xy is free and $xy=u\omega_{j_1}\omega_{j_2}\cdots\omega_{j_l}$ is its unique expression as in Lemma* (1.5), *then one can find $v\in U$ such that $x=u\omega_{j_1}\omega_{j_2}\cdots\omega_{j_m}v$ and $y=v^{-1}\omega_{j_{m+1}}\omega_{j_{m+2}}\cdots\omega_{j_l}$ $(m=l(x))$.*

Now we give a

Proof of Lemma (1.6). The assertions for $i=0$ are trivial, since $UG_lU=G_l$. Also the assertions for $i=2$ follow from the remark preceding Lemma (9.9). So we prove them for $i=1$.

Let $G_1=\sum_j U\omega_j$ and $G_2=\sum_{i_1,i_2}^{(*)} U\omega_{i_1}\omega_{i_2}$ be the decompositions into U-cosets as above. Then the product G_1G_2 in $\mathscr{H}(G, U)$ is expressed as

$$\begin{aligned} G_1G_2 &= \sum_j \sum_{i_1,i_2}^{(*)} U\omega_j\omega_{i_1}\omega_{i_2} \\ &= \sum_{j,i_1,i_2}^{(*)} U\omega_j\omega_{i_1}\omega_{i_2} + \sum^{(1)} U(\omega_j\omega_{i_1})\omega_{i_2} + \sum^{(0)} U(\omega_j\omega_{i_1})\omega_{i_2}, \end{aligned}$$

where $\sum^{(i)}$ is the sum of $U\omega_j\omega_{i_1}\omega_{i_2}$ such that $\omega_j\omega_{i_1} \in G_i$ $(i=0, 1)$. From (9.7), we have $\sum_{j,i_1,i_2}^{(*)}=G_3$. Also from Lemma (1.4), (i) and (iii), it follows that $\sum^{(0)}=q_1q_2G_1$. Now comparing theabove expression for G_1G_2 with that of (1.2), $l=2$, we get $\sum^{(1)}=(q_2-1)G_2$, from which it follows that $\omega_j\omega_{i_1}\omega_{i_2} \in G_2$.

Thus we have proved the assertion (i) for $i=1$, in the case $x=\omega_j$, $y=\omega_{i_1}$, $z=\omega_{i_2} \in \Omega$. The general case is easily reduced to this and Lemma (9.9), since Ω is an arbitrary set of representatives of $U\backslash G_1$. (ii) follows from (i), by taking inverse $(G_l^{-1}=G_l)$. Q.E.D.

§ 10. Construction of a tree $X(q_1, q_2)$

A tree of complete graphs.

Let (G, l) be a pair of a group and a function l on G taking values in non-negative integers, which satisfy (G, l, I). Following an idea of Serre [Ser-1], we define a graph $Z=Z(q_1, q_2)$ on which G acts. Let $VZ:=U\backslash G$ be the homogeneous space of G consisting of the (left) U-cosets. We define a function d_Z: $VZ\times VZ\to N\cup\{0\}$ by

(10.1) $$d_Z(Ux, Uy):=l(xy^{-1}) \quad (x, y \in G).$$

Then it follows immediately from (G, l, I) that d_Z satisfies the following properties.

(10.2) d_Z *is symmetric and G-invariant.*

(10.3) $d_Z(P, R)=0 \Longleftrightarrow P=R$ $(P, R \in VZ)$.

(10.4) *For any* $P \in VZ$, *and* $l \in N\cup\{0\}$, *the cardinality* t_l *of the set* $\{R \in VZ;\ d_Z(P, R)=l\}$ *is finite, and it is independent of* P.

Then we define two points $P, R \in VZ$ to be *adjacent* (or joined by an *edge*), if $d_Z(P, R)=1$. This gives us a combinatorial graph Z on which G acts, having VZ as its set of vertices. Recall that a combinatorial graph is a graph which does not contain any circuit of length $n\leq 2$.

Now we assume that G satisfies the second axiom (G, l, II). Then from (1.3), we have $t_l=(q_1q_2)^{l-1}t$, $t=t_1=q_2(q_1+1)$. Call a circuit of length 3 a *triangle*. Lemma (1.4), (ii) implies that, if $q_2>1$ then the graph Z

contains a triangle; in fact $C:=[U, U\omega, U\rho\omega, U]$ ($\omega, \rho, \rho\omega \in G_1$) is such a triangle. If, on the other hand, $q_2=1$, then our axioms reduce to that of [I-1], in which case (G, l, II) is equivalent to the assertion that Z is a tree (see [Ser-1], p. 117, ex. 2).

We call a circuit $C_n=[P_0, P_1, \cdots, P_n=P_0]$ in a graph Z to be *minimal*, if the only pairs of vertices in $\{P_0, P_1, \cdots, P_{n-1}\}$ adjacent in Z are $\{P_i, P_{i+1}\}$ ($i=0, 1, \cdots, n-1$); or equivalently, if no proper subset of this is the set of vertices of a circuit in Z.

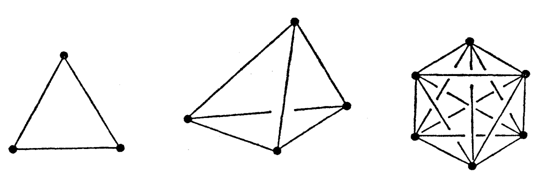
Lemma (10.5). *The graph $Z(q_1, q_2)$ contains no minimal circuit of length $n\geq 4$.*

Proof. Suppose on the contrary that Z contains a minimal circuit $C_n=[P_0, P_1, \cdots, P_n=P_0]$ of length $n\geq 4$. Since the action of G on the set of vertices of Z is transitive, we may assume that $P_0=U$, and write $P_1=Ux_1$, $P_2=Ux_2x_1$, $\cdots$, $P_{n-1}=Ux_{n-1}\cdots x_2x_1$, where $x_1, \cdots, x_{n-1} \in G_1$. Now the assumptions that $n\geq 4$ and that C_n is minimal, imply that $x_ix_{i+1} \in G_2$ for $i=1, \cdots, n-2$. By (9.8), this implies that $l(x_{n-1}\cdots x_2x_1)=d_Z(P_0, P_{n-1})$ $=n-1>1$, a contradiction. Q.E.D.

Lemma (10.6). *If two distinct triangles $\{P_1, P_2, P_3\}$, $\{P_1, P_2, R_3\}$ in Z share an edge $[P_1, P_2]$, then the remaining two vertices are adjacent: $d_Z(P_3, R_3)=1$.*

Proof. Again we may write $P_1=U$, $P_2=U\omega$, $P_3=U\rho\omega$, and $R_3=U\tau\omega$, where ω, ρ, $\rho\omega$, $\tau, \tau\omega \in G_1$. Suppose, on the contrary, that $d_Z(P_3, R_3)=l(\rho\tau^{-1})=2$. Then applying Lemma (1.6), (i) for $x=\rho$, $y=\tau^{-1}$, $z=\tau\omega$, one gets $l(xyz)=l(\rho\omega)=2$, a contradiction. Q.E.D.

Now it is easy to see how our graph $Z(q_1, q_2)$ looks like. The following Lemma shows that it is a '*tree of complete graphs* $K(q_2+1)$'; recall that a complete graph $K(n)$ is a combinatorial graph consisting of n vertices, any two of which are adjacent. Namely, each vertex P in Z is a common vertex of q_1+1 distinct complete graphs isomorphic to $K(q_2+1)$. Call a complete graph isomorphic to $K(4)$ a *tetrahedron* (cf. Fig. (10.7)).



$K(3)$: triangle $K(4)$: tetrahedron $K(n)$: $n=6$

Fig. (10.7)

$Z(q_1, q_2)$: $q_1=1$, $q_2=3$

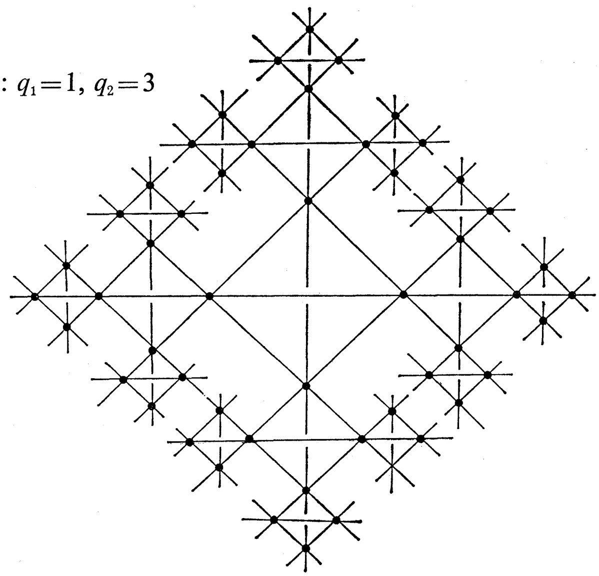


Fig. (10.8)

We say that two (not necessarily distinct) edges to be *related*, if they belong to a tetrahedron in Z (or a triangle if $q_2=2$, or identical if $q_2=1$).

Lemma (10.9). (i) *The above relation is an equivalence relation in the set EZ of edges of Z, which is compatible with the G-action.*

(ii) *For any vertex $P \in VZ$, it induces an equivalence relation in the set $Z(P)$ of vertices of Z adjacent to P, each equivalence class consisting of q_2 vertices.*

Proof. (i) is easily proved by an application of Lemma (10.6). Two vertices $R_1, R_2 \in Z(P)$ are equivalent if and only if the edges $[P, R_1]$, $[P, R_2]$ are related. To prove (ii), we may assume that $P=P_0=U$. Writing $R_1 = U\omega_1$, $R_2=U\omega_2$ ($\omega_1, \omega_2 \in \Omega$), we see that this is the case if and only if $\omega_1\omega_2^{-1} \in U \cup G_1$. For each R_2, the number of such $R_1 \in Z(P)$ is seen from Lemma (1.4) to be $1+(q_2-1)=q_2$. Q.E.D.

Tree $X(q_1, q_2)$.

We now construct a tree $X(q_1, q_2)$ from $Z(q_1, q_2)$, which is of bipartite type of valency (q_1+1, q_2+1), and which is acted upon by our group G. Let $Z^*(q_1, q_2)$ be the barycentric subdivision of the graph $Z=Z(q_1, q_2)$. This means that we add new vertex Q at the middle point of each edge $[P, R]$ of Z, and call it to be adjacent to P, R. The set of added vertices is denoted by V_2Z^*, and the rest is denoted by V_1Z^*. Thus one has

$$VZ^* = V_1Z^* \cup V_2Z^* \text{ (disjoint)}, \quad V_1Z^* \simeq VZ, \quad V_2Z^* \simeq EZ.$$

Clealy the action of G on Z is induced on Z^*, and so is the equivalence relation on $V_2Z^* \simeq EZ$, in such a way that it is compatible with G-action. Therefore one obtains a quotient graph

$$X(q_1, q_2) := Z^*(q_1, q_2)/(\text{equivalence}).$$

Namely we identify the vertices in V_2Z^* which are equivalent; and if two such vertices Q_1, Q_2 are adjacent to a common vertex $P \in V_1Z^*$, we also identify two edges $[P, Q_1]$, $[P\ Q_2]$ (see Fig. (10.10)). Notice, among others, that by this process of making quotient, no two vertices of V_1Z^* ($\simeq VZ$) are identified, as one sees immediately from Lemma (10.9).

As a result, we get a graph $X(q_1, q_2) = (V_1, V_2; E)$ of bipartite type, which is semi-regular of valency (q_1+1, q_2+1), whose set of vertices are divided into two disjoint parts $V_1 := V_1Z^*$ and $V_2 := V_2Z^*/(\text{equivalence})$, and each edge $y = [P, Q] \in E$ joins a vertex $P \in V_1$ and a vertex $Q \in V_2$. Now the following properties (i), (ii), (iii) are easily proved by what we have seen (cf. Fig. (10.10)).

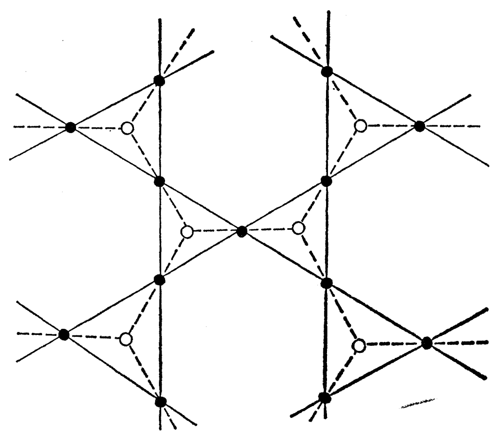


Fig. (10.10)

Theorem (10.11). *Let the notation be as above.*

(i) *The graph $X(q_1, q_2) = (V_1, V_2; E)$ is a connected tree of semi-regular bipartite type with valency (q_1+1, q_2+1): i.e., it is a tree where each vertex $P \in V_1$ is adjacent to q_1+1 vertices in V_2, and each $Q \in V_2$ is adjacent to q_2+1 vertices in V_1.*

(ii) *The distance on $V_1 \simeq VZ$ which is induced from that of the tree*

$X(q_1, q_2)$, coincides with $2d_Z(P, P')$.

(iii) *The group G acts on $X(q_1, q_2)$ in such a way that it has no inversion and that it is transitive on V_1.*

Conversely, suppose that a group G and its action on the connected semi-regular bipartite tree $X=(V_1, V_2, E)$ of valency (q_1+1, q_2+1), satisfying the condition (iii) *above, are given. Then the mapping l: $G \rightarrow N \cup \{0\}$ given by $l(g)=(1/2)d_X(P, Pg)$ ($P \in V_1$) satisfies (G, l, I), (G, l, II).*

Proof. We prove the last assertion. By the assumption we can identify V_1 with $U \backslash G$, where $U:=\mathrm{Stab}(P_0)$, $P_0=U$. The properties (G, l, I) for the mapping l is immediately seen. To show (G, l, II), we note that $\mathscr{H}(G, U)$ is regarded as a subring of $\mathrm{End}(\boldsymbol{Z}[V_1])$, where $\boldsymbol{Z}[V_1]$ is the free $\boldsymbol{Z}$-module over V_1. Namely, a U-double coset $UxU=\bigcup_j Ux_j$ maps $P=Uz$ ($\in V_1$) to the formal sum $\sum_j Ux_j z$. Then one sees that $G_l : P \rightarrow \sum Q$, the sum taken over the set of vertices such that $d_X(P, Q)=2l$. The assertion follows easily from this. Q.E.D.

Corollary (10.12). *Let G be as above.*

(i) *If Γ is a torsion free subgroup of G such that $\Gamma \cap x^{-1}Ux=\{1\}$ for any $x \in G$. Then Γ is a free group.*

(ii) *If, moreover, Γ satisfies $\sharp(U \backslash G/\Gamma)<\infty$, then it has a finite basis of rank $r_\Gamma=h_1q_1-h_2+1$ $(=h_2q_2-h_1+1)$, where $h_1=\sharp(U\backslash G/\Gamma)=\sharp(V_1/\Gamma)$, $h_2=\sharp(V_2/\Gamma)$ are the number of Γ-orbits in V_1, V_2, respectively.*

Proof. This follows from [Ser-1], I.3.3, Theorem 4.4′, together with the relation $(q_1+1)h_1=(q_2+1)h_2$. In fact the assumptions in (i) imply that the (restricted) action of Γ on the tree $X(q_1\, q_2)$ is a free action. To see this, suppose, on the contrary, that $\gamma \in \Gamma$ ($\gamma \neq 1$) has a fixed point $Q \in V_2$. Then it induces a permutation on the finite set of vertices adjacent to Q. Hence some power γ^n of γ has a fixed point $P=Ux \in V_1$. This implies that $\gamma^n \in \Gamma \cap x^{-1}Ux$, a contradiction. Q.E.D.

Notice that the assumption on the torsion-freeness of Γ can be weakened to the one that Γ has no torsion element $\gamma \neq 1$ such that $\gamma^n=1$ for $n=(q_2+1)!$.

§ 11. Graph of groups over a flower

Now we consider the quotient graph $F:=X(q_1, q_2)/G$. Thus we assume that G is a group acting on $X(q_1, q_2)$ as described in (10.11). Then it is clear that F is a finite graph; in fact the image of the vectices of V_1 (resp. V_2) consists of a single (resp. at most $1+q_1$) points. Denote them by $V_1F=\{P\}$ and $V_2F=\{Q_1, \cdots, Q_r\}$. Also denote the image of E by EF.

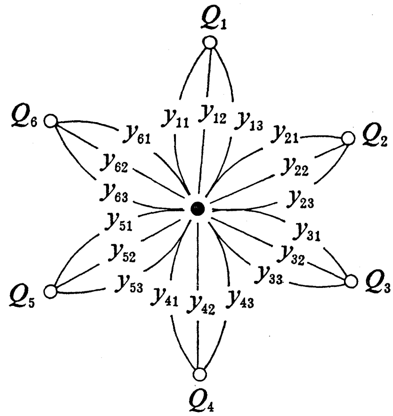


Fig. (11.1)

From its shape, we shall call F *a flower*, and each of its subgraphs consisting of $F^{(i)}:=\{P, Q_i$, all edges $y_{i,j}$ $(1 \leq j \leq t_i)$ connecting them$\}$, *a petal* of the flower F $(1 \leq i \leq r)$.

Let $P_i=P_0 \in V_1$, and let $Q_{i,j} \in V_2$ be the vertex such that the edge $[P_i, Q_{i,j}]$ project to $y_{i,j}$ $(1 \leq j \leq t_i)$. We denote by $U(Q_{i,j})$ the stabilizer in G of $Q_{i,j}$. Similarly, let $Q_i \in V_2$, and let $P_{i,j} \in V_1$ be the vertex such that the edge $[Q_i, P_{i,j}]$ project to $\bar{y}_{i,j}$ $(1 \leq j \leq t_i)$. We denote by $U(P_{i,j})$ the stabilizer in G of $P_{i,j}$. Note that $U \cap U(Q_{i,j})$ (resp. $U(Q_i) \cap U(P_{i,j})$) is the stabilizer of the edge $[P_i, Q_{i,j}]$ (resp. $[Q_i, P_{i,j}]$). Since the two edges $[P_i. Q_{i,j}]$ and $[P_{i,j}, Q_i]$ have the same image $y_{i,j}$ in F, one finds $g_{i,j} \in G$ such that $[P_i, Q_{i,j}]g_{i,j}=[P_{i,j}, Q_i]$, i.e., $P_i g_{i,j}=P_{i,j}$, $Q_i g_{i,j}^{-1}=Q_{i,j}$. It follows that

$$U(Q_i) \cap U(P_{i,j})=g_{i,j}^{-1}(U \cap U(Q_{i,j}))g_{i,j}. \tag{11.2}$$

We refer to the following equalities (11.4), (11.5) as *the regularity condition*.

Proposition (11.3). *Let the notation be as above.*

(i) *The number of petals in* $F=r=\#(V_2/G)$.

(ii) *The number of edges in a petal* $F_i=t_i=\#(V(Q_i; 1)/U(Q_i))$, *and one has*

$$\sum_{j=1}^{t_i} [U(Q_i) : U(Q_i) \cap U(P_{i,j})]=q_2+1 \quad (1 \leq i \leq r). \tag{11.4}$$

(iii) *One has similarly*

$$\sum_{i=1}^{r} \sum_{j=1}^{t_i} [U(P_i) : U(P_i) \cap U(Q_{i,j})]=q_1+1. \tag{11.5}$$

Proof. (i) is clear. To prove (ii), one notes first that for two vertices $Q, Q' \in V(P_i; 1)$, the edges $[P_i, Q]$, $[P_i, Q']$ have the same image in F if and only if there exists $g \in G$ such that $P_i g = P_i$ and $Qg = Q'$, i.e., if and only if they belong to the same $U(P_i)$-orbit. The equality $t_i = \sharp(V(P_i; 1)/U(P_i))$ follows from this, and counting the vertices in $V(P_i; 1)$ in two ways, one gets (11.4). (iii) is proved similarly. Q.E.D.

Recall that a graph of groups $(\boldsymbol{G}, Y)$ is a connected graph Y together with an assignment $\boldsymbol{G}$: $Y \to \{\text{groups}\}$ which assigns to each vertex $P \in VY$ (resp. oriented edge $y = [P, Q] \in EY$) a group $\boldsymbol{G}(P)$ (resp. $\boldsymbol{G}(y)$). Moreover, it is required that $\boldsymbol{G}([P, Q]) = \boldsymbol{G}([Q, P])$, and there are monomorphisms

$$(11.6) \qquad \begin{array}{ll} \boldsymbol{G}([P, Q]) \longrightarrow \boldsymbol{G}(Q), & a \to a^y \\ \parallel & \\ \boldsymbol{G}([Q, P]) \longrightarrow \boldsymbol{G}(P), & a \to a^{\bar{y}} \end{array} \qquad \text{(cf. [Ser-1]).}$$

Suppose that a group G acts on a graph X, and $Y := X/G$ be its quotient graph. Then one gets in the natural way a graph of groups $(\boldsymbol{G}, Y)$, called the *associated graph of groups*. This is exactly what we described above; namely $\boldsymbol{G}(P) = U(P) =$ the stabilizer of P in G, and $\boldsymbol{G}([P, Q]) = \boldsymbol{G}(P) \cap \boldsymbol{G}(Q)$ (note the compatibility condition (11.2)).

A maximal tree T in a graph Y is a subgraph which is a tree, and which is maximal (i.e., containing all vertices of Y). In [Ser-1] the fundamental group $\pi_1(\boldsymbol{G}, Y, T)$ of $(\boldsymbol{G}, Y)$ at T is defined. It is a group with generators $\boldsymbol{G}(P)$ ($P \in VY$) and g_y ($y \in EY$) with the relations

$$(11.7) \qquad \begin{array}{l} g_y^{-1} a^y g_y = a^{\bar{y}}, \quad g_{\bar{y}} = g_y^{-1} \ (y \in EY,\ a \in \boldsymbol{G}(y)), \\ \text{and} \quad g_y = 1 \quad \text{if} \quad y \in ET \ (= \text{edges of } T). \end{array}$$

Then the following fundamental results have been proved in [Ser-1]:

Theorem (11.8) (Serre). *Let $(\boldsymbol{G}, Y)$ be a graph of groups and T a maximal subtree of Y.*

(1) *There exists a tree $\tilde{X}(\boldsymbol{G}, Y, T)$ on which $\pi_1(\boldsymbol{G}, Y, T)$ acts in such a way that $\tilde{X}(\boldsymbol{G}, Y, T)/\pi_1(\boldsymbol{G}, Y, T) \simeq Y$.*

(2) *$\tilde{X}(\boldsymbol{G}, Y, T)$ and $\pi_1(\boldsymbol{G}, Y, T)$ are universal; i.e., for any group G and a connected graph X on which G acts in such a way that $X/G \simeq Y$, and that the associated graph of groups is isomorphic to $(\boldsymbol{G}, Y)$, there are natural surjective morphisms: $\varphi : \pi_1(\boldsymbol{G}, Y, T) \to \boldsymbol{G}$, $\psi : \tilde{X}(G, Y, T) \to X$.*

Theorem (11.9) (Serre). *Suppose that a group G acts on a connected non-empty graph X, and let $(\boldsymbol{G}, Y)$ be its associated graph of groups, where $Y = X/G$. Then the following properties are equivalent:*

(a) *X is a tree.*

(b) ψ: $\tilde{X}(\boldsymbol{G}, Y, T) \longrightarrow X$ *is an isomorphism.*
(c) φ: $\pi_1(\boldsymbol{G}, Y, T) \longrightarrow G$ *is an isomorphism.*

Now it is not difficult to show, by using the explicit construction of $\tilde{X}(\boldsymbol{G}, Y, T)$ described in [Ser-1], § I.5.3, that, if we start from a graph of groups $(\boldsymbol{G}, F)$ over our flower F which satisfies the regularity condition (11.4), (11,5), then $\tilde{X}(\boldsymbol{G}, F, T)$ is isomorphic to $X(q_1, q_2)$. In fact using (11.4) (resp. (11.5)), one can show that any vertex Q (resp. P) lying over a white one $Q_i \in V_2F$ (resp. black one $P_0 \in V_1F$) has q_2+1 edges (resp. q_1+1 edges).

Summing up, we have shown the equivalence of the following classes of groups (with extra structures).

Theorem (11.10) (Main result). *The following three classes of groups with extra data are mutually equivalent:*

(1) *The groups G with a length function $l: G \rightarrow \boldsymbol{N} \cup \{0\}$, satisfying axioms* (G, l, I), (G, l, II).

(2) *The groups G which act on a semi-regular bipartite tree $X(q_1, q_2) = (V_1, V_2; E)$ in such a way that the action is transitive on V_1.*

(3) *The fundamental groups $\pi_1(G, F, T)$ of flowers of groups* $(\boldsymbol{G}, F)$ *with regularity condition* (11.4), (11.5).

Example (11.11). Let G^* be a free group with n generators $\alpha_1, \cdots, \alpha_n$. Denote by $l^*(x)$ the length of the reduced word of $x \in G^*$. Define G to be the subgroup of G^* consisting of x such that $l^*(x) \equiv 0 \pmod 2$, and put $l(x) := (1/2) l^*(x)$. It is well known that G is again a free group, and by Schreier's index theorem, the rank of G is $r_G = (n-1)[G^*: G]+1 = 2n-1$. It is immediately seen that (G, l) satisfies (G, l, I) with $U=\{1\}$. To see the second axiom (G, l, II), we note first that (G^*, l^*) satisfies the corresponding axioms of [I-1], and that $G_l = G^*_{2l}$ for any $l \in \boldsymbol{N} \cup \{0\}$. Namely one has in $\mathscr{H}(G^*, U^*) = \boldsymbol{Z}[G^*]$ (=the group ring of G^*),

$$(11.12) \quad (G_1^*)^2 = G_2^* + (q+1)U^*, \quad G_1^* \cdot G_l^* = G_{l+1} + qG_{l-1}^* \quad (l \geq 2),$$

with $q=2n-1$. Our axiom (G, l, II) with $q_1=q_2=2n-1$ easily follows from these equalities. Let $Z=Z(q_1, q_2)$ be the graph associated with (G, l) (cf. § 10), and let $P_0=U=1$ be the origin of Z. It is also easy to see that two vertices $R, R' \in Z$ are related as in Lemma (10.9), (ii) if and only if their reduced words coincide after the first words. From this it follows that the action of G on the set V_2 of new vertices of $X(q_1, q_2)$ is transitive. On the other hand, since $U=\{1\}$, no two edges adjacent to P_0 are G-equivalent. We have shown that the quotient graph $X(q_1, q_2)/G$ looks like a petal as in the following:

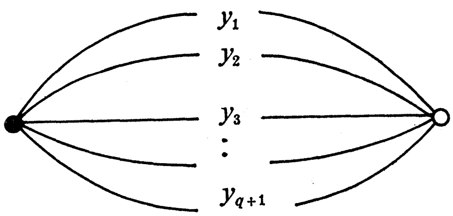


Fig. (11.13)

This example can also be constructed from the graph of groups, associated to the above petal where to both vertices attaches the trivial group (cf. [Ser-1]).

Remark (11.14). As described above, the results of Ihara [I-1] are reproduced if we put $q_1=q_2=q$, and $G_l=G^*_{2l}$, where (G^*, l^*) satisfies the axioms of [I-1]. This explains the relation for $PGL(2, K)$ and $PSL(2, K)$.

§ 12. Tits system and the Hecke algebra

In the rest of this paper, we shall study the steps in which our group G comes to have properties which appear in the theory of Tits system. Thus we assume that G is a group with the conditions described in Theorem (11.10).

Notation being as in § 1, we put $P_0:=U \in V_1$, and choose, once and for all, a vertex $Q_0 \in V_2$ which is adajcent to P_0. Also put, $V:=V_1 \cup V_2$, and for any point $P \in V$, call $U(P)$ the stabilizer of P in G. The action of G on $X(q_1, q_2)$ induces the action of $U(P)$ on each $V(P; l)$. Notice that $U(P_0)=U$, and we have the natural bijection $V(P_0; 2l)/U \simeq U\backslash G_l/U$.

Lemma (12.1). *We have the following inequalities:*

$$\sharp(V_2/G) \leq \sharp(V(P_0: 1)/U) \leq \sharp(V(P_0; 2)/U) \leq \cdots \leq \sharp(V(P_0; l)/U) \leq \cdots$$

Proof. For any vertex $Q \in V_2$, take a vertex $P \in V_1$ which is adjacent to Q. If we write $P=Ux$ $(x \in G)$, then we have $Qx^{-1} \in V(P_0; 1)$. This shows that any G-orbit in V_2 is represented by a vertex in $V(P_0; 1)$, hence the first inequality. The rest follows from the fact that, by mapping the adjacent vertex near to the origin P_0, one has the inverse system

$$V(P_0; 0) \leftarrow V(P_0; 1) \leftarrow V(P_0; 2) \leftarrow \cdots \leftarrow V(P_0; l) \leftarrow \cdots \tag{12.2}$$

where each map is surjective, and U-equivariant. Q.E.D.

Proposition (12 3). *The following conditions are equivalent.*

(a) *G acts transitively on E ($=$the set of edges of $X(q_1, q_2)$)*
(b) *$\#(V(P_0; 1)/U)=1$, i.e., $V(P_0; 1)=Q_0\cdot U$.*
(c) *$G=U_1*_B U_2$ (product with amalgamated subgroup B), where $U_1:=U$, $U_2:=U(Q_0)$, and $B:=U_1\cap U_2$.*

Moreover, it follows under these conditions that the action of U_2 on $V(Q_0; 1)$ is transitive: $\#(V(Q_0; 1)/U_2)=1$.

Proof. The equivalence (a)$\Leftrightarrow$(c) is nothing but Theorems 6, 7 of [Ser-1], § I.4.1. Other part is easily proved as in the above Lemma. (See also the proof of Proposition (12.4)). Q.E.D.

Proposition (12.4). *Suppose that the conditions of the above proposition are satisfied. Then one has the following assertions.*

(i) $\#(U\backslash G_1/U)=1 \Longleftrightarrow \#(B\backslash U_2/B)=2 \Longleftrightarrow \#(V(Q_0; 1)/B)=2$.

(ii) *Under these conditions, one has $s_2^2\in B$ for any $s_2\in U_2-B$. Similar assertions hold for U_1, $s_1\in U_1-B$.*

Proof. (i) Note first that, in general one has the inequalities

$$[U_1: B]\leq 1+q_1, \quad [U_2: B]\leq 1+q_2, \tag{12.5}$$

where both equalities hold if and only if G acts transitively on E. In fact the index $[U_1: B]$ is equal to the number of vertices of the U_1-orbit $Q_0\cdot U_1$ in $V(P_0; 1)$, the latter having $1+q_1$ vertices. And similarly for $[U_2: B]$. Now suppose that G is transitive on E, so that U_1 is transitive on $V(P_0; 1)$. Let $\omega\in G_1$ be such that the vertex P_ω lies in the same component of $X-\{P_0\}$ as Q_0, i.e., $d_X(Q_0, P_\omega)=1$. Then one has

$$\begin{aligned}\#(U\backslash G_1)\geq\#(U\backslash U\omega U)&=\#(U\backslash U\cdot\omega U\omega^{-1})\\&=\#((U\cap\omega U\omega^{-1})\backslash\omega U\omega^{-1})\\&=[U: B]\cdot[B: U\cap\omega^{-1}U\omega].\end{aligned}$$

From this and the equality in (12.5), one sees that

$$G_1=U\omega U \Longleftrightarrow [B: U\cap\omega^{-1}U\omega]=q_2. \tag{$*$}$$

To prove the last equivalence, we notice again that our assumption implies that U_2 is transitive on $V(Q_0; 1)$. Since the stabilizer in U_2 of the vertex P_0 is B, we see that $B\backslash U_2/B$ is in one-to-one correspondence with the orbit space $V(Q_0; 1)/B$, Moreover, since $U\cap\omega^{-1}U\omega$ is the stabilizer of $P_\omega\in V(Q_0: 1)$, we see that the right hand side of $(*)$ is equivalent to the condition that $V(Q_0; 1)-\{P_0\}$ is a single B-orbit, i.e., $\#(B\backslash U_2/B)=2$.

(ii) is an immediate consequence of what we have seen, since s_2 induces the transposition of the two B-orbits in $V(Q_0; 1)$. Q.E.D.

We note that the element s_2 above belongs to G_1. The following Lemma is easily proved by the above arguments and (9.8).

Lemma (12.6). *Suppose that the conditions stated in* (12.4) *are satisfied. Then one has* $(s_1s_2)^l, (s_2s_1)^l \in G_l$ *for any* $l \in N$.

Now we suppose that the conditions in (12.4) are satisfied, and consider the Hecke algebra $\mathscr{H}(G, B)$. Call T_i the element Bs_iB $(i=1, 2)$, and I_B the unit element of $\mathscr{H}(G, B)$.

Proposition (12.7). $T_i^2=(q_i-1)T_i+q_iI_B \quad (i=1, 2)$.

Proof. By our assumption, the set E of the edges of $X(q_1, q_2)$ is identified with $B\backslash G$, and $\mathscr{H}(G, B)$ can be regarded as a subring of $\mathrm{End}(\boldsymbol{Z}[E])$, where $\boldsymbol{Z}[E]$ denotes the free $\boldsymbol{Z}$-module over the set E. Regarding Q_0 as the origin, one can identify E with $V-\{Q_0\}$, by assigning each edge to its end vertex lying on the other side of Q_0. Let $E(Q_0; 1)$ be the set of edges corresponding to $V(Q_0; 1)$. Then from what we have seen in the proof of Proposition (12.4), it follows that $E(Q_0; 1)$ consists of two B-orbits, corresponding to the decomposition $B\backslash U_2/B=B\cup Bs_2B$. This implies that as an element of $\mathrm{End}(\boldsymbol{Z}[E])$, T_2 maps each edge e to the formal sum of the q_2 edges which share the same vertex $Q \in V_2$ with e. The identity for T_2 follows immediately from this fact. The one for T_1 is proved similarly.

Q.E.D.

Notice that $T_1T_2 \neq T_2T_1$. It follows from the above proof that the monomials $I_B, T_1T_2, T_2T_1, \cdots, T_{i_1}T_{i_2}\cdots T_{i_l}$ $(i_m \neq i_{m+1}, m=1, \cdots, l-1)$ are pairwise disjoint, hence they are linearly independent. Also it is easy to prove:

Corollary (12.8). *As elements of* $\mathscr{H}(G, B)$, *one has*

$$G_l=(1+T_1)T_2(T_1T_2)^{l-1}(1+T_1) \quad (l\geq 1).$$

Theorem (12.9). *Notation being as above, suppose that* $q_1, q_2 \geq 2$, *and that the conditions in* (12.4) *are satisfied. Then the following assertions are all equivalent.*

(i) *If* $i_1, \cdots, i_l \in \{1, 2\}$ *and* $i_m \neq i_{m+1}$ $(m=1, 2, \cdots, l-1)$, *then* $T_{i_1}T_{i_2}\cdots T_{i_l}$ *consists of a single B-double coset.*

(ii) $\mathscr{H}(G, B)=\boldsymbol{Z}[T_1, T_2]$.

(iii) $\#(U\backslash G_l/U)=1$ *for any* $l \in N$.

(iv) *U acts transitively on* $V(P_0; l)$ *for any* $l \in N$.

(iv)′ U_2 *acts transitively on* $V(Q_0; l)$ *for any* $l \in N$.

(v) *There exist a subgroup N of G and a set S such that* $(G, \boldsymbol{B}, N, S)$ *is a Tits system.*

Proof. (i)$\Rightarrow$(ii): This follows immediately from (12.8).

(ii)$\Rightarrow$(i): This follows from the obvious relation

$$Bs_{i_1}s_{i_2}\cdots s_{i_l}B\subseteq(Bs_{i_1}B)\cdots(Bs_{i_l}B)=T_{i_1}\cdots T_{i_l},$$

since the left hand side is expressed as a polymomial of T_1, T_2.

(i)$\Rightarrow$(iii): From (12.8), we see that (i) is equivalent to

$$(12.10)\qquad G_l=Bs_2(s_1s_2)^{l-1}B\cup B(s_1s_2)^lB\cup B(s_2s_1)^lB\cup B(s_1s_2)^ls_1B.$$

From this and the fact that $U=B\cup Bs_1B$, it follows that $G_l=U(s_1s_2)^lU$.

(iii)$\Leftrightarrow$(iv): This is immediate from the remark preceding (12.1).

(iv)$\Rightarrow$(i): We shall show that (iv) implies (12.10).

First note that, for any $l\in N$, the set $V(P_0;2l)$ is decomposed according to the distance from Q_0:

$$(12.11)\qquad \begin{aligned}V(P_0;2l)=(V(P_0;2l)\cap V(Q_0;2l-1))\cup(V(P_0;2l)\\ \cap V(Q_0;2l+1)),\end{aligned}$$

where both of these components are stable under B. It is easy to see that $s_1(\in U-B)$ induces the transposition of them. Since U is assumed to be transitive on $V(P_0;2l)$, this implies that B acts on each components transitively, i.e., $\sharp(V(P_0\ 2l)/B)=2$, or equivalently $\sharp(U\backslash G_l/B)=2$. From Lemma (12.6) it follows that

$$(12.12)\qquad G_l=U(s_2s_1)^lB\cup U(s_2s_1)^{l-1}s_2B.$$

Next let $EV(P_0;2l)$ be the set of all edges of $X(q_1,q_2)$ which have as their one of end points the vertices of $V(P_0;2l)$. From our assumption, it is easy to see that $EV(P_0;2l)$ consists of two U-orbits:

$EV^{\pm}(P_0;2l):=\{e\in EV(P_0;2l);$ the other end point $\in V(Q_0;2l\pm1)\}$.

Taking P_0 as the origin of $X(q_1,q_2)$, we can naturally identify E with $V-\{P_0\}$, in such a way that the identification is compatible with U-action. It then follows from (12.3) that one has a commutative diagram

$$(12.13)\qquad \begin{array}{ccc} E & \simeq & B\backslash G\\ \cup & & \cup\\ EV(P_0;2l) & \simeq & B\backslash G_l\end{array}\quad (l\in N),$$

which is compatible with U-action. We then see from (12.12) that

$$EV^+(P_0;2l)\simeq B(s_2s_1)^{-l}U,\quad EV^-(P_0;2l)\simeq Bs_2^{-1}(s_2s_1)^{-l}U.$$

Now it is easy to describe the B-orbits decomposition of $EV^{\pm}(P_0;2l)$. Namely, as in (12.11), each of them is divided into two V-orbits according to the distance between their end points in V_2 and Q. Thus we have

$$\#(B\backslash B(s_2s_1)^{-l}U/B)=\#(B\backslash U(s_2s_1)^lB/B)=2,$$
$$\#(B\backslash Bs_2^{-1}(s_2s_1)^{-l}U/B)=\#(B\backslash Us_2(s_2s_1)^lB/B)=2,$$

hence $\#(B\backslash G_l/B)=4$. Now we see from (12.8) that

(12.14) $$(T_2T_1)^l=B(s_2s_1)^lB,\quad T_1(T_2T_1)^l=Bs_1(s_2s_1)^lB,$$
$$(T_2T_1)^{l-1}T_2=B(s_2s_1)^{l-1}s_2B,\quad (T_1T_2)^2=B(s_1s_2)^lB,$$

for any $l\in N$, hence (12.10).

We have established the equivalence (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv). Since the conditions (i), (ii) are symmetric in q_1, q_2, one sees that (iv)$\Leftrightarrow$(iv)$'$.

(iv), (iv)$'\Leftrightarrow$(v): This has been pointed out in [Ser-1], II.1 ex. However, we sketch briefly how one finds N and S, under the assumptions of (iv), (iv)$'$. Let $L_0^+:=(P_0, Q_0, P_1, Q_1, \cdots)\in \varprojlim V(P_0; l)$ an infinite half line (path) starting from P_0, and let $L_0^-:=(Q_0, P_0, Q_{-1}, P_{-1}, \cdots)\in \varprojlim V(Q_0, l)$ be a similar one starting from Q_0. Put $L_0:=L_0^+\cup L_0^-$. From the assumptions (iv), (iv)$'$, one finds element $s_1\in U_1$ (resp. $s_2\in U_2$) which induces on L_0 the reflection at P_0 (resp. Q_0). Call N the subgroup of G consisting of elements which keep L stable, and put $T:=B\cap N$. Then it is easy to see that $T|L=\{\text{id.}\}$, and T is a normal subgroup of N, and that $N/T=\mathrm{Aut}(L)$ $=\langle s_1, s_2\,|\,s_1^2=s_2^2=\text{id.}\rangle$. Putting $S=\{s_1, s_2 \pmod T\}$, it is not difficult to check the axioms of Tits system (or, BN-pair; see [Bou], [Ser-1], [T-2]. In fact, the only non-trivial part is to show (T-4), which follows from the property (12.14) above. Conversely, if (G, B, N, S) is a Tits system, then it is easy to see that the associated Tits building is isomorphic to our tree $X(q_1, q_2)$ with the same action of G. Therefore the assertion reduces to the well known properties of the Tits system. This completes the proof of Theorem (12.9). Q.E.D.

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