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On pluricanonical systems of algebraic varieties of general type

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Abstract.

We extend Kollár's technique to look for an explicit function $h(n)$ with φ_m birational onto its image for all integers $m \geq h(n)$ and for all n-dimensional nonsingular projective varieties of general type.

§1. Introduction

One of the fundamental problems in birational geometry is to find a constant $r_n > 0$ such that the r_n-canonical map is an Iitaka fibration for any n-dimensional projective variety with positive Kodaira dimension. It is well-known that one may take $r_2 = 5$ (see Bombieri [2]) for surfaces of general type. The existence of r_n for varieties of general type was proved by Hacon–M^ckernan [10], Takayama [19] and Tsuji [20] and, recently, $r_3 \leq 73$ was proved by Chen–Chen [6, Theorem 1.1]. Some other relevant results with regard to the existence of r_n have been already proved by Chen–Hacon [3], Pacienza [15] and Viehweg–Zhang [23]. The following problem, however, is still open:

Problem 1.1. To find an explicit constant μ_n ($n \geq 4$) such that the m-canonical map φ_m is birational onto its image for all $m \geq \mu_n$ and for all n-dimensional projective varieties of general type.

Let V be a n-dimensional nonsingular projective variety of general type. Denote by K_V a canonical divisor on V. A reasonable strategy for studying Problem 1.1 is composed of two steps:

[a] To find a positive integer m_0 such that $h^0(V, m_0K_V) \geq 2$;

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[b] To find an explicit function $g(m_0, n)$ such that the m-canonical map φ_m is birational for all $m \geq g(m_0, n)$.

This strategy works well in dimension 3 (see, for instance, [5, 6]). In fact, since Reid [17] has found the Riemann–Roch formula for minimal 3-folds, we [5, Theorem 1.1] managed to prove $h^0(V, m_0K_V) \geq 2$ by utilizing that formula. On the other hand, Kollár [14] and myself [8] have an effective formula in dimension 3 for Step (b). Generally, for Step [a], since the classification to 4-dimensional terminal singularities is still incomplete, there is no known Riemann–Roch formula for $\chi(mK)$ on minimal varieties. Thus to compute $P_m := h^0(V, m_0K_V)$ is still expected. Though Kollár [14] has essentially solved Step [b], what we are more concerned here is the stable birationality, to be worked out by an improved and generalized technique, of linear systems $|mK + \lceil Q \rceil|$ where Q is any nef $\mathbb{Q}$-divisor. Hence this paper can be regarded as a remark or complementary to Kollár's method. We will build up some new results about induced fibrations from pluricanonical systems.

Assume $P_{m_0}(V) := h^0(V, \mathcal{O}_V(m_0K_V)) \geq 2$ for some positive integer m_0. Set $\varphi_{m_0} := \Phi_{|m_0K_V|}$, the m_0-canonical map of V. We define *the* m_0*-canonical dimension* $\iota := \dim \overline{\varphi_{m_0}(V)}$. Clearly $1 \leq \iota \leq n$. In order to formulate our statements, we introduce the following:

Definition 1.2. Define $\lambda(V)$ to be the smallest positive integer such that $P_{\lambda(V)}(V) \geq 2$ for a given n-dimensional projective variety V of general type. Define $\lambda_n := \sup\{\lambda(V) | \dim(V) = n\}$.

Remark 1.3. According to Hacon–M^{c}kernan [10], Takayama [19] and Tsuji [20], one knows $\lambda_n < +\infty$. Therefore an assumption like $P_{m_0} \geq 2$ is reasonable and natural.

The main result of this paper is the following which, at least, induces new results for the case $n = 4$:

Theorem 1.4. *Let V be a n-dimensional ($n \geq 3$) nonsingular projective variety of general type. Let Q be a nef $\mathbb{Q}$-divisor on V. Assume $P_{m_0} \geq 2$ for some positive integer m_0. Then the linear system $|mK_V + \lceil Q \rceil|$ defines a birational map onto its image for all integers $m \geq \varepsilon(\iota)$ where $\varepsilon(\iota)$ is a function as follows:*

(1) *when $\iota \geq n-2$, $\varepsilon(\iota) = min\{4m_0 + 4, 57\} \cdot (2m_0 + 1)^{n-3} + m_0(n-2) + 2$;*
(2) *when $\iota = n-3$, $\varepsilon(\iota) = 75(2m_0 + 1)^{n-3} + m_0(n-3) + 2$;*
(3) *when $\iota \leq n-4$, $\varepsilon(\iota) = (2m_0+1)^{\iota-1}w_{n-\iota+1} + m_0(\iota-1) + 2$ where $w_{n-\iota+1}$ can be obtained by the number sequence $\{w_t\}_{t=4}^{n-\iota+1}$ with $w_i = \widetilde{\lambda}_i + w_{i-1}(2\widetilde{\lambda}_i + 1)$, $w_4 = 151\widetilde{\lambda}_4 + 75$, $\widetilde{\lambda}_{n-\iota+1} = m_0$ and, for all other i, $\widetilde{\lambda}_i = \lambda_i$.*

By taking $Q = 0$, φ_m is birational for all integers $m \geq \varepsilon(\iota)$.

In particular we have the following:

Corollary 1.5. *Let V be a nonsingular projective n-dimensional ($n \geq 4$) variety of general type. Then φ_m is birational for all integers $m \geq w_n + 2$ where w_n is obtained by the number sequence $\{w_t\}_{t=4}^n$ with $w_i = \lambda_i + w_{i-1}(2\lambda_i + 1)$ and $w_4 = 151\lambda_4 + 75$.*

Corollary 1.6. *(= Corollary 4.3) Let V be a nonsingular projective 4-fold of general type with $P_{m_0} \geq 2$ for some integer $m_0 > 0$. Let Q be any nef $\mathbb{Q}$-divisor on V. Then $\Phi_{|mK_V + \lceil Q \rceil|}$ (in particular, φ_m) is birational for all $m \geq 151m_0 + 77$.*

Theorem 1.4 also implies the following:

Corollary 1.7. *An explicit constant μ_n mentioned in Problem 1.1 can be found by means of Theorem 1.4 if and only if explicit constants ρ_k for all $k \leq n$ can be found such that the pluri-genus $P_{\rho_k} \geq 2$ for all k-dimensional projective varieties of general type.*

From this point of view, a Riemann–Roch formula for $\chi(\mathcal{O}(mK))$ is of key importance just like what Reid has done in [17, last section] for threefolds.

This paper is organized as follows. First we fix the notation for the map φ_{m_0}. Then we systematically study the property, of the induced fibration $f\colon X' \longrightarrow B$, which generalizes known inequalities in 3-dimensional case. In Section 3, we will improve known results on surfaces and 3-folds. Theorems in Section 4 are original. We will prove the main theorem by induction in the last section.

We always use the symbol $\equiv$ to denote numerical equivalence while $\sim$ means linear equivalence.

§2. Properties of canonically induced fibrations

In this paper, V is always a n-dimensional nonsingular projective variety of general type. Let m_0 be a positive integer. Assume that $\Lambda \subset |m_0K_V|$ is a sub-linear system such that $\Phi_\Lambda(V) = 1$ where Φ_Λ is the rational map defined by Λ. We call Λ a *pencil* contained in $|m_0K_V|$. Note that such a pencil always exists. For instance, a 2-dimensional subspace of $H^0(V, m_0K_V)$ corresponds to a pencil. We will study properties of the rational map Φ_Λ in this section.

2.1. Existence of minimal models. By recent works of Birkar–Cascini–Hacon–M^{c}kernan [1], Hacon–M^{c}kernan[12] and Siu [18], V has a minimal model. Again by [12], any fibration $f\colon Y \longrightarrow B$, from a nonsingular variety Y of general type to a smooth curve B, has a relative minimal model. According to the established Minimal Model Program (MMP), one may always assume that a minimal model has at worst $\mathbb{Q}$-factorial terminal singularities.

From this point of view, it suffices to study φ_m on minimal models.

(‡) Throughout X always denotes a minimal model of V.

Because Φ_Λ can be defined on a Zariski open subset of X, we may also regard Λ as a *pencil* on the minimal model X.

2.2. Set up for Φ_Λ. Denote by $\mu\colon V \dashrightarrow X$ the birational contraction map. Because $P_{m_0}(X) = P_{m_0}(V) \geq 2$, we may fix an effective Weil divisor $K_{m_0} \sim m_0 K_X$ on X and a divisor $\widetilde{K}_{m_0} \sim m_0 K_V$ on V. Take successive blow-ups $\pi\colon X' \to X$ along nonsingular centers, such that the following conditions are satisfied:

(i) X' is smooth;
(ii) there is a birational morphism $\pi_V\colon X' \to V$ such that $\mu \circ \pi_V = \pi$;
(iii) the movable part M_0 of $\pi_V^*(\Lambda)$ is base point free and so that $g := \Phi_\Lambda \circ \pi_V$ is a non-constant morphism;
(iv) $\pi^*(K_{m_0}) \cup \pi_V^*(\widetilde{K}_{m_0})$ has simple normal crossing supports;
(v) for certain purpose π even satisfies a couple of extra conditions by further modifying X'. (This condition will be specified in explicit whenever we need it.)

We have a morphism $g\colon X' \longrightarrow W' \subseteq \mathbb{P}^N$. Let $X' \xrightarrow{f} B \xrightarrow{s} W'$ be the Stein factorization of g. We have the following commutative diagram:

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[d, "\pi_V"'] \arrow[dr, "g"] & B \arrow[d, "s"] \\ V \arrow[r, dashed, "\varphi_\Lambda"'] & W'. \end{tikzcd}$$

Denote by $r(X)$ the Cartier index of X. We can write $r(X)K_{X'} = \pi^*(r(X)K_X) + E_\pi$ where E_π is a sum of exceptional divisors. Recall that

$$\pi^*(K_X) := K_{X'} - \frac{1}{r(X)} E_\pi.$$

Clearly, whenever we take the round-up of $m\pi^*(K_X)$ for $m > 0$, we always have $\lceil m\pi^*(K_X)\rceil \le mK_{X'}$.

Denote by $M_{k,X'}$ the movable part of $|kK_{X'}|$ for any positive integer $k > 0$. We may write $m_0K_{X'} =_{\mathbb{Q}} \pi^*(m_0K_X) + E_{\pi,m_0} = M_{m_0,X'} + Z_{m_0}$, where Z_{m_0} is the fixed part of $|m_0K_{X'}|$ and E_{π,m_0} an effective $\mathbb{Q}$-divisor which is a $\mathbb{Q}$-sum of distinct exceptional divisors with regard to π.

Since $M_0 \le M_{m_0,X'} \le \pi^*(m_0K_X)$, we can write $\pi^*(m_0K_X) = M_0 + E'_\Lambda$ where E'_Λ is an effective $\mathbb{Q}$-divisor. By our assumption (Λ is a pencil), B is a nonsingular complete curve. By Bertini's theorem, a general fiber F of the fibration $f\colon X' \longrightarrow B$ is a $(n-1)$-dimensional nonsingular projective variety of general type.

Once a fibration $f\colon X' \longrightarrow B$ is obtained, we may take the relative minimal model $f_0\colon X_0 \longrightarrow B$ of f. Then we can remodify π and π_V again such that a new birational model X'' dominates X' and X_0. Namely assume $\pi'\colon X'' \longrightarrow X'$ and $\pi''\colon X'' \longrightarrow X_0$ are the birational morphisms, set $f'' := f \circ \pi'$, then $f'' = f_0 \circ \pi''$.

Therefore we can make the following:

2.3. Assumption on X'. To avoid too complicated notations, we may assume from the beginning that $X' = X''$ by further birational modifications, i.e. there is a contraction morphism $\theta : X' \longrightarrow X_0$ such that $f = f_0 \circ \theta$ and that f_0 is a relative minimal model of f.

With this assumption, we pick up a general fiber F_0 of f_0 and set $\sigma := \theta|_F$, then $\sigma\colon F \longrightarrow F_0$ is a birational morphism onto the minimal model.

Set $b := g(B)$. We will study the geometry of f according to the value of b. In fact there are two cases:

(i) $b > 0$, $M_0 \sim \sum_{i=1}^{p} F_i \equiv pF$ where the F_i's are different smooth fibers of f for all i and $p \ge 2$;

(ii) $b = 0$, $M_0 \sim pF \le m_0\pi^*(K_X)$ with $p \ge 1$.

2.4. Reduction to problems on X'. As we have seen, there is a birational morphism $\pi_V : X' \longrightarrow V$. Let m be a positive integer and Q a nef $\mathbb{Q}$-divisor on V. Since

$$\pi_{V*}\mathcal{O}_{X'}(mK_{X'} + \pi_V^*(\lceil Q\rceil)) \cong \mathcal{O}_V(mK_V + \lceil Q\rceil)$$

and $mK_{X'} + \pi_V^*(\lceil Q\rceil) \ge mK_{X'} + \lceil Q'\rceil$ where $Q' := \pi_V^*(Q)$ is nef on X', the birationality of $\Phi_{|mK_{X'}+\lceil Q'\rceil|}$ implies that of $\Phi_{|mK_V+\lceil Q\rceil|}$. Furthermore the fact $mK_{X'} \ge m\pi^*(K_X)$ allows us to consider a smaller linear system on X' like:

$$|K_{X'} + \lceil (m-1)\pi^*(K_X) + Q'\rceil|.$$

The 3-dimensional version of the next lemma has appeared as [9, Lemma 3.4].

Lemma 2.5. *Let $f_Y\colon Y \dashrightarrow B_0$ be a rational map onto a smooth curve B_0 where Y is a normal projective minimal variety (i.e. K_Y nef) with at worst terminal singularities. Let $\pi_Y\colon Y' \to Y$ be any birational modification from a nonsingular projective model Y' such that $g_Y := f_Y \circ \pi_Y\colon Y' \longrightarrow B_0$ is a proper morphism. Denote by F_b any irreducible component in a general fiber of g_Y. Assume $g(B_0) > 0$. Then*

$$\mathcal{O}_{F_b}(\pi_Y^*(K_Y)|_{F_b}) \cong \mathcal{O}_{F_b}(\sigma'^*(K_{F_{b,0}}))$$

where $F_{b,0}$ is a minimal model of F_b and there is a contradiction morphism $\sigma'\colon F_b \longrightarrow F_{b,0}$.

Proof. One has a morphism $g_Y\colon Y' \longrightarrow B_0$. By theorems of Shokurov [16] and Hacon–M^{c}kernan [11], each fiber of $\pi_Y\colon Y' \longrightarrow Y$ is rationally chain connected. Therefore, $g_Y(\pi_Y^{-1}(y))$ is a point for all $y \in Y$. Considering the image $G \subset (Y \times B_0)$ of Y' via the morphism $(\pi_Y \times g_y) \circ \triangle_{Y'}$ where $\triangle_{Y'}$ is the diagonal map $Y' \longrightarrow Y' \times Y'$, one knows that G is a projective variety. Let $g_1\colon G \longrightarrow Y$ and $g_2\colon G \longrightarrow B_0$ be two projection maps. Since g_1 is a projective morphism and even a bijective map, g_1 must be both a finite morphism of degree 1 and a birational morphism. Since Y is normal, g_1 must be an isomorphism. So g_Y factors as $f_1 \circ \pi_Y$ where $f_1 := g_2 \circ g_1^{-1}\colon Y \to B_0$ is a well-defined morphism. Let $Y \xrightarrow{f_0} B' \xrightarrow{s'} B_0$ be the Stein factorization of f_1. Set $f' := f_0 \circ \pi_Y$. Then F is a general fiber of f'. Denote by $F_{b,0}$ a general fiber of f_0. Clearly $K_{F_{b,0}} \sim K_Y|_{F_{b,0}}$ is nef and so $F_{b,0}$ is minimal. So it is clear that $\pi_Y^*(K_Y)|_{F_b} \sim \sigma'^*(K_{F_{b,0}})$ where we set $\sigma' := \pi_Y|_{F_b}\colon F_b \longrightarrow F_{b,0}$. Q.E.D.

The above lemma clearly applies to our situation with $b > 0$.

Now we begin to study the case $b = 0$. According to our definition, $M_{m,F}$ denotes the movable part of $|mK_F|$ for any $m > 0$.

Since $B \cong \mathbb{P}^1$, one has $\mathcal{O}_B(1) \hookrightarrow f_*\omega_{X'}^{\otimes m_0}$. Then there is the inclusion:

$$f_*\omega_{X'/B}^{\otimes m} \hookrightarrow f_*\omega_{X'}^{\otimes m(2m_0+1)}$$

for all integers $m > 0$. Because $f_*\omega_{X'/B}^{\otimes m}$ is semi-positive (see Viehweg [22]) and is thus a direct sum of line bundles of non-negative degree, so it is generated by global sections. Therefore any local section of $f_*\omega_{X'/B}^{\otimes m}$ can be extended to a global one of $f_*\omega_{X'}^{\otimes m(2m_0+1)}$. This already means

$$m(2m_0+1)\pi^*(K_X)|_F \geq M_{m(2m_0+1),X'}|_F \geq M_{m,F}.$$

Whenever m is divisible by the Cartier index of F_0, the Base Point Free Theorem says that $M_{m,F} \sim m\sigma^*(K_{F_0})$. Thus we get

$$\pi^*(K_X)|_F \geq \frac{1}{m(2m_0+1)} M_{m(2m_0+1),X'}|_F \geq \frac{1}{2m_0+1}\sigma^*(K_{F_0}). \quad (1)$$

Lemma 2.6. *Keep the same notation as in 2.2. Assume $b=0$ and that a general fiber of f has a Gorenstein minimal model. Then there exists a sequence of positive rational numbers $\{\beta_t\}$, with $\beta_t < \frac{p}{m_0+p}$ and $\beta_t \underset{t\mapsto+\infty}{\mapsto} \frac{p}{m_0+p}$, such that*

$$\pi^*(K_X)|_F - \beta_t\sigma^*(K_{F_0})$$

is $\mathbb{Q}$-linearly equivalent to an effective $\mathbb{Q}$-divisor for all integers $t>0$.

Proof. When $n = \dim(V) = 3$, this is nothing but Theorem [9, Lemma 3.3]. Here we generalize it for any n. One has $\mathcal{O}_B(p) \hookrightarrow f_*\omega_{X'}^{m_0}$ and therefore $f_*\omega_{X'/B}^{t_0p} \hookrightarrow f_*\omega_{X'}^{t_0p+2t_0m_0}$ for any positive integer t_0.

Note that $f_*\omega_{X'/B}^{t_0p}$ is generated by global sections since it is semi-positive according to Viehweg [22]. So any local section can be extended to a global one. On the other hand, whenever t_0 is bigger, $|t_0p\sigma^*(K_{F_0})|$ is free by Base Point Free Theorem and is exactly the movable part of $|t_0pK_F|$ by the ordinary projection formula. Clearly one has the following relation:

$$a_0\pi^*(K_X)|_F \geq M_{t_0p+2t_0m_0,X'}|_F \geq b_0\sigma^*(K_{F_0})$$

where $a_0 := t_0p + 2t_0m_0$ and $b_0 := t_0p$. This means that there is an effective $\mathbb{Q}$-divisor E'_F on F such that

$$a_0\pi^*(K_X)|_F =_{\mathbb{Q}} b_0\sigma^*(K_{F_0}) + E'_F.$$

Thus $\pi^*(K_X)|_F =_{\mathbb{Q}} \frac{p}{p+2m_0}\sigma^*(K_{F_0}) + E_F$ with $E_F = \frac{1}{a_0}E'_F$.

Let us consider the case $p \geq 2$ first.

Assume we have defined a_l and b_l such that the following is satisfied with $l=t$:

$$a_l\pi^*(K_X)|_F \geq b_l\sigma^*(K_{F_0}).$$

We will define a_{t+1} and b_{t+1} inductively such that the above inequality is satisfied with $l=t+1$. One may assume from the beginning (modulo necessary blow-ups) that the fractional part of the support of $a_t\pi^*(K_X)$ is of simple normal crossing. Then the Kawamata–Viehweg vanishing theorem gives the surjective map

$$H^0(K_{X'} + \lceil a_t\pi^*(K_X)\rceil + F) \longrightarrow H^0(F, K_F + \lceil a_t\pi^*(K_X)\rceil|_F).$$

One has the relation

$$\begin{aligned} |K_{X'} + \lceil a_t\pi^*(K_X)\rceil + F||_F &= |K_F + \lceil a_t\pi^*(K_X)\rceil|_F| \\ &\supset |K_F + b_t\sigma^*(K_{F_0})| \\ &\supset |(b_t+1)\sigma^*(K_{F_0})|. \end{aligned}$$

Denote by $M'_{a_t+1,X'}$ the movable part of $|(a_t+1)K_{X'}+F|$. Applying [7, Lemma 2.7], one has $M'_{a_t+1,X'}|_F \geq (b_t+1)\sigma^*(K_{F_0})$. Re-modifying our original π such that $|M'_{a_t+1,X'}|$ is base point free. In particular, $M'_{a_t+1,X'}$ is nef. Since X is of general type $|mK_X|$ gives a birational map whenever m is big enough. Thus we see that $M'_{a_t+1,X'}$ is big if we fix a very big t_0 in advance.

Now the Kawamata–Viehweg vanishing theorem again gives

$$\begin{aligned} |K_{X'} + M'_{a_t+1,X'} + F||_F &= |K_F + M'_{a_t+1,X'}|_F| \\ &\supset |K_F + (b_t+1)\sigma^*(K_{F_0})| \\ &\supset |(b_t+2)\sigma^*(K_{F_0})|. \end{aligned}$$

Repeat the above procedure and denote by $M'_{a_t+u,X'}$ the movable part of $|K_{X'}+M'_{a_t+u-1,X'}+F|$ for integers $u \geq 2$. For the same reason, we may assume $|M'_{a_t+u,X'}|$ is base point free and is thus nef and big. Inductively, for any $u > 0$, one has:

$$M'_{a_t+u,X'}|_F \geq (b_t+u)\sigma^*(K_{F_0}).$$

Applying the vanishing theorem once more, one has

$$\begin{aligned} |K_{X'} + M'_{a_t+u,X'} + F||_F &= |K_F + M'_{a_t+u,X'}|_F| \\ &\supset |K_F + (b_t+u)\sigma^*(K_{F_0})| \\ &\supset |(b_t+u+1)\sigma^*(K_{F_0})|. \end{aligned}$$

Take $u = p-1$. Noting that

$$|K_{X'} + M'_{a_t+p-1,X'} + F| \subset |(a_t+p+m_0)K_{X'}|$$

and applying [7, Lemma 2.7] again, one has

$$a_{t+1}\pi^*(K_X)|_F \geq M_{a_t+p+m_0,X'}|_F \geq M'_{a_t+p,X'}|_F \geq b_{t+1}\sigma^*(K_{F_0})$$

where $a_{t+1} := a_t + p + m_0$ and $b_{t+1} = b_t + p$. Set $\beta_t = \frac{b_t}{a_t}$. Clearly $\lim_{t\mapsto+\infty}\beta_t = \frac{p}{m_0+p}$.

The case $p=1$ can be considered similarly with a simpler induction. We leave it as an exercise. Q.E.D.

Lemma 2.7. *Let $\widetilde{\Lambda} \subset |L|$ be a pencil on V where L is a divisor on V. Let R be a nef and big $\mathbb{Q}$-divisor on V. Assume there is a birational modification $\pi_{\widetilde{\Lambda}} : V' \longrightarrow V$ such that:*

(1) *the fractional part of $\pi_{\widetilde{\Lambda}}^*(R)$ has simple normal crossing supports and the movable part of $\pi_{\widetilde{\Lambda}}^*(\widetilde{\Lambda})$ is base point free;*

(2) $|K_{V'} + \lceil \pi_{\widetilde{\Lambda}}^*(R) \rceil| \neq \emptyset$.

Then $|K_V + \lceil R \rceil + L|$ distinguishes different irreducible elements in the movable part of $\widetilde{\Lambda}$.

Proof. Noting that

$$\pi_{\widetilde{\Lambda}*} \mathcal{O}_{V'}(K_{V'} + \pi_{\widetilde{\Lambda}}^*(\lceil R \rceil) + \pi_{\widetilde{\Lambda}}^*(L)|) \cong \mathcal{O}_V(K_V + \lceil R \rceil + L)$$

and that $\pi_{\widetilde{\Lambda}}^*(\lceil R \rceil) \geq \lceil \pi_{\widetilde{\Lambda}}^*(R) \rceil$, we only need to study the smaller linear system $|K_{V'} + \lceil \pi_{\widetilde{\Lambda}}^*(R) \rceil) + M_{\widetilde{\Lambda}}|$ where $|M_{\widetilde{\Lambda}}|$ is the movable part of $\pi_{\widetilde{\Lambda}}^*(\widetilde{\Lambda})$. By our assumption, $|M_{\widetilde{\Lambda}}|$ is composed with a pencil and $f_{\widetilde{\Lambda}} : V' \longrightarrow B_{\widetilde{\Lambda}}$ is an induced fibration of $\Phi_{|M_{\widetilde{\Lambda}}|}$, where $B_{\widetilde{\Lambda}}$ is a smooth curve.

If $g(B_{\widetilde{\Lambda}}) = 0$, then $|K_{V'} + \lceil \pi_{\widetilde{\Lambda}}^*(R) \rceil + M_{\widetilde{\Lambda}}|$ distinguishes different general fibers of $f_{\widetilde{\Lambda}}$ because $|K_{V'} + \lceil \pi_{\widetilde{\Lambda}}^*(R) \rceil| \neq \emptyset$.

If $g(B_{\widetilde{\Lambda}}) > 0$, we pick up two general fibers $F'_{\widetilde{\Lambda}}$ and $F''_{\widetilde{\Lambda}}$. Then $M_{\widetilde{\Lambda}} - F'_{\widetilde{\Lambda}} - F''_{\widetilde{\Lambda}}$ is nef and the Kawamata–Viehweg vanishing theorem [13, 21] gives

$$H^1(V', K_{V'} + \lceil \pi_{\widetilde{\Lambda}}^*(R) \rceil + M_{\widetilde{\Lambda}} - F'_{\widetilde{\Lambda}} - F''_{\widetilde{\Lambda}}) = 0.$$

Thus follows the following surjective map:

$$\begin{aligned} & H^0(V', K_{V'} + \lceil \pi_{\widetilde{\Lambda}}^*(R) \rceil + M_{\widetilde{\Lambda}}) \\ \longrightarrow \quad & H^0(F'_{\widetilde{\Lambda}}, (K_{V'} + \lceil \pi_{\widetilde{\Lambda}}^*(R) \rceil)|_{F'_{\widetilde{\Lambda}}}) \oplus H^0(F_{\widetilde{\Lambda}''}, (K_{V'} + \lceil \pi_{\widetilde{\Lambda}}^*(R) \rceil)|_{F''_{\widetilde{\Lambda}}}). \end{aligned}$$

Again the assumption $|K_{V'} + \lceil \pi_{\widetilde{\Lambda}}^*(R) \rceil| \neq \emptyset$ implies that $|K_{V'} + \lceil \pi_{\widetilde{\Lambda}}^*(R) \rceil + M_{\widetilde{\Lambda}}|$ (and thus $|K_V + \lceil R \rceil + L|$) distinguishes different irreducible elements in the movable part of $\widetilde{\Lambda}$. We are done. Q.E.D.

The following lemma is tacitly used in our context.

Lemma 2.8. *Let $\bar{Q}$ be any $\mathbb{Q}$-divisor on a nonsingular projective variety Z. Let $\widetilde{\pi} : \widetilde{Z} \longrightarrow Z$ be any birational modification. Assume that $|K_{\widetilde{Z}} + \lceil \widetilde{\pi}^*(\bar{Q}) \rceil|$ gives a birational map. Then $|K_Z + \lceil \bar{Q} \rceil|$ gives a birational map.*

Proof. This is clear due to the fact:

$$\widetilde{\pi}_*\mathcal{O}_{\widetilde{Z}}(K_{\widetilde{Z}} + \widetilde{\pi}^*(\lceil \bar{Q} \rceil)) \cong \mathcal{O}_Z(K_Z + \lceil \bar{Q} \rceil)$$

and $\widetilde{\pi}^*(\lceil \bar{Q} \rceil) \geq \lceil \widetilde{\pi}^*(\bar{Q}) \rceil$. Q.E.D.

§3. $\mathbb{Q}$-divisors on surfaces and threefolds

We leave the proof for the next two results on surfaces as an exercise which is really a standard $\mathbb{Q}$-divisor argument.

Lemma 3.1. *Let S be a nonsingular projective surface of general type. Denote by $\sigma : S \longrightarrow S_0$ the birational contraction onto the minimal model S_0. For any nef and big $\mathbb{Q}$-divisor Q_2 on S, one has*

$$h^0(S, K_S + m\sigma^*(K_{S_0}) + \lceil Q_2 \rceil) > 1$$

under one of the following situations:
(1) $m \geq 2$;
(2) $m = 1$ and $p_g(S) > 0$.

Theorem 3.2. *Keep the same notation as in Lemma 3.1. Then the rational map $\Phi_{|K_S + m\sigma^*(K_{S_0}) + \lceil Q_2 \rceil|}$ is birational in either of the following cases:*
(1) $m \geq 4$;
(2) $m \geq 3$ and $p_g(S) > 0$.

Proposition 3.3. *Assume* $\dim(V) = 3$ *and $P_{m_0}(V) \geq 2$ for some positive integer m_0. Keep the same notation as in 2.2. Let Q_3 be a nef $\mathbb{Q}$-divisor on V and Q_3' a nef $\mathbb{Q}$-divisor on X'. Then $K_{X'} + \lceil q_3\pi^*(K_X) + Q_3' \rceil + F$ is effective for all rational numbers $q_3 > 2m_0 + 2$. Consequently $mK_V + \lceil Q_3 \rceil$ is effective for all integers $m \geq 3m_0 + 4$.*

Proof. The last statement is a direct application of the first one due to 2.4. We prove the first statement.

> *(£) Take further necessary modifications to X' such that the supports of the fractional parts of Q_3' and $\pi_V^*(Q_3)$ are of simple normal crossing. For simplicity we still use X' to denote the final birational model dominating V.*

We have a fibration $f : X' \longrightarrow B$ induced from $|m_0K|$ as in 2.2. We consider the linear system

$$|K_{X'} + \lceil q_3\pi^*(K_X) + Q_3' \rceil + F| \subset |(\lceil q_3 \rceil + m_0 + 1)K_{X'} + \lceil Q_3' \rceil|.$$

Because $q_3\pi^*(K_X)+Q_3'$ is nef and big and the fractional part of Q_3' is of simple normal crossing by our assumption, the vanishing theorem says

$$H^1(X, K_{X'} + \lceil q_3\pi^*(K_X) + Q_3'\rceil) = 0.$$

Thus one has the surjective map:

$$H^0(X, K_{X'} + \lceil q_3\pi^*(K_X) + Q_3'\rceil + F)$$
$$\longrightarrow H^0(F, K_F + \lceil q_3\pi^*(K_X) + Q_3'\rceil|_F).$$

Note that in our case a general fiber of f is a surface of general type which has a Gorenstein minimal model. Thus the conditions in both Lemma 2.5 and Lemma 2.6 are satisfied. If $b > 0$, Lemma 2.5 says that $\pi^*(K_X)|_F \sim \sigma^*(K_{F_0})$ where $\sigma : F \longrightarrow F_0$ is the contraction map. If $g(B) = 0$, Lemma 2.6 says that one can find a very big number s such that

$$\pi^*(K_X)|_F \geq \beta_s\sigma^*(K_{F_0})$$

and β_s is sufficiently near $\frac{p}{m_0+p} \geq \frac{1}{m_0+1}$ and $\beta_s < \frac{p}{m_0+p}$.

Let us put $\alpha_s := \frac{p}{m_0+p} - \beta_s$. Then $\alpha_s \mapsto 0$ whenever $s \mapsto +\infty$.

Whenever $q_3 > 2m_0 + 2$, one has

$$\begin{aligned} q_3\pi^*(K_X)|_F &\geq q_3\beta_s\sigma^*(K_{F_0}) = q_3(\frac{p}{m_0+p} - \alpha_s)\sigma^*(K_{F_0}) \\ &= (2 + (\frac{q_3p - 2m_0 - 2p}{m_0+p} - q_3\alpha_s))\sigma^*(K_{F_0}). \end{aligned}$$

When s is big enough, one sees $\frac{q_3p-2m_0-2p}{m_0+p} - q_3\alpha_s > 0$. We may assume that $q_3\pi^*(K_X)|_F - q_3\beta_s\sigma^*(K_{F_0})$ is $\mathbb{Q}$-linearly equivalent to an effective divisor $R_{q_3,s}$ on F. Then

$$\begin{aligned} \widetilde{R_{q_3,s}} &:= q_3\pi^*(K_X)|_F - R_{q_3,s} - 2\sigma^*(K_{F,0}) \\ &\equiv (\frac{q_3p - 2m_0 - 2p}{m_0+p} - q_3\alpha_s)\sigma^*(K_{F_0}) \end{aligned}$$

is nef and big since $\frac{q_3p-2m_0-2p}{m_0+p} > 0$. Therefore

$$H^0(K_F + \lceil q_3\pi^*(K_X) + Q_3'\rceil|_F) \supset H^0(K_F + 2\sigma^*(K_{F_0}) + \lceil\widetilde{R}_{q_3,n} + Q_3'\rceil) \neq 0$$

by Lemma 3.1. And in fact $h^0(K_F + 2\sigma^*(K_{F_0}) + \lceil\widetilde{R}_{q_3,n} + Q_3'\rceil) > 1$.

Q.E.D.

Remark 3.4. In the proof of Proposition 3.3, if the general fiber of f is a surface with $p_g > 0$, then, according to Lemma 3.1, $K_{X'} + \lceil q_3\pi^*(K_X) + Q_3')\rceil + F$ is effective for all rational numbers $q_3 > m_0 + 1$. And accordingly $mK_V + \lceil Q_3\rceil$ is effective for all integers $m \geq 2m_0 + 3$.

Theorem 3.5. *Assume* $\dim(V) = 3$ *and* $P_{m_0}(V) \geq 2$ *for some positive integer* m_0. *Keep the same notation as in 2.2. Let* Q_3 *be a nef* $\mathbb{Q}$*-divisor on* V *and* Q_3' *a nef* $\mathbb{Q}$*-divisor on* X'. *Then*

(1) $\Phi_{|K_{X'}+\lceil q_3\pi^*(K_X)+Q_3'\rceil+M_0|}$ *is birational for all rational numbers* $q_3 > 4m_0 + 4$. *In particular* $\Phi_{|mK_V+\lceil Q_3\rceil|}$ *is birational for all integers* $m \geq 5m_0 + 6$;

(2) *if the general fiber* F *of* $f\colon X' \longrightarrow B$ *has positive geometric genus,* $\Phi_{|K_{X'}+\lceil q_3\pi^*(K_X)+Q_3'\rceil+M_0|}$ *is birational for all rational numbers* $q_3 > 3m_0+3$. *In particular* $\Phi_{|mK_V+\lceil Q_3\rceil|}$ *is birational for all integers* $m \geq 4m_0 + 5$.

Proof. According to Proposition 3.3 and Remark 3.4,

$$K_{X'} + \lceil q_3\pi^*(K_X) + Q_3'\rceil$$

is always effective under each situation since $m_0\pi^*(K_X) \geq M_0$. Therefore Lemma 2.7 says that $|K_{X'} + \lceil q_3\pi^*(K_X) + Q_3'\rceil + M_0|$ can distinguish different generic irreducible elements of $|M_0|$. Thus it suffices to prove the birationality of $\Phi_{|K_{X'}+\lceil q_3\pi^*(K_X)+Q_3'\rceil+M_0|}|_F$ for a general fiber F of f. The proofs for statements (1) and (2) are similar. We only consider (1) while omitting the proof for (2).

Of course, the first step in utilizing the vanishing theorem is to make the support of the fractional part of $\{Q_3'\}$ to be simple normal crossing. This can be done by re-modifying X'. For simplicity we may assume, from now on, that our X' has the property stated in ($\mathcal{L}$) (see the proof of Proposition 3.3).

The Kawatama–Viehweg vanishing theorem implies, noticing $F|_F \sim 0$, that

$$|K_{X'} + \lceil q_3\pi^*(K_X) + Q_3'\rceil + M_0||_F = |K_F + \lceil q_3\pi^*(K_X) + Q_3'\rceil|_F|.$$

We study a smaller system $|K_F + \lceil q_3\pi^*(K_X)|_F + Q_3'|_F\rceil$. We have already $\pi^*(K_X)|_F \geq \beta_s\sigma^*(K_{F_0})$ and $0 < \alpha_s := \frac{p}{m_0+p} - \beta_s$, $\alpha_s \mapsto 0$ whenever $s \mapsto +\infty$. When $q_3 > 4m_0 + 4$,

$$\begin{aligned} q_3\pi^*(K_X)|_F &\geq q_3\beta_s\sigma^*(K_F) \\ &= 4\sigma^*(K_{F_0}) + t_s\sigma^*(K_{F_0}) \end{aligned}$$

where $t_s := \frac{q_3p-4m_0-4p}{m_0+p} - q_3\alpha_s > 0$ whenever s is big enough. Therefore, by Theorem 3.2 and Lemma 2.8, $|K_S + \lceil 4\sigma^*(K_{F_0}) + t_s\sigma^*(K_{F_0}) + Q_3'\rceil|_F|$ gives a birational map. Being a bigger linear system,

$$|K_F + \lceil q_3\pi^*(K_X) + Q_3'\rceil|_F|$$

also gives a birational map. We are done. Q.E.D.

3.6. Threefolds V with $\chi(\mathcal{O}_V) > 1$. Keep the same notation as in 2.2. As seen in [6, Lemma 2.32], if $\chi(\mathcal{O}_V) > 1$ and $q(V) = 0$, then a general fiber F of $f : X' \longrightarrow B$ has the geometric genus $p_g(F) > 0$.

Lemma 3.7. *Assume* $\dim(V) = 3$. *Then*

$$\lambda(V) \leq \begin{cases} 18 & \text{if } \chi(\mathcal{O}_V) > 1 \text{ and } q(V) = 0; \\ 10 & \text{otherwise.} \end{cases}$$

Proof. The first statement $\lambda(V) \leq 18$ is due to [6, Theorem 4.8].
When $q(V) > 0$, $\lambda(V) \leq 3$ by Chen–Hacon [4].
When $\chi(\mathcal{O}_V) = 1$, $\lambda(V) \leq 10$ by [6, Corollary 3.13].
Finally when $\chi(\mathcal{O}_V) < 0$, $\lambda(V) \leq 3$ is a direct consequence of Reid's plurigenus formula by Reid [17] and by Chen–Zuo [9, Lemma 4.1]. Q.E.D.

From now on, we classify a 3-fold V into two types:

(I) $\chi(\mathcal{O}_V) > 1$ *and* $q(V) = 0$;
(II) *either* $\chi(\mathcal{O}_V) \leq 1$ *or* $q(V) > 0$.

§4. Point separation on 4-folds

Proposition 4.1. *Assume* $n = \dim(V) = 4$ *and* $P_{m_0}(V) \geq 2$ *for some positive integer* m_0. *Keep the same notation as in 2.2. Let* Q_4 *be a nef* $\mathbb{Q}$*-divisor on* V *and* Q_4' *a nef* $\mathbb{Q}$*-divisor on* X'. *Then*

(1) $K_{X'} + \lceil q_4\pi^*(K_X) + Q_4' \rceil + F$ *is an effective divisor for all rational numbers* $q_4 > 74m_0 + 37$;
(2) $mK_V + \lceil Q \rceil$ *is effective for all integers* $m \geq 75m_0 + 39$.

Proof. (2) is a direct result from (1) according to 2.4. We only prove (1). Similar to assumption ($\mathcal{L}$) in the proof of Proposition 3.3, we may assume that X' is good enough.

Because $q_4\pi^*(K_X) + Q_4'$ is nef and big, and its fractional part has normal crossing supports by assumption, the Kawamata–Viehweg vanishing theorem gives the surjective map:

$$H^0(X', K_{X'} + \lceil q_4\pi^*(K_X) + Q_4' \rceil + F)$$
$$\longrightarrow H^0(F, K_F + \lceil q_4\pi^*(K_X) + Q_4' \rceil|_F).$$

By Lemma 2.5 and inequality (1), one has

$$\begin{aligned} \lceil q_4\pi^*(K_X) + Q_4' \rceil|_F &\geq \lceil (q_4\pi^*(K_X) + Q_4')|_F \rceil \\ &\geq \lceil \frac{q_4}{2m_0+1}\sigma^*(K_{F_0}) + Q_4'|_F \rceil. \end{aligned}$$

If F is of type (I), then, by Remark 3.4, 3.6 and Lemma 3.7, we need to set $\frac{q_4}{2m_0+1} > 37 \geq (\lambda(F)+1)+\lambda(F)$, i.e. $q_4 > 74m_0+37$, so that $K_F + \lceil \frac{q_4}{2m_0+1}\sigma^*(K_{F_0}) + Q_4'|_F \rceil$ is an effective divisor on F.

If F is of type (II), then by Proposition 3.3 and Lemma 3.7 we need $\frac{q_4}{2m_0+1} > 32 \geq (2\lambda(F)+2)+\lambda(F)$ so that $K_F + \lceil \frac{q_4}{2m_0+1}\sigma^*(K_{F_0}) + Q_4'|_F \rceil$ is effective.

In a word, when $q_4 > 74m_0+37$, $K_{X'} + \lceil q_4\pi^*(K_X) + Q_4' \rceil + F$ is effective. Q.E.D.

Theorem 4.2. *Assume $n = \dim(V) = 4$ and $P_{m_0} \geq 2$ for some positive integer m_0. Keep the same notation as in 2.2. Let Q_4 be a nef $\mathbb{Q}$-divisor on V and Q_4' a nef $\mathbb{Q}$-divisor on X'. Then*

(1) $|K_{X'} + \lceil q_4\pi^*(K_X) + Q_4' \rceil + M_0|$ *gives a birational map for all rational numbers $q_4 > 150m_0 + 75$;*

(2) $\Phi_{|mK_V + \lceil Q \rceil|}$ *is birational for all integers $m \geq 151m_0 + 77$.*

Proof. Similar to assumption (£), we may assume that X' is good enough (after a necessary modification). Also (2) is a direct result of (1). We only prove (1).

By Proposition 4.1, we see that $K_{X'} + \lceil q_4\pi^*(K_X) + Q_4' \rceil \geq 0$. Lemma 2.7 tells us that we only need to verify the birationality of

$$\Phi_{|K_{X'}+\lceil q_4\pi^*(K_X)+Q_4'\rceil+M_0||_F}$$

for a general fiber F of f. The vanishing theorem gives

$$|K_{X'} + \lceil q_4\pi^*(K_X) + Q_4' \rceil + M_0||_F = |K_F + \lceil q_4\pi^*(K_X) + Q_4' \rceil|_F|$$

noticing $M_0|_F \sim 0$. Lemma 2.5 and inequality (1) imply

$$\lceil q_4\pi^*(K_X) + Q_4' \rceil|_F \geq \lceil \frac{q_4}{2m_0+1}\sigma^*(K_{F_0}) + Q_4'|_F \rceil.$$

Noting that F is a threefold of general type, we still use a similar argument to that in the proof of Proposition 4.1.

If F is of type (I), then, by Theorem 3.5(2), 3.6, Lemma 3.7 and Lemma 2.8, $|K_F + \lceil \frac{q_4}{2m_0+1}\sigma^*(K_{F_0}) + Q_4'|_F \rceil|$ gives a birational map when

$$\frac{q_4}{2m_0+1} > 75 \geq (3\lambda(F)+3)+\lambda(F),$$

i.e. $q_4 > 150m_0+75$.

If F is of type (II), by Theorem 3.5(1) and Lemma 3.7,

$$\Phi_{|K_F+\lceil \frac{q_4}{2m_0+1}\sigma^*(K_{F_0})+Q_4'|_F\rceil|}$$

is birational when $\frac{q_4}{2m_0+1} > 54 \geq (4\lambda(F)+4)+\lambda(F)$, i.e. $q_4 > 108m_0+54$.

To make a conclusion, $|K_{X'} + \lceil q_4\pi^*(K_X) + Q_4'\rceil + M_0|$ gives a birational map for all rational numbers $q_4 > 150m_0 + 75$. Q.E.D.

A direct result of Theorem 4.2 is the following:

Corollary 4.3. *Assume* $\dim(V) = 4$ *and* $P_{m_0} \geq 2$ *for some positive integer* m_0. *Then* φ_m *is birational onto its image for all integers* $m \geq 151m_0 + 77$.

§5. Proof of the main theorem

We organize the proof according to the value of ι.

First we consider the case $\iota \geq n-2$.

Proposition 5.1. *Assume* $n = \dim(V) \geq 3$, $P_{m_0}(V) \geq 2$ *for some positive integer* m_0 *and* $\iota \geq n-2$. *Keep the same notation as in 2.2. Let* Q_n *be any nef* $\mathbb{Q}$*-divisor on* V *and* Q_n' *any nef* $\mathbb{Q}$*-divisor on* X'. *Then*

(1) $K_{X'} + \lceil q_n\pi^*(K_X) + Q_n'\rceil + (n-3)M_0 + F$ *is effective for all rational numbers* $q_n >$ *min*$\{2m_0+2, 22\} \cdot (2m_0+1)^{n-3}$;

(2) $mK_V + \lceil Q_n\rceil$ *is effective for all integers* $m \geq$ *min*$\{2m_0+2, 22\} \cdot (2m_0+1)^{n-3} + m_0(n-2) + 2$.

Proof. Noting that (2) is a direct application of (1), we only prove (1). We are going to do an induction on n.

When $n = 3$, Proposition 3.3 says that the statement is true when $q_3 > 2m_0+2 = (2m_0+2)\cdot(2m_0+1)^{n-3}+m_0(n-3)$. On the other hand, we may replace m_0 with $\lambda(V)$. In fact, Lemma 3.7 gives $\lambda(V) \leq 18$ for type (I) and $\lambda(V) \leq 10$ for type (II). Thus, by Proposition 3.3 and Remark 3.4, $K_{X'} + \lceil q_3\pi^*(K_X) + Q_3'\rceil + F$ is effective whenever $q_3 > 22$. Therefore statement (1) is true for $q_3 > \min\{2m_0+2, 22\}\cdot(2m_0+1)^{n-3} + m_0(n-3)$.

Assume that statement (1) is correct for varieties of dimension $\leq n-1$. Starting with a good model X' satisfying 2.3, we can do the induction. Noticing that F is of dimension $n-1$, we hope to reduce the problem onto F. Since $\iota \geq n-2$, we have $\dim\varphi_{m_0}(F) \geq \dim\Phi_{|M_0|}(F) \geq n-3 = \dim(F)-2$ by the simple additivity property. Because $M_0|_F \leq m_0K_{X'}|_F \sim m_0K_F$, we know $\Phi_{|m_0K_F|}(F) \geq \Phi_{|M_0|_F|}(F) \geq \dim(F)-2$. Because $q_n\pi^*(K_X) + Q_n'$ is nef and big and has simple normal crossing fractional parts, the vanishing theorem gives the surjective map:

$$\begin{aligned} & H^0(X', K_{X'} + \lceil q_n\pi^*(K_X) + Q'_n \rceil + (n-3)M_0 + F) \\ \longrightarrow\ & H^0(F, K_F + \lceil q_n\pi^*(K_X) + Q'_n \rceil|_F + (n-3)M_0|_F) \\ \supset\ & H^0(F, K_F + \lceil q_n\pi^*(K_X)|_F + Q'_n|_F \rceil + (n-3)M_0|_F) \\ \supset\ & H^0(F, K_F + \lceil q'_n\sigma^*(K_{F_0}) + Q'_n|_F \rceil + (n-4)M_0|_F + M_0|_F) \qquad (4) \end{aligned}$$

where $q'_n \geq \frac{q_n}{2m_0+1} > \min\{2m_0+2, 22\} \cdot (2m_0+1)^{(n-1)-3}$ by Lemma 2.5 and inequality (1). Because $M_0|_F \leq m_0 K_F$ and by taking those pencil $\Lambda_F \subset |M_0||_F$, the induction and Lemma 2.8 tell us that $K_F + \lceil q'_n\sigma^*(K_{F_0}) + Q'_n|_F \rceil + ((n-1)-3)M_0|_F + M_0|_F$ is effective whenever $q'_n > \min\{2m_0+2, 22\} \cdot (2m_0+1)^{(n-1)-3}$. We are done. Q.E.D.

Theorem 5.2. *Assume $n = \dim(V) \geq 3$, $P_{m_0}(V) \geq 2$ for some positive integer m_0 and $\iota \geq n-2$. Keep the same notation as in 2.2. Let Q_n be any nef $\mathbb{Q}$-divisor on V and Q'_n any nef $\mathbb{Q}$-divisor on X'. Then*

(1) *$|K_{X'} + \lceil q_n\pi^*(K_X) + Q'_n \rceil + (n-2)M_0|$ gives a birational map for all rational numbers*

$$q_n > min\{4m_0+4, 57\} \cdot (2m_0+1)^{n-3};$$

(2) *$|mK_V + \lceil Q_n \rceil|$ gives a birational map for all integers*

$$m \geq min\{4m_0+4, 57\} \cdot (2m_0+1)^{n-3} + m_0(n-2) + 2.$$

Proof. Again since (2) is a direct application of (1), we only prove (1). We are going to do an induction on n.

Under the assumption $q_n > \min\{4m_0+4, 57\} \cdot (2m_0+1)^{n-3}$, since $m_0\pi^*(K_X) \geq M_0$, we see

$$\begin{aligned} & K_{X'} + \lceil q_n\pi^*(K_X) + Q'_n \rceil + (n-3)M_0 \\ \geq\ & K_{X'} + \lceil \tilde{q}_n\pi^*(K_X) + Q'_n \rceil + (n-3)M_0 + F \geq 0 \end{aligned}$$

by Proposition 5.1 since $\tilde{q}_n := q_n - m_0 > \min\{2m_0+2, 22\} \cdot (2m_0+1)^{n-3}$. According to Lemma 2.7, $|K_{X'} + \lceil q_n\pi^*(K_X) + Q'_n \rceil + (n-2)M_0|$ can distinguish different fibers of f. We are left to show the birationality of the rational map given by $|K_{X'} + \lceil q_n\pi^*(K_X) + Q'_n \rceil + (n-2)M_0||_F$ for a general fiber F of f.

When $n = 3$ and $q_3 > 4m_0 + 4 = (4m_0+4) \cdot (2m_0+1)^{n-3}$, the statement is nothing but Theorem 3.5(1). On the other hand, we may replace m_0 with $\lambda(V)$. In fact, Lemma 3.7 gives $\lambda(V) \leq 18$ for type (I) and $\lambda(V) \leq 10$ for type (II). Then, by Theorem 3.5(1) and (2),

$|K_{X'} + \lceil q_n\pi^*(K_X) + Q'_n\rceil + (n-2)M_0|$ gives a birational map for all rational numbers $q_3 > \max\{57, 44\} = 57$. Therefore statement (1) is true whenever $q_3 > \min\{4m_0 + 4, 57\}$.

Assume that statement (1) is correct for varieties of dimension $\le n-1$. Starting with a good model X' satisfying 2.3, we can do the induction again. Still, we see $\Phi_{|m_0K_F|}(F) \ge \Phi_{|M_0|_F|}(F) \ge \dim(F) - 2$. We hope to reduce to the problem on F. According to the relation (4), we only need to study

$$|J_{n-1}| := |K_F + \lceil q'_n\sigma^*(K_{F_0}) + Q'_n|_F\rceil + (n-4)M_0|_F + M_0|_F|$$

where $q'_n \ge \frac{q_n}{2m_0+1} > \min\{4m_0 + 4, 57\} \cdot (2m_0 + 1)^{(n-1)-3}$ by Lemma 2.5 and inequality (1). Because $M_0|_F \le m_0K_F$ and by taking those pencil $\Lambda_F \subset |M_0||_F$, the induction and Lemma 2.8 tell us that $|K_F + \lceil q'_n\sigma^*(K_{F_0}) + Q'_n|_F\rceil + ((n-1)-2)M_0|_F|$ gives a birational map. Thus $|K_{X'} + \lceil q_n\pi^*(K_X) + Q'_n\rceil + (n-2)M_0||_F$ gives a birational map. We are done. Q.E.D.

Next we discuss the case $\iota = n-3$.

Proposition 5.3. *Assume $n = \dim(V) \ge 4$, $P_{m_0}(V) \ge 2$ for some positive integer m_0 and $\iota \ge n-3$. Keep the same notation as in 2.2. Let Q_n be any nef $\mathbb{Q}$-divisor on V and Q'_n any nef $\mathbb{Q}$-divisor on X'. Then*

(1) *$K_{X'} + \lceil q_n\pi^*(K_X) + Q'_n\rceil + (n-4)M_0 + F$ is effective for all rational numbers $q_n > 37(2m_0+1)^{n-3}$;*

(2) *$mK_V + \lceil Q_n\rceil$ is effective for all integers $m \ge 37(2m_0+1)^{n-3} + m_0(n-3) + 2$.*

Proof. Statement (1) implies (2). So we only prove (1). Similar to the assumption ($\pounds$), we may assume that X' is good enough (modulo blow-ups) for our purpose. We prove by an induction on n.

When $n = 4$, (1) is exactly Proposition 4.1(1).

Assume that (1) is correct for all varieties of dimension $n-1$. Pick a general fiber F of f. Because $M_0|_F \le m_0K_{X'}|_F \sim m_0K_F$, we know $\Phi_{|m_0K_F|}(F) \ge \Phi_{|M_0|_F|}(F) \ge n-4 = \dim(F) - 3$. As long as we take those pencils $\Lambda_F \subset |M_0||_F$ on F, the induction works on F. Thus we restrict everything onto F. By the vanishing theorem, we may get the similar relation to (4):

$$\begin{aligned} &|K_{X'} + \lceil q_n\pi^*(K_X) + Q'_n\rceil + (n-4)M_0 + F||_F \\ \supset\ &|K_F + \lceil q'_n\sigma^*(K_{F_0}) + Q'_n|_F\rceil + ((n-1)-4)M_0|_F + M_0|_F)| \end{aligned}$$

where $q'_n \ge \frac{q_n}{2m_0+1} > 37(2m_0+1)^{(n-1)-3}$ by Lemma 2.5 and inequality (1). The later linear system is non-empty by induction. Therefore $K_{X'}+$

$\lceil q_n\pi^*(K_X) + Q'_n\rceil + (n-4)M_0 + F$ is effective for all rational numbers $q_n > 37(2m_0+1)^{n-3}$. Q.E.D.

Theorem 5.4. *Assume $n = \dim(V) \geq 4$, $P_{m_0}(V) \geq 2$ for some positive integer m_0 and $\iota \geq n-3$. Keep the same notation as in 2.2. Let Q_n be any nef $\mathbb{Q}$-divisor on V and Q'_n any nef $\mathbb{Q}$-divisor on X'. Then*

(1) *$K_{X'} + \lceil q_n\pi^*(K_X) + Q'_n\rceil + (n-4)M_0 + F$ is effective for all rational numbers $q_n > 75(2m_0+1)^{n-3}$;*

(2) *$mK_V + \lceil Q_n\rceil$ is effective for all integers $m \geq 75(2m_0+1)^{n-3} + m_0(n-3) + 2$.*

Proof. The proof is parallel to that of Proposition 5.3. To avoid unnecessary redundancy, we omit the details. Q.E.D.

Definition 5.5. The sequences $\{u_t\}_{t=4}^n$ and $\{w_t\}_{t=4}^n$ are defined by the following rules:

- $\widetilde{\lambda}_n = m_0$ and, for all $i < n$, $\widetilde{\lambda}_i = \lambda_i$;
- $u_4 = 75\widetilde{\lambda}_4 + 37$ and $w_4 = 151\widetilde{\lambda}_4 + 75$;
- for all i, $u_i = \widetilde{\lambda}_i + u_{i-1}(2\widetilde{\lambda}_i + 1)$ and $w_i = \widetilde{\lambda}_i + w_{i-1}(2\widetilde{\lambda}_i + 1)$.

Finally we study the case $\iota \leq n-4$. We begin with the case $\iota = 1$.

Theorem 5.6. *Assume $n = \dim(V) \geq 4$ and $P_{m_0} \geq 2$ for some positive integer m_0 and $\iota \geq 1$. Keep the same notation as in 2.2. Let Q_n be any nef $\mathbb{Q}$-divisor on V and Q'_n any nef $\mathbb{Q}$-divisor on X'. Then*

(1) *$K_{X'} + \lceil \widetilde{q}_n\pi^*(K_X) + Q'_n\rceil$ is effective for all rational numbers $\widetilde{q}_n > u_n$.*

(2) *$|K_{X'} + \lceil \widetilde{q}_n\pi^*(K_X) + Q'_n\rceil|$ gives a birational map for all rational numbers $\widetilde{q}_n > w_n$.*

(3) *$|mK_V + \lceil Q_n\rceil|$ gives a birational map for all integers $m \geq w_n + 2$.*

Proof. Statement (3) is a direct result of (2). So we have to prove (1) and (2).

When $n = 4$, the conditions in (1) and (2) read $\widetilde{q}_4 > u_4 := 75m_0 + 37$ and $\widetilde{q}_4 > w_4 := 151m_0 + 75$. Both the statements are nothing but Proposition 4.1(1) and Theorem 4.2(1), noting that $m_0\pi^*(K_X) \geq M_0 \geq F$. Besides if we take $m_0 = \lambda(V)$, the statements are true for $\widetilde{q}_4 > u_4 := 75\lambda(V) + 37$ and $\widetilde{q}_4 > w_4 := 151\lambda(V) + 75$.

Assume that the statements are correct for $n-1$ dimensional varieties. By definition, $m_0 \geq \lambda(V)$ and $\lambda_n \geq \lambda(V)$. Because $\widetilde{q}_n\pi^*(K_X) \geq (\widetilde{q}_n - m_0)\pi^*(K_X) + M_0$, we will study $|K_{X'} + \lceil(\widetilde{q}_n - m_0)\pi^*(K_X) + Q'_n\rceil +$

$M_0|$. Now the vanishing theorem gives:

$$\begin{aligned} & |K_{X'} + \lceil(\widetilde{q}_n - m_0)\pi^*(K_X) + Q'_n\rceil + M_0||_F \\ = \quad & |K_F + \lceil(\widetilde{q}_n - m_0)\pi^*(K_X) + Q'_n\rceil|_F| \\ \supset \quad & |K_F + \lceil(\widetilde{q}_n - m_0)\pi^*(K_X)|_F + Q'_n|_F\rceil \\ \supset \quad & |K_F + \lceil\frac{\widetilde{q}_n - m_0}{2m_0+1}\sigma^*(K_{F_0}) + Q'_n|_F\rceil|. \end{aligned}$$

Clearly $\dim(F) = n-1$, the induction hypothesis says

$$K_F + \lceil\frac{\widetilde{q}_n - m_0}{2m_0+1}\sigma^*(K_{F_0}) + Q'_n|_F\rceil$$

is effective when $\frac{\widetilde{q}_n - m_0}{2m_0+1} > u_{n-1}$ and $|K_F + \lceil\frac{\widetilde{q}_n - m_0}{2m_0+1}\sigma^*(K_{F_0}) + Q'_n|_F\rceil|$ gives a birational map when $\frac{\widetilde{q}_n - m_0}{2m_0+1} > w_{n-1}$. Both conditions can be replaced by $\widetilde{q}_n > u_n = \widetilde{\lambda}_n + u_{n-1}(2\widetilde{\lambda}_n + 1)$ and $\widetilde{q}_n > w_n = \widetilde{\lambda}_n + w_{n-1}(2\widetilde{\lambda}_n + 1)$, where $\widetilde{\lambda}_n = m_0$. Note however it is enough to take $\widetilde{\lambda}_i = \lambda_i$ for all $i < n$. We are done. Q.E.D.

Theorem 5.7. *Assume $n = \dim(V) \geq 5$ and $P_{m_0} \geq 2$ for some positive integer m_0 and $\iota \leq n-4$. Keep the same notation as in 2.2. Let Q_n be any nef $\mathbb{Q}$-divisor on V and Q'_n any nef $\mathbb{Q}$-divisor on X'. Then*

(1) $K_{X'} + \lceil q_n\pi^*(K_X) + Q'_n\rceil + (\iota-1)M_0$ *is effective for all rational numbers $q_n > (2m_0+1)^{\iota-1}u_{n-\iota+1}$.*
(2) $|K_{X'} + \lceil q_n\pi^*(K_X) + Q'_n\rceil + (\iota-1)M_0|$ *gives a birational map for all rational numbers $q_n > (2m_0+1)^{\iota-1}w_{n-\iota+1}$.*
(3) $|mK_V + \lceil Q_n\rceil|$ *gives a birational map for all integers $m \geq (2m_0+1)^{\iota-1}w_{n-\iota+1} + m_0(\iota-1) + 2$.*

Proof. Statement (3) is direct from (2). So we only need to prove (1) and (2).

When $n = 5$, one necessarily has $\iota = 1$ and the statements are nothing but those in Theorem 5.6.

First We consider statement (1). We may restrict the problem to F by the vanishing theorem. Then, since $M_0|_F \leq m_0\sigma^*(K_{F_0}) \leq m_0K_F$, we may study the linear system

$$|K_F + \lceil\frac{q_n}{2m_0+1}\sigma^*(K_{F_0}) + Q'_n|_F\rceil + (\iota-2)M_0|_F|. \tag{5}$$

Note that $\dim\Phi_{|m_0K_F|}(F) \geq \Phi_{|M_0|}(F) \geq \iota - 1$. Then we can do an induction and repeat this program for finite times. Finally we are reduced to study the non-emptyness of the linear system on W of dimension $n-\iota+1$:

$$|K_W + \lceil \frac{q_n}{(2m_0+1)^{\iota-1}} \tau^*(K_{W_0}) + Q'_n|_W \rceil| \tag{6}$$

where $\tau\colon W \to W_0$ is a contraction morphism to the minimal model. Furthermore $\dim \Phi_{|m_0 K_W|}(W) \geq \dim \Phi_{|M_0|}(W) \geq 1$. Now Theorem 5.6(1) says that $\frac{q_n}{(2m_0+1)^{\iota-1}} > u_{n-\iota+1}$ is enough to secure the non-emptyness of the linear system (6), where $u_{n-\iota+1}$ is obtained by the sequence $\{u_t\}_{t=4}^{n-\iota+1}$ with $u_i = \widetilde{\lambda}_i + u_{i-1}(2\widetilde{\lambda}_i + 1)$, $u_4 = 75\widetilde{\lambda}_4 + 37$, $\widetilde{\lambda}_{n-\iota+1} = m_0$ and, for all other i, $\widetilde{\lambda}_i = \lambda_i$. Therefore statement (1) is correct.

Statement (1) and Lemma 2.7 allow us to reduce the problem onto lower dimensional varieties. Thus what we are left to do is similar to that for statement (1). So after one step restriction, we get the linear system (5) on F. After successive restrictions and inductions, we may obtain the linear system (6) on W of dimension $n-\iota+1$. Now we may apply Theorem 5.6(2) to get the condition $\frac{q_n}{(2m_0+1)^{\iota-1}} > w_{n-\iota+1}$ where $w_{n-\iota+1}$ is obtained by the sequence $\{w_t\}_{t=4}^{n-\iota+1}$ with $w_i = \widetilde{\lambda}_i + w_{i-1}(2\widetilde{\lambda}_i + 1)$, $w_4 = 151\widetilde{\lambda}_4 + 75$, $\widetilde{\lambda}_{n-\iota+1} = m_0$ and, for all other i, $\widetilde{\lambda}_i = \lambda_i$. We are done. Q.E.D.

Finally we propose the following:

Problem 5.8. As we have seen, inequality (1) in Section 2 is the key step to get optimal birationality. Can one find a better constant $\gamma > \frac{1}{2m_0+1}$ such that $\pi^*(K_X)|_F \geq \gamma\sigma^*(K_{F_0})$? When $\dim(V) = 3$, $\gamma = \frac{p}{m_0+p}$ is nearly optimal by virtue of our previous work.

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