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On a Duality of Branching Rules for Affine Lie Algebras

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§ 0. Introduction

Let $\tilde{\mathfrak{g}}$ be an affine (Kac-Moody) Lie algebra, and let $\mathfrak{g}$ be an affine Lie subalgebra thereof. In this paper we consider the irreducible decomposition of an irreducible $\tilde{\mathfrak{g}}$-module $L(\tilde{\Lambda})$ with highest weight $\tilde{\Lambda}$ with regards to the subalgebra $\mathfrak{g}$. More specifically, we are to find the multiplicity $(\tilde{\Lambda}: \Lambda)$ of an irreducible highest weight $\mathfrak{g}$-module $L(\Lambda)$ appearing there, to be called the branching coefficient.

Recent theory of Kac-Peterson [1] [2] enables us to formulate this problem in terms of modular functions as follows. In the decomposition of $L(\tilde{\Lambda})$, the irreducible components appear as a "string" $L(\Lambda-n\delta)$ $(n \in \mathbf{Z})$ in the direction of the null root δ of $\mathfrak{g}$. Introducing the generating function $E_{\tilde{\Lambda}\Lambda}(q)=\sum_{n\in\mathbf{Z}}(\tilde{\Lambda}: \Lambda-n\delta)q^n$, we are led to the identity of characters

$$\text{(0.1)} \qquad \mathrm{ch}_{L(\tilde{\Lambda})}|_{\mathfrak{h}}=\sum_{\Lambda} E_{\tilde{\Lambda}\Lambda}(q)\,\mathrm{ch}_{L(\Lambda)}, \quad q=e^{-\delta}.$$

Here $\mathfrak{h}$ denotes the Cartan subalgebra of $\mathfrak{g}$, and Λ runs over a finite set of dominant integral weights of $\mathfrak{g}$ mod $\mathbf{C}\delta$ having the same level $m=(\Lambda, \delta)$. Kac-Peterson [1] [2] show that, up to a rational power of q, the characters are expressible as quotients of classical theta functions. As a consequence, $e_{\tilde{\Lambda}\Lambda}(\tau)=E_{\tilde{\Lambda}\Lambda}(q)\times$(some power of q) with $q=e^{2\pi i\tau}$ becomes a modular function (this is an object analogous to the string functions in [1] [2].) Thus the problem is to determine the modular functions $e_{\tilde{\Lambda}\Lambda}(\tau)$ defined via the theta function identity (0.1).

For level 1 modules $L(\tilde{\Lambda})$, this has been studied in [3] [4], motivated by its connection with soliton theory. There the functions $e_{\tilde{\Lambda}\Lambda}(\tau)$ have been determined for the pairs $A_{2l-1}^{(1)}\supset C_l^{(1)}$, $A_{2l}^{(1)}\supset A_{2l}^{(2)}$ and $C_{2l}^{(1)}\supset C_l^{(1)}$ in terms of Hecke indefinite modular forms. Since (0.1) involves theta functions of dimension l, it gets complicated for large l. The point in [4] (although somewhat obscure there) was to derive another identity for $e_{\tilde{\Lambda}\Lambda}(\tau)$ that involves theta functions of dimension 1, that is, the level of $\tilde{\Lambda}$.

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The purpose of the present paper is to establish a general duality of this sort, relating the functions $e_{\tilde{\Lambda}\Lambda}(\tau)$ for two pairs of Lie algebras $(\tilde{\mathfrak{g}}, \mathfrak{g})$ vs. $(\tilde{\mathfrak{g}}^{\dagger}, \mathfrak{g}^{\dagger})$. What is characteristic here is that the roles of the rank l and the level m of the representations are interchanged between these two pairs. We note here a similarity between our work and Frenkel's [5] where he considers the branching coefficients with respect to a Heisenberg subalgebra.

Let us illustrate our result in the simpler case of finite dimensional Lie algebras. For fixed positive integers l, m, consider the pairs

$$\tilde{\mathfrak{g}}=\mathfrak{gl}(2l, \mathbf{C})\supset\mathfrak{g}=\mathfrak{sp}(2l, \mathbf{C})$$
$$\tilde{\mathfrak{g}}^{\dagger}=\mathfrak{gl}(m, \mathbf{C})\subset\mathfrak{g}^{\dagger}=\mathfrak{sp}(2m, \mathbf{C}).$$

Their irreducible representations are parametrized by Young diagrams $\tilde{Y}$, Y, $\tilde{Y}^{\dagger}$ and $Y^{\dagger}$, respectively. To be more precise, let $\tilde{Y}$ (resp. $\tilde{Y}^{\dagger}$) stand for the corresponding irreducible $\mathfrak{gl}(2l, \mathbf{C})$-module (resp. $\mathfrak{gl}(m, \mathbf{C})$-module) on which the center of $\mathfrak{gl}(2l, \mathbf{C})$ acts as a scalar $|\tilde{Y}|$=the number of tiles composing $\tilde{Y}$ (resp. $|\tilde{Y}^{\dagger}|-l$). There is a one to one correspondence between diagrams $\tilde{Y}$ contained in a rectangle $R_{2l,m}$ of size $2l\times m$ and $\tilde{Y}^{\dagger}$ in $R_{m,2l}$ of size $m\times 2l$; namely, take the complement and transpose the diagram. In the same way, let $Y\subset R_{l,m}$ correspond to $Y^{\dagger}\subset R_{m,l}$. Under these correspondences, the following duality of branching coefficients holds:

Theorem 1. $(\tilde{Y}: Y)=(Y^{\dagger}: \tilde{Y}^{\dagger})$.

Theorem 1 appears as Proposition 1.3 in Section 1. Note again that the choices of l, m are arbitrary. To prove Theorem 1, we make use of an identity which gives the characters of $\mathfrak{gl}(2l+m, \mathbf{C})$ (resp. $\mathfrak{sp}(2l+2m, \mathbf{C})$) in terms of those of $\mathfrak{gl}(2l, \mathbf{C})$ and $\mathfrak{gl}(m, \mathbf{C})$ (resp. $\mathfrak{sp}(2l, \mathbf{C})$ and $\mathfrak{sp}(2m, \mathbf{C})$). It will be termed the complementary decomposition of characters in the text.

The same method applies to affine Lie algebras as well. Consider now the pairs

$$\tilde{\mathfrak{g}}=A^{(1)}_{2l-1}\supset\mathfrak{g}=C^{(1)}_{l}$$
$$\tilde{\mathfrak{g}}^{\dagger}=A^{(1)}_{m-1}\subset\mathfrak{g}^{\dagger}=C^{(1)}_{m}.$$

The corresponding theta function identities read as follows:

(0.2) $$\tilde{\chi}^{(2l)}_{\tilde{\Lambda}}(\tau, u)=\sum_{\Lambda} e_{\tilde{\Lambda}\Lambda}(\tau)\chi^{(l)}_{\Lambda}(\tau, u)$$

(0.3) $$\sum_{\tilde{\Lambda}^{\dagger}} \hat{\chi}^{(m)}_{\tilde{\Lambda}^{\dagger}}(\tau, u^{\dagger})e^{\dagger}_{\tilde{\Lambda}^{\dagger}\Lambda^{\dagger}}(\tau)=\chi^{(m)}_{\Lambda^{\dagger}}(\tau, u^{\dagger}).$$

Here $\tilde{\Lambda}$ and Λ denote dominant integral weights of $A_{2l-1}^{(1)}$ and $C_l^{(1)}$, both of level m. They are in one-to-one correspondence with $\tilde{\Lambda}^\dagger$ (level $2l$) of $A_{m-1}^{(1)}$ or $\Lambda^\dagger$ (level l) of $C_m^{(1)}$ respectively, by the same correspondence as in Theorem 1. $\bar{\chi}_{\tilde{\Lambda}}^{(2l)}(\tau, u), \chi_{\Lambda}^{(l)}(\tau, u)$ and $\chi_{\Lambda^\dagger}^{(m)}(\tau, u^\dagger)$ signify the suitably normalized characters of $A_{2l-1}^{(1)}$ (restricted to $C_l^{(1)}$), $C_l^{(1)}$ and $C_m^{(1)}$ respectively. Finally

$$\hat{\chi}_{\tilde{\Lambda}^\dagger}^{(m)}(\tau, u^\dagger)=\sum_{\nu=0}^{m-1} \vartheta^{(\nu)}\left(\tau, \frac{1}{m} \sum_{j=1}^{m} u_j^\dagger\right)\eta(\tau)^{-1}\chi_{\sigma^\nu(\tilde{\Lambda}^\dagger)}^{(m-1)}(\tau, u^\dagger)$$

where $\vartheta^{(\nu)}(\tau, u)$ is a theta function of dimension 1 (see (2.12)), $\chi_{\tilde{\Lambda}^\dagger}^{(m-1)}$ denotes the character of $A_{m-1}^{(1)}$, $\eta(\tau)$ the Dedekind eta function, and σ is a diagram automorphism which permutes the vertices of the Dynkin diagram cyclically. In the notations above, we have

Theorem 2. $e_{\tilde{\Lambda}\Lambda}(\tau)=e^\dagger_{\tilde{\Lambda}^\dagger\Lambda^\dagger}(\tau)$.

Note that Theorem 2 reduces to Theorem 1 above in the limit $q=e^{2\pi i\tau}\to 0$.

We shall work out a list of such dual identities for pairs $\tilde{\mathfrak{g}}\supset\mathfrak{g}$, where $\mathfrak{g}=\tilde{\mathfrak{g}}^\sigma$ is the invariant subalgebra of an involutive diagram automorphism σ of $\tilde{\mathfrak{g}}$. This leads in particular to a duality of branching coefficients for the pairs $C_{2l}^{(1)}\supset C_l^{(1)}$ vs. $C_m^{(1)}\subset C_m^{(1)}$ (Proposition 3.11). In most cases, however, an extra factor (independent of weights) enters in the right side of the second identity (0.3). We do not know a simple Lie theoretical interpretation of (0.3) in such cases. Nevertheless it provides an effective way of determining $e_{\tilde{\Lambda}\Lambda}(\tau)$ for small m.

Let us give a brief account on the plan of this paper. In Section 1 we shall treat finite dimensional Lie algebras $\mathfrak{gl}(n, \mathbf{C})$, $\mathfrak{o}(n, \mathbf{C})$ or $\mathfrak{sp}(n, \mathbf{C})$, and explain the complementary decompositions and duality of branching coefficients for them. The essential points of this paper are all contained in this paragraph. Sections 2 and 3 deal with affine Lie algebras. In order to fix notations, we first review in Section 2 on generalities of affine Lie algebras and Kac-Peterson's theory. In the latter part of Section 2, we give the complementary decomposition of characters for affine Lie algebras. The result is listed in Table II. Section 3 is devoted to the duality of $e_{\tilde{\Lambda}\Lambda}(\tau)$ associated with a pair $\tilde{\mathfrak{g}}\supset\tilde{\mathfrak{g}}^\sigma$ (Table IV). As an application, we shall explicitly determine $e_{\tilde{\Lambda}\Lambda}(\tau)$ for small levels m in terms of either Hecke indefinite modular forms or (positive-definite) theta functions of dimension 1. In particular, for $A_{2l+1}^{(2)}\supset A_{2l}^{(2)}$, $B_{l+1}^{(2)}\supset D_{l+1}^{(2)}$ and $D_{l+1}^{(1)}\supset B_l^{(1)}$, $e_{\tilde{\Lambda}\Lambda}(\tau)$ is determined for all m.

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§ 1. The finite dimensional case

In this section we shall deal with classical finite dimensional Lie algebras, and explain complementary decomposition of characters. As an application, we derive a duality of branching rules of irreducible representations between two pairs of Lie algebras.

Let $\mathfrak{g}_n$ be one of the classical complex Lie algebras

$$\mathfrak{gl}(n, \mathbf{C}), \quad \mathfrak{o}(2n+1, \mathbf{C}), \quad \mathfrak{sp}(2n, \mathbf{C}) \quad \text{or} \quad \mathfrak{o}(2n, \mathbf{C}).$$

We say that $\mathfrak{g}_n$ is of type A, B, C, D accordingly. The theory of highest weight tells that an irreducible representation of $\mathfrak{g}_n$ is specified by giving a sequence of integers (or half odd integers) $\lambda=(\lambda_1, \cdots, \lambda_n)$, $\lambda_1>\cdots>\lambda_n$. The corresponding character

$$\chi_\lambda^{(n)}=N_\lambda^{(n)}(z_1, \cdots, z_n)/D^{(n)}(z_1, \cdots, z_n)$$

is explicitly given in terms of determinants (Table I).

In Table I, $|z^{\lambda_1}, \cdots, z^{\lambda_n}|$ (resp. $|z^{\lambda_1}-z^{-\lambda_1}, \cdots, z^{\lambda_n}-z^{-\lambda_n}|$) signifies $\det (z_i^{\lambda_j})_{1\leqq i,j\leqq n}$ (resp. $\det (z_i^{\lambda_j}-z_i^{-\lambda_j})_{1\leqq i,j\leqq n}$).

The irreducible characters of $\mathfrak{gl}(n, \mathbf{C})$ are graphically represented by Young diagrams. When $\lambda_n\geqq 0$, $\chi_\lambda^{(n)}$ is in one-to-one correspondence with a Young diagram Y of signature $(\lambda_1-n+1, \lambda_2-n+2, \cdots, \lambda_n)$. In this case we use also the notation $\chi_Y^{(n)}$ to signify $\chi_\lambda^{(n)}$.

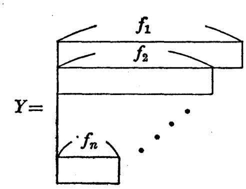


Fig. 1.1

In the general case, take an integer $e\leqq\lambda_n$ and put $f_1=\lambda_1-n+1-e, \cdots, f_n=\lambda_n-e, f_1\geqq\cdots\geqq f_n\geqq 0$. Denoting by Y the diagram of signature $(f_1, \cdots, f_n)$, we have

$$\chi_\lambda^{(n)}(z_1, \cdots, z_n)=(z_1\cdots z_n)^e\chi_Y^{(n)}(z_1, \cdots, z_n).$$

Note that Y is determined from λ up to the "shifting" $e\mapsto e-1, f_i\mapsto f_i+1$.

Table I

Type A:	$N_\lambda^{(n)}(z_1, \cdots, z_n) = \lvert z^{\lambda_1}, z^{\lambda_2}, \cdots, z^{\lambda_n}\rvert$ $\lambda=(\lambda_1, \cdots, \lambda_n) \in \mathbf{Z}^n, \quad \lambda_1 > \cdots > \lambda_n,$ $D^{(n)}(z_1, \cdots, z_n) = N^{(n)}_{n-1, n-2, \cdots, 0}(z_1, \cdots, z_n)$ $= \prod_{1\leqq i<j\leqq n} (z_i - z_j).$
Type B†):	$N_\lambda^{(n)}(z_1, \cdots, z_n) = \lvert z^{\lambda_1} - z^{-\lambda_1}, z^{\lambda_2} - z^{-\lambda_2}, \cdots, z^{\lambda_n} - z^{-\lambda_n}\rvert$ $\lambda=(\lambda_1, \cdots, \lambda_n) \in (\mathbf{Z}+s+1/2)^n, \quad \lambda_1 > \cdots \lambda_n > 0,$ $D^{(n)}(z_1, \cdots, z_n) = N^{(n)}_{n-1/2, n-3/2, \cdots, 1/2}(z_1, \cdots, z_n)$ $= \prod_{i=1}^{n} (z_i^{1/2} - z_i^{-1/2}) \prod_{1\leqq i<j\leqq n} (z_i - z_j)(1 - z_i^{-1} z_j^{-1}).$
Type C:	$N_\lambda^{(n)}(z_1, \cdots, z_n) = \lvert z^{\lambda_1} - z^{-\lambda_1}, z^{\lambda_2} - z^{-\lambda_2}, \cdots, z^{\lambda_n} - z^{-\lambda_n}\rvert$ $\lambda=(\lambda_1, \cdots, \lambda_n) \in \mathbf{Z}^n, \quad \lambda_1 > \cdots > \lambda_n > 0,$ $D^{(n)}(z_1, \cdots, z_n) = N^{(n)}_{n, n-1, \cdots, 1}(z_1, \cdots, z_n)$ $= \prod_{i=1}^{n} (z_i - z_i^{-1}) \prod_{1\leqq i<j\leqq n} (z_i - z_j)(1 - z_i^{-1} z_j^{-1}).$
Type D†):	$N_\lambda^{(n)}(z_1, \cdots, z_n) = \sum_{\substack{\varepsilon_1, \cdots, \varepsilon_n = \pm 1 \\ \varepsilon_1\varepsilon_2\cdots\varepsilon_n = 1}} \lvert z^{\varepsilon_1\lambda_1}, z^{\varepsilon_2\lambda_2}, \cdots, z^{\varepsilon_n\lambda_n}\rvert$ $\lambda=(\lambda_1, \cdots, \lambda_n) \in (\mathbf{Z}+s)^n, \quad \lambda_1 > \cdots > \lambda_{n-1} > \lvert\lambda_n\rvert \geqq 0,$ $D^{(n)}(z_1, \cdots, z_n) = N^{(n)}_{n-1, n-2, \cdots, 0}(z_1, \cdots, z_n)$ $= \prod_{1\leqq i<j\leqq n} (z_i - z_j)(1 - z_i^{-1} z_j^{-1}).$

†) $s=0$ (tensor representation) or $s=1/2$ (spin representation).

Likewise, an irreducible character $\chi_\lambda^{(n)}$ of $\mathfrak{sp}(2n, \mathbf{C})$ is represented by a Young diagram of signature $(f_1, \cdots, f_n)$ with $f_1 = \lambda_1 - n$, $f_2 = \lambda_2 - n + 1$, $\cdots, f_n = \lambda_n - 1$. In this case the correspondence $\chi_\lambda^{(n)} \leftrightarrow Y$ is one-to-one. We write $\chi_Y^{(n)}$ to mean $\chi_\lambda^{(n)}$.

Now let $n = l + m$ with fixed integers $l, m \geqq 1$. There is a natural inclusion $\mathfrak{g}_l \oplus \mathfrak{g}_m \subset \mathfrak{g}_{l+m}$. We shall give a relation among the corresponding characters

$$\chi_\lambda^{(l)}(z_1, \cdots, z_l), \chi_{\lambda\dagger}^{(m)}(w_1, \cdots, w_m) \quad \text{and} \quad \chi_{\lambda\#}^{(l+m)}(z_1, \cdots, z_l, w_1, \cdots, w_m).$$

Let

$$F^{(l,m)}(z_1, \cdots, z_l, w_1, \cdots, w_m)$$
$$= D^{(l+m)}(z_1, \cdots, z_l, w_1, \cdots, w_m) / D^{(l)}(z_1, \cdots, z_l) D^{(m)}(w_1, \cdots, w_m)$$

be the ratio of denominators. Explicitly it is given by

Type A: $F^{(l,m)}(z_1, \cdots, z_l, w_1, \cdots, w_m)=\prod_{i=1}^{l}\prod_{j=1}^{m}(z_i-w_j),$

Type B, C, D: $F^{(l,m)}(z_1, \cdots, z_l, w_1, \cdots, w_m)=\prod_{i=1}^{l}\prod_{j=1}^{m}(z_i-w_j)(1-z_i^{-1}w_j^{-1}).$

Proposition 1.1. (Complementary decomposition). *Notations being as above, we have the following identities.*

Type A, B, C:

$$\sum_{\{\lambda\}\cup\{\lambda^\dagger\}=\{\lambda^\#\}} \operatorname{sgn}(\lambda, \lambda^\dagger)\chi_\lambda^{(l)}(z_1, \cdots, z_l)\chi_{\lambda^\dagger}^{(m)}(w_1, \cdots, w_m) \tag{1.1}$$
$$=F^{(l,m)}(z_1, \cdots, z_l, w_1, \cdots, w_m)\chi_{\lambda^\#}^{(l+m)}(z_1, \cdots, z_l, w_1, \cdots, w_m),$$

Type D:

$$\sum_{\{\lambda\}\cup\{\lambda^\dagger\}=\{\lambda^\#\}} \operatorname{sgn}(\lambda, \lambda^\dagger)(\chi_\lambda^{(l)}(z_1, \cdots, z_l)\chi_{\lambda^\dagger}^{(m)}(w_1, \cdots, w_m) + \chi_{\sigma(\lambda)}^{(l)}(z_1, \cdots, z_l)\chi_{\sigma^\dagger(\lambda^\dagger)}^{(m)}(w_1, \cdots, w_m)) \tag{1.2}$$
$$=F^{(l,m)}(z_1, \cdots, z_l, w_1, \cdots, w_m)\chi_{\lambda^\#}^{(l+m)}(z_1, \cdots, z_l, w_1, \cdots, w_m).$$

Here the sum is taken over the partitions $\{\lambda_1, \cdots, \lambda_l\}\cup\{\lambda_1^\dagger, \cdots, \lambda_m^\dagger\}=\{\lambda_1^\#, \cdots, \lambda_{l+m}^\#\}$ *with* $\lambda_1>\cdots>\lambda_l$, $\lambda_1^\dagger>\cdots>\lambda_m^\dagger$ *and* $\lambda_1^\#>\cdots>\lambda_{l+m}^\#$. *The factor* $\operatorname{sgn}(\lambda, \lambda^\dagger)$ *stands for the sign of the permutation*

$$\begin{pmatrix}\lambda_1^\#, \cdots, \lambda_l^\#, \lambda_{l+1}^\#, \cdots, \lambda_{l+m}^\# \\ \lambda_1, \cdots, \lambda_l, \lambda_1^\dagger, \cdots, \lambda_m^\dagger\end{pmatrix}.$$

In the case of type D, σ *denotes the involution* $(\lambda_1, \cdots, \lambda_n)\mapsto(\lambda_1, \cdots, \lambda_{n-1}, -\lambda_n)$ *which corresponds to the symmetry of the Dynkin diagram of* $\mathfrak{o}(2n, \mathbf{C})$.

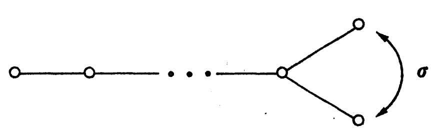


Fig. 1.2

Proof. In view of the explicit representation of characters in Table I, (1.1) and (1.2) are simple consequences of the Laplace expansion formula for determinants. □

Hereafter the formulas (1.1), (1.2) will be referred to as the complementary decomposition of characters.

Consider the special case $\chi_{\lambda^\#}^{(l+m)}\equiv 1$. For the type A or C, the correspondence $\lambda\leftrightarrow\lambda^\dagger$ is neatly described in terms of Young diagrams as follows. Fix a rectangle $R_{l,m}$ of size $l\times m$, and denote by $R_{m,l}$ its transposition. Two diagrams $Y\subset R_{l,m}$ and $Y^\dagger\subset R_{m,l}$ are said to be complementary with respect to $R_{l,m}$ (or $R_{m,l}$), if one is obtained from the other by first taking the complement with respect to $R_{l,m}$ (or $R_{m,l}$) and then transposing the diagram (Fig. 1.3).

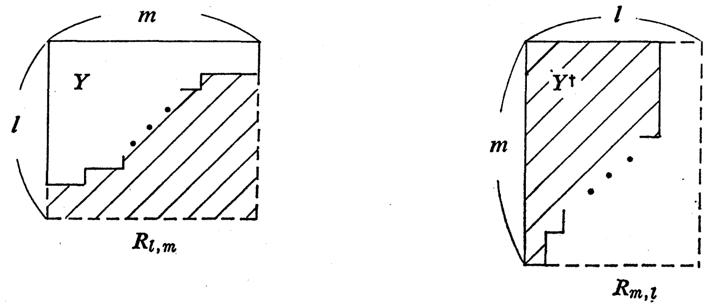


Fig. 1.3

For a Young diagram Y, we denote by $|Y|$ the number of tiles that constitute Y.

Proposition 1.2. *For the characters of type* A *or* C, *we have*

$$\sum_{Y\subset R_{l,m}} (-)^{|Y^\dagger|}\chi_Y^{(l)}(z_1,\cdots,z_l)\chi_{Y^\dagger}^{(m)}(w_1,\cdots,w_m)=F^{(l,m)}(z_1,\cdots,z_l,w_1,\cdots,w_m)$$

where the sum extends over all the diagrams Y contained in $R_{l,m}$, and $Y^\dagger$ denotes its complementary diagram.

Proof. In the formula (1.1), put $\lambda_\nu^\sharp=l+m-\nu$, $\lambda_\nu=f_\nu+l-\nu$ and $\lambda_\nu^\dagger=f_\nu^\dagger+m-\nu$ (resp. $\lambda_\nu^\sharp=l+m+1-\nu$, $\lambda_\nu=f_\nu+l+1-\nu$ and $\lambda_\nu^\dagger=f_\nu^\dagger+m+1-\nu$) for type A (resp. type C). We have then $m\geqq f_1\geqq\cdots\geqq f_l\geqq 0$ and

$$\operatorname{sgn}(\lambda,\lambda^\dagger)=(-)^{\sum_{\nu=1}^{l}(m-f_\nu)}.$$

Clearly the corresponding diagram Y of signature $(f_1,\cdots,f_l)$ satisfies $Y\subset R_{l,m}$, $\operatorname{sgn}(\lambda,\lambda^\dagger)=(-)^{lm-|Y|}$. It is easy to check that the complementary diagram $Y^\dagger$ has the signature $(f_1^\dagger,\cdots,f_m^\dagger)$. □

Now we proceed to considering two pairs of Lie algebras $\tilde{\mathfrak{g}}\supset\mathfrak{g}$ vs. $\tilde{\mathfrak{g}}^\dagger\subset\mathfrak{g}^\dagger$.

The Lie algebra $\mathfrak{sp}(2n,\mathbf{C})$ is realized as the matrix algebra

$$\left\{\begin{bmatrix}A_1 & A_2\\ A_3 & -{}^tA_1\end{bmatrix}\middle|\,A_1, A_2, A_3\colon n\times n \text{ matrices with } A_2={}^tA_2,\ A_3={}^tA_3\right\}.$$

There is thus a natural embedding

(1.3) $$\tilde{\mathfrak{g}}=\mathfrak{gl}(2l, \mathbf{C})\supset\mathfrak{g}=\mathfrak{sp}(2l, \mathbf{C}).$$

Taking $A_2=A_3=0$ in the above, we obtain another embedding

(1.4) $$\tilde{\mathfrak{g}}^{\dagger}=\mathfrak{gl}(m, \mathbf{C})\subset\mathfrak{g}^{\dagger}=\mathfrak{sp}(2m, \mathbf{C}).$$

Consider an irreducible $\tilde{\mathfrak{g}}$-module $L_{\tilde{Y}}$ corresponding to a Young diagram $\tilde{Y}$. By virtue of the complete reducibility of representations, it decomposes into a direct sum of irreducible $\mathfrak{g}$-modules L_Y, also parametrized by Young diagrams Y. Let $(\tilde{Y}: Y)$ denote the multiplicity of L_Y in $L_{\tilde{Y}}$. We have then the following relation of characters

(1.5) $$\chi_{\tilde{Y}}^{(2l)}(z_1, \cdots, z_l, z_l^{-1}, \cdots, z_1^{-1})=\sum_{Y}(\tilde{Y}: Y)\chi_Y^{(l)}(z_1, \cdots, z_l).$$

The non-negative integers $(\tilde{Y}: Y)$ are called the branching coefficients. Likewise, the pair (1.4) gives rise to another relation of characters:

(1.6) $$\sum_{\tilde{Y}^{\dagger}}(w_1 \cdots w_m)^{-l}\chi_{\tilde{Y}^{\dagger}}^{(m)}(w_1, \cdots, w_m)(Y^{\dagger}: \tilde{Y}^{\dagger})=\chi_{Y^{\dagger}}^{(m)}(w_1, \cdots, w_m).$$

In the left hand side, only the characters of the form $(w_1 \cdots w_m)^{-l}\chi_{\tilde{Y}^{\dagger}}^{(m)}$ appear, as we shall see shortly (see (1.7) in the proof of Proposition 1.3.).

Proposition 1.3. *For fixed l, $m\geqq 1$, let $R_{2l,m}$ (resp. $R_{l,m}$) denote a rectangle of size $2l\times m$ (resp. $l\times m$), and let $R_{m,2l}$ (resp. $R_{m,l}$) be its transpose.*

(i) *For $\tilde{Y}\subset R_{2l,m}$, $(\tilde{Y}: Y)=0$ unless $Y\subset R_{l,m}$ and $|\tilde{Y}|\equiv|Y|$* mod 2.

(ii) *For $Y^{\dagger}\subset R_{m,l}$, $(Y^{\dagger}: \tilde{Y}^{\dagger})=0$ unless $\tilde{Y}^{\dagger}\subset R_{m,2l}$ and $|\tilde{Y}^{\dagger}|\equiv|Y^{\dagger}|+lm$* mod 2.

(iii) *We have the duality of branching coefficients*

$$(\tilde{Y}: Y)=(Y^{\dagger}: \tilde{Y}^{\dagger})$$

where the pairs $(\tilde{Y}, \tilde{Y}^{\dagger})$, $(Y, Y^{\dagger})$ are complementary with respect to $R_{2l,m}$ or $R_{l,m}$, respectively.

Proof. We apply Proposition 1.2 to the present situation. We have then

$$\sum_{\tilde{Y}\subset R_{2l,m}}(-)^{|\tilde{Y}^{\dagger}|}\chi_{\tilde{Y}}^{(2l)}(z_1, \cdots, z_l, z_l^{-1}, \cdots, z_1^{-1})(w_1 \cdots w_m)^{-l} \times\chi_{\tilde{Y}^{\dagger}}^{(m)}(w_1, \cdots, w_m)$$
$$=(-)^{lm}\prod_{i=1}^{l}\prod_{j=1}^{m}(z_i-w_j)(1-z_i^{-1}w_j^{-1})$$
$$=(-)^{lm}\sum_{Y\subset R_{l,m}}(-)^{|Y^{\dagger}|}\chi_Y^{(l)}(z_1, \cdots, z_l)\chi_{Y^{\dagger}}^{(m)}(w_1, \cdots, w_m).$$

Substitute (1.5) to the equations above. Since the characters $\chi_Y^{(l)}(z_1, \cdots, z_l)$ are linearly independent, we may equate their coefficients to obtain

$$(1.7) \qquad \sum_{\tilde{Y}\subset R_{2l,m}} (w_1 \cdots w_m)^{-l}\chi_{\tilde{Y}^\dagger}^{(m)}(w_1, \cdots, w_m)(-)^{|\tilde{Y}^\dagger|+lm}(\tilde{Y}: Y) = (-)^{|Y^\dagger|}\chi_{Y^\dagger}^{(m)}(w_1, \cdots, w_m).$$

At the same time, we see that $(\tilde{Y}: Y)=0$ unless $Y\subset R_{l,m}$.

On the other hand, the characters of $\mathfrak{gl}(2l, \mathbf{C})$ or $\mathfrak{sp}(2l, \mathbf{C})$ have the properties

$$\chi_{\tilde{Y}}^{(2l)}(-z_1, \cdots, -z_{2l})=(-)^{|\tilde{Y}|}\chi_{\tilde{Y}}^{(2l)}(z_1, \cdots, z_{2l})$$
$$\chi_Y^{(l)}(-z_1, \cdots, -z_l)=(-)^{|Y|}\chi_Y^{(l)}(z_1, \cdots, z_l).$$

Along with (1.5) they imply

$$(1.8) \qquad (-)^{|\tilde{Y}|}(\tilde{Y}: Y)=(\tilde{Y}: Y)(-)^{|Y|},$$

thereby proving (i). From (1.7) and (1.8) we obtain

$$(1.9) \qquad \sum_{\tilde{Y}^\dagger\subset R_{m,2l}} (w_1 \cdots w_m)^{-l}\chi_{\tilde{Y}^\dagger}^{(m)}(w_1, \cdots, w_m)(\tilde{Y}: Y)=\chi_{Y^\dagger}^{(m)}(w_1, \cdots, w_m).$$

Assertions (ii) and (iii) now follow from (1.6), (1.9) and linear independence of $\chi_{\tilde{Y}^\dagger}^{(m)}(w_1, \cdots, w_m)$. □

Example. $(m=1)$

Let $\tilde{Y}_j$ (resp. Y_k) denote the Young diagrams of signature

$$(\overbrace{1, \cdots, 1}^{j}, \overbrace{0, \cdots, 0}^{2l-j}) \quad (\text{resp. } (\overbrace{1, \cdots, 1}^{k}, \overbrace{0, \cdots, 0}^{l-k})),$$

so that $\tilde{Y}_j^\dagger$ (resp. $Y_k^\dagger$) has the signature $2l-j$ (resp. $l-k$). We have then

$$\chi_{\tilde{Y}_j}^{(2l)}(z_1, \cdots, z_l, z_l^{-1}, \cdots, z_1^{-1})=q_j,$$
$$\chi_{Y_k}^{(l)}(z_1, \cdots, z_l)=q_k-q_{k-2},$$
$$w^{-l}\chi_{\tilde{Y}_j^\dagger}^{(1)}(w)=w^{l-j}$$
$$\chi_{Y_k^\dagger}^{(1)}(w)=(w^{l-k+1}-w^{-l+k-1})/(w-w^{-1})=w^{l-k}+w^{l-k-2}+\cdots+w^{-l+k}.$$

Here we have set

$$\sum_{\nu\geqq 0} q_\nu t^\nu=\prod_{i=1}^{l} (1+z_i t)(1+z_i^{-1}t), \quad q_\nu=q_{2l-\nu}.$$

It follows immediately that

$$(\tilde{Y}_j : Y_k) = (Y_k^\dagger : \tilde{Y}_j^\dagger) = \begin{cases} 1 & (\text{if } k \leqq j \leqq 2l-k,\ k \equiv j \bmod 2) \\ 0 & (\text{otherwise}). \end{cases}$$

Analogous results are available for other types of Lie algebras. Here we consider the two cases $(\tilde{\mathfrak{g}}, \mathfrak{g}) = (\mathfrak{gl}(2l+1, \mathbf{C}), \mathfrak{o}(2l+1, \mathbf{C}))$ or $(\mathfrak{o}(2l+2, \mathbf{C}), \mathfrak{o}(2l+1, \mathbf{C}))$. Using the complementary decomposition (1.1), (1.2) we obtain a duality

$$(\tilde{\lambda} : \lambda) = (\lambda^\dagger : \tilde{\lambda}^\dagger)$$

of "branching coefficients" $(\tilde{\lambda} : \lambda)$, $(\lambda^\dagger : \tilde{\lambda}^\dagger)$ defined as follows.

$\mathfrak{gl}(2l+1, \mathbf{C}) \supset \mathfrak{o}(2l+1, \mathbf{C})$:

(1.10) $$\chi_{\tilde{\lambda}}^{(2l+1)}(z_1, \cdots, z_l, 1, z_l^{-1}, \cdots, z_1^{-1}) = \sum_{\lambda} (\tilde{\lambda} : \lambda) \chi_{\lambda}^{(l)}(z_1, \cdots, z_l)$$

(1.10)† $$\sum_{\tilde{\lambda}^\dagger} (w_1 \cdots w_m)^{-l-1/2} \chi_{\tilde{\lambda}^\dagger}^{(m)}(w_1, \cdots, w_m)(-)^{\sum_{\nu=1}^m (\tilde{\lambda}_\nu^\dagger - \lambda_\nu + 1)}(\lambda^\dagger : \tilde{\lambda}^\dagger) = \chi_{\lambda^\dagger}^{(m)}(w_1, \cdots, w_m) \prod_{j=1}^m (w_j^{1/2} - w_j^{-1/2})$$

where $\chi_{\tilde{\lambda}^\dagger}^{(m)}$ (resp. $\chi_{\lambda^\dagger}^{(m)}$) denotes the character of $\mathfrak{gl}(m, \mathbf{C})$ (resp. $\mathfrak{o}(2m+1, \mathbf{C})$). The suffixes $\tilde{\lambda}$, λ, $\tilde{\lambda}^\dagger$ and $\lambda^\dagger$ range over

$$\{\tilde{\lambda}_1, \cdots, \tilde{\lambda}_{2l+1}\} \cup \{\tilde{\lambda}_1^\dagger, \cdots, \tilde{\lambda}_m^\dagger\} = \{2l+m, \cdots, 1, 0\},$$
$$\{\lambda_1, \cdots, \lambda_l\} \cup \{\lambda_1^\dagger, \cdots, \lambda_m^\dagger\} = \{l+m-1/2, \cdots, 3/2, 1/2\},$$

with $\tilde{\lambda}_1 > \cdots > \tilde{\lambda}_{2l+1}$, $\tilde{\lambda}_1^\dagger > \cdots > \tilde{\lambda}_m^\dagger$, $\lambda_1 > \cdots > \lambda_l$ and $\lambda_1^\dagger > \cdots > \lambda_m^\dagger$.

$\mathfrak{o}(2l+2, \mathbf{C}) \supset \mathfrak{o}(2l+1, \mathbf{C})$:

(1.11) $$\chi_{\tilde{\lambda}}^{(l+1)}(z_1, \cdots, z_l, 1) = \sum_{\lambda} (\tilde{\lambda} : \lambda) \chi_{\lambda}^{(l)}(z_1, \cdots, z_l),$$

(1.11)′ $$\sum_{\tilde{\lambda}^\dagger} (\chi_{\lambda^\dagger}^{(m)}(w_1, \cdots, w_m)(\lambda^\dagger : \tilde{\lambda}^\dagger) + \chi_{\sigma^\dagger(\tilde{\lambda}^\dagger)}^{(m)}(w_1, \cdots, w_m)(\lambda^\dagger : \sigma^\dagger(\tilde{\lambda}^\dagger)))(-)^{\sum_{\nu=1}^m (\tilde{\lambda}_\nu^\dagger - \lambda_\nu^\dagger - 1/2)} = \chi_{\lambda^\dagger}^{(m)}(w_1, \cdots, w_m) \prod_{j=1}^m (w_j^{1/2} - w_j^{-1/2})^2$$

where $\chi_{\tilde{\lambda}^\dagger}^{(m)}$ (resp. $\chi_{\lambda^\dagger}^{(m)}$) denotes the character of $\mathfrak{o}(2m, \mathbf{C})$ (resp. $\mathfrak{o}(2m+1, \mathbf{C})$). With $s=0$ or $1/2$ fixed, the suffixes range over

$$\{\tilde{\lambda}_1, \cdots, \tilde{\lambda}_{l+1}\} \cup \{\tilde{\lambda}_1^\dagger, \cdots, \tilde{\lambda}_m^\dagger\} = \{l+m+s, \cdots, 1+s, s\},$$
$$\{\lambda_1, \cdots, \lambda_l\} \cup \{\lambda_1^\dagger, \cdots, \lambda_m^\dagger\} = \{l+m-1/2+s, \cdots, 3/2+s, 1/2+s\},$$

with $\tilde{\lambda}_1 > \cdots > \tilde{\lambda}_{l+1}$, $\tilde{\lambda}_1^\dagger > \cdots > \tilde{\lambda}_m^\dagger$, $\lambda_1 > \cdots > \lambda_l$ and $\lambda_1^\dagger > \cdots > \lambda_m^\dagger$.

To derive (1.10), (1.11) in the case $s=1/2$, we have chosen as $\chi_{\lambda\#}$ the character of simplest spin representations and used the relation

$$\begin{aligned}\chi^{(n+1)}_{n+1/2,\, n-1/2,\dots,\pm 1/2}&(z_1, \cdots, z_n, 1; \mathfrak{o}(2n+2, \mathbf{C}))\\ &=\prod_{i=1}^{n}(z_i^{1/2}+z_i^{-1/2})\\ &=\chi^{(n)}_{n,n-1,\dots,1}(z_1, \cdots, z_n; \mathfrak{o}(2n+1, \mathbf{C})).\end{aligned}$$

Unlike the case of $\mathfrak{gl}(2l, \mathbf{C})\supset\mathfrak{sp}(2l, \mathbf{C})$, one cannot get rid of the extra factors

$$\prod_{j=1}^{m}(w_j^{1/2}-w_j^{-1/2}) \quad \text{or} \quad \prod_{j=1}^{m}(w_j^{1/2}-w_j^{-1/2})^2$$

in $(1.10)^{\dagger}$, $(1.11)^{\dagger}$. We do not know how to interpret these formulas Lie theoretically.

§ 2. Complementary decomposition of characters

In this section, after preparatory paragraphs on the theory of Kac and Peterson [2], we present several formulas on complementary decomposition of characters of highest weight modules over affine Lie algebras.

Let $A=(a_{ij})_{i,j=0,\dots,l}$ be a generalized Cartan matrix. We assume that A is affine in the sense of [1]. For a later purpose we define the Cartan subalgebra $\mathfrak{h}$ and its dual $\mathfrak{h}^*$ as follows.

Let $\mathfrak{h}$ be a complex vector space of dimension $2(l+1)$ spanned by linearly independent elements h_i and d_i $(i=0, \cdots, l)$. In $\mathfrak{h}^*$ we choose the dual basis Λ_i and δ_i $(i=0, \cdots, l)$;

$$\Lambda_i(h_j)=\delta_{ij}, \quad \Lambda_i(d_j)=0,$$
$$\delta_i(h_j)=0, \quad \delta_i(d_j)=\delta_{ij}.$$

The simple roots $\alpha_i \in \mathfrak{h}^*$ $(i=0, \cdots, l)$ are defined by

$$\alpha_i(h_j)=a_{ji}, \quad \alpha_i(d_j)=\delta_{ij}.$$

The affine Lie algebra $\mathfrak{g}(A)$ is the complex Lie algebra generated by $\mathfrak{h}\cup\{e_i, f_i\}_{i=0,\dots,l}$ and the following defining relations.

$$\begin{aligned}&[e_i, f_j]=\delta_{ij}h_j, \quad (i, j=0, \cdots, l),\\ &[h, e_i]=\alpha_i(h)e_i, \quad [h, f_i]=-\alpha_i(h)f_i, \quad (i=0, \cdots, l; h\in\mathfrak{h}),\\ &[h, h']=0, \quad (h, h'\in\mathfrak{h}),\\ &(\operatorname{ad} e_i)^{1-a_{ij}}e_j=0, \quad (\operatorname{ad} f_i)^{1-a_{ij}}f_j=0, \quad (i, j=0, \cdots, l).\end{aligned}$$

The positive integer l is called the rank of $\mathfrak{g}(A)$.

We define a symmetric bilinear form on $\mathfrak{h}^*$ by

$$(\alpha_i, \alpha_j)=a_i^{-1}a_i^{\vee}a_{ij},$$
$$(\alpha_i, \Lambda_j)=a_i^{-1}a_i^{\vee}\delta_{ij},$$
$$(\Lambda_i, \Lambda_j)=0, \quad (i,j=0, \cdots, l).$$

Here a_i and $a_i^{\vee}$ $(i=0, \cdots, l)$ are the numerical invariants defined in [2]. The null root δ and the canonical central element c are determined by them:

$$\delta=\sum_{i=0}^{l} a_i\alpha_i, \quad c=\sum_{i=0}^{l} a_i^{\vee}h_i.$$

We shall identify $\mathfrak{h}$ with $\mathfrak{h}^*$ by using the above bilinear form. By this identification we have

$$a_i\Lambda_i=a_i^{\vee}d_i, \quad a_i\alpha_i=a_i^{\vee}h_i \quad (i=0, \cdots, l).$$

For $\lambda \in \mathfrak{h}^*$ we denote by $\bar{\lambda}$ the orthogonal projection of λ on $\bar{\mathfrak{h}}^*=\mathbf{C}\alpha_1+\cdots+\mathbf{C}\alpha_l$. We set $\mathfrak{h}_0^*=\mathbf{C}\Lambda_0+\bar{\mathfrak{h}}^*+\mathbf{C}\delta$. This is the orthogonal complement to $\mathbf{C}\delta_1+\cdots+\mathbf{C}\delta_l$.

We denote by $\Delta(\Delta_+)$ the set of (positive) roots. The set $\bar{\Delta}\underset{\text{def}}{=}\Delta\cap\bar{\mathfrak{h}}^*$ can be identified with the set of roots of the finite dimenional complex simple Lie algebra $\overline{\mathfrak{g}(A)}$ whose Dynkin diagram is obtained from that of $\mathfrak{g}(A)$ by removing the 0-th vertex. Affine Lie algebras are classified according to the type of $\overline{\mathfrak{g}(A)}$ and a numerical invariant $k=1, 2, 3$ [2].

The Weyl group W is the subgroup of $O(\mathfrak{h}^*)$ (the orthogonal group) generated by the fundamental reflections r_i $(i=0, \cdots, l)$ defined by

$$r_i(\lambda)=\lambda-\frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)}\alpha_i. \tag{2.1}$$

We note that by requiring $(\Lambda_i, \Lambda_j)=0$ $(i,j=0, \cdots, l)$ the W-invariant and symmetric bilinear form is unique up to constant multiple.

The Weyl group admits the semi-direct product decomposition

$$W=\bar{W}\ltimes T, \tag{2.2}$$

where $\bar{W}$ is the (classical) Weyl group for $\overline{\mathfrak{g}(A)}$ generated by r_i $(i=1, \cdots, l)$ and T is an abelian normal subgroup of W which is isomorphic to a lattice M in $\bar{\mathfrak{h}}^*$ through the following key formula in [2].

$$\alpha \in M\subset\bar{\mathfrak{h}}^*\longmapsto t_\alpha \in T\subset O(\mathfrak{h}^*),$$

(2.3) $$t_\alpha(\lambda)=\lambda+m\alpha-\left(\frac{m}{2}|\alpha|^2+(\lambda,\alpha)\right)\delta,\quad (\lambda\in\mathfrak{h}^*),$$

where $|\alpha|^2=(\alpha,\alpha)$ and $m=(\lambda,\delta)$ (called the level of λ).

The lattice M is given by $M=\mathbf{Z}\bar{h}_0+\mathbf{Z}h_1+\cdots+\mathbf{Z}h_l$ (if $k=1$) or $M=\mathbf{Z}\bar{\alpha}_0+\mathbf{Z}\alpha_1+\cdots+\mathbf{Z}\alpha_l$ (if $k\neq 1$). Note that $M\otimes_{\mathbf{Z}}\mathbf{C}=\bar{\mathfrak{h}}^*\cong\mathbf{C}^l$.

We denote by $P(P_+)$ the set of (dominant) integral weights in $\mathfrak{h}^*$. For $\Lambda\in P_+$ we denote by $L(\Lambda)$ the irreducible highest weight module with the highest weight Λ. The character $\mathrm{ch}_{L(\Lambda)}$ is the following formal sum.

$$\mathrm{ch}_{L(\Lambda)}=\sum_{\lambda\in\mathfrak{h}^*}\dim L(\Lambda)_\lambda e^\lambda$$

where $L(\Lambda)_\lambda=\{v\in L(\Lambda)\,|\,hv=\lambda(h)v \text{ for all } h\in\mathfrak{h}\}$. The Weyl-Kac character formula reads as

(2.4) $$\mathrm{ch}_{L(\Lambda)}=\sum_{w\in W}\det w\, e^{w(\Lambda+\rho)}\Big/\sum_{w\in W}\det w\, e^{w(\rho)},$$

where $\rho\in\mathfrak{h}^*$ is requested to satisfy $\rho(h_i)=1$ $(i=0,\cdots,l)$. In [2] Kac and Peterson have shown that the character is expressible as a quotient of alternating sums of classical theta functions, as we shall summarize below.

We denote by M^* the dual lattice of M; $M^*=\{\alpha\in\bar{\mathfrak{h}}^*\,|\,(\alpha,\beta)\in\mathbf{Z}$ for all $\beta\in M\}$. For $\mu\in M^*$ and $m\in\mathbf{Z}_+$ we define classical theta function $\Theta^M_{\mu,m}(\tau,u)$ $(\tau\in\mathscr{H}_+=\{\tau\in\mathbf{C}\,|\,\mathrm{Im}\,\tau>0\},\ u\in\bar{\mathfrak{h}}^*)$ by

$$\Theta^M_{\mu,m}(\tau,u)=\sum_{\gamma\in M+\mu/m}\mathbf{e}\left[\frac{\tau}{2}m|\gamma|^2+m(\gamma,u)\right],$$

where $\mathbf{e}[*]=\exp(2\pi i*)$. We also define the following alternating sum.

$$A^M_{\mu,m}(\tau,u)=\sum_{w\in W}\det w\,\Theta^M_{w(\mu),m}(\tau,u).$$

For $h=2\pi i(-\tau\Lambda_0+u+\eta)$ with $\tau\in\mathscr{H}_+$, $u\in\bar{\mathfrak{h}}^*$ and $\eta\in\mathbf{C}\delta_0+\cdots+\mathbf{C}\delta_l$ (2.2) and (2.3) imply that

(2.5) $$\sum_{w\in W}\det w\, e^{w(\lambda)(h)}=\mathbf{e}\left[-\left(\frac{|\bar{\lambda}|^2}{2(\lambda,\delta)}+(\lambda,\Lambda_0)\right)\tau+(\lambda,\eta)\right]A^M_{\bar{\lambda},(\lambda,\delta)}(\tau,u).$$

Now, for $\Lambda\in P_+$ we define the normalized character $\chi_\Lambda(\tau,u)$ by

$$\chi_\Lambda(\tau,u)=A^M_{\overline{\Lambda+\rho},m+g}(\tau,u)/A^M_{\bar{\rho},g}(\tau,u)$$

with $m=(\Lambda,\delta)$ and $g=(\rho,\delta)$. Then we have

$$\mathrm{ch}_{L(\Lambda)}(h)=\mathbf{e}[-s_\Lambda\tau+(\Lambda,\eta)]\chi_\Lambda(\tau,u),$$

where

$$s_\Lambda=\frac{|\overline{\Lambda+\rho}|^2}{2(m+g)}-\frac{|\bar{\rho}|^2}{2g}+(\Lambda,\Lambda_0).$$

The following is well-known as the denominator formula.

$$\sum_{w\in W}\det w\, e^{w(\rho)}=e^\rho\prod_{\alpha\in\Delta_+}(1-e^{-\alpha})^{\mathrm{mult}\,\alpha}.$$

This enables us to rewrite $A^M_{\rho,g}(\tau,u)$ as a product of the Dedekind η-function and the following theta functions of dimension 1. (See Table II.)

$$\eta(\tau)=q^{1/24}\prod_{n\geq 0}(1-q^{n+1}),$$

$$\theta(\tau,u)=\sum_{n\in\mathbf{Z}}(-)^n\,\mathbf{e}\left[\frac{1}{2}\left(n+\frac{1}{2}\right)^2\tau+\left(n+\frac{1}{2}\right)u\right]$$

$$=z^{1/2}q^{1/8}\prod_{n\geq 0}(1-q^{n+1})(1-zq^{n+1})(1-z^{-1}q^n),$$

$$\hat{\theta}(\tau,u)=\sum_{n\in\mathbf{Z}}(-)^n\,\mathbf{e}\left[\frac{1}{2}n^2\tau+nu\right]$$

$$=\prod_{n\geq 0}(1-q^{n+1})(1-zq^{n+1/2})(1-z^{-1}q^{n+1/2}),$$

where $q=e^{2\pi i\tau}$ and $z=e^{2\pi iu}$. In general, for $\lambda\in\mathbf{Z}$ and $N\in\mathbf{Z}_+$ we define

$$\vartheta_{\lambda,N}(\tau,u)=\sum_{n\in\mathbf{Z}}(-)^{Nn}\,\mathbf{e}\left[\frac{N}{2}\left(n+\frac{\lambda}{N}\right)^2\tau+(Nn+\lambda)u\right],$$

$$\hat{\vartheta}_{\lambda,N}(\tau,u)=\sum_{n\in\mathbf{Z}}(-)^{(N+1)n}\,\mathbf{e}\left[\frac{N}{2}\left(n+\frac{\lambda}{N}\right)^2\tau+(Nn+\lambda)u\right].$$

In what follows we shall consider a triple of affine Lie algebras, say $\mathfrak{g}(A)$, $\mathfrak{g}(A^\dagger)$ and $\mathfrak{g}(A^\sharp)$. The concepts corresponding to $\mathfrak{g}(A^\dagger)$ or $\mathfrak{g}(A^\sharp)$ will be distinguished by the mark $\dagger$ or $\sharp$, respectively. For instance, the set of roots for $\mathfrak{g}(A)$, $\mathfrak{g}(A^\dagger)$ and $\mathfrak{g}(A^\sharp)$ are denoted by Δ, $\Delta^\dagger$ and $\Delta^\sharp$, respectively. The only exception is (,), which we use in common for the sake of simplicity.

First we consider the pair $(\mathfrak{g}(A),\mathfrak{g}(A^\sharp))$. We assume that there exists an embedding $\iota\colon\mathfrak{h}_0^*\hookrightarrow\mathfrak{h}_0^{*\sharp}$ satisfying

$$\iota(\Lambda_0)=\Lambda_0^\sharp,\quad \iota(\delta)=\delta^\sharp,\quad \iota(\bar{\mathfrak{h}}^*)\subset\bar{\mathfrak{h}}^{*\sharp},$$

$$\iota(\mathbf{Z}h_0+\cdots+\mathbf{Z}h_l)\subset\mathbf{Z}h_0^\sharp+\cdots+\mathbf{Z}h_{l^\sharp}^\sharp,\quad \iota(\Delta)\subset\Delta^\sharp,\tag{2.6}$$

$$(\lambda,\lambda')=(\iota(\lambda),\iota(\lambda'))\quad\text{for }\lambda,\lambda'\in\mathfrak{h}_0^*.$$

The natural injective homomorphism $W \hookrightarrow W^{\#}$ is also denoted by ι. Then we have

(2.7) $$\iota(w)\cdot\iota(\lambda)=\iota\cdot w(\lambda) \qquad \text{for } w\in W \text{ and } \lambda\in\mathfrak{h}_0^*.$$

The transposed mapping of ι is denoted by π: Namely we have $(\lambda^{\#}, \iota(\lambda)) =(\pi(\lambda^{\#}), \lambda)$ for $\lambda^{\#}\in\mathfrak{h}_0^{*\#}$ and $\lambda\in\mathfrak{h}_0^*$. Note that $\pi(\Lambda_0^{\#})=\Lambda_0$, $\pi(\delta^{\#})=\delta$ and for $\lambda\in\bar{\mathfrak{h}}^*$ $\pi\cdot\iota(\lambda)=\lambda$.

Proposition 2.1. *For $w\in W$ and $\lambda^{\#}\in\mathfrak{h}_0^{*\#}$ we have*

(2.8) $$\iota(w)(\lambda^{\#})=\lambda^{\#}+\iota(w\cdot\pi(\lambda^{\#})-\pi(\lambda^{\#})).$$

Proof. Because of (2.7) it is sufficient to prove this for $w=r_i$. Using (2.1) we have

$$\begin{aligned}\iota(r_i)(\lambda^{\#})&=\lambda^{\#}-\frac{2(\lambda^{\#}, \iota(\alpha_i))}{(\iota(\alpha_i), \iota(\alpha_i))}\iota(\alpha_i),\\ &=\lambda^{\#}-\frac{2(\pi(\lambda^{\#}), \alpha_i)}{(\alpha_i, \alpha_i)}\iota(\alpha_i),\\ &=\lambda^{\#}+\iota(r_i\cdot\pi(\lambda^{\#})-\pi(\lambda^{\#})).\end{aligned}$$ □

Let ω be the projection $\mathfrak{h}_0^*=\mathbf{C}\Lambda_0\oplus\bar{\mathfrak{h}}^*\oplus\mathbf{C}\delta\rightarrow\mathbf{C}\Lambda_0\oplus\bar{\mathfrak{h}}^*$, and set $\hat{\pi}=\omega\cdot\pi$. Since $\pi(\lambda^{\#})-\hat{\pi}(\lambda^{\#})\in\mathbf{C}\delta$ and $w(\delta)=\delta$, (2.8) can be rewritten as

$$\iota(w)(\lambda^{\#})=\lambda^{\#}+\iota(w\cdot\hat{\pi}(\lambda^{\#})-\hat{\pi}(\lambda^{\#})).$$

We note that if $\lambda^{\#}\in P^{\#}\cap\mathfrak{h}_0^{*\#}$ then $\hat{\pi}(\lambda^{\#})\in P\cap\mathfrak{h}_0^*$.

Now we assume that there exists another embedding $\iota^{\dagger}$: $\mathfrak{h}_0^{*\dagger}\hookrightarrow\mathfrak{h}_0^{*\#}$ satisfying (2.6) with ι, Λ_0, δ, etc. replaced by $\iota^{\dagger}$, $\Lambda_0^{\dagger}$, $\delta^{\dagger}$, etc. We assume that

(2.9) $$(\iota(\lambda), \iota^{\dagger}(\lambda^{\dagger}))=0 \qquad \text{for } \lambda\in\bar{\mathfrak{h}}^* \text{ and } \lambda^{\dagger}\in\bar{\mathfrak{h}}^{*\dagger}.$$

Proposition 2.2. *For $w\in W$, $w^{\dagger}\in W^{\dagger}$ and $\lambda^{\#}\in\mathfrak{h}_0^{*\#}$ we have*

$$\iota(w)\cdot\iota^{\dagger}(w^{\dagger})(\lambda^{\#})=\lambda^{\#}+\iota(w\cdot\hat{\pi}(\lambda^{\#})-\hat{\pi}(\lambda^{\#}))+\iota^{\dagger}(w^{\dagger}\hat{\pi}^{\dagger}(\lambda^{\#})-\hat{\pi}^{\dagger}(\lambda^{\#})).$$

In particular, $\iota(w)$ and $\iota^{\dagger}(w^{\dagger})$ are commutative.

Proof. For $\lambda^{\dagger}\in\mathfrak{h}_0^{*\dagger}$ such that $(\lambda^{\dagger}, \delta^{\dagger})=0$, (2.9) implies $\hat{\pi}\iota^{\dagger}\lambda^{\dagger}=0$. Applying this to $\lambda^{\dagger}=w^{\dagger}\cdot\hat{\pi}^{\dagger}(\lambda^{\#})-\hat{\pi}^{\dagger}(\lambda^{\#})$ we obtain

$$\begin{aligned}\iota(w)\cdot\iota^{\dagger}(w^{\dagger})(\lambda^{\#})-\lambda^{\#}&=\iota(w)\cdot\iota^{\dagger}(w^{\dagger}\cdot\hat{\pi}^{\dagger}(\lambda^{\#})-\hat{\pi}^{\dagger}(\lambda^{\#}))+\iota(w\cdot\hat{\pi}(\lambda^{\#})-\hat{\pi}(\lambda^{\#})),\\ &=\iota^{\dagger}(w^{\dagger}\cdot\hat{\pi}^{\dagger}(\lambda^{\#})-\hat{\pi}^{\dagger}(\lambda^{\#}))+\iota(w\cdot\hat{\pi}(\lambda^{\#})-\hat{\pi}(\lambda^{\#})).\end{aligned}$$ □

Proposition 2.3. *For $\lambda^\# \in \mathfrak{h}_0^{*\#}$ such that $(\lambda^\#, \delta^\#) = m > 0$ and $(\lambda^\#, \Lambda_0^\#) = 0$ we have*

$$A^{M\#}_{\lambda\#,m}(\tau, \iota(u) + \iota^\dagger(u^\dagger)) \tag{2.10}$$
$$= \sum_{w\# \in \iota(W) \times \iota\dagger(W\dagger) \backslash W\#} \det w^\# \, \mathbf{e}\left[\frac{\tau}{2m}(|\overline{w^\#(\lambda^\#)}|^2 - |\bar{\pi} w^\#(\lambda^\#)|^2 - |\bar{\pi}^\dagger w^\#(\lambda^\#)|^2)\right]$$
$$\times A^M_{\overline{\bar{\pi} w\#(\lambda\#)},m}(\tau, u) A^M_{\overline{\bar{\pi}\dagger w\#(\lambda\#)},m}(\tau, u^\dagger).$$

Proof. Using Proposition 2.2 we have

$$\sum_{w\# \in W\#} \det w^\# \, e^{w\#(\lambda\#)} = \sum_{w\# \in \iota(W) \times \iota\dagger(W\dagger) \backslash W\#} \det w^\# \, e^{w\#(\lambda\#)}$$
$$\times \sum_{w \in W} \det w \, e^{\iota(w \cdot \bar{\pi} \cdot w\#(\lambda\#) - \bar{\pi} \cdot w\#(\lambda\#))}$$
$$\times \sum_{w\dagger \in W\dagger} \det w^\dagger \, e^{\iota\dagger(w\dagger \cdot \bar{\pi}\dagger \cdot w\#(\lambda\#) - \bar{\pi}\dagger \cdot w\#(\lambda\#))}.$$

Evaluating this identity at $2\pi i(-\tau \Lambda_0^\# + \iota(u) + \iota^\dagger(u^\dagger))$ we obtain (2.10). □

An element $\lambda \in \mathfrak{h}^*$ is called regular if for any $\alpha \in \Delta$ such that $\bar{\alpha} \neq 0$ $(\lambda, \alpha) \neq 0$. If $\lambda \in \mathfrak{h}^*$ is regular there exists a unique $w \in W$ such that $w(\lambda)$ is dominant. We introduce the following notations.

$$\operatorname{sgn} \lambda = \det w, \quad [\lambda] = w(\lambda).$$

Then (2.10) can be rewritten as follows.

Proposition 2.4 (Complementary decomposition). *Let $\Lambda^\#$ be a dominant integral weight in $\mathfrak{h}_0^{*\#}$ such that $(\Lambda^\#, \delta^\#) = m^\# \in \mathbf{Z}_+$ and $(\Lambda^\#, \Lambda_0^\#) = 0$. We choose $\rho^\# \in \mathfrak{h}_0^{*\#}$ so that $(\rho^\#, \Lambda_0^\#) = 0$. Then we have*

$$\frac{A^{M\#}_{\rho\#,g\#}(\tau, \iota(u) + \iota^\dagger(u^\dagger))}{A^M_{\rho,g}(\tau, u) A^{M\dagger}_{\rho\dagger,g\dagger}(\tau, u^\dagger)} \chi_{\Lambda\#}(\tau, \iota(u) + \iota^\dagger(u^\dagger))$$
$$= \sum_{w\# \in \iota(W) \times \iota\dagger(W\dagger) \backslash W\#} \det w^\# \operatorname{sgn} \bar{\pi} w^\#(\Lambda^\# + \rho^\#) \operatorname{sgn} \bar{\pi}^\dagger w^\#(\Lambda^\# + \rho^\#) \tag{2.11}$$
$$\times \mathbf{e}\left[\frac{\tau}{2(m^\# + g^\#)}(|\overline{w^\#(\Lambda^\# + \rho^\#)}|^2 - |\bar{\pi} \cdot w^\#(\Lambda^\# + \rho^\#)|^2 - |\bar{\pi}^\dagger \cdot w^\#(\Lambda^\# + \rho^\#)|^2)\right]$$
$$\times \chi_{[\bar{\pi} \cdot w\#(\Lambda\# + \rho\#)] - \rho}(\tau, u) \chi_{[\bar{\pi}\dagger \cdot w\#(\Lambda\# + \rho\#)] - \rho\dagger}(\tau, u^\dagger),$$

where $g = (\rho, \delta)$, $g^\dagger = (\rho^\dagger, \delta^\dagger)$ and $g^\# = (\rho^\#, \delta^\#)$.

Now we write this formula explicitly in the following cases of $(\mathfrak{g}(A), \mathfrak{g}(A^\dagger), \mathfrak{g}(A^\#))$.

Case (1) $(A_l^{(1)}, A_{m-1}^{(1)}, A_{l+m}^{(1)})$

Case (2) $(C_l^{(1)}, C_m^{(1)}, C_{l+m}^{(1)})$

Case (3) $(A_{2l}^{(2)}, A_{2m}^{(2)}, A_{2(l+m)}^{(2)})$

Case (4) $(D_{l+1}^{(2)}, D_{m+1}^{(2)}, D_{l+m+1}^{(2)})$

Case (5) $(A_{2l-1}^{(2)}, A_{2m-1}^{(2)}, A_{2(l+m)-1}^{(2)})$

Case (6) $(B_l^{(1)}, B_m^{(1)}, B_{l+m}^{(1)})$

Case (7) $(D_l^{(1)}, D_m^{(1)}, D_{l+m}^{(1)})$

In Table II we list basic data and the complementary decomposition for Cases (1)–(7), respectively.

In 1° the Dynkin diagram is given. In 2° δ, c and g are given. In 3° α_i, h_i, $\bar{\Lambda}_i$ and $\bar{\rho}$ are written in terms of the orthogonal basis ε_i ([6]). In 4° the Weyl group $W \cong \overline{W} \ltimes M$ is given. In 5° the set of dominant integral weight of level m is described. In 6° the product form of the denominator formula is given. Finally the complementary decomposition is given in 7°. For definiteness, the rank l of the affine Lie algebra is explicitly exhibited in the notation of the normalized characters.

Compared to other cases, Case (1) is complicated because $|\overline{w^\#(\lambda^\#)}|^2 - |\bar{\pi}\cdot w^\#(\lambda^\#)|^2 - |\bar{\pi}^\dagger \cdot w^\#(\lambda^\#)|^2$ does not vanish and $\#(\iota(W)\times\iota^\dagger(W^\dagger)\backslash W^\#)$ is infinite. The following give the decomposition for Case (1) with $\Lambda^\# = 0$ in a much more convenient form than that in Table II.

We set $N = l+m+1$. We denote by $\mathscr{P}$ or $\mathscr{P}_0$ the following subset of $\mathfrak{S}_N$.

$$\mathscr{P} = \{w^\# \in \mathfrak{S}_N \mid w^\#(1) < \cdots < w^\#(l+1),\ w^\#(l+2) < \cdots < w^\#(N)\},$$
$$\mathscr{P}_0 = \{w^\# \in \mathscr{P} \mid w^\#(l+1) = N\}.$$

For $w^\# \in \mathscr{P}$ we set

$$\lambda_i = \frac{N+1}{2} - w^\#(i), \quad (i = 1, \cdots, l+1),$$
$$\lambda_j^\dagger = \frac{N+1}{2} - w^\#(l+1+j), \quad (j = 1, \cdots, m).$$

Let Λ be a weight in $\mathfrak{h}^*$ satisfying

$$(\Lambda, \delta) = m, \quad \overline{\Lambda + \rho} = \sum_{i=1}^{l+1}\left(\lambda_i - \frac{1}{l+1}\sum_{i'=1}^{l+1}\lambda_{i'}\right)\varepsilon_i,$$

and let $\Lambda^\dagger$ be a weight in $\mathfrak{h}^{*\dagger}$ satisfying

$$(\lambda^\dagger, \delta^\dagger) = l+1, \quad \overline{\Lambda^\dagger + \rho^\dagger} = \sum_{j=1}^{m}\left(\lambda_j^\dagger - \frac{1}{m}\sum_{j'=1}^{m}\lambda_{j'}^\dagger\right)\varepsilon_j^\dagger.$$

We denote by Y the Young diagram with the signature $(\lambda_1-\lambda_{l+1}-l, \cdots, \lambda_l-\lambda_{l+1}-1)$. The following are in one to one correspondence.

(i) $w^\# \in \mathscr{P}_0$,

(ii) $\Lambda \in P_+ \bmod \mathbf{C}\delta_0+\cdots+\mathbf{C}\delta_l$ s.t. $(\Lambda, \delta)=m$,

(iii) $Y\subset R_{l+1,m}$: the rectangle of size $(l+1)\times m$ s.t. the $(l+1)$-th row of Y is void,

(iv) $Y^\dagger\subset R_{m,l+1}$: the rectangle of size $m\times(l+1)$ s.t. the first column of $Y^\dagger$ is full.

In fact, the correspondences (i)↔(ii) and (ii)↔(iii) are given above, and the correspondence (iii)↔(iv) is given in Section 1.

Note that for $w^\# \in \mathscr{P}_0$, $\det w^\#=(-)^{|Y^\dagger|}$. We set $\chi_Y^{(l)}(\tau, u)=\chi_\Lambda^{(l)}(\tau, u)$.

Let $\hat{u}^\dagger=\sum_{j=1}^m \hat{u}_j^\dagger \varepsilon_j^\dagger$ such that $|\hat{u}^\dagger| \underset{\text{def}}{=} \sum_{j=1}^m \hat{u}_j^\dagger$ is not necessarily zero. We set $u^\dagger=\sum_{j=1}^m u_j^\dagger \varepsilon_j^\dagger \in \mathfrak{h}^{*\dagger}$ with $u_j^\dagger=\hat{u}_j^\dagger-(1/m)|\hat{u}^\dagger|$.

We denote by σ the isomorphism of $P^\dagger \bmod \mathbf{C}\delta_0+\cdots+\mathbf{C}\delta_{m-1}$ such that

$$\sigma(\Lambda_0)=\Lambda_1,\quad \sigma(\Lambda_1)=\Lambda_2,\ \cdots,\ \sigma(\Lambda_{m-1})=\Lambda_0.$$

For $Y^\dagger\subset R_{m,l+1}$ we define

$$(2.12)\qquad \hat{\chi}_{Y^\dagger}^{(m)}(\tau, \hat{u}^\dagger)=\frac{1}{\eta(\tau)}\sum_{q=0}^{m-1}(-)^{(l+1)q}\vartheta_{(l+1)q-((l+1)m/2)+|Y^\dagger|,(l+1)m}\left(\tau, \frac{|\hat{u}^\dagger|}{m}\right) \times\chi_{\sigma^q(\Lambda^\dagger)}^{(m-1)}(\tau, u^\dagger).$$

Let $A^\dagger=(a_{ij}^\dagger)_{i,j=1,\cdots,m}$ be an $m\times m$ matrix with $a_{ij}^\dagger=N\delta_{ij}-1$. The inverse $A^{\dagger-1}=(\check{a}_{ij}^\dagger)_{i,j,=1,\cdots,m}$ is given by

$$\check{a}_{ij}^\dagger=\frac{1}{N}\left(\delta_{ij}+\frac{1}{l+1}\right).$$

We define an m vector $\mu=(\mu_j)_{j=1,\cdots,m}$ by

$$\mu_j=\frac{1}{N}\left(\lambda_j^\dagger+\frac{1}{l+1}\sum_{j'=1}^m \lambda_{j'}^\dagger\right).$$

The following identity will be shown in the proof of the following proposition.

$$(2.13)\qquad \begin{aligned}&\eta(\tau)^{m(3-m)/2}\prod_{1\le i<j\le m}\theta(\tau, \hat{u}_i^\dagger-\hat{u}_j^\dagger)\hat{\chi}_{Y^\dagger}^{(m)}(\tau, \hat{u}^\dagger)\\ &=\sum_{w^\dagger\in\mathfrak{S}_m}\det w^\dagger \sum_{\substack{n_j\in\mathbf{Z}\\ r_j=n_j+\mu_{w^\dagger(j)}}}(-)^{(l+1)m\sum_{j=1}^m n_j}\\ &\qquad\times\mathbf{e}\left[\frac{\tau}{2}\sum_{i,j=1}^m a_{ij}^\dagger r_i r_j+\sum_{i,j=1}^m a_{ij}^\dagger r_i \hat{u}_j^\dagger\right].\end{aligned}$$

Proposition 2.5. *The notations are as above. We set*

$$u=\sum_{i=1}^{l+1} u_i\varepsilon_i \quad \text{and} \quad \hat{u}^\dagger=\sum_{j=1}^{m} \hat{u}_j^\dagger\varepsilon_j^\dagger.$$

We assume that $\sum_{i=1}^{l+1} u_i=0$. *Then we have*

$$\eta(\tau)^{-(l+1)m}\sum_{\substack{1\le i\le l+1\\ 1\le j\le m}} \theta(\tau, u_i-\hat{u}_j^\dagger)=\sum_{\mathscr{P}_0}(-)^{|Y^\dagger|}\chi_Y^{(l)}(\tau, u)\hat{\chi}_{Y^\dagger}^{(m)}(\tau, \hat{u}^\dagger). \tag{2.14}$$

Proof. The denominator formula for $A_{l+m}^{(1)}$ implies

$$\begin{aligned}
&\eta^{(l+m)(1-l-m)/2}\prod_{1\le i<j\le l+1}\theta(\tau, u_i-u_j)\\
&\qquad\times\prod_{1\le i<j\le m}\theta(\tau, \hat{u}_i^\dagger-\hat{u}_j^\dagger)\prod_{\substack{1\le i\le l+1\\ 1\le j\le m}}\theta(\tau, u_i-\hat{u}_j^\dagger)\\
&=\sum_{(w^\#, r)\in\mathscr{P}\times\mathbf{Z}}\det w^\#\\
&\quad\times\sum_{w\in\mathfrak{S}_{l+1}}\det w\sum_{l_i\in\mathbf{Z}, \sum_{i=1}^{l+1} l_i=r}\mathbf{e}\left[\frac{\tau}{2N}\sum_{i=1}^{l+1}(Nl_i+\lambda_{w(i)})^2\right.\\
&\qquad\qquad\left.+\sum_{i=1}^{l+1}(Nl_i+\lambda_{w(i)})u_i\right]\\
&\quad\times\sum_{w^\dagger\in\mathfrak{S}_m}\det w^\dagger\sum_{m_j\in\mathbf{Z}, \sum_{j=1}^{m} m_j=-r}\mathbf{e}\left[\frac{\tau}{2N}\sum_{j=1}^{m}(Nm_j+\lambda_{w^\dagger(j)}^\dagger)^2\right.\\
&\qquad\qquad\left.+\sum_{j=1}^{m}(Nm_j+\lambda_{w^\dagger(j)}^\dagger)\hat{u}_j^\dagger\right].
\end{aligned}\tag{2.15}$$

We set

$$\nu=Nr+\lambda_1+\cdots+\lambda_{l+1}=Nr-\lambda_1^\dagger-\cdots-\lambda_m^\dagger,$$

$$\lambda_i^\circ=\lambda_i-\frac{\nu}{l+1}, \quad (i=1, \cdots, l+1),$$

$$\lambda_j^{\dagger\circ}=\lambda_j^\dagger-\frac{\nu}{l+1}, \quad (i=1, \cdots, m).$$

Then we have

$$\begin{aligned}
&\sum_{i=1}^{l+1}(Nl_i+\lambda_{w(i)}^\circ)=0,\\
&\sum_{i=1}^{l+1}(Nl_i+\lambda_{w(i)})^2=\sum_{i=1}^{l+1}(Nl_i+\lambda_{w(i)}^\circ)^2+\frac{\nu^2}{l+1},\\
&\sum_{i=1}^{l+1}(Nl_i+\lambda_{w(i)})u_i=\sum_{i=1}^{l+1}(Nl_i+\lambda_{w(i)}^\circ)u_i.
\end{aligned}\tag{2.16}$$

Hence the right hand side of (2.15) can be rewritten as

$$\begin{aligned}(2.17)\quad &\sum_{(w^\sharp, r)\in\mathscr{P}\times\mathbf{Z}} \det w^\sharp \times \sum_{w\in\mathfrak{S}_{l+1}} \det w \sum_{l_i\in\mathbf{Z}, \Sigma_{i=1}^{l+1} l_i = r} \mathbf{e}\Big[\frac{\tau}{2N}\sum_{i=1}^{l+1}(Nl_i+\lambda^\circ_{w(i)})^2 \\ &\qquad + \sum_{i=1}^{l+1}(Nl_i+\lambda^\circ_{w(i)})u_i\Big] \\ &\times \sum_{w^\dagger\in\mathfrak{S}_m} \det w^\dagger \sum_{m_j\in\mathbf{Z}, \Sigma_{j=1}^m m_j=-r} \mathbf{e}\Big[\frac{\tau}{2}\sum_{i,j=1}^m \check{a}^\dagger_{ij}\Big(m_i+\frac{\lambda^{\dagger\circ}_{w^\dagger(i)}}{N}\Big) \\ &\qquad \times\Big(m_j+\frac{\lambda^{\dagger\circ}_{w^\dagger(j)}}{N}\Big) + \sum_{i,j=1}^m \check{a}^\dagger_{ij}\Big(m_i+\frac{\lambda^{\dagger\circ}_{w^\dagger(i)}}{N}\Big)\hat{u}^\dagger_j\Big].\end{aligned}$$

We shall define a bijective mapping $\kappa\colon \mathscr{P}\times\mathbf{Z}\to\mathscr{P}\times\mathbf{Z}$ so that the following is bijective.

$$(2.18)\qquad \begin{array}{ccc} \mathscr{P}_0\times\mathbf{Z} & \longrightarrow & \mathscr{P}\times\mathbf{Z} \\ \cup & & \cup \\ (w^\sharp, s) & \longmapsto & \kappa^s(w^\sharp, 0) \end{array}$$

Note that $w^\sharp\in\mathscr{P}$ is in one to one correspondence with such a partition

$$\{\lambda_i\}_{i=1,\dots,l+1}\cup\{\lambda^\dagger_j\}_{j=1\dots,m}=\Big\{\frac{N-1}{2}, \frac{N-3}{2}, \cdots, \frac{1-N}{2}\Big\}$$

that satisfies $\lambda_1>\cdots>\lambda_{l+1}$ and $\lambda^\dagger_1>\cdots>\lambda^\dagger_m$. We define r', $\{\lambda'_i\}_{i=1,\dots,l+1}$ and $\{\lambda^{\dagger\prime}_j\}_{j=1,\dots,m}$ corresponding to $\kappa(w^\sharp, r)$ as follows. If $w^\sharp(1)=1$ then we set

$$\begin{aligned} &r'=r+1, \\ &\lambda'_i=\begin{cases}\lambda_{i+1}+1 & (i=1,\cdots,l) \\ \lambda_1+1-N & (i=l+1),\end{cases} \\ &\lambda^{\dagger\prime}_j=\lambda^\dagger_j+1 \qquad (j=1,\cdots,m).\end{aligned}$$

If $w^\sharp(1)\neq 1$ then we set

$$\begin{aligned} &r'=r, \\ &\lambda'_i=\lambda_i+1 \qquad (i=1,\cdots,l+1), \\ &\lambda^{\dagger\prime}_j=\begin{cases}\lambda^\dagger_{j+1}+1 & (j=1,\cdots,m-1) \\ \lambda^\dagger_1+1-N & (j=m).\end{cases}\end{aligned}$$

Then ν, $\{\lambda^\circ_i\}_{i=1,\dots,l+1}$ and $\{\lambda^{\dagger\circ}_j\}_{j=1,\dots,m}$ will change as follows.

$$\nu'=\nu+1,$$

Case $w^{\#}(1)=1$.

$$\lambda_i^{\circ\prime}=\begin{cases}\lambda_{i+1}^{\circ} & (i=1,\cdots,l)\\ \lambda_1^{\circ}-N & (i=l+1),\end{cases}$$
$$\lambda_j^{\dagger\circ\prime}=\lambda_j^{\dagger\circ} \qquad (j=1,\cdots,m).$$

Case $w^{\#}(1)\neq 1$.

$$\lambda_i^{\circ\prime}=\lambda_i^{\circ} \qquad (i=1,\cdots,l+1),$$
$$\lambda_j^{\dagger\circ\prime}=\begin{cases}\lambda_{j+1}^{\dagger\circ} & (j=1,\cdots,m-1)\\ \lambda_1^{\dagger\circ}-N & (j=m).\end{cases}$$

In particular

$$-r'+\frac{1}{N}\sum_{j=1}^{m}\lambda_j^{\dagger\circ\prime}=-r+\frac{1}{N}\sum_{j=1}^{m}\lambda_j^{\dagger\circ}-1.$$

Namely, in the correspondence of (2.18) we have

$$s=r-\frac{1}{N}\sum_{j=1}^{m}\lambda_j^{\dagger\circ}.$$

The above consideration enables us to rewrite (2.17) so that the sum over $\mathscr{P}\times\mathbf{Z}$ reads as the sum over $\mathscr{P}_0\times\mathbf{Z}$, which gives rise to the right hand side of (2.14) in the form of (2.13) (up to a trivial factor).

To show (2.13) we introduce

$$\hat{\lambda}_j^{\dagger}=\lambda_j^{\dagger}+\frac{\nu}{m}.$$

The right hand side of (2.15) can be rewritten as

$$\begin{aligned}&\sum_{(w^{\#},r)\in\mathscr{P}\times\mathbf{Z}}\det w^{\#}\sum_{w\in\mathfrak{S}_{l+1}}\det w^{\dagger}\sum_{l_i\in\mathbf{Z},\sum_{i=1}^{l+1}l_i=r}\mathbf{e}\Big[\frac{\tau}{2N}\sum_{i=1}^{l+1}(Nl_i+\lambda_{w(i)}^{\circ})^2\\&\qquad\qquad+\sum_{i=1}^{l+1}(Nl_i+\lambda_{w(i)}^{\circ})u_i\Big]\\ &\times\sum_{w^{\dagger}\in\mathfrak{S}_m}\det w^{\dagger}\sum_{m_j\in\mathbf{Z},\sum_{j=1}^{m}m_j=-r}\mathbf{e}\Big[\frac{\tau}{2N}\sum_{j=1}^{m}(Nm_j+\hat{\lambda}_{w^{\dagger}(j)}^{\dagger})^2\\&\qquad\qquad+\sum_{j=1}^{m}(Nm_j+\hat{\lambda}_{w^{\dagger}(j)}^{\dagger})\hat{u}_j^{\dagger}\Big]\,\mathbf{e}\Big[\frac{\tau}{2(l+1)m}\nu^2-\frac{\nu}{m}|\hat{u}^{\dagger}|\Big].\end{aligned}\tag{2.19}$$

The change of $\hat{\lambda}_j^{\dagger}$ caused by κ is as follows.

Case $w^{\#}(1)=1$.

$$\hat{\lambda}_j^{\dagger\prime}=\hat{\lambda}_j^{\dagger}+\frac{N}{m}\quad(j=1,\cdots,m).$$

Case $w^{\#}(1)\neq 1$.

$$\hat{\lambda}_j^{\dagger\prime}=\begin{cases}\hat{\lambda}_{j+1}^{\dagger}+\dfrac{N}{m} & (j=1,\cdots,m-1)\\[2mm] \hat{\lambda}_1^{\dagger}+\dfrac{N}{m}-N & (j=m).\end{cases}$$

Let $\Lambda^{\dagger}$ be a weight in $\mathfrak{h}^{*\dagger}$ of level $l+1$ such that

$$\overline{\Lambda^{\dagger}+\rho^{\dagger}}=\sum_{j=1}^{m}\hat{\lambda}_j^{\dagger}\varepsilon_j^{\dagger}.$$

Then we have

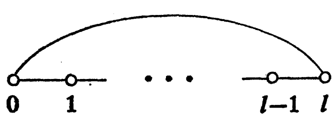
$$\overline{\sigma^{-1}(\Lambda^{\dagger})+\rho^{\dagger}}=\sum_{j=1}^{m-1}\left(\lambda_{j+1}^{\dagger}+\frac{N}{m}\right)\varepsilon_j^{\dagger}+\left(\hat{\lambda}_1^{\dagger}+\frac{N}{m}-N\right)\varepsilon_m^{\dagger}.$$

Hence, rewriting (2.19) so that the sum over $\mathscr{P}\times\mathbf{Z}$ reads as the sum over $\mathscr{P}_0\times\mathbf{Z}$ again, we obtain (2.14) with $\chi_{Y^{\dagger}}^{(m)}(\tau,\hat{u}^{\dagger})$ given in the form (2.12). □

Table II

(1) $A_l^{(1)}$

1°



2° $\delta=\alpha_0+\cdots+\alpha_l,\quad c=h_0+\cdots+h_l,\quad g=l+1$

3° $(\varepsilon_i,\varepsilon_j)=\delta_{ij}\quad(i,j=1,\cdots,l+1)$,

$$\alpha_i=h_i=\begin{cases}\delta-\varepsilon_1+\varepsilon_{l+1} & (i=0)\\ \varepsilon_i-\varepsilon_{i+1} & (i=1,\cdots,l)\end{cases}$$

$$\bar{\Lambda}_i=\varepsilon_1+\cdots+\varepsilon_i-\frac{i}{l+1}(\varepsilon_1+\cdots+\varepsilon_{l+1})\quad(i=0,\cdots,l)$$

$$\bar{\rho}=\frac{l}{2}\varepsilon_1+\left(\frac{l}{2}-1\right)\varepsilon_2+\cdots-\frac{l}{2}\varepsilon_{l+1}$$

4° $\bar{W}\cong\mathfrak{S}_{l+1}$,

$$M=\left\{\sum_{i=1}^{l+1}\nu_i\varepsilon_i\,\middle|\,\nu_i\in\mathbf{Z},\ \sum_{i=1}^{l+1}\nu_i=0\right\}.$$

5° $\overline{\Lambda+\rho}=\lambda_1\varepsilon_1+\cdots+\lambda_{l+1}\varepsilon_{l+1}$,

$\lambda_1>\cdots>\lambda_{l+1},\quad \lambda_i-\lambda_j\in\mathbf{Z},\quad \sum_{i=1}^{l+1}\lambda_i=0,\quad l+m\geq\lambda_1-\lambda_{l+1}.$

The following isomorphism of $\mathfrak{h}^*$ is denoted by σ.

$\sigma(\Lambda_0)=\Lambda_1,\quad \sigma(\Lambda_1)=\Lambda_2,\ \cdots,\ \sigma(\Lambda_l)=\Lambda_0,$

$\sigma(\delta_0)=\delta_1,\quad \sigma(\delta_1)=\delta_2,\ \cdots,\ \sigma(\delta_l)=\delta_0.$

6° $A_\rho(\tau,u)=\eta(\tau)^{(1/2)l(1-l)}\prod_{1\leq i<j\leq l+1}\theta(\tau,u_i-u_j)$

where $u=\sum_{i=1}^{l+1}u_i\varepsilon_i$.

7° $$\eta(\tau)^{1-(l+1)m}\prod_{\substack{1\leq i\leq l+1\\ 1\leq j\leq m}}\theta(\tau,u_i-u_j^\dagger)\chi_{\tilde\Lambda}^{(l+m)}(\tau,\iota(u)+\iota^\dagger(u^\dagger))$$
$$=\sum_{s\in\mathbf{Z}}\ \sum_{\tilde w\in\mathfrak{S}_{l+m+1}}{}'\ \det\tilde w(-)^{(l+m-1)s}$$
$$\times\mathbf{e}\left[\frac{\tau N}{2}\left(\frac{1}{l+1}+\frac{1}{m}\right)\left(s+\frac{\tilde\lambda}{N}\right)^2+N\left(\frac{1}{l+1}\sum_{j=1}^{l+1}u_j-\frac{1}{m}\sum_{j=1}^{m}u_j^\dagger\right)\right.$$
$$\left.\times\left(s+\frac{\tilde\lambda}{N}\right)\right]\chi^{(l)}_{\sigma^s(\Lambda)}(\tau,u)\chi^{(m)}_{\sigma^{\dagger-s}(\Lambda^\dagger)}(\tau,u^\dagger).$$

The sum $\sum'_{\tilde w\in\mathfrak{S}_{l+m+1}}$ extends over such $\tilde w\in\mathfrak{S}_{l+m+1}$ that

$$\tilde w(1)<\cdots<\tilde w(l+1)\quad\text{and}\quad\tilde w(l+2)<\cdots<\tilde w(l+m+1),$$

and $\iota(\varepsilon_i)=\tilde\varepsilon_i$ $(i=1,\cdots,l+1)$, $\iota^\dagger(\varepsilon_j^\dagger)=\tilde\varepsilon_{j+l+1}$ $(j=1,\cdots,m)$. Define $\tilde\lambda_i$ $(i=1,\cdots,l+m+1)$ so that $\overline{\tilde\Lambda+\tilde\rho}=\sum_{i=1}^{l+m+1}\tilde\lambda_i\tilde\varepsilon_i$. The data N, $\tilde\lambda$, Λ and $\Lambda^\dagger$ are determined by $\tilde w$ and $\tilde\Lambda$ as follows.

$$N=l+m+(\tilde\Lambda,\tilde\delta)+1,$$
$$\tilde\lambda=\sum_{i=1}^{l+1}\tilde\lambda_{\tilde w(i)}=-\sum_{j=1}^{m}\tilde\lambda_{\tilde w(l+1+j)},$$
$$\Lambda=(m+(\tilde\Lambda,\tilde\delta))\Lambda_0+\sum_{i=1}^{l+1}\left(\tilde\lambda_{\tilde w(i)}-\frac{\tilde\lambda}{l+1}\right)\varepsilon_i-\bar\rho,$$
$$\Lambda^\dagger=(l+(\tilde\Lambda,\tilde\delta)+1)\Lambda_0+\sum_{j=1}^{m}\left(\tilde\lambda_{\tilde w(l+1+j)}+\frac{\tilde\lambda}{m}\right)\varepsilon_j^\dagger-\bar\rho^\dagger.$$

(2) $C_l^{(1)}$

1° 

2° $\delta=\alpha_0+2\alpha_1+\cdots+2\alpha_{l-1}+\alpha_l,\quad c=h_0+\cdots+h_l,\quad g=l+1$

3° $(\varepsilon_i,\varepsilon_j)=\frac{1}{2}\delta_{ij}\quad(i,j=1,\cdots,l),$

$$\alpha_0 = h_0 = \delta - 2\varepsilon_1,$$
$$\alpha_i = h_i/2 = \varepsilon_i - \varepsilon_{i+1} \quad (i=1, \cdots, l-1)$$
$$\alpha_l = h_l = 2\varepsilon_l.$$
$$\bar{\Lambda}_i = \varepsilon_1 + \cdots + \varepsilon_i \quad (i=0, \cdots, l)$$
$$\bar{\rho} = l\varepsilon_1 + \cdots + \varepsilon_l.$$

4° $\bar{W} \cong \mathfrak{S}_l \ltimes \{\pm 1\}^l,$
$$M = \left\{ \sum_{i=1}^{l} \nu_i \varepsilon_i \,|\, \nu_i \in 2\mathbf{Z} \right\}$$

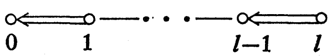
5° $\overline{\Lambda+\rho} = \lambda_1\varepsilon_1 + \cdots + \lambda_l\varepsilon_l,$
$$l+m \geq \lambda_1 > \cdots > \lambda_l \geq 1, \quad \forall \lambda_i \in \mathbf{Z}.$$

6° $A_\rho(\tau, u) = \eta(\tau)^{l(1-l)} \prod_{i=1}^{l} \theta(\tau, 2u_j) \prod_{1\leq i<j\leq l} \theta(\tau, u_i+u_i)\theta(\tau, u_i-u_j),$

where $u = \sum_{i=1}^{l} 2u_i\varepsilon_i.$

7°† $\eta(\tau)^{-2lm} \sum_{\substack{1\leq i\leq l \\ 1\leq j\leq m}} \theta(\tau, u_i+u_j^\dagger)\theta(\tau, u_i-u_j^\dagger)\chi_{\Lambda}^{(l+m)}(\tau, \iota(u)+\iota^\dagger(u^\dagger))$
$$= \sum_{\tilde{w}\in\mathfrak{S}_{l+m}}{}' \det \tilde{w}\chi_{\Lambda}^{(l)}(\tau, u)\chi_{\Lambda^\dagger}^{(m)}(\tau, u^\dagger).$$

(3) $A_{2l}^{(2)}$

1° 

2° $\delta = 2\alpha_0 + \cdots + 2\alpha_{l-1} + \alpha_l, \quad c = h_0 + 2h_1 + \cdots 2h_l, \quad g = 2l+1$

3° $(\varepsilon_i, \varepsilon_j) = \delta_{ij} \quad (i, j=1, \cdots, l),$
$$\alpha_0 = h_0/2 = \delta/2 - \varepsilon_i,$$
$$\alpha_i = h_i = \varepsilon_i - \varepsilon_{i+1} \quad (i=1, \cdots, l-1),$$
$$\alpha_l = 2h_l = 2\varepsilon_l.$$
$$\bar{\Lambda}_i = \varepsilon_1 + \cdots + \varepsilon_i \quad (i=0, \cdots, l),$$
$$\bar{\rho} = l\varepsilon_1 + \cdots + \varepsilon_l.$$

4° $\bar{W} \cong \mathfrak{S}_l \ltimes \{\pm 1\}^l,$
$$M = \left\{ \sum_{i=1}^{l} \nu_i \varepsilon_i \,|\, \nu_i \in \mathbf{Z} \right\}.$$

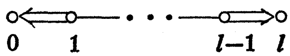
5° $\overline{\Lambda+\rho} = \lambda_1\varepsilon_1 + \cdots + \lambda_l\varepsilon_l,$
$$l + \left[\frac{m}{2}\right] \geq \lambda_1 > \cdots > \lambda_l \geq 1, \quad \forall \lambda_i \in \mathbf{Z}.$$

6° $A_\rho(\tau, u) = \eta(\tau)^{l(1-l)}\eta(2\tau)^{-l} \prod_{1\leq i<j\leq l} \theta(\tau, u_i+u_j)\theta(\tau, u_i-u_j)$
$$\times \prod_{1\leq i\leq l} \theta(2\tau, 2u_i)\hat{\theta}(\tau, u_i),$$

where $u=\sum_{i=1}^{l} u_i\varepsilon_i$.

$7^{\circ\dagger)}$ $$\eta(\tau)^{-2lm}\prod_{\substack{1\le i\le l\\1\le j\le m}}\theta(\tau,u_i+u_j^\dagger)\theta(\tau,u_i-u_j^\dagger)\chi_{\Lambda}^{(l+m)}(\tau,\iota(u)+\iota^\dagger(u^\dagger))$$
$$=\sum_{\tilde{w}\in\mathfrak{S}_{l+m}}{}' \det\tilde{w}\chi_{\Lambda}^{(l)}(\tau,u)\chi_{\Lambda^\dagger}^{(m)}(\tau,u^\dagger).$$

(4) $D_{l+1}^{(2)}$

1° 

2° $\delta=\alpha_0+\cdots+\alpha_l,\quad c=h_0+2h_1+\cdots+2h_{l-1}+h_l,\quad g=2l,$

3° $(\varepsilon_i,\varepsilon_j)=2\delta_{ij}\quad (i,j=1,\cdots,l),$

$\alpha_0=h_0=\delta-\varepsilon_1,$

$\alpha_i=2h_i=\varepsilon_i-\varepsilon_{i+1}\quad (i=1,\cdots,l-1),$

$\alpha_l=h_l=\varepsilon_l.$

$\bar{\Lambda}_i=\varepsilon_1+\cdots+\varepsilon_i\quad (i=1,\cdots,l-1),$

$2\bar{\Lambda}_l=\varepsilon_1+\cdots+\varepsilon_l,$

$\bar{\rho}=(l-\frac{1}{2})\varepsilon_1+\cdots+\frac{1}{2}\varepsilon_l$

4° $\bar{W}\cong\mathfrak{S}_l\ltimes\{\pm 1\}^l,$

$M=\left\{\sum_{i=1}^{l}\nu_i\varepsilon_i \,|\, \nu_i\in\mathbf{Z}\right\}$

5° $\overline{\Lambda+\rho}=\lambda_1\varepsilon_1+\cdots+\lambda_l\varepsilon_l,$

$2l+m-1\ge 2\lambda_1>\cdots>2\lambda_l\ge 1,$

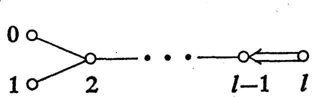
$\forall\lambda_i\in\mathbf{Z}$ or $\forall\lambda_i\in\mathbf{Z}+\frac{1}{2}.$

6° $$A_\rho(\tau,u)=\eta(\tau)^{1-l}\eta(2\tau)^{-(1-l)^2}\prod_{1\le i<j\le l}\theta(2\tau,u_i+u_j)\theta(2\tau,u_i-u_j)$$
$$\times\prod_{1\le i\le l}\theta(\tau,u_i),$$

where $u=\sum_{i=1}^{l}\frac{u_i}{2}\cdot\varepsilon_i$.

$7^{\circ\dagger)}$ $$\eta(\tau)^{-1}\eta(2\tau)^{1-2lm}\prod_{\substack{1\le i\le l\\1\le j\le m}}\theta(2\tau,u_i+u_j^\dagger)\theta(2\tau,u_i-u_j^\dagger)$$
$$\times\chi_{\Lambda}^{(l+m)}(\tau,\iota(u)+\iota^\dagger(u^\dagger))=\sum_{\tilde{w}\in\mathfrak{S}_{l+m}}{}'\det\tilde{w}\chi_{\Lambda}^{(l)}(\tau,u)\chi_{\Lambda^\dagger}^{(m)}(\tau,u^\dagger).$$

(5) $A_{2l-1}^{(2)}$

1° 

2° $\delta=\alpha_0+\alpha_1+2\alpha_2+\cdots+2\alpha_{l-1}+\alpha_l.$

$c=h_0+h_1+2h_2+\cdots+2h_l,\quad g=2l.$

3° $(\varepsilon_i,\varepsilon_j)=\delta_{ij}\quad (i,j=1,\cdots,l),$

$\alpha_0=h_0=\delta-\varepsilon_1-\varepsilon_2,$

$\alpha_i=h_i=\varepsilon_i-\varepsilon_{i+1}\quad (i=1,\cdots,l-1),$

$\alpha_l=2h_l=2\varepsilon_l.$

$\bar{\Lambda}_i=\varepsilon_1+\cdots+\varepsilon_i\quad (i=1,\cdots,l).$

$\bar{\rho}=l\varepsilon_1+\cdots+\varepsilon_l.$

4° $\bar{W}\cong\mathfrak{S}_l\ltimes\{\pm1\}^l,$

$$M=\left\{\sum_{i=1}^{l}\nu_i\varepsilon_i \,\middle|\, \sum_{i=1}^{l}\nu_i\in 2\mathbf{Z},\ \nu_i\in\mathbf{Z}\right\}$$

5°[††] $\overline{\Lambda+\rho}=\lambda_1\varepsilon_1+\cdots+\lambda_l\varepsilon_l,$

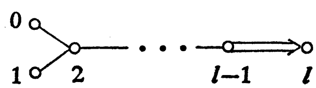
$2l+m-1\geq\lambda_1+\lambda_2,\quad \lambda_1>\cdots>\lambda_l\geq 1,\quad \forall\lambda_i\in\mathbf{Z}.$

6° $$A_\rho(\tau,u)=\eta(\tau)^{-(1-l)^2}\eta(2\tau)^{1-l}\times\prod_{1\leq i<j\leq l}\theta(\tau,u_i+u_j)\theta(\tau,u_i-u_j)\prod_{1\leq i\leq l}\theta(2\tau,2u_i),$$

where $u=\sum_{i=1}^{l}u_i\varepsilon_i.$

7°[†] $$\eta(\tau)^{1-2lm}\eta(2\tau)^{-1}\prod_{\substack{1\leq i\leq l\\1\leq j\leq m}}\theta(\tau,u_i+u_j^\dagger)\theta(\tau,u_i-u_j^\dagger)\chi_{\Lambda}^{(l+m)}(\tau,\iota(u)+\iota^\dagger(u^\dagger))$$
$$=\sum_{\tilde{w}\in\mathfrak{S}_{l+m}}{}'\det\tilde{w}\sum_{(\sigma,\sigma^\dagger)=(\mathrm{id},\mathrm{id}),(\sigma_1,\sigma_1^\dagger)}\chi_{\sigma(\Lambda)}^{(l)}(\tau,u)\chi_{\sigma^\dagger(\Lambda^\dagger)}^{(m)}(\tau,u^\dagger).$$

(6) $B_l^{(1)}$

1° 

2° $\delta=\alpha_0+\alpha_1+2\alpha_2+\cdots+2\alpha_l,$

$c=h_0+h_1+2h_2+\cdots+2h_{l-1}+h_l,\quad g=2l-1.$

3° $(\varepsilon_i,\varepsilon_j)=\delta_{ij}\quad (i,j=1,\cdots,l).$

$\alpha_0=h_0=\delta-\varepsilon_1-\varepsilon_2,$

$\alpha_i=h_i=\varepsilon_i-\varepsilon_{i+1}\quad (i=1,\cdots,l-1),$

$\alpha_l=\dfrac{h_l}{2}=\varepsilon_l.$

$\bar{\Lambda}_i=\varepsilon_1+\cdots+\varepsilon_i\quad (i=1,\cdots,l-1),$

$\bar{\Lambda}_l=\frac{1}{2}(\varepsilon_1+\cdots+\varepsilon_l),$

$\bar{\rho}=(l-\frac{1}{2})\varepsilon_1+\cdots+\frac{1}{2}\varepsilon_l.$

4° $\overline{W} \cong \mathfrak{S}_l \ltimes \{\pm 1\}^l$,

$$M = \left\{ \sum_{i=1}^{l} \nu_i \varepsilon_i \,\middle|\, \sum_{i=1}^{l} \nu_i \in 2\mathbf{Z},\ \forall \nu_i \in \mathbf{Z} \right\}.$$

5°†† $\overline{\Lambda+\rho} = \lambda_1 \varepsilon_1 + \cdots + \lambda_l \varepsilon_l$,

$2l+m-2 \geq \lambda_1 + \lambda_2, \quad \lambda_1 > \cdots > \lambda_l \geq \frac{1}{2}$,

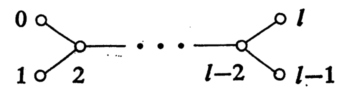
$\forall \lambda_i \in \mathbf{Z}$ or $\forall \lambda_i \in \mathbf{Z} + \frac{1}{2}$.

6° $$A_\rho(\tau, u) = \eta(\tau)^{l(1-l)} \prod_{1 \leq i < j \leq l} \theta(\tau, u_i + u_j)\theta(\tau, u_i - u_j) \prod_{1 \leq j \leq l} \theta(\tau, u_j),$$

where $u = \sum_{i=1}^{l} u_i \varepsilon_i$.

7° $$\eta(\tau)^{-2lm} \prod_{\substack{1 \leq i \leq l \\ 1 \leq j \leq m}} \theta(\tau, u_i + u_j^\dagger)\theta(\tau, u_i - u_j^\dagger) \chi_\Lambda^{(l+m)}(\tau, \iota(u) + \iota^\dagger(u^\dagger))$$
$$= \sum_{\tilde{w} \in \mathfrak{S}_{l+m}}{}' \det \tilde{w} \sum_{(\sigma, \sigma^\dagger) = (\mathrm{id}, \mathrm{id}), (\sigma_1, \sigma_1^\dagger)} \chi_{\sigma(\Lambda)}^{(l)}(\tau, u) \chi_{\sigma^\dagger(\Lambda^\dagger)}^{(m)}(\tau, u^\dagger).$$

(7) $D_l^{(1)}$

1° 

2° $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \cdots + 2\alpha_{l-2} + \alpha_{l-1} + \alpha_l$,

$c = h_0 + h_1 + 2h_2 + \cdots + 2h_{l-2} + h_{l-1} + h_l, \quad g = 2l-2$.

3° $(\varepsilon_i, \varepsilon_j) = \delta_{ij} \quad (i, j = 1, \cdots, l)$.

$\alpha_0 = h_0 = \delta - \varepsilon_1 - \varepsilon_2$,

$\alpha_i = h_i = \varepsilon_i - \varepsilon_{i+1} \quad (i = 1, \cdots, l-1)$,

$\alpha_l = h_l = \varepsilon_{l-1} + \varepsilon_l$.

$\overline{\Lambda}_i = \varepsilon_1 + \cdots + \varepsilon_i \quad (i = 1, \cdots, l-2)$,

$\overline{\Lambda}_{l-1} = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-1} - \varepsilon_l)$,

$\overline{\Lambda}_l = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{l-1} + \varepsilon_l)$,

$\overline{\rho} = (l-1)\varepsilon_1 + \cdots + \varepsilon_{l-1}$.

4° . $\overline{W} \cong \mathfrak{S}_l \ltimes \{\pm 1\}^{l-1}$,

$$M = \left\{ \sum_{i=1}^{l} \nu_i \varepsilon_i \,\middle|\, \sum_{i=1}^{l} \nu_i \in 2\mathbf{Z},\ \forall \nu_i \in \mathbf{Z} \right\}$$

5°†† $\overline{\Lambda+\rho} = \lambda_1 \varepsilon_1 + \cdots + \lambda_l \varepsilon_l$,

$2l+m-3 \geq \lambda_1 + \lambda_2, \quad \lambda_1 > \cdots > \lambda_l, \quad \lambda_{l-1} + \lambda_l \geq 1$,

$\forall \lambda_i \in \mathbf{Z}$ or $\forall \lambda_i \in \mathbf{Z} + \frac{1}{2}$.

The following isomorphism of $\mathfrak{h}^*$ is denoted by σ_2.

$\sigma_2(\Lambda_i) = \Lambda_i, \quad i = 1, \cdots, l-2, \quad \sigma_2(\Lambda_{l-1}) = \Lambda_l, \quad \sigma_2(\Lambda_l) = \Lambda_{l-1}$,

$$\sigma_2(\delta_i)=\delta_i,\quad i=1,\cdots,l-2,\quad \sigma_2(\delta_{l-1})=\delta_l,\quad \sigma_2(\delta_l)=\delta_{l-1}.$$

6° $$A_\rho(\tau,u)=\eta(\tau)^{2l-l^2}\prod_{1\le i<j\le l}\theta(\tau,u_i+u_j)\theta(\tau,u_i-u_j),$$

where $u=\sum_{i=1}^{l}u_i\varepsilon_i$.

7°†) $$\eta(\tau)^{-2lm}\prod_{\substack{1\le i\le l\\ 1\le j\le m}}\theta(\tau,u_i+u_j^\dagger)\theta(\tau,u_i-u_j^\dagger)\chi_{\tilde{\Lambda}}^{(l+m)}(\tau,\iota(u)+\iota^\dagger(u^\dagger))$$
$$=\sum_{\tilde{w}\in\mathfrak{S}_{l+m}}{}' \det\tilde{w}$$
$$\times\sum_{(\sigma,\sigma\dagger)=(\mathrm{id},\mathrm{id}),(\sigma_1,\sigma_1\dagger),(\sigma_2,\sigma_2\dagger),(\sigma_1\sigma_2,\sigma_1\dagger\sigma_2\dagger)}\chi_{\sigma(\Lambda)}^{(l)}(\tau,u)\chi_{\sigma\dagger(\Lambda\dagger)}^{(m)}(\tau,u^\dagger).$$

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†) The sum $\sum'_{\tilde{w}\in\mathfrak{S}_{l+m}}$ extends over such $\tilde{w}\in\mathfrak{S}_{l+m}$ that $\tilde{w}(1)<\cdots<\tilde{w}(l)$ and $\tilde{w}(l+1)<\cdots<\tilde{w}(l+m)$, and $\iota(\varepsilon_i)=\tilde{\varepsilon}_i$ $(i=1,\cdots,l)$ and $\iota^\dagger(\varepsilon_i^\dagger)=\tilde{\varepsilon}_{l+i}$ $(i=1,\cdots,m)$. Define $\tilde{\lambda}_i$ $(i=1,\cdots,l+m)$ so that $\overline{\tilde{\Lambda}+\tilde{\rho}}=\sum_{i=1}\tilde{\lambda}_i\tilde{\varepsilon}_i$. The data Λ and $\Lambda^\dagger$ are determined by $\tilde{w}$ and $\tilde{\Lambda}$ as follows.

In (2)
$$\Lambda=(m+(\tilde{\Lambda},\tilde{\delta}))\Lambda_0+\sum_{i=1}^{l}\tilde{\lambda}_{\tilde{w}(i)}\varepsilon_i-\bar{\rho},$$
$$\Lambda^\dagger=(l+(\tilde{\Lambda},\tilde{\delta}))\Lambda_0+\sum_{i=1}^{m}\tilde{\lambda}_{\tilde{w}(l+1+i)}\varepsilon_i^\dagger-\bar{\rho}^\dagger.$$

In (3)–(7)
$$\Lambda=(2m+(\tilde{\Lambda},\tilde{\delta}))\Lambda_0+\sum_{i=1}^{l}\tilde{\lambda}_{\tilde{w}(i)}\varepsilon_i-\bar{\rho},$$
$$\Lambda^\dagger=(2l+(\tilde{\Lambda},\tilde{\delta}))\Lambda_0+\sum_{i=1}^{m}\tilde{\lambda}_{\tilde{w}(l+1+i)}\varepsilon_i^\dagger-\bar{\rho}^\dagger.$$

††) The following isomorphism of $\mathfrak{h}^*$ is denoted by σ_1.
$$\sigma_1(\Lambda_0)=\Lambda_1,\quad \sigma_1(\Lambda_1)=\Lambda_0,\quad \sigma_1(\Lambda_i)=\Lambda_i,\quad (i=2,\cdots,l),$$
$$\sigma_1(\delta_0)=\delta_1,\quad \sigma_1(\delta_1)=\delta_0,\quad \sigma_1(\delta_i)=\delta_i,\quad (i=2,\cdots,l).$$

§ 3. Duality of branching coefficients

Let $\mathfrak{g}(\tilde{A})$ be an affine Lie algebra, and let $\tilde{I}$ denote the set of vertices of its Dynkin diagram. An automorphism σ of $\mathfrak{g}(\tilde{A})$ is called a diagram automorphism if there exists a permutation $\hat{\sigma}$ of $\tilde{I}$ such that $\sigma(\tilde{x}_{\tilde{\imath}})=\tilde{x}_{\hat{\sigma}(\tilde{\imath})}$ holds for all $\tilde{\imath}\in\tilde{I}$ and $\tilde{x}=\tilde{e},\tilde{f},\tilde{h},\tilde{d}$. Here and in what follows, the quantities related to $\mathfrak{g}(\tilde{A})$ will be marked by $\sim$. We define the contragredient action of σ on $\tilde{\mathfrak{h}}^*$. One checks readily the following.

Proposition 3.1.

(i) $\tilde{a}_{\hat{\sigma}(\tilde{\imath})\hat{\sigma}(\tilde{\jmath})}=\tilde{a}_{\tilde{\imath}\tilde{\jmath}},\quad \tilde{a}_{\hat{\sigma}(\tilde{\imath})}=\tilde{a}_{\tilde{\imath}},\quad \tilde{a}^\vee_{\hat{\sigma}(\tilde{\imath})}=\tilde{a}^\vee_{\tilde{\imath}}.$

(ii) $\sigma(\tilde{\Lambda}_{\tilde{\imath}})=\tilde{\Lambda}_{\hat{\sigma}(\tilde{\imath})},\quad \sigma(\tilde{\delta}_{\tilde{\imath}})=\tilde{\delta}_{\hat{\sigma}(\tilde{\imath})},\quad \sigma(\tilde{\alpha}_{\tilde{\imath}})=\tilde{\alpha}_{\hat{\sigma}(\tilde{\imath})}.$

(iii) $(\sigma(h), \sigma(h'))=(h, h'), \quad h, h' \in \tilde{\mathfrak{h}}^*$.

(iv) $\sigma r_i \sigma^{-1}=r_{\hat{\sigma}(i)}$.

Let the orbits of $\hat{\sigma}$ in $\tilde{I}$ be indexed by a set I, and let $\pi: \tilde{I} \to I$ be the map which sends $\tilde{i} \in \tilde{I}$ into its orbit by $\hat{\sigma}$. Then $\sharp I=1$ if and only if $\mathfrak{g}(\tilde{A})=A_l^{(1)}$ and ord $\hat{\sigma}=l+1$. In the sequel we exclude this case, and assume that $\sharp I \geqq 2$. For each $i \in I$, there are then two possibilities:

(1) $\pi^{-1}(i)$ consists of disconnected vertices,

(2) $\pi^{-1}(i)=\{\tilde{i}, \tilde{i}'\}$, and $\tilde{i}, \tilde{i}'$ are connected by a simple line segment.

We set $t(i)=1$ or 2 accordingly. Define $A=(a_{ij})_{i,j \in I}$ by

$$a_{ij}=t(i) \sum_{\tilde{i} \in \pi^{-1}(i)} \tilde{a}_{\tilde{i}\tilde{j}}$$

where $\tilde{j} \in \pi^{-1}(j)$ (any $\tilde{j}$ will do).

Proposition 3.2. *A is an affine generalized Cartan matrix.*

Proposition 3.3. *The invariant subalgebra*

$$\mathfrak{g}(\tilde{A})^{\sigma}=\{X \in \mathfrak{g}(\tilde{A}) \mid \sigma(X)=X\}$$

is isomorphic to $\mathfrak{g}(A)$. Its Chevalley basis is given by the rule

$$e_i=\sqrt{t(i)} \sum_{\tilde{i} \in \pi^{-1}(i)} \tilde{e}_{\tilde{i}}, \qquad f_i=\sqrt{t(i)} \sum_{\tilde{i} \in \pi^{-1}(i)} \tilde{f}_{\tilde{i}},$$

$$h_i=t(i) \sum_{\tilde{i} \in \pi^{-1}(i)} \tilde{h}_{\tilde{i}}, \qquad d_i=\sum_{\tilde{i} \in \pi^{-1}(i)} \tilde{d}_{\tilde{i}}.$$

With the identification $\mathfrak{h} \cong \mathfrak{h}^*$, $\tilde{\mathfrak{h}} \cong \tilde{\mathfrak{h}}^*$, denote by $\iota: \mathfrak{h}^* \hookrightarrow \tilde{\mathfrak{h}}^*$ the imbedding map and by $\pi: \tilde{\mathfrak{h}}^* \to \mathfrak{h}^*$ the restriction map.

Proposition 3.4.

(i) $\iota(\Lambda_i)=b_i \sum_{\tilde{i} \in \pi^{-1}(i)} \tilde{\Lambda}_{\tilde{i}}, \quad \iota(\alpha_i)=b_i t(i) \sum_{\tilde{i} \in \pi^{-1}(i)} \tilde{\alpha}_{\tilde{i}}$ *where* $b_i=a_i^{-1}a_i^\vee/(\tilde{a}_{\tilde{i}}^{-1}\tilde{a}_{\tilde{i}}^\vee)$ *for* $\tilde{i} \in \pi^{-1}(i)$.

(ii) $\pi(\tilde{\Lambda}_{\tilde{i}})=t(i)\Lambda_i, \quad \pi(\tilde{\alpha}_{\tilde{i}})=\alpha_i$ *with* $i=\pi(\tilde{i})$.

(iii) *There is an imbedding of Weyl groups* $\iota: W \hookrightarrow \tilde{W}$ *which satisfies* (2.7) *and*

$$\iota(r_i)=\begin{cases} \prod_{\tilde{i} \in \pi^{-1}(i)} \tilde{r}_{\tilde{i}} & (t(i)=1) \\ \tilde{r}_{\tilde{i}}\tilde{r}_{\tilde{i}'}\tilde{r}_{\tilde{i}}=\tilde{r}_{\tilde{i}'}\tilde{r}_{\tilde{i}}\tilde{r}_{\tilde{i}'} & (t(i)=2, \pi^{-1}(i)=\{\tilde{i}, \tilde{i}'\}). \end{cases}$$

Propositions 3.2–3.4 are verified by case-checking or by using the definition and Proposition 3.1. We omit the proof.

In the sequel we give a standard numbering to $\tilde{I}$ and I (cf. Table II). Writing

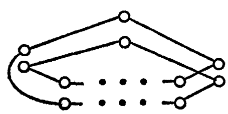
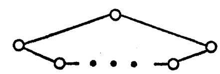
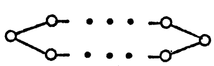
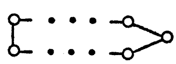
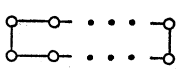
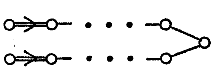
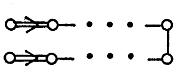
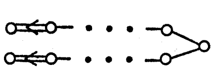
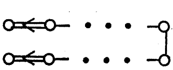
$$(3.1)\qquad \begin{aligned} &\iota(\Lambda_0)=p\tilde{\Lambda}_0+\overline{\iota(\Lambda_0)}+\eta, \qquad \eta \in \bigoplus_{\tilde{i}\in\tilde{I}} \mathbf{C}\tilde{\delta}_{\tilde{i}} \\ &\iota(\delta)=p'\tilde{\delta} \end{aligned}$$

we have $p=a_0^{-1}\tilde{a}_{\tilde{i}}\#\pi^{-1}(0)$ $(\tilde{i}\in\pi^{-1}(0))$ and $p'=t(0)$.

Proposition 3.5. *The invariant bilinear forms* $(\ ,\ )_{\tilde{\mathfrak{h}}^*}$, $(\ ,\ )_{\mathfrak{h}^*}$ *on* $\tilde{\mathfrak{h}}^*$ *and* $\mathfrak{h}^*$ *are related by*

$$(\iota h, \iota h')_{\tilde{\mathfrak{h}}^*}=pp'(h, h')_{\mathfrak{h}^*}, \qquad h, h' \in \mathfrak{h}^*.$$

Table

	$\mathfrak{g}(\tilde{A})$		$\mathfrak{g}(A)$		p
(1)	$A^{(1)}_{2l+1}$		$A^{(1)}_{l}$		2
(2)	$A^{(1)}_{2l-1}$		$C^{(1)}_{l}$		1
(3)	$A^{(1)}_{2l}$		$A^{(2)}_{2l}$		1
(4)	$A^{(1)}_{2l+1}$		$D^{(2)}_{l+1}$		2
(5)	$C^{(1)}_{2l}$		$C^{(1)}_{l}$		2
(6)	$C^{(1)}_{2l+1}$		$A^{(2)(\dagger)}_{2l}$		2
(7)	$D^{(2)}_{2l+1}$		$A^{(2)}_{2l}$		1
(8)	$D^{(2)}_{2l+2}$		$D^{(2)}_{l+1}$		2

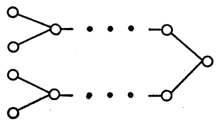
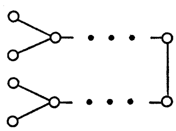
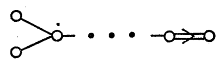
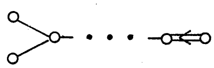
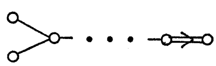
Proof. Since $W \subset \tilde{W}$, the bilinear form $\mathfrak{h}^* \ni h, h' \mapsto (\iota h, \iota h')_{\tilde{\mathfrak{h}}^*}$ is W-invariant. It vanishes on $\bigoplus_{i \in I} \mathbf{C}\Lambda_i$ by Proposition 3.4. Hence it is a constant multiple of $(\ ,\ )_{\mathfrak{h}^*}$. The constant is determined by taking $h=\Lambda_0$, $h'=\delta$. □

In Table III, we give a list of pairs $\mathfrak{g}(\tilde{A}) \supset \mathfrak{g}(A)$ that arise from an involutive diagram automorphism (i.e. $\sigma^2 = \mathrm{id.}$). Here we do not consider those related to exceptional Lie algebras.

III.

p'	$\overline{\iota(\Lambda_0)}$	$\lvert\overline{\iota(\Lambda_0)}\rvert^2$	$\overline{\iota(u)}$
1	$\frac{1}{2}\sum_{i=1}^{l+1}\tilde{\varepsilon}_i - \frac{1}{2}\sum_{i=l+2}^{2l+2}\tilde{\varepsilon}_i$	$\frac{1}{2}(l+1)$	$\sum_{i=1}^{l+1} u_i\tilde{\varepsilon}_i + \sum_{i=1}^{l+1} u_i\tilde{\varepsilon}_{l+i+1}$
1	0	0	$\sum_{i=1}^{l} u_i\tilde{\varepsilon}_i - \sum_{i=1}^{l} u_i\tilde{\varepsilon}_{2l+1-i}$
2	$\frac{1}{2(2l+1)}\sum_{i=1}^{2l+1}\tilde{\varepsilon}_i - \frac{1}{2}\tilde{\varepsilon}_{2l+1}$	$\frac{l}{2(2l+1)}$	$\sum_{i=1}^{l} u_i\tilde{\varepsilon}_i - \sum_{i=1}^{l} u_i\tilde{\varepsilon}_{2l+1-i}$
2	$\frac{1}{2l+2}\sum_{i=1}^{2l+2}\tilde{\varepsilon}_i - \tilde{\varepsilon}_{2l+2}$	$\frac{2l+1}{2l+2}$	$\sum_{i=1}^{l} u_i\tilde{\varepsilon}_i - \sum_{i=1}^{l} u_i\tilde{\varepsilon}_{2l+2-i}$
1	$\sum_{i=1}^{2l}\tilde{\varepsilon}_i$	l	$\sum_{i=1}^{l} 2u_i\tilde{\varepsilon}_i - \sum_{i=1}^{l} 2u_i\tilde{\varepsilon}_{2l+1-i}$
2	$2\sum_{i=1}^{l}\tilde{\varepsilon}_i + \tilde{\varepsilon}_{l+1}$	$2l+\frac{1}{2}$	$-\sum_{i=1}^{l} 2u_i\tilde{\varepsilon}_{l+1-i} + \sum_{i=1}^{l} 2u_i\tilde{\varepsilon}_{l+1+i}$
1	$\frac{1}{4}\sum_{i=1}^{2l}\tilde{\varepsilon}_i$	$\frac{1}{4}l$	$\sum_{i=1}^{l}\frac{1}{2}u_i\tilde{\varepsilon}_i - \sum_{i=1}^{l}\frac{1}{2}u_i\tilde{\varepsilon}_{2l+1-i}$
1	$\frac{1}{2}\sum_{i-1}^{2l+1}\tilde{\varepsilon}_i$	$l+\frac{1}{2}$	$\sum_{i=1}^{l}\frac{1}{2}u_i\tilde{\varepsilon}_i - \sum_{i=1}^{l}\frac{1}{2}u_i\tilde{\varepsilon}_{2l+2-i}$

(Cotinued from

	$\mathfrak{g}(\tilde{A})$		$\mathfrak{g}(A)$		p
(9)	$D_{l+1}^{(1)}$		$B_l^{(1)}$		1
(10)	$D_{2l}^{(1)}$		$A_{2l-1}^{(2)}$		2
(11)	$D_{2l+1}^{(1)}$		$B_l^{(1)}$		2
(12)	$A_{2l+1}^{(2)}$		$A_{2l}^{(2)}$		1
(13)	$B_{l+1}^{(1)}$		$D_{l+1}^{(2)}$		2

(†) The 0-th vertex is placed at the right end.

Remark 1. In Table III, we set

$$u=\begin{cases}\sum_{i=1}^{l+1} u_i\varepsilon_i, \quad \sum_{i=1}^{l+1} u_i=0 & (\mathfrak{g}(A)=A_l^{(1)})\\ \sum_{i=1}^{l} 2u_i\varepsilon_i & (\mathfrak{g}(A)=C_l^{(1)})\\ \sum_{i=1}^{l} \frac{1}{2}u_i\varepsilon_i & (\mathfrak{g}(A)=D_{l+1}^{(2)})\\ \sum_{i=1}^{l} u_i\varepsilon_i & (\text{otherwise}).\end{cases}$$

Remark 2. In all cases except (6), we have $\pi(0)=0\in I$, so that $\iota(\overline{\mathfrak{h}^*})\subset\overline{\tilde{\mathfrak{h}}^*}$ and $(\overline{\iota(\Lambda_0)}, \overline{\iota(u)})=(\iota(\Lambda_0), \iota(u))=0$.

Consider now an irreducible highest weight module $L(\tilde{\Lambda})$ of $\mathfrak{g}(\tilde{A})$ with dominant integral highest weight $\tilde{\Lambda}\in\tilde{P}_+$.

Proposition 3.6. *Regarded as a* $\mathfrak{g}(A)$*-module,* $L(\tilde{\Lambda})$ *decomposes into a*

Table III)

p'	$\overline{\iota(\Lambda_0)}$	$\lvert\overline{\iota(\Lambda_0)}\rvert^2$	$\overline{\iota(u)}$
1	0	0	$\sum_{i=1}^{l} u_i\tilde{\varepsilon}_i$
1	$\frac{1}{2}\sum_{i=1}^{2l}\tilde{\varepsilon}_i$	$\frac{1}{2}l$	$\sum_{i=1}^{l} u_i\tilde{\varepsilon}_i-\sum_{i=1}^{l} u_i\tilde{\varepsilon}_{2l+1-i}$
1	$\frac{1}{2}\sum_{i=1}^{2l+1}\tilde{\varepsilon}_i$	$\frac{1}{2}l+\frac{1}{4}$	$\sum_{i=1}^{l} u_i\tilde{\varepsilon}_i-\sum_{i=1}^{l} u_i\tilde{\varepsilon}_{2l+2-i}$
1	$\frac{1}{2}\tilde{\varepsilon}_1$	$\frac{1}{4}$	$\sum_{i=1}^{l} u_i\tilde{\varepsilon}_{i+1}$
1	$\tilde{\varepsilon}_1$	1	$\sum_{i=1}^{l} u_i\tilde{\varepsilon}_{i+1}$

direct sum of irreducible highest weight $\mathfrak{g}(A)$*-modules.*

Proof. The proposition follows from the complete reducibility theorem ([2] Proposition 2.9) for $\mathfrak{g}(A)$-modules, once the conditions of this theorem are verified. The only non-trivial check is that we have $\dim \mathfrak{u}(\mathbf{C}f_i)v<\infty$ for each $i\in I$ and $v\in L(\tilde{\Lambda})$, where $\mathfrak{u}(\mathfrak{a})$ stands for the universal enveloping algebra of $\mathfrak{a}$. In view of Proposition 3.3 it suffices to show $\dim \mathfrak{u}(\oplus_{\tilde{i}\in\tilde{I}'}\mathbf{C}\tilde{f}_{\tilde{i}})v<\infty$ for any proper subset $\tilde{I}'$ of $\tilde{I}$. Without loss of generality, we may assume that $\tilde{I}'$ is connected. Then $\{\tilde{e}_{\tilde{i}}, \tilde{f}_{\tilde{i}}, \tilde{h}_{\tilde{i}}\}_{\tilde{i}\in\tilde{I}'}$ generates a finite dimensional simple Lie algebra $\mathfrak{g}'\subset\mathfrak{g}(\tilde{A})$. Clearly the complete reducibility theorem is applicable to the $\mathfrak{g}'$-module $L(\tilde{\Lambda})$. Hence it is a direct sum of finite dimensional irreducible $\mathfrak{g}'$-modules, and the proof is over. □

Let us paraphrase Proposition 3.6 in terms of characters. Set $h=-2\pi i(\tau\Lambda_0-u)\in\mathfrak{h}^*$ with $\tau\in\mathscr{H}_+$ and $u\in\bar{\mathfrak{h}}^*$. Let $\tilde{\lambda}\in\tilde{\mathfrak{h}}^*$ be such that $m=(\tilde{\lambda},\tilde{\delta})_{\tilde{\mathfrak{h}}^*}>0$. If we evaluate the sum $\sum_{\tilde{\iota}\in\tilde{T}}e^{\tilde{\iota}\tilde{\lambda}}$ at $\iota(h)$, we get

$$\mathbf{e}\left[-\left(p\frac{|\bar{\tilde{\lambda}}|^2}{2m}+\frac{|\overline{\iota(\Lambda_0)}|^2}{2p}m+(\tilde{\lambda}, p\tilde{\Lambda}_0+\eta)\right)\tau\right]$$
$$\times \sum_{\gamma \in \bar{M}+m^{-1}\bar{\tilde{\lambda}}-p^{-1}\overline{\iota(\Lambda_0)}} \mathbf{e}\left[\frac{mp}{2}|\gamma|^2\tau+m(\gamma, \overline{\iota(u)})\right]$$

where we have used (2.3) together with $\iota(u)-\overline{\iota(u)} \in \mathbf{C}\tilde{\delta}$, $(\tilde{\delta}, \eta)=0$ for $\eta \in \bigoplus_{i\in I}\mathbf{C}\tilde{\delta}_i$ and that $(\overline{\iota(\Lambda_0)}, \overline{\iota(u)})+(p\tilde{\Lambda}_0, \iota(u))=(\iota(\Lambda_0), \iota(u))=0$. In view of this formula, we normalize the restriction of the character of $\mathfrak{g}(\tilde{A})$ as follows (cf. (2.5)).

$$\chi_{\tilde{\Lambda}}(\tau, u)=\mathbf{e}\left[\left(p\tilde{s}_{\tilde{\Lambda}}+\frac{m}{2p}|\overline{\iota(\Lambda_0)}|^2+(\tilde{\Lambda}, \eta)\right)\tau\right]\mathrm{ch}_{L(\tilde{\Lambda})}(\iota(h))$$
$$=\mathbf{e}\left[\frac{m}{2p}|\overline{\iota(\Lambda_0)}|^2\tau-\frac{m}{p}(\overline{\iota(\Lambda_0)}, \overline{\iota(u)})\right]\chi_{\tilde{\Lambda}}(p\tau, \overline{\iota(u)}-\tau\overline{\iota(\Lambda_0)})$$
$$=A^{\bar{M}}_{\bar{\tilde{\Lambda}}+\bar{\rho}-((m+\tilde{g})/p)\overline{\iota(\Lambda_0)}, m+\tilde{g}}(p\tau, \overline{\iota(u)})/A^{\bar{M}}_{\bar{\rho}-(\tilde{g}/p)\overline{\iota(\Lambda_0)}, \tilde{g}}(p\tau, \overline{\iota(u)}).$$

Here $m=(\tilde{\Lambda}, \tilde{\delta})$, $\tilde{g}=(\tilde{\rho}, \tilde{\delta})$ with $(\ ,\ )=(\ ,\ )_{\tilde{\mathfrak{h}}^*}$ and $\chi_{\tilde{\Lambda}}$ denotes the (unspecialized) normalized character of $\mathfrak{g}(\tilde{A})$.

Proposition 3.7. *There exist holomorphic functions $e_{\tilde{\Lambda}\Lambda}(\tau)$ on $\mathscr{H}_+$ such that*

$$\chi_{\tilde{\Lambda}}(\tau, u)=\sum_{\Lambda} e_{\tilde{\Lambda}\Lambda}(\tau)\chi_{\Lambda}(\tau, u) \tag{3.2}$$

holds. Here the sum runs over the finite set $\Lambda \in P_+$ mod $\bigoplus_{i\in I}\mathbf{C}\delta_i$, $(\Lambda, \delta)=p'(\tilde{\Lambda}, \tilde{\delta})$.

Proof. Set $V=\{v \in L(\tilde{\Lambda})\,|\,e_i v=0\ (i \in I)\}$, and let $V=\bigoplus_{\lambda\in\mathscr{L}}V_\lambda$, $V_\lambda=\{v \in V\,|\,hv=\lambda(h)v$ for $h \in \mathfrak{h}\}$ be its weight space decomposition with $\mathscr{L}=\{\lambda \in P_+\,|\,V_\lambda\neq 0\}$. Note that $(\lambda, \delta)=(\pi(\tilde{\Lambda}), \delta)=p'(\tilde{\Lambda}, \tilde{\delta})$ holds for all $\lambda \in \mathscr{L}$, since $(\alpha_i, \delta)=0$ $(i \in I)$. We have $L(\tilde{\Lambda})\cong\bigoplus_{\lambda\in\mathscr{L}}L(\lambda)^{\dim V_\lambda}$. Given $\pi(\tilde{\Lambda}) \in P_+$, let us fix a system of representatives $\mathscr{R}$ of the set

$$\{\Lambda \in P_+ \bmod \bigoplus_{i\in I}\mathbf{C}\delta_i\,|\,(\Lambda, \delta)=p'(\tilde{\Lambda}, \tilde{\delta}),\ \Lambda\equiv\pi(\tilde{\Lambda}) \bmod Q\}$$

where $Q=\bigoplus_{i\in I}\mathbf{Z}\alpha_i$. Since $(\bigoplus_{i\in I}\mathbf{C}\delta_i)\cap Q=\mathbf{Z}\delta$, an element $\lambda \in \mathscr{L}$ can be represented as $\lambda=\Lambda-n\delta$ with some $\Lambda \in \mathscr{R}$ and $n \in \mathbf{Z}$. Put

$$E_{\tilde{\Lambda}\Lambda}(q)=\sum_{n\in\mathbf{Z}}\dim V_{\Lambda-n\delta}\cdot q^n.$$

We have then

(3.3) $$\pi(\mathrm{ch}_{L(\tilde{\Lambda})})=\sum_{\Lambda\in\mathfrak{R}} E_{\tilde{\Lambda}\Lambda}(e^{-\delta})\,\mathrm{ch}_{L(\Lambda)}.$$

This implies that the series $E_{\tilde{\Lambda}\Lambda}(e^{-\delta})$ is majorized by

$$\sum_{\mu:\text{weight of } L(\tilde{\Lambda})} \text{mult}\,\mu\cdot|e^{-\pi\mu}|$$

up to a power of $e^{-\delta}$, and hence $E_{\tilde{\Lambda}\Lambda}(q)$ converges absolutely for $|q|<1$ ([2]). Evaluating both sides of (3.3) at $h=-2\pi i(\tau\Lambda_0-u)$, we get (3.2) with the definition

$$e_{\tilde{\Lambda}\Lambda}(\tau)=\mathbf{e}\left[\left(p\tilde{s}_{\tilde{\Lambda}}-s_{\Lambda}+\frac{m}{2p}|\overline{\iota(\Lambda_0)}|^2+(\tilde{\Lambda},\eta)\right)\tau\right]E_{\tilde{\Lambda}\Lambda}(e^{2\pi i\tau}).$$

Note that $\tilde{\chi}_{\tilde{\Lambda}}(\tau, u)$ (resp. $\chi_{\Lambda}(\tau, u)$) depends only on $\tilde{\Lambda}$ mod $\bigoplus_{i\in\tilde{I}}\mathbf{C}\tilde{\delta}_i$ (resp. Λ mod $\bigoplus_{i\in I}\mathbf{C}\delta_i$), and so does $e_{\tilde{\Lambda}\Lambda}(\tau)$. □

Remark. From the proof above, we have

$$e_{\tilde{\Lambda}\Lambda}(\tau)=0 \quad \text{if } \pi(\tilde{\Lambda})\not\equiv\Lambda \bmod Q.$$

Propositon 3.8. $e_{\sigma(\tilde{\Lambda})\Lambda}(\tau)=e_{\tilde{\Lambda}\Lambda}(\tau)$.

Proof. The proposition follows from the fact $\tilde{\chi}_{\sigma(\tilde{\Lambda})}(\tau, u)=\tilde{\chi}_{\tilde{\Lambda}}(\tau, u)$, which we shall show below. Without loss of generality we may assume $\tilde{\Lambda}, \tilde{\rho}\in\tilde{\mathfrak{h}}_0^*$. In this case $(\tilde{\Lambda},\eta)=0=(\sigma(\tilde{\Lambda}),\eta)$, and $|\tilde{\Lambda}+\tilde{\rho}|^2=|\bar{\tilde{\Lambda}}+\bar{\tilde{\rho}}|^2$. Since σ is an isometry (Proposition 3.1 (iii)), we have

(3.4) $$\tilde{s}_{\sigma(\tilde{\Lambda})}=\tilde{s}_{\tilde{\Lambda}}.$$

On the other hand, σ normalizes the Weyl group $\tilde{W}$ of $\mathfrak{g}(\tilde{A})$ (Proposition 3.1 (iv)). Hence

(3.5) $$\mathrm{ch}_{L(\tilde{\Lambda})}(h)=\mathrm{ch}_{L(\sigma(\tilde{\Lambda}))}(\sigma(h))$$

holds for any $h\in\tilde{\mathfrak{h}}^*$. (3.4) and (3.5) prove our assertion. □

As we have seen in the classical case (§ 1), the complementary decomposition enables us to derive an identity "dual" to (3.2) that involves the same functions $e_{\tilde{\Lambda}\Lambda}(\tau)$. The results corresponding to the pairs in Table III are listed in Table IV below. In deriving them we have made use of the following product formulas for some special characters:

$$A_{2l}^{(2)}:\quad \chi_{\Lambda_0}(\tau, u)=\eta(\tau)^{-l}\prod_{i=1}^{l}\hat{\theta}\left(\tau, u_i+\frac{1}{2}\right).$$

$$D_{l+1}^{(2)}:\quad \chi_{\Lambda_0}(\tau, u)=\eta(\tau)^{-1}\eta(2\tau)^{1-l}\prod_{i=1}^{l}\hat{\theta}\Big(2\tau, u_i+\frac{1}{2}\Big),$$

$$\chi_{\Lambda_l}(\tau, u)=\eta(\tau)^{-1}\eta(2\tau)^{1-l}\prod_{i=1}^{l}\Big(-i\theta\Big(2\tau, u_i+\frac{1}{2}\Big)\Big),$$

$$\chi_{\Lambda_0+\Lambda_l}(\tau, u)=\eta(\tau)^{1-l}\eta(2\tau)\prod_{i=1}^{l}\Big(-i\theta\Big(\tau, u_i+\frac{1}{2}\Big)\Big).$$

$$B_l^{(1)}:\quad \chi_{\Lambda_l}(\tau, u)=\eta(\tau)^{-1-l}\eta(2\tau)\sum_{i=1}^{l}\theta\Big(\tau, u_i+\frac{1}{2}\Big).$$

In fact, each of them is reduced to one of the denominator formulas for affine Lie algebras. That the corresponding numerators have product formulas for arbitrary types of specializations was found by Wakimoto [7]. We also utilized the following special cases of branching rules, which can be checked by using principal specializations.

$A_{2l+1}^{(2)}\supset A_{2l}^{(2)}$

$$\bar{\chi}_{\bar{\Lambda}_0}(\tau, u)=\frac{\eta(2\tau)}{\eta(\tau)}\chi_{\Lambda_0}(\tau, u).$$

$B_{l+1}^{(1)}\supset D_{l+1}^{(2)}$

$$\bar{\chi}_{\bar{\Lambda}_0}(\tau, u)=\frac{\eta(4\tau)}{\eta(2\tau)}\chi_{\Lambda_0}(\tau, u),$$

$$\bar{\chi}_{\bar{\Lambda}_{l+1}}(\tau, u)=\frac{\eta(2\tau)^2}{\eta(\tau)\eta(4\tau)}\chi_{\Lambda_l}(\tau, u),$$

$$\bar{\chi}_{\bar{\Lambda}_0+\bar{\Lambda}_{l+1}}(\tau, u)=\frac{\eta(2\tau)}{\eta(\tau)}\chi_{\Lambda_0+\Lambda_l}(\tau, u).$$

$D_{l+1}^{(1)}\supset B_l^{(1)}$

$$\bar{\chi}_{\bar{\Lambda}_{l+1}}(\tau, u)=\frac{\eta(2\tau)}{\eta(\tau)}\chi_{\Lambda_l}(\tau, u),$$

$$\bar{\chi}_{\bar{\Lambda}_0}(\tau, u)=\frac{\eta(\tau)}{\eta(2\tau)}\sum_{\sigma=\mathrm{id},\sigma_1}\xi_\sigma(\tau)\chi_{\sigma(\Lambda_0)}(\tau, u),$$

$$\bar{\chi}_{\bar{\Lambda}_0+\bar{\Lambda}_l}(\tau, u)=\sum_{\sigma=\mathrm{id},\sigma_1}\xi_\sigma(\tau)\chi_{\sigma(\Lambda_0)+\Lambda_l}(\tau, u),$$

where

$$\xi_\sigma(\tau)=\frac{\eta(\tau)}{2}\left(\frac{1}{\eta\left(\frac{\tau}{2}\right)}+\operatorname{sgn}\sigma\frac{\mathbf{e}[1/48]}{\eta\left(\frac{\tau+1}{2}\right)}\right)\quad\text{with } \operatorname{sgn}\sigma=\begin{cases}1 & \sigma=\mathrm{id}\\ -1 & \sigma=\sigma_1.\end{cases}$$

For each pair $(\tilde{\mathfrak{g}}, \mathfrak{g})$ in Table IV, the following data are given.

1° Correspondence of weights $\tilde{\Lambda}\leftrightarrow\Lambda$ for $(\tilde{\mathfrak{g}}, \mathfrak{g})$.

2° Correspondence of weights $\tilde{\Lambda}^\dagger\leftrightarrow\Lambda^\dagger$ for the complementary pair $(\tilde{\mathfrak{g}}^\dagger, \mathfrak{g}^\dagger)$.

3° Theta function indentity associated with $(\tilde{\mathfrak{g}}^\dagger, \mathfrak{g}^\dagger)$ (the one dual to (3.2)).

4° Explicit formulas for $e_{\tilde{\lambda}\Lambda}(\tau)$ for small m.

In 1° and 2°, $\tilde{\Lambda}$ etc. are given in terms of the coefficients $\tilde{\lambda}_i$ of $\tilde{\varepsilon}_i$ etc. as in Table II. The summation in 3° is taken over the partitions $\tilde{P}$ given in 1°.

Let us recall here the definition of Hecke indefinite modular forms that appear in 4°. Let $B(\gamma,\gamma)$ be a binary quadratic form of signature (1,1) defined on a lattice $L\subset\mathbf{R}^2$ of rank 2. We assume $B(\gamma,\gamma)\in 2\mathbf{Z}\,(\gamma\in L)$. Set $L^*=\{\gamma\in\mathbf{R}^2\,|\,B(\gamma,\gamma')\in\mathbf{Z}$ for all $\gamma'\in L\}$, $G=\{g\in O(B)\,|\,gL\subset L\}$ and $G_0=\{g\in G\cap SO_0(B)\,|\,g$ leaves L^*/L pointwise fixed$\}$. Let further $B(\gamma,\gamma)=l_1(\gamma)l_2(\gamma)$ be a factorization into real linear forms $l_i(\gamma)$. A Hecke indefinite modular form with characteristic $\mu\in L^*$ is by definition

$$\theta^B_{L,\mu}(\tau)=\sum_{\substack{\gamma\in G_0\backslash(L+\mu)\\ B(\gamma,\gamma)>0}}\operatorname{sgn} l_1(\gamma)\cdot\mathbf{e}\left[\frac{\tau}{2}B(\gamma,\gamma)\right].$$

For our purpose, the following three types are of interest. In all cases we set $L=\mathbf{Z}^2$, $\gamma=\begin{pmatrix}x\\y\end{pmatrix}\in\mathbf{Z}^2$, and take $l_i(\gamma)$ so that $l_1(\gamma)>0$ for $x\gg 0$.

(i) $B(\gamma,\gamma)=2(l+2)x^2-2ly^2$,

$$H^l_{jk}(\tau)=\theta^B_{L,\mu}(\tau),\quad \mu=\begin{bmatrix}\dfrac{k+1}{2(l+2)}\\[2mm] \dfrac{j}{2l}\end{bmatrix}.$$

(ii) $B'(\gamma,\gamma)=2(l+2)x^2-8(l+1)y^2$,

$$H'^l_{jk}(\tau)=\theta^{B'}_{L,\mu}(\tau),\quad \mu=\begin{bmatrix}\dfrac{1}{2}+\dfrac{k+1}{2(l+2)}\\[2mm] \dfrac{1}{4}+\dfrac{j+1}{4(l+1)}\end{bmatrix}.$$

(iii) $B''(\gamma,\gamma)=8(l+1)x^2-2ly^2$,

$$H''^l_{jk}(\tau)=\theta^{B''}_{L,\mu}(\tau),\quad \mu=\begin{bmatrix}\dfrac{1}{4}+\dfrac{k}{4(l+1)}\\[2mm] \dfrac{1}{2}+\dfrac{j}{2l}\end{bmatrix}.$$

The cases (i) and (ii) are discussed in [2] and [4], respectively. For (iii), the generators of G are given by

$$\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} \text{ and } a=\begin{pmatrix} 2l+1 & l \\ 4l+4 & 2l+1 \end{pmatrix}.$$

A fundamental domain of $G \cap SO_0(B'')$ (resp. G_0) is given by

$$F=\{(x, y) \in \mathbf{R}^2 \mid -2|x|<|y| \leqq 2|x|\} \text{ (resp. } F \cup a(F)).$$

There are cases where the complementary identity in Table IV admits a simple interpretation. For the pair (2) or (5), the Lie algebras $\tilde{\mathfrak{g}}^\dagger$, $\mathfrak{g}^\dagger$ of the complementary pair are related as

(2)† $\tilde{\mathfrak{g}}^\dagger = A^{(1)}_{m-1} \subset \mathfrak{g}^\dagger = C^{(1)}_m$,

(5)† $\tilde{\mathfrak{g}}^\dagger = C^{(1)}_m \subset \mathfrak{g}^\dagger = C^{(1)}_m$.

These embeddings are given by realization in terms of Laurent polynomials $\mathfrak{sl}(m, \mathbf{C}) \otimes \mathbf{C}[t, t^{-1}] \subset \mathfrak{sp}(2m, \mathbf{C}) \otimes \mathbf{C}[t, t^{-1}]$ (cf. § 1), $\mathfrak{sp}(2m, \mathbf{C}) \otimes \mathbf{C}[t^2, t^{-2}] \subset \mathfrak{sp}(2m, \mathbf{C}) \otimes \mathbf{C}[t, t^{-1}]$, which are to be extended suitably to $\mathbf{C}c \oplus (\oplus \mathbf{C}d_i)$.

Proposition 3.10. *For the pairs* (2)† *or* (5)†, *an irreducible* $\mathfrak{g}^\dagger$*-module* $L(\Lambda^\dagger)$ *is completely reducible as a* $\tilde{\mathfrak{g}}^\dagger$*-module.*

Proof. Since the proofs are similar, we do only for (2)†. The Chevalley basis for $\tilde{\mathfrak{g}}^\dagger = A^{(1)}_{m-1}$ reads

$$\tilde{e}^\dagger_0 = [[\cdots[e^\dagger_0, e^\dagger_1], e^\dagger_2], \cdots, e^\dagger_m], \quad \tilde{f}^\dagger_0 = [[\cdots[f^\dagger_0, f^\dagger_1], f^\dagger_2], \cdots, f^\dagger_m],$$
$$\tilde{h}^\dagger_0 = h^\dagger_0 + h^\dagger_m + c^\dagger, \quad \tilde{d}^\dagger_0 = d^\dagger_0$$

and $\tilde{x}^\dagger_i = x^\dagger_i$ $(1 \leqq i \leqq m-1,\ x = e, f, h, d)$. It suffices to prove that

$$\dim_{\mathbf{C}} \mathfrak{u}(\mathbf{C}\tilde{f}^\dagger_0)v < \infty \quad \text{for any } v \in L(\Lambda^\dagger).$$

There is a well defined action of the Weyl group $W^\dagger$ of $\mathfrak{g}^\dagger = C^{(1)}_m$ both on $L(\Lambda^\dagger)$ and on $\mathfrak{g}^\dagger$. One checks that $r^\dagger_0 f^\dagger_0 = -e^\dagger_0$, $r^\dagger_0 f^\dagger_1 = [f^\dagger_1, f^\dagger_0]$ and $r_0 f^\dagger_i = f^\dagger_i$ $(2 \leqq i \leqq m)$, from which follows $r^\dagger_0 \tilde{f}^\dagger_0 = [\cdots[[f^\dagger_1, f^\dagger_2], f^\dagger_3], \cdots, f^\dagger_m]$. Since this expression involves only $f^\dagger_i$ with $1 \leqq i \leqq m$, our assertion follows from the proof of Proposition 3.6. □

There are thus theta function identities associated with the pairs (2†), (5†), which turn out to be of the forms

$$\sum_{\tilde{\Lambda}^\dagger} \hat{\chi}^{(m)}_{\tilde{\Lambda}^\dagger}(\tau, u^\dagger_1, \cdots, u^\dagger_m) e^\dagger_{\tilde{\Lambda}^\dagger \Lambda^\dagger}(\tau) = \chi^{(m)}_{\Lambda^\dagger}(\tau, u^\dagger_1, \cdots, u^\dagger_m)$$

$$\sum_{\tilde{\Lambda}^\dagger} \chi^{(m)}_{\tilde{\Lambda}^\dagger}(2\tau, u^\dagger_1, \cdots, u^\dagger_m) e^\dagger_{\tilde{\Lambda}^\dagger \Lambda^\dagger}(\tau) = \chi^{(m)}_{\Lambda^\dagger}(\tau, u^\dagger_1, \cdots, u^\dagger_m)$$

respectively. Comparing these with Table IV, we conclude the following.

Proposition 3.11. *For the pairs* (2) *or* (5), *we have the duality*

$$e_{\tilde{\Lambda}\Lambda}(\tau)=e^{\dagger}_{\tilde{\Lambda}^{\dagger}\Lambda^{\dagger}}(\tau)$$

where the correspondence of weights $\tilde{\Lambda}\leftrightarrow\tilde{\Lambda}^{\dagger}$, $\Lambda\leftrightarrow\Lambda^{\dagger}$ *is described as in Table* IV.

Remark. In terms of Young diagrams, the correspondence for (2)$\leftrightarrow$(2)† is the same as in Section 1, Proposition 1.3 (take complement with respect to a rectangle and transpose the diagram). For (5)$\leftrightarrow$(5)†, one simply transposes the diagram without taking the complement.

Table IV.

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(1) $A^{(1)}_{2l+1}\supset A^{(1)}_{l}$

1°. $A^{(1)}_{2l+1}$: $(\tilde{\Lambda},\tilde{\delta})=m$

$A^{(1)}_{m-1}$: $(\tilde{\Lambda}^{\dagger},\tilde{\delta}^{\dagger})=2l+2$

$\tilde{P}$: $\{\tilde{\lambda}_i-\tilde{\lambda}_{2l+2}\}_{1\leqq i\leqq 2l+1}\cup\{\tilde{\mu}^{\dagger}_j\}_{1\leqq j\leqq m}=\{2l+1+m,\cdots,2,1\}$

with $\tilde{\lambda}^{\dagger}_j=\tilde{\mu}^{\dagger}_j-\dfrac{1}{m}(\tilde{\mu}^{\dagger}_1+\cdots+\tilde{\mu}^{\dagger}_m)$, $\tilde{\mu}^{\dagger}_1>\cdots>\tilde{\mu}^{\dagger}_m$.

2°. $A^{(1)}_{l}$: $(\Lambda,\delta)=m$

$A^{(1)}_{m-1}$: $(\Lambda^{\dagger},\delta^{\dagger})=l+1$

P: $\{\lambda_i-\lambda_{l+1}\}_{1\leqq i\leqq l}\cup\{\mu^{\dagger}_j\}_{1\leqq j\leqq m}=\{l+m,\cdots,2,1\}$

with $\lambda^{\dagger}_j=\mu^{\dagger}_j-\dfrac{1}{m}(\mu^{\dagger}_1+\cdots+\mu^{\dagger}_m)$, $\mu^{\dagger}_1>\cdots>\mu^{\dagger}_m$.

3° $\sum_{\tilde{P}}(-)^{\varepsilon(\hat{\Lambda}^{\dagger},\Lambda^{\dagger})}\hat{\chi}^{(m)}_{\tilde{\Lambda}^{\dagger}}(2\tau,u^{\dagger}_1,\cdots,u^{\dagger}_m)e_{\tilde{\Lambda}\Lambda}(\tau)$

$$=\mathbf{e}\left[\frac{1}{8}(l+1)m\tau-\frac{1}{2}(l+1)\sum_{j=1}^{m}u^{\dagger}_j\right]\hat{\chi}^{(m)}_{\Lambda^{\dagger}}\left(\tau,u^{\dagger}_1-\frac{\tau}{2},\cdots,u^{\dagger}_m-\frac{\tau}{2}\right),$$

$\varepsilon(\hat{\Lambda}^{\dagger},\Lambda^{\dagger})=\sum_{j=1}^{m}(\tilde{\mu}^{\dagger}_j+\mu^{\dagger}_j)$.

4°. $(m=1)$

$$e_{\tilde{\Lambda}_j\Lambda_k}(\tau)=\begin{cases}\eta(\tau)^{-1}\eta(2\tau) & (j=k\text{ or }j=k+l+1)\\ 0 & (\text{otherwise})\end{cases}$$

$(0\leqq j\leqq 2l+1,\ 0\leqq k\leqq l)$.

(2) $A^{(1)}_{2l-1}\supset C^{(1)}_{l}$

1°. $A^{(1)}_{2l-1}$: $(\tilde{\Lambda},\tilde{\delta})=m$

$A_{m-1}^{(1)}$: $(\tilde{\Lambda}^\dagger, \tilde{\delta}^\dagger)=2l$

$\tilde{P}$: $\{\tilde{\lambda}_i-\tilde{\lambda}_{2l}\}_{1\leq i\leq 2l-1}\cup\{\tilde{\mu}_j^\dagger\}_{1\leq j\leq m}=\{2l-1+m, \cdots, 2, 1\}$

with $\tilde{\lambda}_j^\dagger=\tilde{\mu}_j^\dagger-\frac{1}{m}(\tilde{\mu}_1^\dagger+\cdots+\tilde{\mu}_m^\dagger)$, $\tilde{\mu}_1^\dagger>\cdots>\tilde{\mu}_m^\dagger$.

2°. $C_l^{(1)}$: $(\Lambda, \delta)=m$

$C_m^{(1)}$: $(\Lambda^\dagger, \delta^\dagger)=l$

P: $\{\lambda_i\}_{1\leq i\leq l}\cup\{\lambda_j^\dagger\}_{1\leq j\leq m}=\{l+m, \cdots, 2, 1\}$.

3°. $\sum_{\tilde{P}}\hat{\chi}_{\tilde{\Lambda}^\dagger}^{(m)}(\tau, u_1^\dagger, \cdots, u_m^\dagger)e_{\tilde{\Lambda}\Lambda}(\tau)=\chi_{\Lambda^\dagger}^{(m)}(\tau, u_1, \cdots, u_m)$.

4°. $(m=1)$

$e_{\tilde{\Lambda}_j\Lambda_k}(\tau)=\eta(\tau)^{-2}H_{jk}^l(\tau)\quad(0\leqq j\leqq 2l-1,\ 0\leqq k\leqq l)$.

(3) $A_{2l}^{(1)}\supset A_{2l}^{(2)}$

1°. $A_{2l}^{(1)}$: $(\tilde{\Lambda}, \tilde{\delta})=2m$

$A_{m-1}^{(1)}$: $(\tilde{\Lambda}^\dagger, \tilde{\delta}^\dagger)=2l+1$

$\tilde{P}$: $\{\tilde{\lambda}_i-\tilde{\lambda}_{2l+1}\}_{1\leq i\leq 2l}\cup\{\tilde{\mu}_j^\dagger\}_{1\leq j\leq m}=\{2l+m, \cdots, 2, 1\}$

with $\tilde{\lambda}_j^\dagger=\tilde{\mu}_j^\dagger-\frac{1}{m}(\tilde{\mu}_1^\dagger+\cdots+\tilde{\mu}_m^\dagger)$, $\tilde{\mu}_1^\dagger>\cdots>\tilde{\mu}_m^\dagger$.

2°. $A_{2l}^{(2)}$: $(\Lambda, \delta)=2m$

$A_{2m}^{(2)}$: $(\Lambda^\dagger, \delta^\dagger)=2l$

P: $\{\lambda_i\}_{1\leq i\leq l}\cup\{\lambda_j^\dagger\}_{1\leq j\leq m}=\{l+m, \cdots, 2, 1\}$.

3°.
$$\sum_{\tilde{P}}(-)^{\varepsilon(\tilde{\Lambda}^\dagger,\Lambda^\dagger)}\hat{\chi}_{\tilde{\Lambda}^\dagger}^{(m)}(\tau, u_1^\dagger, \cdots, u_m^\dagger)e_{\tilde{\Lambda}\Lambda}(\tau)$$
$$=\mathbf{e}\left[\frac{m\tau}{8(2l+1)}+\frac{1}{2}\sum_{j=1}^m u_j^\dagger\right]$$
$$\times\chi_{\Lambda^\dagger}^{(m)}\left(\tau, u_1^\dagger+\frac{\tau}{2(2l+1)}, \cdots, u_m^\dagger+\frac{\tau}{2(2l+1)}\right)$$
$$\times\prod_{j=1}^m\left(\eta(\tau)^{-1}\theta\left(\tau, u_j^\dagger+\frac{\tau}{2(2l+1)}\right)\right),$$

$$\varepsilon(\tilde{\Lambda}^\dagger, \Lambda^\dagger)=\sum_{j=1}^m(\tilde{\mu}_j^\dagger+\lambda_j^\dagger)+(l+1)m.$$

4°.[†] $(m=1)$

$$e_{\tilde{\Lambda}_j\Lambda_k'}(\tau)=\begin{cases}\eta(2\tau)^{-2}H_{j+l+1,\,k+l+1}^{2l+1}(2\tau) & (j\equiv k \bmod 2)\\ \eta(2\tau)^{-2}H_{l-j,k+l+1}^{2l+1}(2\tau) & (j\not\equiv k \bmod 2)\end{cases}$$

$(0\leqq j\leqq 2l,\ 0\leqq k\leqq l)$.

(4) $A_{2l+1}^{(1)} \supset D_{l+1}^{(2)}$

1°. $A_{2l+1}^{(1)}: (\tilde{\Lambda}, \tilde{\delta}) = m$

$A_{m-1}^{(1)}: (\tilde{\Lambda}^{\dagger}, \tilde{\delta}^{\dagger}) = 2l+2$

$\tilde{P}: \{\tilde{\lambda}_i - \tilde{\lambda}_{2l+2}\}_{1 \leqq i \leqq 2l+1} \cup \{\tilde{\mu}_j^{\dagger}\}_{1 \leqq j \leqq m} = \{2l+1+m, \cdots, 2, 1\}$

with $\tilde{\lambda}_j^{\dagger} = \tilde{\mu}_j^{\dagger} - \frac{1}{m}(\tilde{\mu}_1^{\dagger} + \cdots + \tilde{\mu}_m^{\dagger}), \quad \tilde{\mu}_1^{\dagger} > \cdots > \tilde{\mu}_m^{\dagger}$.

2°. $D_{l+1}^{(2)}: (\Lambda, \delta) = 2m$

$D_{m+1}^{(2)}: (\Lambda^{\dagger}, \delta^{\dagger}) = 2l$

$P: \{\lambda_i\}_{1 \leqq i \leqq l} \cup \{\lambda_j^{\dagger}\}_{1 \leqq j \leqq m} = \left\{l+m+\frac{1}{2}, \cdots, \frac{3}{2}, \frac{1}{2}\right\}$.

3°.
$$\sum_{\tilde{P}} (-)^{\varepsilon(\tilde{\Lambda}^{\dagger}, \Lambda^{\dagger})} \hat{\chi}_{\tilde{\Lambda}^{\dagger}}^{(m)}(2\tau, u_1^{\dagger}, \cdots, u_m^{\dagger}) e_{\tilde{\Lambda}\Lambda}(\tau)$$
$$= \mathbf{e}\left[\frac{m\tau}{8(l+1)} + \frac{1}{2}\sum_{j=1}^{m} u_j^{\dagger}\right] \chi_{\Lambda^{\dagger}}^{(m)}\left(\tau, u_1^{\dagger} + \frac{\tau}{2l+2}, \cdots, u_m^{\dagger} + \frac{\tau}{2l+2}\right)$$
$$\times \frac{\eta(\tau)}{\eta(2\tau)} \prod_{j=1}^{m} \left(\eta(\tau)^{-1}\theta\left(\tau, u_j^{\dagger} + \frac{\tau}{2l+2}\right)\right),$$
$$\varepsilon(\tilde{\Lambda}^{\dagger}, \Lambda^{\dagger}) = \sum_{j=1}^{m} \left(\tilde{\mu}_j^{\dagger} + \lambda_j^{\dagger} + \frac{1}{2}\right) + lm.$$

4°.[†] $(m=1)$

$$e_{\tilde{\Lambda}_j \Lambda_k'}(\tau) = \begin{cases} 1 & (j=k \text{ or } j=2l+1-k) \\ 0 & (\text{otherwise}), \end{cases}$$
$$(0 \leqq j \leqq 2l+1, \ 0 \leqq k \leqq l).$$

(5) $C_{2l}^{(1)} \supset C_l^{(1)}$

1°. $C_{2l}^{(1)}: (\tilde{\Lambda}, \tilde{\delta}) = m$

$C_m^{(1)}: (\tilde{\Lambda}^{\dagger}, \tilde{\delta}^{\dagger}) = 2l$

$\tilde{P}: \{\tilde{\lambda}_i\}_{1 \leqq i \leqq 2l} \cup \{2l+m+1-\tilde{\lambda}_j^{\dagger}\}_{1 \leqq j \leqq m} = \{2l+m, \cdots, 2, 1\}$.

2°. $C_l^{(1)}: (\Lambda, \delta) = m$

$C_m^{(1)}: (\Lambda^{\dagger}, \delta^{\dagger}) = l$

$P: \{\lambda_i\}_{1 \leqq i \leqq l} \cup \{l+m+1-\lambda_j^{\dagger}\}_{1 \leqq j \leqq m} = \{l+m, \cdots, 2, 1\}$.

3°. $\sum_{\tilde{P}} \chi_{\tilde{\Lambda}^{\dagger}}^{(m)}(2\tau, u_1^{\dagger}, \cdots, u_m^{\dagger}) e_{\tilde{\Lambda}\Lambda}(\tau) = \chi_{\Lambda^{\dagger}}^{(m)}(\tau, u_1^{\dagger}, \cdots, u_m^{\dagger})$.

4°. $(m=1)$

$$e_{\tilde{\Lambda}_j \Lambda_k}(\tau) = \eta(\tau)^{-1}\eta(2\tau)^{-1} H_{jk}^{\prime l}(\tau) \quad (0 \leqq j \leqq 2l, \ 0 \leqq k \leqq l).$$

(6) $C_{2l+1}^{(1)} \supset A_{2l}^{(2)}$

1°. $C_{2l+1}^{(1)}: (\tilde{\Lambda}, \tilde{\delta}) = m$

$C_m^{(1)}$: $(\tilde{\Lambda}^\dagger, \tilde{\delta}^\dagger)=2l+1$

$\tilde{P}$: $\{\tilde{\lambda}_i\}_{1\leq i\leq 2l+1}\cup\{\tilde{\lambda}_j^\dagger\}_{1\leq j\leq m}=\{2l+m+1, \cdots, 2, 1\}$.

2°. $A_{2l}^{(2)}$: $(\Lambda, \delta)=2m$

$A_{2m}^{(2)}$: $(\Lambda^\dagger, \delta^\dagger)=2l$

P: $\{\lambda_i\}_{1\leq i\leq l}\cup\{\lambda_j^\dagger\}_{1\leq j\leq m}=\{l+m, \cdots, 2, 1\}$.

3°.
$$\sum_{\tilde{P}}(-)^{\varepsilon(\tilde{\Lambda}^\dagger,\Lambda^\dagger)}\chi_{\tilde{\Lambda}^\dagger}^{(m)}(2\tau, u_1^\dagger, \cdots, u_m^\dagger)e_{\tilde{\Lambda}\Lambda}(\tau)$$
$$=\chi_{\Lambda^\dagger}^{(m)}(\tau, u_1^\dagger, \cdots, u_m^\dagger)\prod_{j=1}^{m}(\eta(\tau)^{-1}\hat{\theta}(\tau, u_j^\dagger)),$$
$$\varepsilon(\tilde{\Lambda}^\dagger, \Lambda^\dagger)=\sum_{j=1}^{m}(\tilde{\lambda}_j^\dagger+\lambda_j^\dagger).$$

4°.[†] $(m=1)$
$$e_{\tilde{\Lambda}_j\Lambda_k'}(\tau)=\begin{cases}\eta(\tau)^{-1}\eta(2\tau) & (j=l-k \text{ or } l+1+k)\\ 0 & (\text{otherwise}),\end{cases}$$
$$(0\leq j\leq 2l+1, 0\leq k\leq l).$$

(7) $D_{2l+1}^{(2)}\supset A_{2l}^{(2)}$ $(s, s'=0$ or $1)$

1°. $D_{2l+1}^{(2)}$: $(\tilde{\Lambda}, \tilde{\delta})=2m+s+2s'$

$D_{m+1}^{(2)}$: $(\tilde{\Lambda}^\dagger, \tilde{\delta}^\dagger)=4l+s+2s'$

$\tilde{P}$: $\{\tilde{\lambda}_i\}_{1\leq i\leq 2l}\cup\{\tilde{\lambda}_j^\dagger\}_{1\leq j\leq m}$
$$=\left\{2l+m-\frac{1}{2}+\frac{s'}{2}, \cdots, \frac{3+s'}{2}, \frac{1+s'}{2}\right\}.$$

2°. $A_{2l}^{(2)}$: $(\Lambda, \delta)=2m+s+2s'$

$A_{2(m+s')}^{(2)}$: $(\Lambda^\dagger, \delta^\dagger)=2l+s$

P: $\{\lambda_i\}_{1\leq i\leq l}\cup\{\lambda_j^\dagger\}_{1\leq j\leq m+s'}=\{l+m+s', \cdots, 2, 1\}$.

3°.
$$\sum_{\tilde{P}}(-)^{\varepsilon(\tilde{\Lambda}^\dagger,\Lambda^\dagger)}\chi_{\tilde{\Lambda}^\dagger}^{(m)}(\tau, u_1^\dagger, \cdots, u_m^\dagger)e_{\tilde{\Lambda}\Lambda}(\tau)$$
$$=\begin{cases}\mathbf{e}\left[\frac{1}{4}lm\tau+l\sum_{j=1}^{m}u_j^\dagger\right]\chi_{\Lambda^\dagger}^{(m)}\left(\tau, u_1^\dagger+\frac{\tau}{2}, \cdots, u_m^\dagger+\frac{\tau}{2}\right)\frac{\eta(2\tau)}{\eta(\tau)} \\ \qquad\qquad (s=0, s'=0) \\ \mathbf{e}\left[\frac{1}{4}l(m+1)\tau+l\sum_{j=1}^{m}u_j^\dagger\right] \\ \quad\times\chi_{\Lambda^\dagger}^{(m+1)}\left(\tau, u_1^\dagger+\frac{\tau}{2}, \cdots, u_m^\dagger+\frac{\tau}{2}, \frac{1}{2}+\frac{\tau}{2}\right) \\ \quad\times\frac{\eta(2\tau)^2}{\eta(\tau)^2}\prod_{j=1}^{m}\left(-i\eta(\tau)^{-1}\theta\left(\tau, u_j^\dagger+\frac{1}{2}\right)\right) \quad (s=0, s'=1)\end{cases}$$

$$\left|\begin{array}{l}\mathbf{e}\left[\frac{1}{4}\left(l+\frac{1}{2}\right)m\tau+\left(l+\frac{1}{2}\right)\sum_{j=1}^{m}u_j^\dagger\right]\\ \qquad\times\chi_{\tilde{A}^\dagger}^{(m)}\left(\tau, u_1^\dagger+\frac{\tau}{2}, \cdots, u_m^\dagger+\frac{\tau}{2}\right)\\ \qquad\times\frac{\eta(2\tau)^2}{\eta(\tau)^2}\prod_{j=1}^{m}\left(\frac{\eta(\tau)\hat{\theta}(2\tau, u_j^\dagger+\frac{1}{2})}{-i\eta(2\tau)\theta(\tau, u_j^\dagger+\frac{1}{2})}\right) \quad (s=1, s'=0),\end{array}\right.$$

$$\varepsilon(\tilde{\Lambda}^\dagger, \Lambda^\dagger)=\sum_{j=1}^{m}\left(\tilde{\lambda}_j^\dagger+\frac{1-s'}{2}\right)+\sum_{j=1}^{m+s'}\lambda_j^\dagger+(m+s')s'+lm.$$

4°.[†] $(m=1)$

$$e_{\tilde{\Lambda}_j', \Lambda_k'}(\tau)=\begin{cases}\eta(\tau)^{-1}\eta(2\tau)^{-1}H^{2l+1}_{j+l+1, k+l+1}(2\tau) & (j\equiv k \bmod 2)\\ \eta(\tau)^{-1}\eta(2\tau)^{-1}H^{2l+1}_{l-j, k+l+1}(2\tau) & (j\not\equiv k \bmod 2)\end{cases}$$

$(0\leqq j\leqq 2l,\ 0\leqq k\leqq l)$,

$$e_{\tilde{\Lambda}_0+\tilde{\Lambda}_{2l}, \Lambda_k'}(\tau)=\eta(2\tau)^2\eta(\tau)^{-3}\mathbf{e}\left[-\frac{2k+1}{4}\right]\vartheta_{2k+1, 4l+6}\left(\frac{\tau}{2}, \frac{1}{4}\right)$$

$(0\leqq k\leqq l)$.

(8) $D^{(2)}_{2l+2}\supset D^{(2)}_{l+1}$ $(s, s'=0 \text{ or } 1)$

1°. $D^{(2)}_{2l+2}$: $(\tilde{\Lambda}, \tilde{\delta})=2m+s+2s'$

$D^{(2)}_{m+1}$: $(\tilde{\Lambda}^\dagger, \tilde{\delta}^\dagger)=4l+2+s+2s'$

$\tilde{P}$: $\{\tilde{\lambda}_i\}_{1\leqq i\leqq 2l+1}\cup\{\tilde{\lambda}_j^\dagger\}_{1\leqq j\leqq m}$

$$=\left\{2l+m+\frac{1+s'}{2}, \cdots, \frac{3+s'}{2}, \cdots, \frac{1+s'}{2}\right\}.$$

2°. $D^{(2)}_{l+1}$: $(\Lambda, \delta)=2m+s+2s'$

$D^{(2)}_{m+s'+1}$: $(\Lambda^\dagger, \delta^\dagger)=2l+s$

P: $\{\lambda_i\}_{1\leqq i\leqq l}\cup\{\lambda_j^\dagger\}_{1\leqq j\leqq m+s'}=\left\{l+m+s'-\frac{1}{2}, \cdots, \frac{3}{2}, \frac{1}{2}\right\}.$

3°. $\sum_{\tilde{P}}(-)^{\varepsilon(\tilde{\Lambda}^\dagger, \Lambda^\dagger)}\chi_{\tilde{\Lambda}^\dagger}^{(m)}(2\tau, u_1^\dagger, \cdots, u_m^\dagger)e_{\tilde{\Lambda}\Lambda}(\tau)$

$$=\begin{cases}\mathbf{e}\left[\frac{1}{2}lm\tau+l\sum_{j=1}^{m}u_j^\dagger\right]\chi_{\Lambda^\dagger}^{(m)}(\tau, u_1^\dagger+\tau, \cdots, u_m^\dagger+\tau)\\ \qquad\times\frac{\eta(4\tau)\eta(\tau)}{\eta(2\tau)^2}\prod_{j=1}^{m}(\eta(2\tau)^{-1}\hat{\theta}(2\tau, u_j^\dagger)) \quad (s=0, s'=0)\\ \mathbf{e}\left[\frac{1}{2}l(m+1)\tau+l\sum_{j=1}^{m}u_j^\dagger\right]\\ \qquad\times\chi_{\Lambda^\dagger}^{(m+1)}\left(\tau, u_1^\dagger+\tau, \cdots, u_m^\dagger+\tau, \frac{1}{2}+\tau\right)\end{cases}$$

$$\left|\begin{array}{l} \times\dfrac{\eta(2\tau)}{\eta(\tau)}\prod_{j=1}^{m}\left(-i\eta(2\tau)^{-2}\theta\left(2\tau,u_j^\dagger+\dfrac{1}{2}\right)\hat{\theta}(2\tau,u_j^\dagger)\right) \\ \hfill (s=0,\ s'=1) \\ \mathbf{e}\left[\dfrac{1}{2}\left(l+\dfrac{1}{2}\right)m\tau+\left(l+\dfrac{1}{2}\right)\sum_{j=1}^{m}u_j^\dagger\right] \\ \quad\times\chi_{\Lambda^\dagger}^{(m)}(\tau,u_1^\dagger+\tau,\cdots,u_m^\dagger+\tau)\dfrac{\eta(4\tau)\eta(\tau)}{\eta(2\tau)^2} \\ \quad\times\prod_{j=1}^{m}\left(-\dfrac{\hat{\theta}(2\tau,u_j^\dagger)\hat{\theta}(4\tau,u_j^\dagger+\frac{1}{2})}{i\eta(4\tau)\theta(2\tau,u_j^\dagger+\frac{1}{2})}\right)\quad (s=1,\ s'=0), \end{array}\right.$$

$$\varepsilon(\tilde{\Lambda}^\dagger,\Lambda^\dagger)=\sum_{j=1}^{m}\left(\tilde{\lambda}_j^\dagger+\frac{1-s'}{2}\right)+\sum_{j=1}^{m+s'}\left(\lambda_j^\dagger+\frac{1}{2}\right)+ms'+s'+lm.$$

4°.[†] $(m=1)$

$$e_{\tilde{\Lambda}_j',\Lambda_k'}(\tau)=\begin{cases}\eta(2\tau)^{-1}\eta(4\tau) & (j=k \text{ or } 2l+1-k)\\ 0 & (\text{otherwise})\end{cases}$$

$$(0\leqq j\leqq 2l+1,\ 0\leqq k\leqq l),$$

$$e_{\tilde{\Lambda}_0+\tilde{\Lambda}_{2l+1},\Lambda_k'}(\tau)=\eta(2\tau)^{-1}\vartheta_{2k+1,4l+4}(\tau,0)\quad (0\leqq k\leqq l),$$

$$e_{\tilde{\Lambda}_0+\tilde{\Lambda}_j',\Lambda_0+\Lambda_k'}(\tau)=\eta(2\tau)^{-1}\eta(4\tau)^{-1}(H'^{4l+4}_{2j+1,2k+1}(2\tau)-H'^{4l+4}_{2j+1,2k+4l+7}(2\tau))$$

$$(0\leqq j\leqq 2l+1,\ 0\leqq k\leqq l).$$

(9) $D_{l+1}^{(1)}\supset B_l^{(1)}$ $(s'=0 \text{ or } 1,\ s=0 \text{ or } 1)$

1°. $D_{l+1}^{(1)}$: $(\tilde{\Lambda},\tilde{\delta})=2m+s+s'$

$D_m^{(1)}$: $(\tilde{\Lambda}^\dagger,\tilde{\delta}^\dagger)=2l+2+s+s'$

$$\tilde{P}:\ \{\tilde{\lambda}_i\}_{1\leqq i\leqq l+1}\cup\{\tilde{\lambda}_j^\dagger\}_{1\leqq j\leqq m}=\left\{l+m+\frac{s}{2},\cdots,\frac{s}{2}\right\}$$

2°. $B_l^{(1)}$: $(\Lambda,\delta)=2m+s+s'$

$B_m^{(1)}$: $(\Lambda^\dagger,\delta^\dagger)=2l+s+s'$

$$P:\ \{\lambda_i\}_{1\leqq i\leqq l}\cup\{\lambda_j^\dagger\}_{1\leqq j\leqq m}=\left\{l+m+\frac{s-1}{2},\cdots,\frac{s+1}{2}\right\}$$

3°.
$$\sum_{\tilde{P}}\ \sum_{(\sigma,\sigma^\dagger)=(\mathrm{id},\mathrm{id}),(\tilde{\sigma}_1,\tilde{\sigma}_1^\dagger),(\tilde{\sigma}_2,\tilde{\sigma}_2^\dagger),(\tilde{\sigma}_1\tilde{\sigma}_2,\tilde{\sigma}_1^\dagger\tilde{\sigma}_2^\dagger)}\chi^{(m)}_{\sigma^\dagger(\tilde{\Lambda}^\dagger)}(\tau,u^\dagger)(-)^{\varepsilon(\tilde{\Lambda}^\dagger,\Lambda^\dagger)}e_{\sigma(\tilde{\Lambda})\Lambda}(\tau)$$

$$=\left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{s'-s}\frac{\prod_{1\leqq j\leqq m}\theta(\tau,u_j^\dagger)^2}{\eta(\tau)^{2m}}\sum_{\sigma^\dagger=\mathrm{id},\sigma_1^\dagger}\xi_{\sigma^\dagger}(\tau)\chi^{(m)}_{\sigma^\dagger(\Lambda^\dagger)}(\tau,u^\dagger).$$

$\tilde{\sigma}_i$, $\tilde{\sigma}_i^\dagger$, $\sigma_i^\dagger$ are the automorphisms in Table II,

$$\varepsilon(\tilde{\Lambda}^\dagger,\Lambda^\dagger)=\sum_{1\leqq j\leqq m}\left(\tilde{\lambda}_j^\dagger-\lambda_j^\dagger-\frac{1}{2}\right)$$

$$\xi_\sigma(\tau)=\begin{cases} 1 & s'=0,\ \lambda_1^\dagger \neq l+m+\dfrac{s-1}{2} \\ 1/2 & s'=0,\ \lambda_1^\dagger = l+m+\dfrac{s-1}{2} \\ \dfrac{\eta(\tau)}{2}\left(\dfrac{1}{\eta(\tau/2)}+\dfrac{\mathrm{e}[1/48]}{\eta((\tau+1)/2)}\right) & s'=1,\ \sigma=\mathrm{id} \\ \dfrac{\eta(\tau)}{2}\left(\dfrac{1}{\eta(\tau/2)}-\dfrac{\mathrm{e}[1/48]}{\eta((\tau+1)/2)}\right) & s'=1,\ \sigma=\sigma_1^\dagger \end{cases}$$

4°. $$\left(\frac{\eta(\eta)}{\eta(2\tau)}\right)^{s-s'}\eta(\tau)^m e(\tau)$$
$$=C\cdot\sum_{\sigma^\dagger=\mathrm{id},\sigma_1^\dagger}\operatorname{sgn}\sigma^\dagger\operatorname{sgn}\tilde{\sigma}\sum_{w\in\mathfrak{S}_m\ltimes\{\pm1\}^m}\xi_{\sigma^\dagger}(\tau)\operatorname{sgn}w$$
$$\times\Theta^M_{(N+1)w\overline{(\sigma^\dagger(\Lambda^\dagger)+\rho^\dagger)}-N\overline{(\tilde{\sigma}^\dagger(\tilde{\Lambda}^\dagger)+\tilde{\rho})},N(N+1)}(\tau,0)$$

where

$$e(\tau)=e_{\tilde{\sigma}(\tilde{\Lambda})\Lambda}(\tau),\quad(\tilde{\sigma},\tilde{\sigma}^\dagger)=(\mathrm{id},\mathrm{id})\text{ or }(\tilde{\sigma}_1,\tilde{\sigma}_1^\dagger)\quad\text{if }\tilde{\sigma}_1^\dagger(\tilde{\Lambda}^\dagger)\neq\tilde{\Lambda}^\dagger$$
$$e(\tau)=e_{\tilde{\Lambda}\Lambda}(\tau)+e_{\tilde{\sigma}_1(\tilde{\Lambda})\Lambda}(\tau),\quad(\tilde{\sigma},\tilde{\sigma}^\dagger)=(\mathrm{id},\mathrm{id})\quad\text{if }\tilde{\sigma}_1^\dagger(\tilde{\Lambda}^\dagger)=\tilde{\Lambda}^\dagger$$
$$C=\begin{cases}1 & (\tilde{\sigma}_2^\dagger(\tilde{\Lambda}^\dagger)\neq\tilde{\Lambda}^\dagger)\\ 1/2 & (\tilde{\sigma}_2^\dagger(\tilde{\Lambda}^\dagger)=\tilde{\Lambda}^\dagger)\end{cases}$$
$$N=2l+2m-1+s+s',$$
$$\operatorname{sgn}\sigma^\dagger=\begin{cases}1 & \sigma^\dagger=\mathrm{id}\\ -1 & \sigma^\dagger=\sigma_1^\dagger\end{cases}$$
$$\operatorname{sgn}\tilde{\sigma}=\begin{cases}1 & \tilde{\sigma}=\mathrm{id}\\ -1 & \tilde{\sigma}=\tilde{\sigma}_1\end{cases}$$
$$\operatorname{sgn}w=\begin{cases}\det w & s=1\\ \det w_1 & s=0,\quad w=(w_1,\varepsilon)\in\mathfrak{S}_m\ltimes\{\pm1\}^m.\end{cases}$$

Θ^M signifies the classical theta function defined in Section 2 on the lattice

$$M=\left\{\sum_{j=1}^m\nu_j\varepsilon_j^\dagger\,\middle|\,\sum_{j=1}^m\nu_j\in2\mathbf{Z},\ \forall\nu_j\in\mathbf{Z}\right\}$$
$$\cong\left\{\sum_{j=1}^m\nu_j\tilde{\varepsilon}_j^\dagger\,\middle|\,\sum_{j=1}^m\nu_j\in2\mathbf{Z},\ \forall\nu_j\in\mathbf{Z}\right\}$$
(identified through $\varepsilon_j^\dagger=\tilde{\varepsilon}_j^\dagger$ for $\forall j$)

(10) $D_{2l}^{(1)}\supset A_{2l-1}^{(2)}$

1°. $D_{2l}^{(1)}$; $(\tilde{\Lambda},\tilde{\delta})=2m$

$D_m^{(1)}$: $(\tilde{\Lambda}^\dagger,\tilde{\delta}^\dagger)=4l$

$\tilde{P}$: $\{\tilde{\lambda}_i\}_{1\leq i\leq 2l}\cup\{\tilde{\lambda}_j^\dagger\}_{1\leq j\leq m}=\{2l+m-1,\cdots,1,0\}$.

2°. $A_{2l-1}^{(2)}$: $(\Lambda, \delta)=2m$

$A_{2m-1}^{(2)}$: $(\Lambda^{\dagger}, \delta^{\dagger})=2l$

P: $\{\lambda_i\}_{1\leqq i\leqq l}\cup\{\lambda_j^{\dagger}\}_{1\leqq j\leqq m}=\{l+m, \cdots, 2, 1\}$.

3°. $$\sum_{\tilde{P}} \sum_{(\sigma,\sigma^{\dagger})=(id,id),(\bar{\sigma}_1,\bar{\sigma}_1^{\dagger}),(\bar{\sigma}_2,\bar{\sigma}_2^{\dagger}),(\bar{\sigma}_1\bar{\sigma}_2,\bar{\sigma}_1^{\dagger}\bar{\sigma}_2^{\dagger})} \times\chi_{\sigma^{\dagger}(\tilde{\Lambda}^{\dagger})}^{(m)}(2\tau, u_1^{\dagger}, \cdots, u_m^{\dagger})(-)^{\varepsilon(\tilde{\Lambda}^{\dagger},\Lambda^{\dagger})}e_{\sigma(\tilde{\Lambda})\Lambda}(\tau)$$

$$=\frac{\eta(2\tau)}{\eta(\tau)}\mathbf{e}\left[\frac{1}{4}lm\tau+l\sum_{j=1}^{m}u_j^{\dagger}\right]C\sum_{\sigma^{\dagger}=\mathrm{id},\sigma_1}\chi_{\sigma^{\dagger}(\Lambda^{\dagger})}^{(m)}\left(\tau, u_1^{\dagger}+\frac{\tau}{2}, \cdots, u_m^{\dagger}+\frac{\tau}{2}\right)$$

$$\varepsilon(\tilde{\Lambda}^{\dagger}, \Lambda^{\dagger})=\sum_{j=1}^{m}(\tilde{\lambda}_j^{\dagger}+\lambda_j^{\dagger}-1)+lm$$

$$C=\begin{cases}1 & \sigma_1^{\dagger}(\Lambda^{\dagger})\neq\Lambda^{\dagger}\\ 1/2 & \sigma_1^{\dagger}(\Lambda^{\dagger})=\Lambda^{\dagger}\end{cases}$$

4°.[†] $(m=1)$

$$e_{\tilde{\Lambda}_j'\Lambda_k'}(\tau)=\xi_k\eta(\tau)^{-1}\eta(2\tau)^{-1}H_{jk}^{\prime\prime l}(\tau)\quad(1\leqq j\leqq 2l-1)$$

$$e_{2\tilde{\Lambda}_0,\Lambda_k'}(\tau)+e_{2\tilde{\Lambda}_1,\Lambda_k'}(\tau)=\xi_k\eta(\tau)^{-1}\eta(2\tau)^{-1}H_{0k}^{\prime\prime l}(\tau)$$

$$e_{2\tilde{\Lambda}_{2l},\Lambda_k'}(\tau)+e_{2\tilde{\Lambda}_{2l-1},\Lambda_k'}(\tau)=\xi_k\eta(\tau)^{-1}\eta(2\tau)^{-1}H_{0k}^{\prime\prime l}(\tau)$$

where

$$\xi_k=\begin{cases}1 & (0<k\leqq l)\\ 1/2 & (k=0).\end{cases}$$

(11) $D_{2l+1}^{(1)}\supset B_l^{(1)}$

1°. $D_{2l+1}^{(1)}$: $(\tilde{\Lambda}, \tilde{\delta})=2m$

$D_m^{(1)}$: $(\tilde{\Lambda}^{\dagger}, \tilde{\delta}^{\dagger})=2(2l+1)$

$\tilde{P}$: $\{\tilde{\lambda}_i\}_{1\leqq i\leqq 2l+1}\cup\{\tilde{\lambda}_j^{\dagger}\}_{1\leqq j\leqq m}=\{2l+m, \cdots, 1, 0\}$.

2°. $B_l^{(1)}$: $(\Lambda, \delta)=2m$

$B_m^{(1)}$: $(\Lambda^{\dagger}, \delta^{\dagger})=2l$

P: $\{\lambda_i\}_{1\leqq i\leqq l}\cup\{\lambda_j^{\dagger}\}_{1\leqq j\leqq m}=\left\{l+m-\frac{1}{2}, \cdots, \frac{3}{2}, \frac{1}{2}\right\}$.

3°. $$\sum_{\tilde{P}} \sum_{(\sigma,\sigma^{\dagger})=(id,id),(\bar{\sigma}_1,\bar{\sigma}_1^{\dagger}),(\bar{\sigma}_2,\bar{\sigma}_2^{\dagger}),(\bar{\sigma}_1\bar{\sigma}_2,\bar{\sigma}_1^{\dagger}\bar{\sigma}_2^{\dagger})} \times\chi_{\sigma^{\dagger}(\tilde{\Lambda}^{\dagger})}^{(m)}(2\tau, u_1^{\dagger}, \cdots, u_m^{\dagger})(-)^{\varepsilon(\tilde{\Lambda}^{\dagger},\Lambda^{\dagger})}e_{\sigma(\tilde{\Lambda})\Lambda}(\tau)$$

$$=\mathbf{e}\left[\frac{1}{4}lm\tau+l\sum_{j=1}^{m}u_j^{\dagger}\right]C\sum_{\sigma^{\dagger}=\mathrm{id},\sigma_1^{\dagger}}\chi_{\sigma^{\dagger}(\Lambda^{\dagger})}^{(m)}\left(\tau, u_1^{\dagger}+\frac{\tau}{2}, \cdots, u_m^{\dagger}+\frac{\tau}{2}\right)$$

$$\times\prod_{j=1}^{m}(\eta(\tau)^{-1}\hat{\theta}(\tau, u_j^{\dagger}))$$

$$\varepsilon(\tilde{\Lambda}^\dagger, \Lambda^\dagger) = \sum_{j=1}^{m} \left(\tilde{\lambda}_j^\dagger + \lambda_j^\dagger - \frac{1}{2} \right) + lm$$

$$C = \begin{cases} 1 & \sigma_1^\dagger(\Lambda^\dagger) \neq \Lambda^\dagger \\ 1/2 & \sigma_1^\dagger(\Lambda^\dagger) = \Lambda^\dagger. \end{cases}$$

4°.[†] $(m=1)$

$$e_{\tilde{\Lambda}_k{}'\Lambda_k{}'}(\tau) = e'_{2l+1-k,\Lambda_k{}'}(\tau) = \eta(\tau)^{-1}\eta(2\tau) \quad (1 \leqq k \leqq l)$$

$$e_{2\tilde{\Lambda}_0 2\Lambda_k}(\tau) + e_{2\tilde{\Lambda}_1 2\Lambda_k}(\tau) = e_{2\tilde{\Lambda}_{2l} 2\Lambda_k}(\tau) + e_{2\tilde{\Lambda}_{2l+1} 2\Lambda_k}(\tau) = \eta(\tau)^{-1}\eta(2\tau)$$

$(k=0 \text{ or } 1)$

all other $e_{\tilde{\sigma}(\tilde{\Lambda}_{j'}), \sigma(\Lambda_{k'})}(\tau) = 0 \quad (\tilde{\sigma} = \mathrm{id}, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_1\tilde{\sigma}_2;\ \sigma = \mathrm{id}, \sigma_1).$

(12) $A_{2l+1}^{(2)} \supset A_{2l}^{(2)} \quad (s' = 0 \text{ or } 1)$

1°. $A_{2l+1}^{(2)}: (\tilde{\Lambda}, \tilde{\delta}) = 2m + s'$

$A_{2m-1}^{(2)}: (\tilde{\Lambda}^\dagger, \tilde{\delta}^\dagger) = 2l + 2 + s'$

$\tilde{P}: \{\tilde{\lambda}_i\}_{1 \leqq i \leqq l+1} \cup \{\tilde{\lambda}_j^\dagger\}_{1 \leqq j \leqq m} = \{l + m + 1, \cdots, 1\}$

2°. $A_{2l}^{(2)}: (\Lambda, \delta) = 2m + s'$

$A_{2m}^{(2)}: (\Lambda^\dagger, \delta^\dagger) = 2l + s'$

$P: \{\lambda_i\}_{1 \leqq i \leqq l} \cup \{\lambda_j^\dagger\}_{1 \leqq j \leqq m} = \{l + m, \cdots, 1\}$

3° $\displaystyle\sum_{\tilde{P}} \sum_{\sigma = \mathrm{id}, \tilde{\sigma}_1 \dagger} \chi_{\sigma(\tilde{\Lambda}^\dagger)}^{(m)}(\tau, u^\dagger)(-)^{\varepsilon(\tilde{\Lambda}^\dagger, \Lambda^\dagger)} e_{\tilde{\Lambda}\Lambda}(\tau)$

$$= \left(\frac{\eta(\tau)}{\eta(2\tau)} \right)^{1-s'} \frac{\prod_{1 \leqq j \leqq m} \hat{\theta}(\tau, u_j^\dagger)^2}{\eta(\tau)^{2m}} \chi_{\Lambda^\dagger}^{(m)}(\tau, u^\dagger),$$

$\tilde{\sigma}_1^\dagger: \tilde{\lambda}_1^\dagger \leftrightarrow 2(l + m + 1) + s' - \tilde{\lambda}_1^\dagger,$

$\varepsilon(\tilde{\Lambda}^\dagger, \Lambda^\dagger) = \sum_{1 \leqq j \leqq m} (\tilde{\lambda}_j^\dagger - \lambda_j^\dagger)$

4°. $$\eta(\tau)^m \left(\frac{\eta(\tau)}{\eta(2\tau)} \right)^{s'} e_{\tilde{\Lambda}\Lambda}(\tau) = C \cdot \det \Big(\sum_{\varepsilon = \pm 1} \varepsilon \vartheta_{(N+1)\varepsilon\lambda_j^\dagger - N\tilde{\lambda}_k^\dagger, N(N+1)}^{(s')}(\tau, 0) \Big)$$

$(j, k = 1, \cdots, m)$

$N = 2l + 2m + 1 + s'$

$$C = \begin{cases} 1 & \tilde{\sigma}_1^\dagger(\tilde{\Lambda}^\dagger) \neq \tilde{\Lambda}^\dagger \\ 1/2 & \tilde{\sigma}_1^\dagger(\tilde{\Lambda}^\dagger) = \tilde{\Lambda}^\dagger \end{cases}$$

$$\vartheta^{(s')} = \begin{cases} \hat{\vartheta} & s' = 0 \\ \vartheta & s' = 1 \end{cases}$$

(13) $B_{l+1}^{(1)} \supset D_{l+1}^{(2)} \quad (s' = 0 \text{ or } 1,\ s = 0 \text{ or } 1)$

1°. $B_{l+1}^{(1)}: (\tilde{\Lambda}, \tilde{\delta}) = 2m + s + s'$

$B_m^{(1)}: (\tilde{\Lambda}^\dagger, \tilde{\delta}^\dagger) = 2l + 2 + s + s'$

$$\tilde{P}\colon \{\tilde{\lambda}_i\}_{1\leq i\leq l+1}\cup\{\tilde{\lambda}_j^\dagger\}_{1\leq j\leq m}=\left\{l+m+\frac{s+1}{2},\ \cdots,\ \frac{s+1}{2}\right\}$$

2° $D_{l+1}^{(2)}$: $(\Lambda, \delta)=2m+s+s'$

$D_{m+1}^{(2)}$: $(\Lambda^\dagger, \delta^\dagger)=2l+s+s'$

$$P\colon \{\lambda_i\}_{1\leq j\leq l}\cup\{\lambda_j^\dagger\}_{1\leq j\leq m}=\left\{l+m+\frac{s-1}{2},\ \cdots,\ \frac{s+1}{2}\right\}$$

3°.
$$\sum_{\tilde{P}}\sum_{\sigma=\mathrm{id},\tilde{\sigma}_1^\dagger}\chi^{(m)}_{\sigma(\tilde{\Lambda}^\dagger)}(2\tau, u^\dagger)(-)^{\varepsilon(\tilde{\Lambda}^\dagger,\Lambda^\dagger)}e_{\tilde{\Lambda}\Lambda}(\tau)$$
$$=\left(\frac{\eta(4\tau)}{\eta(2\tau)}\right)^{s'-s}\left(\frac{\eta(\tau)}{\eta(2\tau)}\right)^{1-s}\frac{\prod_{1\leq j\leq m}\hat{\theta}(2\tau, u_j^\dagger)^2}{\eta(2\tau)^{2m}}\chi^{(m)}_{\Lambda^\dagger}(\tau, u^\dagger),$$

$\tilde{\sigma}_1^\dagger$: $\tilde{\lambda}_1^\dagger\leftrightarrow 2(l+m)+s+s'+1-\tilde{\lambda}_1^\dagger$,

$\varepsilon(\tilde{\Lambda}^\dagger, \Lambda^\dagger)=\sum_{1\leq j\leq m}(\tilde{\lambda}_j^\dagger-\lambda_j^\dagger)$

4°.
$$\eta(2\tau)^m\left(\frac{\eta(2\tau)}{\eta(4\tau)}\right)^{s'-s}\left(\frac{\eta(2\tau)}{\eta(\tau)}\right)^{-s}e_{\tilde{\Lambda}\Lambda}(\tau)$$
$$=C\cdot\det\left((-)^{(1-\varepsilon)(1-s)/2}\sum_{\varepsilon=\pm1}\varepsilon\vartheta^{(s+s')}_{(N+1)\varepsilon\lambda_j^\dagger-N\tilde{\lambda}_k^\dagger,N(N+1)}(2\tau, 0)\right)$$
$$(j, k=1, \cdots, m)$$

$N=2l+2m+s+s'$

$$C=\begin{cases}1 & \tilde{\sigma}_1^\dagger(\tilde{\Lambda}^\dagger)\neq\tilde{\Lambda}^\dagger\\ 1/2 & \tilde{\sigma}_1^\dagger(\tilde{\Lambda}^\dagger)=\tilde{\Lambda}^\dagger\end{cases}$$

$$\vartheta^{(s+s')}=\begin{cases}\hat{\vartheta} & s+s'\colon \text{odd}\\ \vartheta & s+s'\colon \text{even}\end{cases}$$

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†) Λ'_j signifies the following weight.

$A_{2l}^{(2)}$: $2\Lambda_0$ $(j=0)$, Λ_j $(1\leqq j\leqq l)$

$D_{l+1}^{(2)}$: $2\Lambda_0$ $(j=0)$, Λ_j $(1\leqq j\leqq l-1)$, $2\Lambda_l$ $(j=l)$

$D_l^{(1)}$: $2\Lambda_0$ $(j=0)$, $\Lambda_0+\Lambda_1$ $(j=1)$, Λ_j $(2\leqq j\leqq l-2)$, $\Lambda_{l-1}+\Lambda_l$ $(j=l-1)$, $2\Lambda_l$ $(j=l)$

$A_{2l-1}^{(2)}$: $2\Lambda_0$ $(j=0)$, $\Lambda_0+\Lambda_1$ $(j=1)$, Λ_j $(2\leqq j\leqq l)$

$B_l^{(1)}$: $2\Lambda_0$ $(j=0)$, $\Lambda_0+\Lambda_1$ $(j=1)$, Λ_j $(2\leqq j\leqq l-1)$, $2\Lambda_l$ $(j=l)$.

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