Part B

Advanced Theory

Chapter VI
The Fine Structure Theory

The basic ideas of the fine structure theory have already been outlined in IV.4. In this chapter we develop rigorously the material sketched there. We commence with a certain class of set functions – the rudimentary functions – and then, with the aid of these functions we shall define a new hierarchy of constructible sets, namely the Jensen hierarchy, $(J_\alpha \mid \alpha \in \mathrm{On})$. This hierarchy has all of the important properties of the usual L_α-hierarchy, with the difference that each level in the Jensen hierarchy has many of the properties of the limit levels of the L_α-hierarchy (notably amenability). The Jensen hierarchy is thus a more convenient hierarchy as far as a detailed examination of individual levels is concerned. Certainly it is possible to carry out a comparable study of the sets L_α, but only at the cost of some considerable (though in a sense "trivial") technical difficulties. Intuitively, we may regard J_α as a slightly expanded version of L_α which is closed under simple set functions such as ordered pairs, etc. This is not totally accurate (as we shall see), but it should serve the reader well enough until a more complete understanding is achieved.

1. Rudimentary Functions

The definition of rudimentary functions has already been given in Chapter IV, but is repeated here for convenience.

A function $f\colon V^n \to V$ is said to be *rudimentary* (*rud* for short) iff it is generated by the following schemas:

(i) $f(x_1, \ldots, x_n) = x_i \qquad (1 \leqslant i \leqslant n)$;

(ii) $f(x_1, \ldots, x_n) = \{x_i, x_j\} \qquad (1 \leqslant i, j \leqslant n)$;

(iii) $f(x_1, \ldots, x_n) = x_i - x_j \qquad (1 \leqslant i, j \leqslant n)$;

(iv) $f(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n))$, where $h, g_1, \ldots, g_k$ are rudimentary;

(v) $f(y, x_2, \ldots, x_n) = \bigcup_{z \in y} g(z, x_2, \ldots, x_n)$, where g is rudimentary.

Notice that in the above definition we have made use of proper classes, which is not strictly allowable in ZF set theory. There are two ways of avoiding this, both of relevance to our later development. Firstly, since any "rudimentary function" from V^n to V will be built up from functions of types (i)–(iii) in the above list by means of finitely many applications of the composition rules (iv) and (v), we could replace any mention of the "function" by the LST formula which is implicit in the construction of the function via these schemas. In other words, we are just making use of our usual (and we hope familiar by now) conventions concerning proper classes in ZF set theory (Chapter I). An alternative approach is to regard the "V" in the above definition as being some set (e.g. *a* V_α) which is large enough to contain all of the sets which we are interested in at any one time, in which case the rudimentary functions defined are genuine functions (i.e. they are *sets*). Since the untimate goal in set theory is to study the properties of *sets*, this second approach is clearly adequate. Nevertheless we choose to take the "class function" approach as basic for one important reason: it emphasises the *uniformity* of the rudimentary functions; how their construction is quite independent of any particular set domain under consideration.

A similar situation has already arisen in Chapter II. When we studied the L_α-hierarchy, we proved "global" results concerning the logical complexity of the LST formulas which define the L_α-hierarchy, as well as "local" results concerning the definability (using the language $\mathscr{L}_V$) of the hierarchy within given levels of the hierarchy (I.2.6 and I.2.7 provide good examples of this parallel development). Here, the rudimentary functions are used (instead of the language $\mathscr{L}_V$) to define the Jensen hierarchy of constructible sets: global results will be proved using *class* rudimentary "functions" (which correspond to LST formulas), and local results will be proved using *set* rudimentary functions (which are genuine sets, as are the formulas of $\mathscr{L}_V$).

From now on, except for occasional remarks, we leave it to the reader to supply the relevant "rigorisation" of our development in the appropriate fashion.

To continue with our definition then, if A is a class we say that a function $f\colon V^n \to V$ is *rudimentary relative to A* (*A-rud* for short) iff it is generated by schemas (i)–(v) above and the schema:

(vi) $\quad f(x_1, \dots, x_n) = A \cap x_i \quad (1 \leqslant i \leqslant n).$

If p is a set, we say that a function $f\colon V^n \to V$ is *rudimentary in parameter p* (or simply *rudimentary in p*) iff it is generated by schemas (i)–(v) and the schema:

(vii) $\quad f(x_1, \dots, x_n) = p.$

By a *rudimentary definition* of a rudimentary function f we mean a sequence $f_0, \dots, f_n$ of functions such that $f_n = f$ and for each $i \leqslant n$, f_i is obtained from $f_0, \dots, f_{i-1}$ by means of a single application of one of the schemas (i)–(v) above. Similarly for an *A-rud definition* of an A-rud function and a *rud in p definition* of a function rud in p.

A class $R \subseteq V^n$ is said to be *rudimentary* iff there is a rudimentary function $f\colon V^n \to V$ such that

$$R = \{(x_1, \dots, x_n) \mid f(x_1, \dots, x_n) \neq \emptyset\}.$$

Similarly for an *A-rud* class and a *rud in p* class.

The following lemma lists some of the basic properties of rudimentary functions. In each case, the simple proof is given in parentheses alongside the statement of the result.

1.1 Lemma.

(1) *The function* id (*the identity function*) *is rud.* (By schema (i).)

(2) *The function* $f(x) = \bigcup x$ *is rud.* (By schema (v) together with (1) above.)

(3) *The function* $f(x, y) = x \cup y$ *is rud.* ($f(x, y) = \bigcup\{x, y\}$, so use schema (ii) and (2) above.)

(4) *The function* $f(x_1, \ldots, x_n) = \{x_1, \ldots, x_n\}$ *is rud.* (By schema (ii), the function $g(x_1, \ldots, x_n) = \{x_n\}$ is rud for each n. But,

$$\{x_1, \ldots, x_{n+1}\} = \{x_1, \ldots, x_n\} \cup \{x_{n+1}\}.$$

So argue by induction on n, using schema (iv), together with (3).)

(5) *The function* $f(x_1, \ldots, x_n) = (x_1, \ldots, x_n)$ *is rud.* (By definition,

$$(x_1, \ldots, x_n) = \{\{x_1\}, \{x_1, (x_2, \ldots, x_n)\}\}.$$

So argue by induction on n, using schemas (ii) and (iv).)

(6) *The function* $f_m(x) = m$ *is rud for each* $m \in \omega$. (We have:

$$f_0(x) = 0 = x - x; \quad f_1(x) = 1 = \{0\}; \quad f_2(x) = 2 = \{0, 1\}; \quad \text{etc.}$$

So use schemas (iii) and (iv), together with (4), and proceed by induction on m.)

(7) *The relations* $(x \notin y)$ *and* $(x \neq y)$ *are rud.* (We have:

$$(x \notin y) \leftrightarrow \{x\} - y \neq \emptyset;$$
$$(x \neq y) \leftrightarrow (x - y) \cup (y - x) \neq \emptyset.$$

The result is clear now in view of earlier results.)

(8) *If* $f(y, \vec{x})$ *is rud, so is the function* $g(y, \vec{x}) = (f(z, \vec{x}) \mid z \in y)$. (Use schema (v), together with previous results and the identity

$$g(y, \vec{x}) = \bigcup_{z \in y} \{(f(z, \vec{x}), z)\}.)$$

(9) *If* $f\colon V^n \to V$ *is rud and* $R \subseteq V^n$ *is rud, then* $g\colon V^n \to V$ *is rud, where we set*

$$g(\vec{x}) = \begin{cases} f(\vec{x}), & \text{if } \quad R(\vec{x}), \\ \emptyset, & \text{if } \neg R(\vec{x}). \end{cases}$$

(Let r be a rud function such that

$$R(\vec{x}) \leftrightarrow r(\vec{x}) \neq \emptyset.$$

Then

$$g(\vec{x}) = \bigcup_{y \in r(\vec{x})} f(\vec{x}).)$$

(10) *Let χ_R be the characteristic function of R. Then R is rud iff χ_R is rud.* (If χ_R is rud, then since

$$R(\vec{x}) \leftrightarrow \chi_R(\vec{x}) \neq \emptyset,$$

R is rud, by definition. Conversely, if R is rud, then χ_R is rud by (6) and (9).)

(11) *R is rud iff $\neg R$ is rud.* (By (10), since $\chi_R(\vec{x}) = 1 - \chi_{\neg R}(\vec{x})$.)

(12) *The relations $(x \in y)$ and $(x = y)$ are rud.* (By (7) and (11).)

(13) (*Definition by Cases*) *Let f_i: $V^n \to V$ be rud for $i = 1,, \dots, m$. Let $R_i \subseteq V^n$ be rud for $i = 1, \dots, m$, and such that $R_i \cap R_j = \emptyset$ for $i \neq j$ and $R_1 \cup \dots \cup R_m = V^n$. Define f: $V^n \to V$ by*

$$f(\vec{x}) = f_i(\vec{x}) \leftrightarrow R_i(\vec{x}).$$

Then f is rud. (For each $i = 1, \dots, m$, set

$$f_i'(\vec{x}) = \begin{cases} f_i(\vec{x}), & \text{if} \quad R_i(\vec{x}) \\ \emptyset, & \text{if} \neg R_i(\vec{x}). \end{cases}$$

By (9), each f_i' is rud. But then f is rud, since

$$f(\vec{x}) = f_1'(\vec{x}) \cup \dots \cup f_m'(\vec{x}).)$$

(14) *If $R(z, \vec{x})$ is rud, so is the function*

$$f(y, \vec{x}) = y \cap \{z \mid R(z, \vec{x})\}.$$

(Set

$$h(z, \vec{x}) = \begin{cases} \{z\}, & \text{if} \quad R(z, \vec{x}) \\ \emptyset, & \text{if} \neg R(z, \vec{x}). \end{cases}$$

By (9), h is rud. Hence f is rud, since

$$f(y, \vec{x}) = \bigcup_{z \in y} h(z, \vec{x}).)$$

(15) *Let $R(z, \vec{x})$ be rud and such that for any $\vec{x}$ there is at most one z such that $R(z, \vec{x})$. Then f is rud, where we define*

$$f(y, \vec{x}) = \begin{cases} \text{that } z \in y \text{ such that } R(z, \vec{x}), \text{ if such a } z \text{ exists}, \\ \emptyset, \text{ if no such } z \text{ exists}. \end{cases}$$

(By (14) and the identity

$$f(y, \vec{x}) = \bigcup (y \cap \{z \mid R(z, \vec{x})\}).)$$

(16) *If* $R(y, \vec{x})$ *is rud, so are* $(\exists z \in y)\, R(z, \vec{x})$ *and* $(\forall z \in y)\, R(z, \vec{x})$. (Let r be a rud function such that

$$R(y, \vec{x}) \leftrightarrow r(y, \vec{x}) \neq \emptyset.$$

Then

$$(\exists z \in y)\, R(z, \vec{x}) \leftrightarrow \bigcup_{z \in y} r(z, \vec{x}) \neq \emptyset,$$

so $(\exists z \in y)\, R(z, \vec{x})$ is rud. The second part now follows using (11).)

(17) *The function* $f(x) = \bigcap x$ *is rud.* (Use (12), (16) and (14) and the identity

$$f(x) = (\bigcup x) \cap \{z \mid (\forall y \in x)(z \in y)\}.)$$

(18) *The function* $f(x, y) = x \cap y$ *is rud.* (Because $f(x, y) = \bigcap \{x, y\}$.)

(19) *If* $R_i \subseteq V^n$ *are rud for* $i = 1, \ldots, m$, *then* $S = R_1 \cup \ldots \cup R_m$ *and* $T = R_1 \cap \ldots \cap R_m$ *are rud.* (Let $r_i = \chi_{R_i}$ for each i. Then

$$S(\vec{x}) \leftrightarrow r_1(\vec{x}) \cup \ldots \cup r_m(\vec{x}) \neq \emptyset,$$

$$T(\vec{x}) \leftrightarrow r_1(\vec{x}) \cup \ldots \cup r_m(\vec{x}) \neq \emptyset.$$

The result follows easily now.)

(20) *The functions* $(x)_0$ *and* $(x)_1$ *are rud.* (For example,

$$(x)_0 = \begin{cases} \text{that } z \in \bigcup x \text{ such that } (\exists v \in \bigcup x)(x = (z, v)), \text{ if such a } z \text{ exists}, \\ \emptyset, \text{ if no such } z \text{ exists}. \end{cases}$$

Now use (15).)

(21) *Define*

$$x(y) = \begin{cases} \textit{that } z \in \bigcup \bigcup x \textit{ such that } (z, y) \in x, \textit{ if there is a unique such } z, \\ \emptyset, \textit{ if there is no unique such } z. \end{cases}$$

Then the function $f(x, y) = x(y)$ *is rud.* (By Definition by cases.)

(22) *The functions* $\mathrm{dom}(x)$ *and* $\mathrm{ran}(x)$ *are rud.* (We have:

$$\mathrm{dom}(x) = \{z \in \bigcup \bigcup x \mid (\exists w \in \bigcup \bigcup x)((w, z) \in x)\};$$

$$\mathrm{ran}(x) = \{z \in \bigcup \bigcup x \mid (\exists w \in \bigcup \bigcup x)((z, w) \in x)\}.)$$

(23) *The function* $f(x, y) = x \times y$ *is rud.* (By the identity

$$x \times y = \bigcup_{u \in x} \bigcup_{v \in y} \{(u, v)\}.)$$

(24) *The function* $f(x, y) = x \restriction y$ *is rud.* (By the identity

$$x \restriction y = x \cap (\mathrm{ran}(x) \times y).)$$

(25) *The function* $f(x, y) = x''y$ *is rud.* (Since $x''y = \mathrm{ran}(x \restriction y)$.)

(26) *The function* x^{-1} *is rud.* (By definition,

$$x^{-1} = u''(x \cap (\mathrm{ran}(x) \times \mathrm{dom}(x))), \qquad \text{where}$$
$$u(z) = ((z)_1, (z)_0).) \quad \square$$

By now, the reader may well have observed that all of the results in the above lemma are valid if we replace "rud" by "Σ_0". (In class terms, a function is said to be Σ_0 iff it is of the form

$$\{(y, \vec{x}) \mid \Phi(y, \vec{x})\},$$

where Φ is a Σ_0 formula of LST. In set theoretic terms, a function f is said to be Σ_0 iff there is a Σ_0 formula φ of $\mathscr{L}$ such that for any $\vec{x}, y$, if M is a transitive set such that $\vec{x}, y \in M$, then

$$f(\vec{x}) = y \leftrightarrow \models_M \varphi(\mathring{y}, \vec{\mathring{x}}).$$

By I.9.15, these notions are, in a sense, "equivalent".) However, it is not the case that the class of rud functions is the same as the class of Σ_0 functions. As we shall show presently, the rud functions form a proper subcollection of the Σ_0 functions. Strange as it may at first seem, in the case of relations, the notions of being rud and of being Σ_0 do coincide (as we prove later). The reason why there is no paradox here is that, whereas a function f is Σ_0 just in case it is of the form $\{(y, \vec{x}) \mid \Phi(y, \vec{x})\}$, where Φ is a Σ_0 formula of LST (so the fundamental concept is that of a relation, functions being treated as simply special kinds of relation), a function f is rud iff it can be built up using the schemas for rud functions (i.e. the fundamental concept is that of a function, and relations are effectively identified with their characteristic function).

In order to show that every rud function is Σ_0, it is convenient to introduce the following auxiliary notion.

Say a function $f: V^n \to V$ is *simple* iff, whenever $R(z, \vec{y})$ is a Σ_0 relation, the relation $R(f(\vec{x}), \vec{y})$ is also Σ_0.

The following lemma shows that simplicity is characterised by two special cases of the simplicity requirement.

1.2 Lemma. *A function $f: V^n \to V$ is simple iff:*

(i) *the predicate $z \in f(\vec{x})$ is Σ_0; and*

(ii) *whenever $A(z)$ is a Σ_0 predicate, so too is $(\exists z \in f(\vec{x}))\, A(z)$.*

Proof. ($\to$) Trivial, since (i) and (ii) are special cases of the simplicity requirements.

($\leftarrow$). Using (i) and (ii) we shall prove by induction on the logical complexity of R that if $R(z, \vec{y})$ is Σ_0, so too is $R(f(\vec{x}), \vec{y})$.

(a) Suppose first that $R(z, \vec{y})$ has the form $(z = y_i)$. Then

$$R(f(\vec{x}), \vec{y}) \leftrightarrow f(\vec{x}) = y_i$$
$$(\forall z \in f(\vec{x}))(z \in y_i) \wedge (\forall z \in y_i)(z \in f(\vec{x})).$$

By (i), the clause $(\forall z \in y_i)(z \in f(\vec{x}))$ is Σ_0, and by (ii) the clause $(\forall z \in f(\vec{x}))(z \in y_i)$ is Σ_0. Hence $R(f(\vec{x}), \vec{y})$ is Σ_0.

(b) Now suppose $R(z, \vec{y})$ has the form $(z \in y_i)$. Then

$$R(f(\vec{x}), \vec{y}) \leftrightarrow f(\vec{x}) \in y_i$$
$$\leftrightarrow (\exists z \in y_i)(f(\vec{x}) = z).$$

By part (a) above, the clause $(f(\vec{x}) = z)$ is Σ_0. Hence $R(f(\vec{x}), \vec{y})$ is Σ_0.

(c) Suppose that $R(z, \vec{y})$ has the form $(y_i \in z)$. Then

$$R(f(\vec{x}), \vec{y}) \leftrightarrow y_i \in f(\vec{x}).$$

This is Σ_0 by (i).

That takes care of all the primitive (i.e. atomic) cases.

(d) If $R(z, \vec{y})$ has the form $S(z, \vec{y}) \wedge T(z, \vec{y})$ the induction step is immediate.

(e) If $R(z, \vec{y})$ has the form $\neg S(z, \vec{y})$ the induction step is also immediate.

(f) Suppose that $R(z, \vec{y})$ has the form $(\exists u \in y_i) S(u, z, \vec{y})$. Then

$$R(f(\vec{x}), \vec{y}) \leftrightarrow (\exists u \in y_i) S(u, f(\vec{x}), \vec{y}),$$

and the induction step follows at once.

(g) Finally, suppose that $R(z, \vec{y})$ has the form $(\exists u \in z) S(u, z, \vec{y})$. Then

$$R(f(\vec{x}), \vec{y}) \leftrightarrow (\exists u \in f(\vec{x})) S(u, f(\vec{x}), \vec{y}),$$

and the induction step follows from (ii). □

1.3 Lemma. *If f is rud, then f is simple. Hence all rud functions are Σ_0.*

Proof. Let f be rud, and let $f_0, \ldots, f_n$ be a rud definition of f. Using 1.2, we shall prove by induction on $i \leqslant n$ that f_i is simple. (Such a proof is said to be "by induction on a rud definition of f".)

It is clear that schemas (i), (ii) and (iii) for rud functions all give simple functions. (In each case it is trivial to check conditions (i) and (ii) of 1.2.)

To handle schema (iv) we use the definition of simplicity. Let

$$f(\vec{x}) = h(g_1(\vec{x}), \ldots, g_k(\vec{x})),$$

where $h, g_1, \ldots, g_k$ are already known to be simple. Let $R(z, \vec{y})$ be Σ_0. Define S by

$$S(z_1, \ldots, z_k, \vec{y}) \leftrightarrow R(h(z_1, \ldots, z_k), \vec{y}).$$

Since h is simple, S is Σ_0. But

$$R(f(\vec{x}), \vec{y}) \leftrightarrow S(g_1(\vec{x}), \ldots, g_k(\vec{x}), \vec{y}).$$

So, as $g_1, \ldots, g_k$ are simple it follows (in k steps) that $R(f(\vec{x}), \vec{y})$ is Σ_0.

Finally, for schema (v) we use 1.2 again. Suppose that

$$f(y, x_2, \ldots, x_n) = \bigcup_{u \in y} g(u, x_2, \ldots, x_n),$$

where g is known to be simple. Then

$$z \in f(y, \vec{x}) \leftrightarrow (\exists u \in y)(z \in g(u, \vec{x})).$$

Since g is simple, by 1.2(i) the clause $(z \in g(u, \vec{x}))$ is Σ_0. Hence $(z \in f(y, \vec{x}))$ is Σ_0. Again, if $A(z)$ is Σ_0, then

$$(\exists z \in f(y, \vec{x}))\, A(z) \leftrightarrow (\exists u \in y)(\exists z \in g(u, \vec{x}))\, A(z).$$

Since g is simple, by 1.2(ii) the clause $(\exists z \in g(u, \vec{x}))\, A(z)$ is Σ_0. Hence $(\exists z \in f(y, \vec{x}))\, A(z)$ is Σ_0. The proof is complete. □

That the converse to 1.3 is false will follow from the following result.

1.4 Lemma (Finite Rank Property). *Let $f: V^n \to V$ be rud. Then there is a $p \in \omega$ such that for all $x_1, \ldots, x_n$,*

$$\text{rank}(f(x_1, \ldots, x_n)) < \max(\text{rank}(x_1), \ldots, \text{rank}(x_n)) + p.$$

Proof. By induction on a rud definition of f. The details are trivial. □

Consider now the constant function $f: V \to V$ defined by

$$f(x) = \omega \qquad (\text{all } x).$$

By 1.4, f cannot be rud. But f has the Σ_0 definition

$$y = f(x) \leftrightarrow \text{On}(y) \wedge \lim(y) \wedge (\forall z \in y)(\text{succ}(z) \vee z = \emptyset).$$

But as the next lemma shows, the *graph* of f (i.e. the set $\{(y, x) \mid y = f(x)\}$) is rud.

1.5 Lemma. *Let $R \subseteq V^n$. Then R is rud iff it is Σ_0.*

Proof. If R is rud, then χ_R is rud, so by 1.4 χ_R is Σ_0, so R is Σ_0. Conversely, by 1.1, parts (11), (12), (16), (19), the class of all Σ_0 relations is a subclass of the class of all rud relations. □

A useful consequence of 1.5 is that, because of 1.1 (14), if $R(y, \vec{x})$ is a Σ_0 relation, then the function

$$f(y, \vec{x}) = \{z \in y \mid R(z, \vec{x})\}$$

is rud. We utilise this fact in our next, highly relevant result.

A class M is said to be *rudimentary closed* iff $f''M^n \subseteq M$ for all rud functions $f\colon V^n \to V$ (all n).

1.6 Lemma. *Let M be a transitive set containing ω. If M is rud closed then it is amenable.*

Proof. Recall that a transitive set M is amenable iff:

(i) $(\forall x, y \in M)(\{x, y\} \in M)$;
(ii) $(\forall x \in M)(\bigcup x \in M)$;
(iii) $\omega \in M$;
(iv) $(\forall x, y \in M)(x \times y \in M)$;
(v) if $R \subseteq M$ is $\Sigma_0(M)$, then $(\forall u \in M)(R \cap u \in M)$.

Assume that M is rud closed. By definition, the functions $f(x, y) = \{x, y\}$ and $f(x) = \bigcup x$ are rud, so M satisfies (i) and (ii) above. And by the hypotheses of the lemma, M satisfies (iii). By 1.1 (23), the function $f(x, y) = x \times y$ is rud, so (iv) is valid. That leaves us with (v). Let $R \subseteq M$ be $\Sigma_0(M)$. Suppose R is $\Sigma_0^M((p_1, \ldots, p_n))$. Let S be Σ_0^M such that

$$(\forall x \in M)[R(x) \leftrightarrow S(x, p_1, \ldots, p_n)].$$

Since S is Σ_0, it is rud (by 1.5). So by 1.1 (14), the function

$$f(u, x_1, \ldots, x_n) = u \cap \{x \mid S(x, x_1, \ldots, x_n)\}$$

is rud. Hence as $p_1, \ldots, p_n \in M$ and M is rud closed,

$$u \in M \to f(u, p_1, \ldots, p_n) \in M.$$

In other words,

$$u \in M \to u \cap R \in M,$$

as required. (Notice that we have here made use of "localised" versions of 1.5 and 1.1.) $\square$

The converse to 1.6 is false. But by strenthening amenability clause (v) a little, it is possible to obtain a complete characterisation of rud closure in amenability like terms. We leave this as an exercise for the reader. (Hint: See what is required in order to prove the "converse" to 1.6.)

The *rudimentary closure* of a set X is the smallest rudimentary closed set Y such that $X \subseteq Y$. It is immediate that the rudimentary closure of X is of the form

$$\{f(\vec{x}) \mid f \text{ is rud and } \vec{x} \in X\}.$$

1.7 Lemma. *If U is transitive, then the rud closure of U is transitive.*

Proof. Let W be the rud closure of U. We prove by induction on a rud definition of f that for any rud function $f\colon V^n \to V$ and any $x_1, \ldots, x_n \in W$,

$$(*) \qquad TC(x_1) \subseteq W \wedge \ldots \wedge TC(x_n) \subseteq W \to TC(f(x_1, \ldots, x_n)) \subseteq W.$$

Since U is transitive and, as noted above,

$$W = \{f(\vec{x}) \mid f \text{ is rud and } \vec{x} \in U\}.$$

this proves the lemma.

If $f(x_1, \ldots, x_n) = x_i$, (*) is a propositional tautology.

If $f(x_1, \ldots, x_n) = \{x_i, x_j\}$, then

$$TC(f(x_1, \ldots, x_n)) = TC(\{x_i, x_j\}) = \{x_i, x_j\} \cup TC(x_i) \cup TC(x_j),$$

and (*) is immediate.

If $f(x_1, \ldots, x_n) = x_i - x_j$, then

$$TC(f(x_1, \ldots, x_n)) = TC(x_i - x_j) \subseteq TC(x_i),$$

and again (*) is immediate.

If $f(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n))$, where $h, g_1, \ldots, g_k$ are rudimentary and where (*) holds for $h, g_1, \ldots, g_k$, then (*) for f follows from the application of (*) first to each of $g_1, \ldots, g_k$ and then to h.

Finally, suppose $f(y, x_2, \ldots, x_n) = \bigcup_{z \in y} g(z, x_2, \ldots, x_n)$, where g is rudimentary and where (*) holds for g. If $TC(y) \subseteq W$, then $TC(z) \subseteq W$ for all $z \in y$, so by applying (*) to $g(z, x_2, \ldots, x_n)$ for each $z \in y$ we get (*) for f by taking the union according to the definition of f.

The proof is complete. □

We consider now the notion of relatively rudimentary functions. We show that these reduce, in a natural way, to combinations of rud functions and the function $f(x) = A \cap x$.

1.8 Lemma. *Let $A \subseteq V$. If $f\colon V^n \to V$ is A-rud, then f is expressible, in a uniform way with respect to any given A-rud definition of f, as a combination of rud functions and the function $a(x) = A \cap x$.*

Proof. Let $P(f)$ mean that f is expressible as a composition of rud functions and the function a defined above. We shall show that if f is A-rud, then $P(f)$. The proof is by induction on a rud definition of f. (The uniformity will be an immediate consequence of the proof.)

Clauses (i), (ii), (iii), and (vi) in the definition of A-rud functions cause no difficulties in the induction. And clause (iv) is taken care of by virtue of the fact that a composition of compositions is itself a composition. The only trickly step is the proof that if $P(g)$ holds and f is defined by

$$f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x}),$$

then $P(f)$ holds. We do this by induction on the "complexity" of g. More precisely, let $P_0(h)$ mean that h is rud, and, inductively, let $P_{n+1}(h)$ mean that

$$h(\vec{x}) = h_0(\vec{x}, A \cap h_1(\vec{x}), \ldots, A \cap h_m(\vec{x}))$$

for some $h_0, h_1, \ldots, h_m$ such that $P_0(h_0)$ and $P_n(h_1), \ldots, P_n(h_m)$ are all valid. By the definition of P, it is clear that

$$P(h) \leftrightarrow \exists n P_n(h).$$

So it suffices to prove that $R(n)$ holds for all n, where $R(n)$ means:

$$\text{if } P_n(g) \text{ and } f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x}), \text{ then } P(f).$$

We do this by induction on n.

For $n = 0$ there is nothing to prove, since in this case f is itself rud. So suppose that $n > 0$ and that $R(n - 1)$ holds. Let g be given such that $P_n(g)$. Thus

$$g(z, \vec{x}) = h_0(z, \vec{x}, A \cap h_1(z, \vec{x}), \ldots, A \cap h_m(z, \vec{x})),$$

where $P_0(h_0)$ and $P_{n-1}(h_1), \ldots, P_{n-1}(h_m)$. Set

$$\tilde{g}(z, \vec{x}, u) = h_0(z, \vec{x}, u \cap h_1(z, \vec{x}), \ldots, u \cap h_m(z, \vec{x})).$$

Clearly, $P_{n-1}(\tilde{g})$. Set

$$\tilde{f}(y, \vec{x}, u) = \bigcup_{z \in y} \tilde{g}(z, \vec{x}, u),$$

$$\tilde{h}(y, \vec{x}) = [\bigcup_{z \in y} h_1(z, \vec{x})] \cup \ldots \cup [\bigcup_{z \in y} h_m(z, \vec{x})].$$

By $R(n - 1)$, both $P(\tilde{f})$ and $P(\tilde{h})$. But

$$f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x}) = \tilde{f}(y, \vec{x}, A \cap \tilde{h}(y, \vec{x})).$$

This proves $R(n)$. $\square$

A structure of the form $\mathbf{M} = \langle M, A \rangle$, where $A \subseteq M$, is said to be *rud closed* iff $f'' M^n \subseteq M$ for all A-rud functions f: $V^n \to V$ (all n).

1.9 Lemma. *Let* $= \langle M, A \rangle$, *where* M *is transitive and* $A \subseteq M$. *Then* $\mathbf{M}$ *is a rud closed structure iff* M *is a rud closed set and the structure* $\mathbf{M}$ *is amenable.*

Proof. A direct consequence of 1.6 and 1.8. $\square$

1.10 Lemma. *Let $A \subseteq V$. If $f\colon V^n \to V$ is A-rud, then $f \restriction M^n$ is uniformly $\Sigma_1^{\langle M, A \cap M\rangle}$ for all transitive, rud closed structures $\langle M, A \cap M\rangle$.*

Proof. By 1.3 and 1.8. $\square$

The following lemma shows that, in a certain, obvious sense, the rud functions have a finite "basis". In the statement of the lemma, we allow the use of "dummy variables" so that, for later convenience, all of the "basis" functions are binary.

1.11 Lemma (The Basis Lemma). *Every rudimentary function is a composition of some or all of the following rudimentary functions*:

$$F_0(x, y) = \{x, y\};$$
$$F_1(x, y) = x - y;$$
$$F_2(x, y) = x \times y;$$
$$F_3(x, y) = \{(u, z, v) \mid z \in x \wedge (u, v) \in y\};$$
$$F_4(x, y) = \{(u, v, z) \mid z \in x \wedge (u, v) \in y\};$$
$$F_5(x, y) = \bigcup x;$$
$$F_6(x, y) = \operatorname{dom}(x);$$
$$F_7(x, y) = \in \cap (x \times x);$$
$$F_8(x, y) = \{x''\{z\} \mid z \in y\}.$$

Proof. It is easily seen that each of the above functions is rudimentary. Hence if $\mathscr{C}$ denotes the class of all functions obtainable from $F_0, \ldots, F_8$ by composition, then every function in $\mathscr{C}$ is rudimentary. We prove the converse, that every rudimentary function is a member of $\mathscr{C}$.

If φ is an $\mathscr{L}$-formula and $x_0, \ldots, x_n$ are variables of $\mathscr{L}$, say $x_0 = v_{i(0)}, \ldots, x_n = v_{i(n)}$, we usually write $\varphi(x_0, \ldots, x_n)$ to indicate that the free variables of φ are all amongst $x_0, \ldots, x_n$. Let us call the expression "$\varphi(x_0, \ldots, x_n)$" a *representation* of φ. Thus, any $\mathscr{L}$-formula has infinitely many representations: if the free variables of φ are all amongst $v_0, \ldots, v_n$, then

$$\varphi(v_0, \ldots, v_n), \qquad \varphi(v_0, \ldots, v_n, v_{n+1}), \qquad \varphi(v_0, \ldots, v_n, v_{n+10}, v_{n+3})$$

are all representations of φ.

For each representation $\varphi(x_0, \ldots, x_n)$ of an $\mathscr{L}$-formula φ we define a function $t_{\varphi(x_0, \ldots, x_n)}$ as follows:

$$t_{\varphi(x_0, \ldots, x_n)}(u) = \{(a_0, \ldots, a_n) \mid a_0, \ldots, a_n \in u \wedge \models_u \varphi(\mathring{a}_0, \ldots, \mathring{a}_n)\}.$$

As a first step towards proving the lemma, we show that $t_{\varphi(x_0, \ldots, x_n)} \in \mathscr{C}$ for any $\varphi(x_0, \ldots, x_n)$. The proof is by induction on the construction of φ.

(a) Suppose that $\varphi(x_0, \ldots, x_n)$ is the formula $(x_i \in x_j)$, where $0 \leqslant i < j \leqslant n$. Thus

$$\begin{aligned} t_{\varphi(x_0,\ldots,x_n)}(u) &= \{(a_0, \ldots, a_n) \mid a_0, \ldots, a_n \in u \wedge \models_u (\mathring{a}_i \in \mathring{a}_j)\} \\ &= \{(a_0, \ldots, a_n) \mid a_0, \ldots, a_n \in u \wedge a_i \in a_j\}. \end{aligned}$$

The main complicating factor is the presence of the "superfluous" variables $x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n$. This is where we use the functions F_3 and F_4. (Remember that, by definition,

$$(x_0, \ldots, x_n) = (x_0, (x_1, \ldots, x_n)) = (x_0, (x_1, (x_2, \ldots, x_n))) = \ldots\ldots.)$$

We shall assume that $0 < i, i + 1 < j, j < n$. This is the most complicated case, with "superfluous" variables in all possible locations. All other cases are degenerate versions of this one. Let us write $G^0(x, y)$ for $F_2(x, y)$ and, inductively, $G^{m+1}(x, y)$ for $F_2(x, G^m(x, y))$. Thus $G^m \in \mathscr{C}$ for all m. Note that, in particular, $G^m(u, u) = u^{m+2}$ for all m. Let

$$H(u) = F_4(G^{n-j-2}(u, u), F_7(u, u)).$$

Then $H \in \mathscr{C}$. But we have

$$\begin{aligned} H(u) &= F_4(u^{n-j}, \in \cap\, u^2) \\ &= \{(a, b, c) \mid c \in u^{n-j} \wedge (a, b) \in (\in \cap\, u^2)\} \\ &= \{(a, b, c) \mid c \in u^{n-j} \wedge a, b \in u \wedge a \in b\}. \end{aligned}$$

Thus,

$$\begin{aligned} F_3(u, H(u)) &= \{(d, e, f) \mid e \in u \wedge (d, f) \in H(u)\} \\ &= \{(a, e, (b, c)) \mid e \in u \wedge (a, b, c) \in H(u)\} \\ &= \{(a, e, b, c) \mid e \in u \wedge (a, b, c) \in H(u)\}. \end{aligned}$$

Similarly,

$$F_3(u, F_3(u, H(u))) = \{(a, e, f, b, c) \mid e, f \in u \wedge (a, b, c) \in H(u)\}.$$

So if we write $F_x(y)$ for $F_3(x, y)$ we have

$$F_u^{j-i-1}(H(u)) = \{(a, e_1, \ldots, e_{j-i-1}, b, c) \mid e_1, \ldots, e_{j-i-1} \in u \wedge (a, b, c) \in H(u)\}.$$

Then

$$\begin{aligned} G^i(u, F_u^{j-i-1}(H(u))) &= \{(f_1, \ldots, f_{i-1}, a, e_1, \ldots, e_{j-i-1}, b, c) \mid \\ &\qquad f_1, \ldots, f_{i-1}, e_1, \ldots, e_{j-i-1} \in u \wedge (a, b, c) \in H(u)\} \\ &= t_{\varphi(x_0,\ldots,x_n)}(u). \end{aligned}$$

Thus $t_{\varphi(x_0,\ldots,x_n)} \in \mathscr{C}$.

(b) Suppose that $\varphi = \psi \vee \theta$ and that $t_{\psi(x_0,\ldots,x_n)}$, $t_{\theta(x_0,\ldots,x_n)} \in \mathscr{C}$. Then $t_{\varphi(x_0,\ldots,x_n)} \in \mathscr{C}$, because,

$$\begin{aligned} t_{\varphi(x_0,\ldots,x_n)}(u) &= t_{\psi(x_0,\ldots,x_n)}(u) \cup t_{\theta(x_0,\ldots,x_n)}(u) \\ &= F_5(F_0(t_{\psi(x_0,\ldots,x_n)}(u), t_{\theta(x_0,\ldots,x_n)}(u)), u). \end{aligned}$$

(c) Suppose that $\varphi = \neg\, \psi$ and that $t_{\psi(x_0,\ldots,x_n)} \in \mathscr{C}$. Then $t_{\varphi(x_0,\ldots,x_n)} \in \mathscr{C}$ because

$$t_{\varphi(x_0,\ldots,x_n)}(u) = u - t_{\psi(x_0,\ldots,x_n)}(u) = F_1(u, t_{\psi(x_0,\ldots,x_n)}(u)).$$

(d) If $\varphi = \psi \wedge \theta$ and $t_{\psi(x_0,\ldots,x_n)}, t_{\theta(x_0,\ldots,x_n)} \in \mathscr{C}$, then $t_{\varphi(x_0,\ldots,x_n)} \in \mathscr{C}$ by (b) and (c).

(e) If $\varphi = \exists y \psi$ and $t_{\psi(y,x_0,\ldots,x_n)} \in \mathscr{C}$, then $t_{\varphi(x_0,\ldots,x_n)} \in \mathscr{C}$, because

$$t_{\varphi(x_0,\ldots,x_n)}(u) = \operatorname{dom}(t_{\psi(y,x_0,\ldots,x_n)}(u) = F_6(t_{\psi(y,x_0,\ldots,x_n)}(u), u).$$

(f) If $\varphi = \forall y \psi$ and $t_{\psi(y,x_0,\ldots,x_n)} \in \mathscr{C}$, then $t_{\varphi(x_0,\ldots,x_n)} \in \mathscr{C}$ by (e) and (c).

(g) If $\varphi(x_0, \ldots, x_n)$ is the formula $(x_i = x_j)$, where $0 \leqslant i, j \leqslant n$, then $t_{\varphi(x_0,\ldots,x_n)} \in \mathscr{C}$. To see this, let $\theta(y, x_0, \ldots, x_n)$ be the formula

$$(y \in x_i) \leftrightarrow (y \in x_j).$$

By (a), together with (b), (c), (d), $t_{\theta(y,x_0,\ldots,x_n)} \in \mathscr{C}$. Let $\psi(x_0, \ldots, x_n)$ be the formula $\forall y \theta(y, x_0, \ldots, x_n)$. By (f), $t_{\psi(x_0,\ldots,x_n)} \in \mathscr{C}$. But clearly,

$$\models_u \varphi(\mathring{a}_0, \ldots, \mathring{a}_n) \quad \text{iff } \models_{u \cup (\cup u)} \psi(\mathring{a}_0, \ldots, \mathring{a}_n).$$

Thus,

$$\begin{aligned} t_{\varphi(x_0,\ldots,x_n)}(u) &= \{(a_0, \ldots, a_n) \mid a_0, \ldots, a_n \in u \wedge \models_{u \cup (\cup u)} \psi(\mathring{a}_0, \ldots, \mathring{a}_n)\} \\ &= u^{n+1} \cap t_{\psi(x_0,\ldots,x_n)}(u \cup (\textstyle\bigcup u)). \end{aligned}$$

But we saw in (a) that the function $F(u) = u^{n+1}$ is in $\mathscr{C}$ (if $n = 0$, use $F(u) = u - (u - u)$ instead), and by F_5, F_0 the function $F(u) = u \cup (\bigcup u)$ is in $\mathscr{C}$. Thus $t_{\varphi(x_0,\ldots,x_n)} \in \mathscr{C}$.

(h) Now suppose that $\varphi(x_0, \ldots, x_n)$ is the formula $(x_i \in x_j)$ where $0 \leqslant j < i \leqslant n$. To see that $t_{\varphi(x_0,\ldots,x_n)} \in \mathscr{C}$, argue as follows. Let $\psi(y, z, x_0, \ldots, x_n)$ be the formula

$$(y \in z) \wedge (y = x_i) \wedge (z = x_j).$$

By (a), (g), (d), $t_{\psi(y,z,x_0,\ldots,x_n)} \in \mathscr{C}$. But clearly,

$$\models_u \varphi(\mathring{a}_0, \ldots, \mathring{a}_n) \quad \text{iff } \models_u \exists y \exists z \psi(y, z, \mathring{a}_0, \ldots, \mathring{a}_n).$$

So by (e), $t_{\varphi(x_0,\ldots,x_n)} \in \mathscr{C}$.

By (a), (h), (g), (b), (c), (d), we see that $t_{\varphi(x_0,\ldots,x_n)} \in \mathscr{C}$ whenever φ is a quantifier free formula of $\mathscr{L}$. Hence by (e), (f), $t_{\varphi(x_0,\ldots,x_n)} \in \mathscr{C}$ for any $\mathscr{L}$-formula φ.

As the next step towards proving the lemma, for any $f: V^n \to V$ we define $f^*: V \to V$ by

$$f^*(u) = f'' u^n.$$

We prove that if f is rudimentary, then $f^* \in \mathscr{C}$. The proof is by induction on a rudimentary definition of f.

(a) Suppose that $f(x_1, \ldots, x_n) = x_i$. Then

$$f^*(u) = f'' u^n = u - (u - u)$$

and so $f^* \in \mathscr{C}$.

(b) Suppose $f(x_1, \ldots, x_n) = x_i - x_j$. Then

$$f^*(u) = f'' u^n = \{x - y \mid x, y \in u\}.$$

Let $\varphi(z, y, x)$ be the formula $z \in (x - y)$. Let

$$\begin{aligned} F(u) &= t_{\varphi(z,x,y)}(u \cup (\bigcup u)) \cap (\bigcup u \times u^2) \\ &= \{(z, x, y) \mid x, y \in u \wedge z = x - y\}. \end{aligned}$$

Since $t_{\varphi(z,x,y)} \in \mathscr{C}$ we have $F \in \mathscr{C}$. But then $f^* \in \mathscr{C}$, since

$$\begin{aligned} F_8(F(u), u^2) &= \{F(u)'' \{a\} \mid a \in u^2\} \\ &= \{F(u)'' \{(x, y)\} \mid x, y \in u\} \\ &= \{\{z\} \mid x, y \in u \wedge z = x - y\} \\ &= \{\{x - y\} \mid x, y \in u\} \\ &= f^*(u). \end{aligned}$$

(c) Let $f(x_1, \ldots, x_n) = \{x_i, x_j\}$. Then

$$f^*(u) = \{\{x, y\} \mid x, y \in u\} = \bigcup (u^2),$$

so $f^* \in \mathscr{C}$.

(d) Let $f(x_1, \ldots, x_n) = h(g_1(x_1, \ldots, x_n), \ldots, g_k(x_1, \ldots, x_n))$, where $h, g_1, \ldots, g_k$ are rudimentary and $h^*, g_1^*, \ldots, g_k^* \in \mathscr{C}$. Let

$$G(u) = g_1^*(u) \cup \ldots \cup g_k^*(u), \;\; H(u) = h^*(G(u)), \;\; K(u) = u^n \cup G(u) \cup H(u).$$

By our assumptions, $G, H, K \in \mathscr{C}$.

By 1.3 there is a Σ_0-formula $\Phi(z_1, \ldots, z_k, x_1, \ldots, x_n)$ of LST such that

$$\begin{aligned} &\Theta(z_1, \ldots, z_k, x_1, \ldots, x_n) \quad \text{iff} \\ &z_1 = g_1(x_1, \ldots, x_n) \wedge \ldots \wedge z_k = g_k(x_1, \ldots, x_n) \end{aligned}$$

and a Σ_0-formula $\Psi(y, z_1, \ldots, z_k)$ of LST such that

$$\Psi(y, z_1, \ldots, z_k) \quad \text{iff } y = h(z_1, \ldots, z_k).$$

Let r exceed the number of quantifiers which occur in Θ and Ψ, and define D by

$$D(u) = u \cup (\bigcup u) \cup (\bigcup \bigcup u) \cup \ldots \cup (\bigcup^r u).$$

Then $D \in \mathscr{C}$, and moreover, by I.9.15, if θ, ψ are the $\mathscr{L}$-analogues of Θ, Ψ, then for any set u and any $y, z_1, \ldots, z_k, x_1, \ldots, x_n \in u$,

$$\Theta(z_1, \ldots, z_k, x_1, \ldots, x_n) \quad \text{iff } \models_{D(u)} \theta(\mathring{z}_1, \ldots, \mathring{z}_k, \mathring{x}_1, \ldots, \mathring{x}_n)$$

and

$$\Psi(y, z_1, \ldots, z_k) \quad \text{iff } \models_{D(u)} \psi(\mathring{y}, \mathring{z}_1, \ldots, \mathring{z}_k).$$

(Strictly speaking, I.9.15 is not adequate for the above, since this would require $D(u)$ to be transitive. However, as is easily seen, the choice of the integer r above makes $D(u)$ resemble a transitive set sufficiently for the proof of I.9.15 to go through for the formulas concerned here.) Let $\varphi(y, x_1, \ldots, x_n)$ be the $\mathscr{L}$-formula

$$\exists z_1 \ldots z_k[\theta(z_1, \ldots, z_k, x_1, \ldots, x_n) \wedge \psi(y, z_1, \ldots, z_k)].$$

Now, $K(u)$ consists of u^n, together with all values of $g_1, \ldots, g_k$ on u and all values of f on u. Thus by the definition of φ,

$$\begin{aligned} &t_{\varphi(y, x_1, \ldots, x_n)}(D \circ K(u)) \cap (f''u^n \times u^n) \\ &\quad = \{(f(x_1, \ldots, x_n), x_1, \ldots, x_n) \mid x_1, \ldots, x_n \in u\}. \end{aligned}$$

Thus

$$f^*(u) = \bigcup F_8(t_{\varphi(y, x_1, \ldots, x_n)}(D \circ K(u)) \cap (H(u) \times u^n), u^n).$$

This shows that $f^* \in \mathscr{C}$.

(e) Suppose that $f(y, x_1, \ldots, x_n) = \bigcup_{v \in y} g(v, x_1, \ldots, x_n)$, where g is rudimentary and $g^* \in \mathscr{C}$. By 1.3 there is a Σ_0-formula $\Phi(z, y, x_1, \ldots, x_n)$ of LST such that

$$\Phi(z, y, x_1, \ldots, x_n) \quad \text{iff } (\exists v \in y)\,[z \in g(v, x_1, \ldots, x_n)].$$

Suppose that Φ has fewer than r quantifiers, and define D as in the above case (d). Then, if φ is the $\mathscr{L}$-analogue of Φ, we have, as above, for any $z, y, x_1, \ldots, x_n \in u$,

$$\Phi(z, y, x_1, \ldots, x_n) \quad \text{iff } \models_{D(u)} \varphi(\mathring{z}, \mathring{y}, \mathring{x}_1, \ldots, \mathring{x}_n).$$

Then

$$\begin{aligned} t_{\varphi(z, y, x_1, \ldots, x_n)}(D(u)) &= \{(z, y, x_1, \ldots, x_n) \mid z, y, x_1, \ldots, x_n \in D(u) \\ &\qquad \wedge (\exists v \in y)\,(z \in g(v, x_1, \ldots, x_n))\} \\ &= \{(z, y, x_1, \ldots, x_n) \mid z, y, x_1, \ldots, x_n \in D(u) \\ &\qquad \wedge z \in f(y, x_1, \ldots, x_n)\}. \end{aligned}$$

So

$$F_8(t_{\varphi(z,y,x_1,\ldots,x_n)}(D(u)), u^{n+1})$$
$$= \{\{z\} \mid y, x_1, \ldots, x_n \in u \wedge z \in f(y, x_1, \ldots, x_n)\}.$$

Thus

$$f^*(u) = \bigcup F_8(t_{\varphi(z,y,x_1,\ldots,x_n)}(D(u)), u^{n+1}),$$

which shows that $f^* \in \mathscr{C}$.

We have proved that $f^* \in \mathscr{C}$ for any rudimentary function f. We are now able to complete the proof of the lemma. Let $f\colon V^n \to V$ be a given rudimentary function. We prove that $f \in \mathscr{C}$. Define $\tilde{f}\colon V \to V$ by

$$\tilde{f}(x) = \begin{cases} f(z_1, \ldots, z_n), & \text{if } x = (z_1, \ldots, z_n) \\ \emptyset, & \text{in all other cases.} \end{cases}$$

By 1.1(9), $\tilde{f}$ is rudimentary. So by the above, $\tilde{f}^* \in \mathscr{C}$. Moreover, $g \in \mathscr{C}$, where we define $g\colon V^n \to V$ by

$$g(x_1, \ldots, x_n) = \{(x_1, \ldots, x_n)\}.$$

(By repeated use of F_0.) But

$$\begin{aligned} f(x_1, \ldots, x_n) &= \bigcup \{f(x_1, \ldots, x_n)\} = \bigcup \{\tilde{f}((x_1, \ldots, x_n))\} \\ &= \bigcup \tilde{f}''\{(x_1, \ldots, x_n)\} = \bigcup \tilde{f}^*((x_1, \ldots, x_n)) \\ &= \bigcup \tilde{f}^*(g(x_1, \ldots, x_n)). \end{aligned}$$

Thus $f \in \mathscr{C}$, and we are done. □

As an immediate corollary of 1.8 and 1.11, we have:

1.12 Lemma (Extended Basis Lemma). *Let $A \subseteq V$, and define F_9 by*

$$F_9(x, y) = A \cap x.$$

Then every A-rudimentary function may be expressed as a composition of some or all of the A-rud functions $F_0, \ldots, F_9$. □

Lemma 1.14 below provides an immediate application of the above basis result. It concerns the semantics of the languages $\mathscr{L}_V(A)$. These languages (or rather more general languages $\mathscr{L}_V(A_1, \ldots, A_k)$) were defined in I.9. As was mentioned there, the basic syntactics and semantics of these languages differs only in a trivial way from that of the language $\mathscr{L}_V$, and so there is no need to spend any time on such a development. Suffice it to say that, what comes out of it is the following. There is a Σ_1 formula $Sat^A(u, a, \varphi)$ of LST (in three variables, u, a, φ) which says that:

> "u is a non-empty set" $\wedge$ "$a \subseteq u$" $\wedge$ "φ is a sentence of $\mathscr{L}_u(A)$ which is true in the structure $\langle u, a\rangle$ under the canonical interpretation".

Just as in I.9.10, we get:

1.13 Lemma. *The* LST *formula* $Sat^A(u, a, \varphi)$ *is* Δ_1^{BS}. $\square$

As usual, we usually write $\models_{\langle u,a\rangle} \varphi$ rather than $Sat^A(u, a, \varphi)$. For any $n \in \omega$, we denote by $\models^{\Sigma_n}_{\langle u,a\rangle}$ the restriction of the relation $\models_{\langle u,a\rangle}$ to the Σ_n sentences of $\mathscr{L}_u(A)$.

The following lemma will provide us with an analogue to II.6.3 for the Jensen hierarchy of constructible sets, defined in the next section.

1.14 Lemma. $\models^{\Sigma_0}_{[M,A\rangle}$ *is uniformly* $\Sigma_1^{\langle M,A\rangle}$ *for transitive, rud closed structures* $\langle M, A\rangle$.

Proof. Consider the language Γ_M which consists of the variables v_n, $n \in \omega$, of $\mathscr{L}$ (i.e. $v_n = (2, n)$), the constant symbols $\mathring{x}(= (3, x))$, for each $x \in M$, and the binary function symbols $\mathring{F}_0, \ldots, \mathring{F}_9$. (More formally, for each $i = 0, \ldots, 9$, $\mathring{F}_i(x, y)$ denotes the set $(0, i, x, y)$.) The syntax of Γ_M is particularly simple. Each variable and each constant of $\mathscr{L}_M$ is a *term* of Γ_M, and if t_1, t_2 are terms of Γ_M, then $\mathring{F}_0(t_1, t_2), \ldots, \mathring{F}_9(t_1, t_2)$ are all terms of Γ_M. Note that each term of Γ_M is an element of M. A *constant* term is one which contains no variables. Each constant term, t, of Γ_M has an obvious interpretation in $\langle M, A\rangle$, where we let x interpret $\mathring{x}$ for each $x \in M$ and F_i interpret $\mathring{F}_i$ for each $i = 0, \ldots, 9$. Since $\langle M, A\rangle$ is rud closed, the interpretation, $t^{\langle M,A\rangle}$ of each constant term t is an element of M. Clearly, for each constant term t and each $x \in M$, we have:

$$\begin{aligned}
&x = t^{\langle M,A\rangle} \quad \text{iff}\\
&\exists f \exists g[\mathrm{Finseq}(f) \wedge \mathrm{Finseq}(g) \wedge \mathrm{dom}(f) = \mathrm{dom}(g) \wedge g(\mathrm{dom}(g) - 1)\\
&\quad = x \wedge (\forall i \in \mathrm{dom}(f))\, [\mathrm{Const}_M(f(i)) \vee (\exists j, k \in i)\, [f(i)\\
&\quad = \mathring{F}_0(f(j), f(k)) \vee \ldots \vee f(i) = \mathring{F}_9(f(j), f(k))] \wedge (\forall i \in \mathrm{dom}(f))\\
&\quad\;\; [\mathrm{Const}_M(f(i)) \rightarrow g(i) = (f(i))_1] \wedge (\forall i \in \mathrm{dom}(f))\, (\forall j, k \in i)\\
&\quad\;\; [(f(i) = \mathring{F}_0(f(j), f(k)) \rightarrow g(i) = F_0(g(j), g(k))) \wedge \ldots \wedge (f(i)\\
&\quad = \mathring{F}_9(f(j), f(k)) \rightarrow g(i) = F_9(g(j), g(k)))]]].
\end{aligned}$$

Now, if such f, g as above exist, they will certainly be elements of M. Moreover, by 1.10, each of the functions $F_0, \ldots, F_9$ is (uniformly) $\Sigma_1^{\langle M,A\rangle}$. Hence the above equivalence shows that the relation $x = t^{<M,A\rangle}$ (as a relation of x, t) is (uniformly) $\Sigma_1^{\langle M,A\rangle}$. The idea now is to utilise this fact by associating with each Σ_0 sentence φ of $\mathscr{L}_M(A)$ a constant term t_φ of Γ_M so that:

(i) the map $\varphi \mapsto t^\varphi$ is $\Sigma_1^{\langle M,A\rangle}$ (uniformly);

(ii) $\models_{\langle M,A\rangle} \varphi$ iff $t_\varphi^{\langle M,A\rangle} = 1$.

In fact, in order to do this, we need to define t_φ for any formula φ, not just sentences. (This is why we allow variables in the language Γ_M.)

As our starting point we take 1.5. This tells us that if $R(\vec{x})$ is a Σ_0 relation, there is a rud function $f(\vec{x})$ such that

$$R(\vec{x}) \leftrightarrow f(\vec{x}) = 1.$$

By 1.11, we know that the function f here may be expressed as a composition of the basic functions $F_0, \ldots, F_9$. Now, the existence of the function f is established by proceeding inductively on the logical structure of R, using 1.1(11), (12), (16), (19), and the proof of 1.11 is (essentially) by induction on a rudimentary definition of f. And by virtue of 1.8, we can extend all of this to allow for the unary predicate A, introducing the extra basic function F_9. So by examining the inductive proofs of 1.5, 1.11, and 1.8, we obtain the required map $\varphi \mapsto t_\varphi$.

We proceed inductively following the logical construction of the Σ_0 formula φ, using the techniques of 1.1, 1.11, and 1.8. Now, if you have spent any time on the proofs of these results, particularly 1.11, you will appreciate that it would be pointless trying to write out explicitly the definition of the function $\varphi \mapsto t_\varphi$. But it should be clear that the following is the case.

From I.9 (extended to the language $\mathscr{L}_M(A)$) we know that there are Σ_0 formulas $F_\in$, $F_=$, F_A, $F_\wedge$, $F_\neg$, $F_\exists$ of LST such that (see, in particular, I.9.3):

$$F_\in(\theta, x, y) \leftrightarrow \theta \text{ is the } \mathscr{L}_M(A) \text{ formula } (x \in y);$$
$$F_=(\theta, x, y) \leftrightarrow \theta \text{ is the } \mathscr{L}_M(A) \text{ formula } (x = y);$$
$$F_A(\theta, x) \leftrightarrow \theta \text{ is the } \mathscr{L}_M(A) \text{ formula } \mathring{A}(x);$$
$$F_\wedge(\theta, \varphi, \psi) \leftrightarrow \theta \text{ is the } \mathscr{L}_M(A) \text{ formula } (\varphi \wedge \psi);$$
$$F_\neg(\theta, \varphi) \leftrightarrow \theta \text{ is the } \mathscr{L}_M(A) \text{ formula } (\neg \varphi);$$
$$F_\exists(\theta, u, \varphi) \leftrightarrow \theta \text{ is the } \mathscr{L}_M(A) \text{ formula } (\exists u \varphi).$$

These LST formulas simply describe the way in which the formulas of $\mathscr{L}_M(A)$ are constructed. Implicit in the proofs of 1.1, 1.11, and 1.8 is the fact that there are Σ_0 formulas $G_\in$, $G_=$, G_A, $G_\wedge$, $G_\neg$, $G_\exists$ of LST such that:

$$G_\in(t, x, y) \leftrightarrow t = t_{(x \in y)};$$
$$G_=(t, x, y) \leftrightarrow t = t_{(x = y)};$$
$$G_A(t, x) \leftrightarrow t = t_{A(x)};$$
$$G_\wedge(t, t_\varphi, t_\psi) \leftrightarrow t = t_{(\varphi \wedge \psi)};$$
$$G_\neg(t, t_\varphi) \leftrightarrow t = t_{(\neg \varphi)};$$
$$G_\exists(t, t_\varphi) \leftrightarrow t = t_{(\exists y \in x)\varphi},$$

where for each φ, t_φ is a term of Γ_M which satisfies (ii) above. These G-formulas describe the way in which the terms t_φ must be combined (together with specific of the function symbols $\mathring{F}_0, \ldots, \mathring{F}_9$) to make (ii) valid, and thus correspond to the induction steps of the proofs of 1.1, 1.11, and 1.7 (all rolled into one).

It follows that there is a Σ_1 formula H of LST such that

$$H(t, \varphi) \leftrightarrow \text{“}\varphi \text{ is a } \Sigma_0 \text{ formula of } \mathscr{L}_M(A)\text{”} \wedge t = t_\varphi.$$

In essence (though not totally accurate), $H(t, \varphi)$ is as follows (see I.9, in particular I.9.6):

$$\exists f \exists g[\text{Build}(f, \varphi) \wedge \text{Finseq}(g) \wedge \text{dom}(g) = \text{dom}(f)$$
$$\wedge\ (\forall i \in \text{dom}(f))((F_\in(f(i), x, y) \to G_\in(g(i), x, y)))$$
$$\wedge \ldots \wedge (F_\exists(f(i), u, f(j)) \to G_\exists(g(i), u, g(j)))].$$

Notice that if $H(t, \varphi)$ is true, it is always possible to find such f, g as above in M. Consequently, if $h(t, \varphi)$ denotes the $\mathscr{L}$-analogue of the LST formula $H(t, \varphi)$, we have, by I.9.15, for any $x, y \in M$,

$$H(y, x) \leftrightarrow \models_M h(\mathring{y}, \mathring{x}).$$

This proves (i) and (ii) and thus completes the proof of the lemma. □

The following result, which will provide us with an analogue of II.6.4 for the Jensen hierarchy, is deduced from 1.14 in exactly the same way that II.6.4 was deduced from II.6.3:

1.15 Lemma. *For any $n \geqslant 1$, the relation $\models^{\Sigma_n}_{\langle M, A\rangle}$ is uniformly $\Sigma_n^{\langle M, A\rangle}$ for all transitive rud closed structures $\langle M, A\rangle$.* □

For any set U, we define the set $\text{rud}(U)$ to be the rudimentary closure of the set $U \cup \{U\}$, i.e. the smallest rud closed set that contains U as a subset and as an element. Notice that by 1.7, we have:

1.16 Lemma. *If U is transitive, then* $\text{rud}(U)$ *is transitive.*

Proof. Immediate, since if U is transitive, then $U \cup \{U\}$ is transitive. □

We shall use the function $\text{rud}(U)$ in order to define the Jensen hierarchy. The following lemmas will be of use to us in this connection. The first of them will enable us to compare the rates of growth of the two constructible hierarchies. The other two will help us to define well-orderings of the levels of the Jensen hierarchy.

1.17 Lemma[7]**.** *Let U be a transitive set. Then*

$$\text{rud}(U) \cap \mathscr{P}(U) = \text{Def}(U).$$

In fact

$$\Sigma_0(\text{rud}(U)) \cap \mathscr{P}(U) = \text{Def}(U).$$

Proof. We commence by proving that

$$(*) \qquad \Sigma_0(U \cup \{U\}) \cap \mathscr{P}(U) = \text{Def}(U).$$

First of all let $A \in \text{Def}(U)$. Thus for some formula $\varphi(x)$ of $\mathscr{L}_U$,

$$A = \{x \in U \mid \models_U \varphi(\mathring{x})\}.$$

7 In the statement of this lemma we extend our notation a little by using $\Sigma_n(M)$ to mean the set of all $\Sigma_n(M)$ subsets of M. This notational extension will be used several times from now on.

Let $\psi(x)$ be the formula of $\mathscr{L}_{U \cup \{U\}}$ obtained from $\varphi(x)$ by binding all unbounded quantifiers by $\mathring{U}$. Clearly, for any $x \in U$.

$$\models_U \varphi(\mathring{x}) \quad \text{iff } \models_{U \cup \{U\}} \psi(\mathring{x}).$$

Thus

$$A = \{x \in U \cup \{U\} \mid \models_{U \cup \{U\}} \mathring{x} \in \mathring{U} \wedge \psi(\mathring{x})\} \in \Sigma_0(U \cup \{U\}) \cap \mathscr{P}(U).$$

Conversely, let $A \in \Sigma_0(U \cup \{U\}) \cap \mathscr{P}(U)$. Thus for some Σ_0 formula $\varphi(x)$ of $\mathscr{L}_{U \cup \{U\}}$,

$$A = \{x \in U \mid \models_{U \cup \{U\}} \varphi(\mathring{x})\}.$$

To show that $A \in \mathrm{Def}(U)$, it suffices to show that for any Σ_0 formula $\varphi(\vec{x})$ of $\mathscr{L}_{U \cup \{U\}}$, there is a formula $\varphi^*(\vec{x})$ of $\mathscr{L}_U$ such that for any $\vec{x} \in U$

$$\models_{U \cup \{U\}} \varphi(\vec{\mathring{x}}) \quad \text{iff } \models_U \varphi^*(\vec{\mathring{x}}).$$

The proof of the above is by induction on φ. Suppose first that φ is primitive. If φ does not involve $\mathring{U}$, take $\varphi^* = \varphi$, in which case the result is clear. Suppose that φ involves $\mathring{U}$. If φ is of the form $(a \in \mathring{U})$ where $a \in Vbl \cup \mathrm{Const}_U$, take φ^* to be $(a = a)$. If φ is of the form $(\mathring{U} = \mathring{U})$, take φ^* to be $\forall x(x = x)$. In all other cases, take φ^* to be $\exists x(x \neq x)$. It is easily seen that φ^* is as required. In case $\varphi = \psi \wedge \theta$ now, we take $\varphi^* = (\psi^*) \wedge (\theta^*)$, and in case $\varphi = \neg \psi$, we take $\varphi^* = \neg (\psi^*)$. Suppose next that φ is of the form $(\exists x \in a)\, \psi$, where $a \in Vbl \cup \mathrm{Const}_U$. In this case take φ^* to be $(\exists x \in a)\,(\psi^*)$. Finally, suppose that φ is of the form $(\exists x \in \mathring{U})\, \psi$. Then we take φ^* to be $\exists x(\psi^*)$. The result is clear now.

By $(*)$, in order to prove the first part of the lemma, it suffices to show that

$$\Sigma_0(U \cup \{U\}) \cap \mathscr{P}(U) = \mathrm{rud}(U) \cap \mathscr{P}(U).$$

First of all, let $A \in \Sigma_0(U \cup \{U\}) \cap \mathscr{P}(U)$. Thus for some Σ_0 formula $\varphi(x)$ of $\mathscr{L}_{U \cup \{U\}}$,

$$A = \{x \in U \mid \models_{U \cup \{U\}} \varphi(\mathring{x})\}.$$

By Σ_0-absoluteness,

$$A = \{x \in U \mid \models_{\mathbf{rud}(U)} \varphi(\mathring{x})\}.$$

But $\mathrm{rud}(U)$ is amenable (by 1.6). Thus by definition of amenability, $A \in \mathrm{rud}(U)$. For the converse, let $A \in \mathrm{rud}(U) \cap \mathscr{P}(U)$. Then for some rudimentary function f and some $a \in U$, $A = f(a, U)$. Now by 1.3 and 1.2 (or rather by localised versions of them where V is taken to be the transitive, rudimentary closed set $\mathrm{rud}(U)$), there is a Σ_0 formula φ of $\mathscr{L}$ such that for any $x \in \mathrm{rud}(U)$,

$$x \in f(a, U) \quad \text{iff } \models_{\mathbf{rud}(U)} \varphi(\mathring{x}, \mathring{a}, \mathring{U}).$$

Thus

$$A = \{x \in U \mid \models_{\mathbf{rud}(U)} \varphi(\mathring{x}, \mathring{a}, \mathring{U})\}.$$

By Σ_0-absoluteness it follows that

$$A = \{x \in U \mid \models_{U \cup \{U\}} \varphi(\mathring{x}, \mathring{a}, \mathring{U})\}.$$

Hence $A \in \Sigma_0(U \cup \{U\})$.

For the second part of the lemma it suffices to prove that

$$\Sigma_0(\mathrm{rud}(U)) \cap \mathscr{P}(U) \subseteq \mathrm{rud}(U).$$

Let $A \in \Sigma_0(\mathrm{rud}(U)) \cap \mathscr{P}(U)$. Then for some Σ_0 formula $\varphi(x)$ of $\mathscr{L}_{\mathrm{rud}(U)}$,

$$A = \{x \in U \mid \models_{\mathrm{rud}(U)} \varphi(\mathring{x})\}.$$

So as $\mathrm{rud}(U)$ is amenable, $A \in \mathrm{rud}(U)$. The proof is complete. $\square$

By tracing through the proof of the above lemma, we see that we have in fact proved the following result:

1.18 Lemma. *Let $\varphi(y, \vec{x})$ be a Σ_0 formula of $\mathscr{L}$. Then there is a formula $\psi(\vec{x})$ of $\mathscr{L}$ such that for any transitive, rudimentary closed set U,*

$$(\forall \vec{x} \in U)\,[\models_{\mathrm{rud}(U)} \varphi(U, \vec{x}) \quad \text{iff } \models_U \psi(\vec{x})]. \quad \square$$

This lemma will be of use to us in dealing with successor levels of the Jensen hierarchy of constructible sets. (See, for example, the proof of V.5.18.) The following consequence of 1.18 will be required in Chapter VIII.

1.19 Lemma. *Let M, N be transitive, rud closed sets, and let*

$$\sigma: M \prec N.$$

Then there is a unique embedding

$$\tilde{\sigma}: \mathrm{rud}(M) \prec_1 \mathrm{rud}(N)$$

such that $\sigma \subseteq \tilde{\sigma}$.

Proof. We show first that if f, g are rudimentary functions and $x, y \in M$ are such that $f(M, x) = g(M, y)$, then $f(N, \sigma(x)) = g(N, \sigma(y))$.

By 1.3 (and Σ_0 absoluteness), let φ be a Σ_0 formula of $\mathscr{L}$ such that for any transitive, rud closed set U and any $a, b, c \in U$.

$$f(a, b) = g(a, c) \leftrightarrow \models_U \exists z\, \varphi(z, \mathring{a}, \mathring{b}, \mathring{c}).$$

Then we have

$$\models_{\mathrm{rud}(M)} \exists z\, \varphi(z, \mathring{M}, \mathring{x}, \mathring{y}).$$

So for some $z \in \text{rud}(M)$,

$$\models_{\text{rud}(M)} \varphi(\mathring{z}, \mathring{M}, \mathring{x}, \mathring{y}).$$

For some rudimentary function h and some $w \in M$, we have $z = h(M, w)$. So

(1) $$\models_{\text{rud}(M)} \varphi(h(M, w)^{\circ}, \mathring{M}, \mathring{x}, \mathring{y}).$$

Since h is rudimentary, hence simple, the formula $\varphi(h(M, w), M, x, y)$ is in fact Σ_0 in the variables M, w, x, y. So by 1.18 there is an $\mathscr{L}$-formula ψ, which depends upon φ but not upon M, such that (1) is equivalent to

(2) $$\models_M \psi(\mathring{w}, \mathring{x}, \mathring{y})$$

(for any such M). Applying σ to (2) we get

(3) $$\models_N \psi(\mathring{\sigma}(w), \mathring{\sigma}(x), \mathring{\sigma}(y)).$$

But the equivalence of (1) and (2) holds for N as well as M. Hence by (3), we get

(4) $$\models_{\text{rud}(N)} \varphi(h(N, \sigma(w))^{\circ}, \mathring{N}, \mathring{\sigma}(x), \mathring{\sigma}(y)).$$

Hence

(5) $$\models_{\text{rud}(N)} \exists z \varphi(z, \mathring{N}, \mathring{\sigma}(x), \mathring{\sigma}(y)).$$

So by choice of φ, we conclude that $f(N, \sigma(x)) = g(N, \sigma(y))$, as required.

By the above result we may define a function $\tilde{\sigma}: \text{rud}(M) \to \text{rud}(N)$ by setting

$$\tilde{\sigma}(f(M, x)) = f(N, \sigma(x))$$

for all rudimentary functions f and all $x \in M$. We show that $\tilde{\sigma}$ is Σ_1 elementary. (Uniqueness of $\tilde{\sigma}$ will then be immediate, of course, since any Σ_1 elementary embedding which extends σ must satisfy the above defining equation.)

Let $\varphi(x, y)$ be a Σ_0 formula of $\mathscr{L}$. Suppose first that for some $x \in \text{rud}(M)$,

$$\models_{\text{rud}(M)} \exists y \varphi(\mathring{x}, y).$$

Pick $y \in \text{rud}(M)$ such that

$$\models_{\text{rud}(M)} \varphi(\mathring{x}, \mathring{y}).$$

There are rudimentary functions f, g and elements $\bar{x}, \bar{y} \in M$ such that $x = f(M, \bar{x})$, $y = g(M, \bar{y})$. Thus

(*) $$\models_{\text{rud}(M)} \varphi(f(M, \bar{x})^{\circ}, g(M, \bar{y})^{\circ}).$$

Since f, g are simple the formula $\varphi(f(M, \bar{x}), g(M, \bar{y}))$ is Σ_0 in variables M, $\bar{x}$, $\bar{y}$. So there is an $\mathscr{L}$-formula ψ, independent of M, such that $(*)$ is equivalent to

$$(**) \qquad \models_M \psi(\mathring{\bar{x}}, \mathring{\bar{y}}).$$

Applying σ to $(**)$, we get

$$\models_N \psi(\sigma(\mathring{\bar{x}}), \sigma(\mathring{\bar{y}})).$$

Since the equivalence of $(*)$ and $(**)$ is valid for N in place of M, we get

$$\models_{\mathrm{rud}(N)} \varphi(f(N, \sigma(\bar{x}))^\circ, g(N, \sigma(\bar{y}))^\circ),$$

i.e.

$$\models_{\mathrm{rud}(N)} \varphi(\tilde{\sigma}(\mathring{x}), \tilde{\sigma}(\mathring{y})).$$

Thus

$$\models_{\mathrm{rud}(N)} \exists y \, \varphi(\tilde{\sigma}(\mathring{x}), y).$$

This is what we set out to prove.

Conversely, suppose that $x \in \mathrm{rud}(M)$ is such that

$$\models_{\mathrm{rud}(N)} \exists y \, \varphi(\tilde{\sigma}(\mathring{x}), y).$$

Let $x = f(M, \bar{x})$, where f is rudimentary and $\bar{x} \in M$. Pick $y \in \mathrm{rud}(N)$ so that

$$\models_{\mathrm{rud}(N)} \varphi(\tilde{\sigma}(\mathring{x}), \mathring{y}).$$

Let $y = g(N, \bar{y})$ where g is rudimentary and $\bar{y} \in N$. Then

$$(+) \qquad \models_{\mathrm{rud}(N)} \varphi(f(N, \sigma(\bar{x}))^\circ, g(N, \bar{y})^\circ).$$

As above, let ψ be an $\mathscr{L}$-formula such that $(+)$ is equivalent to

$$(++) \qquad \models_N \psi(\sigma(\mathring{\bar{x}}), \mathring{\bar{y}})$$

for any such N. We have (since $(+)$ is valid)

$$\models_N \exists y' \, \psi(\sigma(\mathring{\bar{x}}), y').$$

So, as $\sigma: M \prec N$,

$$\models_M \exists y' \, \psi(\mathring{\bar{x}}, y').$$

So for some $y' \in M$,

$$\models_M \psi(\mathring{\bar{x}}, \mathring{y}').$$

By the equivalence of $(+)$ and $(++)$ applied to M, we get

$$\vDash_{\mathrm{rud}(M)} \varphi(\mathring{f(M, \vec{x})}, \mathring{g(M, \vec{y}')}).$$

Thus

$$\vDash_{\mathrm{rud}(M)} \exists y\, \varphi(\mathring{\vec{x}}, y),$$

and the proof is complete. □

The following lemma provides us with a useful hierarchy for the construction of $\mathrm{rud}(U)$ from U.

1.20 Lemma. *There is a rudimentary function* $\mathbf{S}$ *such that whenever* U *is transitive,*

$$U \cup \{U\} \subseteq \mathbf{S}(U) \quad \text{and} \quad \mathrm{rud}(U) = \bigcup_{n<\omega} \mathbf{S}^n(U).$$

Proof. Set

$$\mathbf{S}(U) = [U \cup \{U\}] \cup \left[\bigcup_{i=0}^{8} F_i''(U \cup \{U\})^2\right].$$

The result follows from 1.11. □

1.21 Lemma. *There is a rudimentary function* $\mathbf{Wo}$ *such that whenever* u *is transitive and* r *is a well-ordering of* u, $\mathbf{Wo}(u, r)$ *is an end-extension of* r *which well-orders* $\mathbf{S}(u)$.

Proof. The idea is roughly the same as in II.4.4. Since

$$\mathbf{S}(u) = [u \cup \{u\}] \cup \left[\bigcup_{i=0}^{8} F_i''(u \cup \{u\})^2\right],$$

r induces, via the functions $F_0, \dots, F_8$, a well-ordering of $\mathbf{S}(u)$. The function $\mathbf{Wo}$ will be rudimentary because of 1.1(14) and 1.5, since we shall obtain $\mathbf{Wo}$ by the definition

$$\mathbf{Wo}(u, r) = \mathbf{S}(u)^2 \cap \{(x, y) \mid \Phi(u, r, x, y)\},$$

where Φ is a Σ_0 formula of LST (see below).

Before we formulate Φ precisely, let us indicate what this formula is intended to say. Let $\tilde{r}$ denote the ordering r with u added as a greatest element. To see if $\Phi(u, r, x, y)$, we first check if $x, y \in u \cup \{u\}$, in which case we order x, y according to r, i.e. $\Phi(u, r, x, y)$ iff $x \tilde{r} y$. If $x \in u \cup \{u\}$ and $y \notin u \cup \{u\}$, then $\Phi(u, r, x, y)$ unconditionally holds. If $x \notin u \cup \{u\}$ and $y \in u \cup \{u\}$, then $\neg \Phi(u, r, x, y)$. Now suppose that $x, y \notin u \cup \{u\}$. First we see if the least i for which $x \in F_i''(u \cup \{u\})^2$ is smaller than the least i for which $y \in F_i''(u \cup \{u\})^2$, in which case $\Phi(u, r, x, y)$. If the two indices here are ordered in the opposite way, then $\neg \Phi(u, r, x, y)$. Otherwise, let i be the common least index here, and proceed as follows. Let x_1 be the $\tilde{r}$-least element of $u \cup \{u\}$ for which $x \in F_i''(\{x_1\} \times (u \cup \{u\}))$, and let y_1 be defined analogously for y. If $x_1 \tilde{r} y_1$, then $\Phi(u, r, x, y)$, and if $y_1 \tilde{r} x_1$, then $\neg \Phi(u, r, x, y)$. Otherwise, $x_1 = y_1$, and we define x_2 to be the $\tilde{r}$-least member of $u \cup \{u\}$ such that $x = F_i(x_1, x_2)$ and define y_2 for y, y_1 analogously, and set $\Phi(u, r, x, y)$ iff $x_2 \tilde{r} y_2$.

Precisely, $\Phi(u, r, x, y)$ is the following Σ_0 formula of LST (which we write in an abbreviated form for clarity):

$$\begin{aligned}
&[(x \in u) \wedge (y \in u) \wedge (x r y)] \vee [(x \in u) \wedge (y \notin u)] \\
&\quad \vee [(x = u) \wedge (y \notin u) \wedge (y \neq u)] \vee \bigvee_{i=0}^{8} [(x \notin u) \wedge (x \neq u) \wedge (y \notin u) \\
&\quad \wedge (y \neq u) \wedge \bigwedge_{j<i} (x \notin F_j''(u \cup \{u\})^2 \wedge y \notin F_j''(u \cup \{u\})^2) \\
&\quad \wedge [(x \in F_i''(u \cup \{u\})^2 \wedge y \notin F_i''(u \cup \{u\})^2) \vee (\exists x_1, x_2 \in u \cup \{u\}) \\
&\qquad [x = F_i(x_1, x_2) \wedge (\forall y_1, y_2 \in u \cup \{u\})(y_1 r x_1 \vee y_1 = x_1 \\
&\quad \rightarrow y \neq F_i(y_1, y_2)] \vee (\exists x_1 \in u \cup \{u\})(\exists y_1, y_2 \in u \cup \{u\}) \\
&\qquad [x = F_i(x_1, y_1) \wedge y = F_i(x_1, y_2) \wedge (\forall z_1, z_2 \in u \cup \{u\})(z_1 r x_1 \\
&\quad \vee (z_1 \in u \wedge x_1 = u) \rightarrow x \neq F_i(z_1, z_2) \wedge y \neq F_i(z_1, z_2)) \\
&\quad \wedge (y_1 r y_2 \vee (y_1 \in u \wedge y_2 \notin u))]].
\end{aligned}$$

In connection with the above formula, the following points should be noted. $\bigvee_{i=0}^{8}$ denotes the disjunction of nine formulas for $i = 0, \ldots, 8$, and $\bigwedge_{j<i}$ is the conjunction of i formulas for $j = 0, \ldots, i-1$. In the case $i = 0$, the conjunction $\bigwedge_{j<i}$ should be dropped, whereas for $i = 1$ the conjunction is a degenerate one consisting of a single formula only. Expressions such as $x \in F_i''(u \cup \{u\})^2$ should be written as

$$(\exists y \in u \cup \{u\})(\exists z \in u \cup \{u\})(x = F_i(y, z)).$$

Since the function $u \cup \{u\}$ is simple, quantifiers of the forms $(\exists x \in u \cup \{u\})$ and $(\forall x \in u \cup \{u\})$ are allowed in a Σ_0 formula of course.

An examination of the above formula $\Phi(u, r, x, y)$ should complete the proof of the lemma now. $\square$

To complete this section we prove a result which we shall need in order to prove the Condensation Lemma for the Jensen hierarchy.

1.22 Lemma. *Let M be a transitive, rudimentary closed set, and let $X \prec_1 M$. Then X is rudimentary closed and $\langle X, \in\rangle$ satisfies the Axiom of Extensionality. Let $\pi: \langle X, \in\rangle \cong \langle W, \in\rangle$, where W is transitive. If $f: M^n \rightarrow M$ is rudimentary, then for all $\vec{x} \in X$, $\pi(f(\vec{x})) = f(\pi(\vec{x}))$.*

Proof. Since M is transitive, $\langle M, \in\rangle$ satisfies the Axiom of Extensionality. So for any $x, y \in X$,

$$\models_M [x \neq y \leftrightarrow \exists z(z \in x \leftrightarrow z \notin y)].$$

Thus if $x \neq y$, then since $X \prec_1 M$, we have

$$\models_X \exists z(z \in x \leftrightarrow z \notin y).$$

Hence

$$\models_X [x \neq y \leftrightarrow \exists z(z \in x \leftrightarrow z \notin y)].$$

and so $\langle X, \in \rangle$ satisfies the Axiom of Extensionality. And by 1.3, X is, of course, rudimentary closed, so in particular, if $f\colon M^n \to M$ is rudimentary, then $f(\vec{x}) \in X$ whenever $\vec{x} \in X$. We shall prove by means of an induction on a rudimentary definition of f that $\pi(f(\vec{x})) = f(\pi(\vec{x}))$ for all $\vec{x} \in X$. Cases (i) through (iv) of the rudimentary function schemata cause no problems in this induction, as is easily seen. For case (v), suppose that $f(y, \vec{x}) = \bigcup_{z \in y} g(z, \vec{x})$, where g is rudimentary and for $z, \vec{x} \in X$, it is the case that $\pi(g(z, \vec{x})) = g(\pi(z), \pi(\vec{x}))$. Let $y, \vec{x} \in X$. We show that $\pi(f(y, \vec{x})) = f(\pi(y), \pi(\vec{x}))$.

By definition of π,

$$\pi(f(y, \vec{x})) = \pi''[f(y, \vec{x}) \cap X].$$

And by definition of f,

$$\begin{aligned} f(\pi(y), \pi(\vec{x})) &= \bigcup \{g(z, \pi(\vec{x})) \mid z \in \pi(y)\} \\ &= \bigcup \{g(z, \pi(\vec{x})) \mid z \in \pi''(y \cap X)\} \\ &= \bigcup \{g(\pi(z), \pi(\vec{x})) \mid z \in y \cap X\} \\ &= \bigcup \{\pi(g(z, \vec{x})) \mid z \in y \cap X\}. \end{aligned}$$

So it suffices to show that

$$\pi''[f(y, \vec{x}) \cap X] = \bigcup \{\pi(g(z, \vec{x})) \mid z \in y \cap X\}.$$

Suppose first that $v \in \pi''[f(y, \vec{x}) \cap X]$, say $v = \pi(u)$ where $u \in f(y, \vec{x}) \cap X$. Since $u \in f(y, \vec{x})$, we have $(\exists z \in y)(u \in g(z, \vec{x}))$. But this sentence is $\Sigma_1^M(\{u, y, \vec{x}\})$ and $u, y, \vec{x} \in X \prec_1 M$, so $(\exists z \in y \cap X)(u \in g(z, \vec{x}))$. Hence $v = \pi(u) \in \bigcup \{\pi(g(z, \vec{x})) \mid z \in y \cap X\}$.

Now suppose that $v \in \bigcup \{\pi(g(z, \vec{x})) \mid z \in y \cap X\}$. Pick $z \in y \cap X$ such that $v \in \pi(g(z, \vec{x}))$. Then $v \in \pi''[g(z, \vec{x}) \cap X]$, so for some $u \in g(z, \vec{x}) \cap X$, we have $v = \pi(u)$. But then $u \in \bigcup \{g(z, \vec{x}) \mid z \in y\}$ and $u \in X$, so $u \in f(y, \vec{x}) \cap X$, which gives $v = \pi(u) \in \pi''[f(y, \vec{x}) \cap X]$. The proof is complete. □

2. *The Jensen Hierarchy of Constructible Sets*

The *Jensen hierarchy*, $(J_\alpha \mid \alpha \in \mathrm{On})$, is defined by the following recursion:

$$\begin{aligned} J_0 &= \emptyset; \\ J_{\alpha+1} &= \mathrm{rud}(J_\alpha); \\ J_\lambda &= \bigcup_{\alpha < \lambda} J_\alpha, \quad \text{if } \lim(\lambda). \end{aligned}$$

2.1 Lemma.

(i) *Each* J_α *is transitive.*

(ii) $\alpha \leqslant \beta$ *implies* $J_\alpha \subseteq J_\beta$.

(iii) $\mathit{rank}(J_\alpha) = J_\alpha \cap \mathrm{On} = \omega\alpha$.

Proof. (i) By 1.16.

(ii) Immediate.

(iii) By induction on α. For $\alpha = 0$ the result is trivial. Limit stages in the induction are immediate. For successor steps, we use the finite rank property of rudimentary functions (1.4) to show that

$$\mathrm{rank}(J_{\alpha+1}) = \mathrm{rank}(\mathrm{rud}(J_\alpha)) = \mathrm{rank}(J_\alpha) + \omega.$$

The details are left to the reader. $\square$

Note in particular that in passing from J_α to $J_{\alpha+1}$, exactly ω new ordinals appear: $\omega\alpha, \omega\alpha + 1, \omega\alpha + 2, \ldots, \omega\alpha + n, \ldots, (n \in \omega)$, whereas by 1.17,

$$J_{\alpha+1} \cap \mathscr{P}(J_\alpha) = \mathrm{Def}(J_\alpha).$$

Thus, although $J_{\alpha+1}$ only contains those subsets of J_α which are J_α-definable, these sets appear in a hierarchy which is "stretched" from one level of rank, as is the case with the usual constructible hierarchy, to ω levels of rank. Moreover, this stretched hierarchy is closed under many simple set-theoretic functions such as ordered pairs, union, cartesian product, etc.

To facilitate our handling of the Jensen hierarchy, we define a sub-hierarchy as follows.

$$\begin{aligned} S_0 &= \emptyset; \\ S_{\alpha+1} &= \mathbf{S}(S_\alpha); \\ S_\lambda &= \bigcup_{\alpha<\lambda} S_\alpha, \qquad \text{if } \lim(\lambda). \end{aligned}$$

Clearly, the sets J_α are just the limit levels of this new hierarchy. In fact:

2.2 Lemma.

(i) $\alpha \leqslant \beta$ *implies* $S_\alpha \subseteq S_\beta$;

(ii) $J_\alpha = \bigcup_{\nu<\omega\alpha} S_\nu = S_{\omega\alpha}$.

Proof. (i) Immediate.

(ii) By induction. The only non-trivial step is the successor step. Here we have:

$$\begin{aligned} J_{\alpha+1} &= \mathrm{rud}(J_\alpha) = \bigcup_{n\in\omega} \mathbf{S}^n(J_\alpha) = \bigcup_{n\in\omega} \mathbf{S}^n(S_{\omega\alpha}) = \bigcup_{n\in\omega} S_{\omega\alpha+n} = S_{\omega\alpha+\omega} \\ &= S_{\omega(\alpha+1)}. \quad \square \end{aligned}$$

We shall use the S-hierarchy in order to assist our detailed study of the Jensen hierarchy. But before we commence this study, let us digress for a moment to examine the relationship between the Jensen hierarchy and the usual constructible hierarchy. (In particular, we have not yet proved that the Jensen hierarchy does consist only of constructible sets, and that all constructible sets do appear in the Jensen hierarchy.)

Will, we have $J_0 = L_0 = \emptyset$, of course. And it is easily seen that $J_1 = H_\omega = L_\omega$. In view of these two facts, and our knowledge that $J_\alpha \cap \mathrm{On} = \omega\alpha$ and $L_\alpha \cap \mathrm{On} = \alpha$ for all α, one might be tempted into thinking that $J_\alpha = L_{\omega\alpha}$ for all α. This is not the case, however. (The proof that the above equality is false makes a good little exercise for the reader.) Nevertheless, we do have $J_\alpha = L_\alpha$ whenever $\omega\alpha = \alpha$. As a first step towards proving this, we have:

2.3 Lemma. *For all α, $L_a \subseteq J_\alpha$ and $L_\alpha, (L_\beta \mid \beta \leqslant \alpha) \in J_{\alpha+1}$.*

Proof. We first of all prove that:

$$(*) \qquad u \in J_\alpha \to \mathrm{Def}(u) \subseteq J_\alpha.$$

For $\alpha = 0$ there is nothing to prove, and for $\alpha = 1$ the result is trivial, since $J_1 = H_\omega$, so we shall assume that $\alpha > 1$ from now on. During the proof of 1.11, we showed that for any representation $\varphi(x_0, \dots, x_n)$ of an $\mathscr{L}$-formula φ, the function $t_{\varphi(x_0,\dots,x_n)}$ is rudimentary, where

$$t_{\varphi(x_0,\dots,x_n)}(u) = \{(x_0, \dots, x_n) \mid x_0, \dots, x_n \in u \wedge \models_u \varphi(\mathring{x}_0, \dots, \mathring{x}_n)\}.$$

It follows that the functions $d_{\varphi(x_0,\dots,x_n)}$ are rudimentary, where we define

$$d_{\varphi(x_0,\dots,x_n)}(u, x_1, \dots, x_n) = \begin{cases} \{x_0 \in u \mid \models_u \varphi(\mathring{x}_0, \dots, \mathring{x}_n)\}, & \text{if } x_1, \dots, x_n \in u \\ \emptyset, & \text{otherwise.} \end{cases}$$

Since J_α is rudimentary closed, for each $\varphi(x_0, \dots, x_n)$ we have

$$u, x_1, \dots, x_n \in J_\alpha \to d_{\varphi(x_0,\dots,x_n)}(u, x_1, \dots, x_n) \in J_\alpha.$$

But for any set u,

$$\mathrm{Def}(u) = \{d_{\varphi(x_0,\dots,x_n)}(u, x_1, \dots, x_n) \mid \varphi(x_0, \dots, x_n) \text{ is a representation of an } \mathscr{L}\text{-formula } \varphi \text{ and } x_1, \dots, x_n \in u\}.$$

Thus $u \in J_\alpha$ implies $\mathrm{Def}(u) \subseteq J_\alpha$, which proves (*).

We prove the lemma by induction now. For $\alpha = 0$ there is nothing to prove. For the successor case, suppose we know that $L_\alpha \subseteq J_\alpha$, $L_\alpha \in J_{\alpha+1}$, and $(L_\beta \mid \beta \leqslant \alpha) \in J_{\alpha+1}$. Since $L_\alpha \in J_{\alpha+1}$, (*) tells us at once that $L_{\alpha+1} = \mathrm{Def}(L_\alpha) \subseteq J_{\alpha+1}$. We show next that $L_{\alpha+1} \in \mathrm{Def}(J_{\alpha+1})$, whence $L_{\alpha+1} \in J_{\alpha+2}$, of course. Well, we have

$$\begin{aligned}
L_{\alpha+1} &= \{x \subseteq L_\alpha \mid (\exists \varphi)(\exists(\vec{a}))[\mathrm{Fml}_\emptyset(\varphi) \wedge \vec{a} \in L_\alpha \\
&\quad \wedge (\forall z \in L_\alpha)(z \in x \leftrightarrow \models_{L_\alpha} \varphi(\mathring{z}, \vec{\mathring{a}}))]\} \\
&= \{x \in J_{\alpha+1} \mid x \in L_\alpha \wedge (\exists \varphi)(\exists(\vec{a}))[\mathrm{Fml}_\emptyset(\varphi) \wedge \vec{a} \in L_\alpha \\
&\quad \wedge (\forall z \in L_\alpha)(z \in x \leftrightarrow \models_{L_\alpha} \varphi(\mathring{z}, \vec{\mathring{a}}))]\} \\
&= \{x \in L_{\alpha+1} \mid x \subseteq L_\alpha \wedge (\exists \varphi)(\exists(\vec{a}))[\mathrm{Fml}_\emptyset(\varphi) \wedge \vec{a} \in L_\alpha \\
&\quad \wedge (\forall z \in L_\alpha)(z \in x \leftrightarrow \mathrm{Sat}(L_\alpha, \mathrm{Sub}(\varphi, \vec{v}, \mathring{z}, \vec{\mathring{a}})))]\},
\end{aligned}$$

where for clarity we have abused slightly the notation developed in II.2, using Sub as a function rather than as a relation. Now, for amenable sets M, the predicate $\mathrm{Fml}_\emptyset(-)$ is Δ_1^M (by II.2.4), the function Sub is Δ_1^M (by II.2.7), and the predicate Sat is Δ_1^M (by II.2.8). But $J_{\alpha+1}$ is rudimentary closed, and hence amenable. Moreover, the set $\mathrm{Fml}_\emptyset$ is a subset of $J_{\alpha+1}$. Hence by Δ_1-absoluteness,

$$\begin{aligned}
L_{\alpha+1} = \{x \in J_{\alpha+1} \models_{J_{\alpha+1}} \text{“}(x \subseteq L_\alpha) \wedge (\exists \varphi)(\exists(\vec{a}))[\mathrm{Fml}_\emptyset(\varphi) \wedge \vec{a} \in L_\alpha \\
\wedge (\forall z \in L_\alpha)(z \in x \leftrightarrow \mathrm{Sat}(L_\alpha, \mathrm{Sub}(\varphi, \vec{v}, \mathring{z}, \vec{\mathring{a}})))]\text{”}\}.
\end{aligned}$$

Hence $L_{\alpha+1} \in \mathrm{Def}(J_{\alpha+1})$, giving $L_{\alpha+1} \in J_{\alpha+2}$, as required. Finally, we have

$$(L_\beta \mid \beta \leqslant \alpha + 1) = (L_\beta \mid \beta < \alpha) \cup \{(L_{\alpha+1}, \alpha + 1)\},$$

so by induction hypothesis and the fact that $L_{\alpha+1} \in J_{\alpha+2}$, since $J_{\alpha+2}$ is rudimentary closed, we see that $(L_\beta \mid \beta \leqslant \alpha + 1) \in J_{\alpha+2}$.

There remains the limit case of the induction. Suppose that $\alpha > 0$ is a limit ordinal, and that for all $\beta < \alpha$, $L_\beta \subseteq J_\beta$ and L_β, $(L_\gamma \mid \gamma \leqslant \beta) \in J_{\beta+1}$. So, as J_α is transitive, $L_\beta \subseteq J_\alpha$ for all $\beta < \alpha$. Hence $L_\alpha = \bigcup_{\beta<\alpha} L_\beta \subseteq J_\alpha$. Again,

$$L_\alpha = \{x \in J_\alpha \mid (\exists \nu < \alpha)(x \in L_\nu)\},$$

so we have

$$\begin{aligned}
L_\alpha = \{x \in J_\alpha \mid (\exists f)[&f \text{ is a function } \wedge \mathrm{dom}(f) \in \alpha \wedge f(0) = \emptyset \\
&\wedge (\forall \nu \in \mathrm{dom}(f))[(\lim(\nu) \to f(\nu) = \bigcup_{\tau<\nu} f(\tau)) \wedge (\mathrm{succ}(\nu) \\
&\to f(\nu) = \mathrm{Def}(f(\nu - 1)))] \wedge x \in \mathrm{ran}(f)]\}.
\end{aligned}$$

But by induction hypothesis, $(L_\gamma \mid \gamma \leqslant \beta) \in J_\alpha$ for all $\beta < \alpha$, so the quantifier $(\exists f)$ in the above can be restricted to J_α (without affecting the meaning). Moreover, the unbounded quantifiers involved in the definition of the function Def can also be restricted to J_α, since they only refer to elements of $\bigcup \mathrm{ran}(f)$ (see the proof of II.2.12). Hence, if φ is the $\mathscr{L}$-formula which we have just been (implicitly) discussing, we have

$$L_\alpha = \{x \in J_\alpha \mid \models_{J_\alpha} \varphi(\mathring{x})\}.$$

Thus $L_\alpha \in \mathrm{Def}(J_\alpha) \subseteq J_{\alpha+1}$. Similar considerations lead to the conclusion that $(L_\beta \mid \beta < \alpha) \in \mathrm{Def}(J_\alpha)$, and so

$$(L_\beta \mid \beta \leqslant \alpha) = (L_\beta \mid \beta < \alpha) \cup \{(L_\alpha, \alpha)\} \in J_{\alpha+1}.$$

The lemma is proved. $\square$

Using 2.3, we may now show that

$$L = \bigcup_{\alpha \in \mathrm{On}} J_\alpha.$$

In fact we show that the sets J_α and L_α are equal for many ordinals α.

2.4 Lemma.

(i) $L_\alpha \subseteq J_\alpha \subseteq L_{\omega\alpha}$.

(ii) $J_\alpha = L_\alpha$ *iff* $\omega\alpha = \alpha$.

(iii) $L = \bigcup_{\alpha \in \mathrm{On}} J_\alpha$.

Proof. Clearly, (i) $\rightarrow$ (ii) $\rightarrow$ (iii). We prove (i). By 2.3, we know already that $L_\alpha \subseteq J_\alpha$. We show that $J_\alpha \subseteq L_{\omega\alpha}$. As a first step we prove that

(*) for all α: $u \in L_\alpha \rightarrow \mathbf{S}(u) \in L_{\alpha+5}$.

It is easily seen that for each $i = 0, \ldots, 8$,

$$x, y \in L_\alpha \rightarrow F_i(x, y) \in L_{\alpha+4}.$$

Thus if $u \in L_\alpha$, we have $\mathbf{S}(u) \subseteq L_{\alpha+4}$. So, by Σ_0-absoluteness,

$$\mathbf{S}(u) = \{x \in L_{\alpha+4} \mid \models_{L_{\alpha+4}} \text{“}(x \in u) \wedge (\exists v, w \in u \cup \{u\}) \\ [x = F_0(v, w) \vee \ldots \vee x = F_8(v, w)]\text{”}\}.$$

Hence $\mathbf{S}(u) \in \mathrm{Def}(L_{\alpha+4}) = L_{\alpha+5}$, which proves (*).

In order to prove that $J_\alpha \subseteq L_{\omega\alpha}$, since $L_{\omega\alpha}$ is transitive and $J_\alpha = \bigcup_{\nu < \omega\alpha} S_\nu$, it suffices to show that $S_\nu \in L_{\omega\alpha}$ for all $\nu < \omega\alpha$. By (*), $\mathbf{S}'' L_{\omega\alpha} \subseteq L_{\omega\alpha}$. In particular, $L_{\omega\alpha}$ is rudimentary closed and (by 1.3) there is a Σ_0 formula $\varphi(v_0, v_1)$ of $\mathscr{L}$, independent of α, such that for $x, y \in L_{\omega\alpha}$,

$$y = \mathbf{S}(x) \quad \text{iff } \models_{L_{\omega\alpha}} \varphi(\mathring{y}, \mathring{x}).$$

By induction on α we prove the following result:

$P(\alpha)$: if $\nu < \omega\alpha$, then S_ν, $(S_\tau \mid \tau \leqslant \nu) \in L_{\omega\alpha}$ and the sequence $(S_\nu \mid \nu < \omega\alpha)$ is uniformly $\Sigma_1^{L_{\omega\alpha}}$.

This, of course, will complete the proof of the lemma.

Let $\theta(f)$ be the following Σ_0 formula of $\mathscr{L}$ (to define the hierarchy $(S_\nu \mid \nu \in \mathrm{On})$):

$$\text{“}f \text{ is a function”} \wedge \text{“dom}(f) \text{ is an ordinal”} \wedge f(0) = \emptyset$$
$$\wedge\ (\forall \nu \in \mathrm{dom}(f))\,[(\mathrm{succ}(\nu) \to \varphi(f(\nu), f(\nu - 1)))$$
$$\wedge\ (\lim(\nu) \to f(\nu) = \bigcup_{\tau \in \nu} f(\tau)))].$$

By our above remarks, it is clear that for any α and any $\nu < \omega\alpha$, if

$$\models_{L_{\omega\alpha}} \exists f[\theta(f) \wedge y = f(\nu)],$$

then $y = S_\nu$. We prove the part of $P(\alpha)$ concerning Σ_1 definability by showing that, in fact, for any α and any $\nu < \omega\alpha$,

$$y = S_\nu \quad \text{iff } \models_{L_{\omega\alpha}} \exists f[\theta(f) \wedge y = f(\nu)].$$

Now for the proof of $P(\alpha)$. For $\alpha = 0$ there is nothing to prove. Now assume $P(\alpha)$. Then, in particular, $(S_\tau \mid \tau < \omega\alpha)$ is $\Sigma_1^{L_{\omega\alpha}}$, and hence is an element of $L_{\omega\alpha+1}$. Thus $J_\alpha = \bigcup_{\tau < \omega\alpha} S_\tau \in L_{\omega\alpha+2} \subseteq L_{\omega(\alpha+1)}$. For any $n < \omega$, since $L_{\omega\alpha}$ is rudimentary closed, we have $S_{\omega\alpha+n} = \mathbf{S}^n(J_\alpha) \in L_{\omega(\alpha+1)}$. Thus $S_\nu \in L_{\omega(\alpha+1)}$ for all $\nu < \omega(\alpha + 1)$. Again, for any $n < \omega$, $(S_\tau \mid \tau \leqslant \omega\alpha + n) = (S_\tau \mid \tau < \omega\alpha) \cup \{(S_{\omega\alpha+m}, \omega\alpha + m) \mid m \leqslant n\}$, so as $L_{\omega(\alpha+1)}$ is rudimentary closed, $(S_\tau \mid \tau \leqslant \omega\alpha + n) \in L_{\omega(\alpha+1)}$, and so $(S_\tau \mid \tau \leqslant \nu) \in L_{\omega(\alpha+1)}$ for all $\nu < \omega(\alpha + 1)$. Finally, to show that for any $\nu < \omega(\alpha + 1)$,

$$y = S_\nu \quad \text{iff } \models_{L_{\omega(\alpha+1)}} \exists f[\theta(f) \wedge y = f(\nu)],$$

it clearly suffices to show that whenever $\nu < \omega(\alpha + 1)$ and $y = S_\nu$, then there is an $f \in L_{\omega(\alpha+1)}$ such that

$$\models_{L_{(\alpha+1)}} \theta(f) \wedge y = f(\nu).$$

But $(S_\tau \mid \tau \leqslant \nu) \in L_{\omega(\alpha+1)}$ is such an f, so we are done.

Finally, assume $\delta > 0$ is a limit ordinal and that $P(\alpha)$ holds for all $\alpha < \delta$. It is then trivial that $S_\nu, (S_\tau \mid \tau \leqslant \nu) \in L_{\omega\delta}$ for all $\nu < \omega\delta$. And since $(S_\tau \mid \tau \leqslant \nu) \in L_{\omega\delta}$ for all $\nu < \omega\delta$, the same argument as above shows that for $\nu < \omega\delta$,

$$x = S_\nu \quad \text{iff } \models_{L_{\omega\delta}} \exists f[\theta(f) \wedge y = f(\nu)].$$

The proof is complete. $\square$

Returning now to our study of the Jensen hierarchy itself, the same argument as in 2.4 above shows that

2.5 Lemma. *The sequence $(S_\nu \mid \nu < \omega\alpha)$ is uniformly $\Sigma_1^{J_\alpha}$ for all α.* $\square$

2.6 Corollary. *The sequence* $(J_\nu \mid \nu < \alpha)$ *is uniformly* $\Sigma_1^{J_\alpha}$ *for all* α.

Proof. Clearly, the sequence $(\omega\nu \mid \nu < \alpha)$ is uniformly $\Sigma_1^{J_\alpha}$ for all α, so the result follows easily from 2.5. □

2.7 Lemma. *There are well-orderings* $<_\nu^S$ *of the sets* S_ν *such that*:

(i) $\nu_1 < \nu_2$ *implies* $<_{\nu_1}^S \subseteq <_{\nu_2}^S$;
(ii) $<_{\nu+1}^S$ *is an end-extension of* $<_\nu^S$;
(iii) *the sequence* $(<_\nu^S \mid \nu < \omega\alpha)$ *is uniformly* $\Sigma_1^{J_\alpha}$ *for all* α.

Proof. We use 1.21. Set $<_0^S = \emptyset$, and, by recursion, let

$$<_{\nu+1}^S = \mathbf{Wo}(S_\nu, <_\nu^S),$$
$$<_\lambda^S = \bigcup_{\nu<\lambda} <_\nu^S, \qquad \text{if } \lim(\lambda).$$

Then (i) and (ii) are immediate, whilst (iii) is proved by an argument as in 2.4 and 2.5. □

2.8 Lemma. *There are well-orderings* $<_\alpha$ *of the sets* J_α *such that*:

(i) $\alpha_1 < \alpha_2$ *implies* $<_{\alpha_1} \subseteq <_{\alpha_2}$;
(ii) $<_{\alpha+1}$ *is an end-extension of* $<_\alpha$;
(iii) *the sequence* $(<_\beta \mid \beta < \alpha)$ *is uniformly* $\Sigma_1^{J_\alpha}$ *for all* α;
(iv) $<_\alpha$ *is uniformly* $\Sigma_1^{J_\alpha}$ *for all* α;
(v) *the function* $\mathrm{pr}_\alpha(x) = \{z \mid z <_\alpha x\}$ *is uniformly* $\Sigma_1^{J_\alpha}$ *for all* α.

Proof. Set $<_\alpha = <_{\omega\alpha}^S$ for all α. Then (i)–(iii) are immediate by 2.7. For (iv), note simply that $x <_\alpha y$ iff $(\exists \nu \in J_\alpha)(x <_\nu^S y)$. For (v), note that

$$y = \mathrm{pr}_\alpha(x) \quad \text{iff } (\exists \nu \in J_\alpha)(x \in S_\nu \wedge y = \{z \mid z <_\nu^S x\})$$

and that $\nu < \omega\alpha$ implies $<_\nu^S \in J_\alpha$, and use 2.5 and 2.7. □

By 2.4 we can define a well-ordering $<_J$ of L by setting

$$<_J = \bigcup_{\alpha \in \mathrm{On}} <_\alpha.$$

Then, as was the case with the well-ordering $<_L$, $<_J$ is a Σ_1 well-ordering of L.

2.9 Lemma (Condensation Lemma). *Let* α *be any ordinal. Let* $X \prec_1 J_\alpha$. *Then there is a unique ordinal* β *and a unique isomorphism* π *such that*:

(i) $\pi\colon X \cong J_\beta$;
(ii) $\pi(\nu) \leqslant \nu$ *for all* $\nu \in X \cap \omega\alpha$;
(iii) $\pi(x) \leqslant_J x$ *for all* $x \in X$;
(iv) *if* $Y \subseteq X$ *is transitive, then* $\pi \restriction Y = \mathrm{id} \restriction Y$.

Proof. By 1.22 there are unique π, W such that $\pi\colon X \cong W$, where W is transitive. Let $\beta = \pi''(X \cap \omega\alpha)$. We show that $W = J_\beta$, which proves (i). First we establish a simple claim.

Claim. $\gamma \in X \cap \omega\alpha \to [S_\gamma \in X \wedge \pi(S_\gamma) = S_{\pi(\gamma)}]$.

We prove the claim by induction on γ. Clearly, $0 \in X \cap \omega\alpha$, $S_0 = \emptyset \in X$, and $\pi(S_0) = \pi(\emptyset) = \emptyset = S_0 = S_{\pi(0)}$. Suppose now that $\gamma = \delta + 1$ and we have proved the claim below γ. Since $\gamma \in X$, we have $\delta \in X$ also. And by 2.5, we have $S_\gamma, S_\delta \in X$. Using 1.22 now, together with the induction hypothesis,

$$\pi(S_\gamma) = \pi(S_{\delta+1}) = \pi(\mathbf{S}(S_\delta)) = \mathbf{S}(\pi(S_\delta)) = \mathbf{S}(S_{\pi(\delta)}) = S_{\pi(\delta)+1} = S_{\pi(\gamma)}.$$

Finally, suppose that $\gamma > 0$ is a limit ordinal and we have proved the claim below γ. Notice that $\models_{J_\alpha} \lim(\gamma)$, so $\lim(\operatorname{otp}(X \cap \gamma))$, so $\lim(\pi(\gamma))$. Now, $S_\gamma = \bigcup_{\delta<\gamma} S_\delta$, so $\pi(S_\gamma) = \pi''(S_\gamma \cap X) = \pi''(\bigcup_{\delta<\gamma}(S_\delta \cap X))$, so it suffices to show that $S_{\pi(\gamma)} = \pi''(\bigcup_{\delta<\gamma}(S_\delta \cap X))$. First of all, let $x \in S_{\pi(\gamma)}$. Thus for some $\xi < \pi(\gamma)$, $x \in S_\xi$. But $\operatorname{ran}(\pi)$ is transitive, so $\xi = \pi(\zeta)$ for some $\zeta \in X \cap \gamma$. Thus by induction hypothesis, $x \in S_{\pi(\zeta)} = \pi(S_\zeta) = \pi''(S_\zeta \cap X) \subseteq \pi''(\bigcup_{\delta<\gamma}(S_\delta \cap X))$. Conversely, let $x \in \pi''(\bigcup_{\delta<\gamma}(S_\delta \cap X))$. Thus $x = \pi(y)$, where $y \in \bigcup_{\delta<\gamma}(S_\delta \cap X)$. Now, $\models_{J_\alpha}(\exists\delta < \gamma)(y \in S_\delta)$, so as $y, \gamma \in X \prec_1 J_\alpha$, we have $\models_X(\exists\delta < \gamma)(y \in S_\delta)$, so we can pick $\delta \in X \cap \gamma$ with $y \in S_\delta$. Then by induction hypothesis, $x = \pi(y) \in \pi(S_\delta) = S_{\pi(\delta)}$. But $\pi(\delta) < \pi(\gamma)$. Hence $x \in S_{\pi(\gamma)}$. This proves the claim.

Using the claim, it is now easy to prove that $W = S_{\omega\beta} = J_\beta$. Suppose first that $w \in W$. Thus $w = \pi(x)$ for some $x \in X$. Now $\models_{J_\alpha}(\exists\gamma)(x \in S_\gamma)$, so as $x \in X \prec_1 J_\alpha$, we have $\models_X(\exists\gamma)(x \in S_\gamma)$. So pick $\gamma \in X \cap \omega\alpha$ with $x \in S_\gamma$. Then $w = \pi(x) \in \pi(S_\gamma) = S_{\pi(\gamma)} \subseteq S_{\omega\beta} = J_\beta$. Conversely, suppose that $y \in J_\beta$. Then $y \in S_\gamma$ for some $\gamma < \omega\beta$. But $\gamma = \pi(\delta)$ for some $\delta \in X \cap \omega\alpha$. Thus $y \in S_{\pi(\delta)} = \pi(S_\delta) = \pi''(S_\delta \cap X)$, whence $y \in \operatorname{ran}(\pi) = W$.

That proves part (i) of the lemma. Part (iv) holds by definition of π. And (ii) follows from (iii). So we need to prove (iii). Notice that as $<_\alpha$ is uniformly $\Sigma_1^{J_\alpha}$, we have

$$x <_\alpha y \quad \text{iff } \pi(x) <_\beta \pi(y).$$

Suppose that $x <_J \pi(x)$ for some $x \in X$. Let x be the $<_J$-least such. Since $\pi(x) \in J_\beta$, we must have $x \in J_\beta$ here, so $x = \pi(x')$ for some $x' \in X$. But $\pi(x') = x <_J \pi(x)$ so $x' <_J x$. Thus by choice of x, $\pi(x') \leqslant_J x'$. But this means that $x \leqslant_J x'$, which is absurd. The lemma is proved. $\square$

3. The Σ_1 Skolem Function

The general notion of a Σ_n skolem function was already introduced in II.6. Recall that if $\mathbf{M} = \langle M, (A_i)_{i<\omega}\rangle$, where M is an amenable set and $A_i \subseteq M$, then by a Σ_n-*skolem function* for $\mathbf{M}$ we mean a $\Sigma_n^M(\{p\})$ function h (for some $p \in M$) with

$\mathrm{dom}(h) \subseteq \omega \times M$, such that whenever $p \in \Sigma_n^M(\{x, p\})$ for some $x \in M$, then $\exists y P(y) \to (\exists i \in \omega) P(h(i, x))$. (In which case we say that p is a *good* parameter for h.)

In this section we shall be concerned with structures of the form $\langle M, A\rangle$, where $A \subseteq M$. Notice that if M is rudimentary closed, it is amenable. Hence we may reformulate II.6.1 through II.6.3 as follows.

If h is a function with $\mathrm{dom}(h) \subseteq \omega \times M$, and if $X \subseteq M$, then we shall denote by $h^*(X)$ the set $h''(\omega \times X)$. In what follows we assume $n \geqslant 1$.

3.1 Lemma. *Let $\langle M, A\rangle$ be transitive and rudimentary closed. Let h be a Σ_n skolem function for $\langle M, A\rangle$. If $x \in M$, then $x \in h^*(\{x\}) \prec_n \langle M, A\rangle$. (More precisely, $\langle h^*(\{x\}), A \cap h^*(\{x\})\rangle \prec_n \langle M, A\rangle$.)* $\square$

3.2 Lemma. *Let $\langle M, A\rangle$, h be as above. Let $q \in M$, and let $X \subseteq M$ be closed under ordered pairs. Then $X \cup \{q\} \subseteq h^*(X \times \{q\}) \prec_n \langle M, A\rangle$.* $\square$

3.3 Lemma. *Let $\langle M, A\rangle$, h be as above. Let $X \subseteq M$, and suppose that $h^*(X)$ is closed under ordered pairs. Then $X \subseteq h^*(X) \prec_n \langle M, A\rangle$.* $\square$

Now, in II.6.6 we showed that each limit $L_\alpha(\alpha > \omega)$ has a Σ_1 skolem function. And an entirely parallel proof will show that each $J_\alpha(\alpha > 1)$ has a Σ_1 skolem function. But as our discussion in section 1 indicate, we require slightly more than this. We need to know that each amenable structure $\langle J_\alpha, A\rangle$ has a (uniform) Σ_1 skolem function, and that even in the absence of amenability, the definition of this skolem function still defines a function having "skolem-like" properties. This is where 1.15 comes in. By 1.15 (together with 1.9) we have:

3.4 Lemma. *Let $n \geqslant 1$. If $\alpha > 1$ and $\langle J_\alpha, A\rangle$ is amenable, then $\models^{\Sigma_n}_{\langle J_\alpha, A\rangle}$ is (uniformly) $\Sigma_n^{\langle J_\alpha, A\rangle}$.* $\square$

We now fix, once and for all, some simple enumeration $(\varphi_i \mid i < \omega)$ of all the formulas of $\mathscr{L}(A)$ of the form

$$\varphi_i = \varphi_i(v_0, v_1) = \exists v_2 \bar{\varphi}_i(v_0, v_1, v_2),$$

where $\bar{\varphi}_i$ is Σ_0. The exact definition of this enumeration is not important. All we need to know is that it is $\Delta_1^{J_1}$, which will be the case for any "effective" enumeration. We leave it to the reader to supply any details felt neccessary.

Fix $\langle J_\alpha, A\rangle$ now. For $i \in \omega$ and $x \in J_\alpha$, set:

$$r_{\alpha, A}(i, x) \simeq \text{the } <_J\text{-least } z \in J_\alpha \text{ such that } \models_{\langle J_\alpha, A\rangle} \bar{\varphi}_i((\mathring{z})_0, \mathring{x}, (\mathring{z})_1)$$
$$h_{\alpha, A}(i, x) \simeq (r_{\alpha, A}(i, x))_0.$$

Thus, for $i \in \omega$ and $x, y \in J_\alpha$:

$$y = h_{\alpha, A}(i, x) \leftrightarrow \text{there is a } z \in J_\alpha \text{ such that } (z)_0 = y \text{ and } z \text{ is the } <_J\text{-least } z \text{ in } J_\alpha \text{ such that } \models_{\langle J_\alpha, A\rangle} \bar{\varphi}_i((\mathring{z})_0, \mathring{x}, (\mathring{z})_1).$$

In other words:

$$y = h_{\alpha,A}(i, x) \leftrightarrow \exists z \exists w \ [(z)_0 = y \wedge w = \{v \mid v <_J z\} \\ \wedge \models_{\langle J_\alpha, A\rangle} [\bar{\varphi}_i((\mathring{z})_0, \mathring{x}, (\mathring{z})_1) \wedge (\forall v \in \mathring{w}) \neg \bar{\varphi}_i((v)_0, \mathring{x}, (v)_1)]].$$

Let θ be the canonical Σ_0 formula such that for all $\alpha > 1$ and all $z \in J_\alpha$,

$$w = \{v \mid v <_J z\} \leftrightarrow \models_{J_\alpha} \exists t \theta(\mathring{w}, \mathring{z}, t).$$

(See 2.8(v).)

Then we have:

$$y = h_{\alpha,A}(i, x) \leftrightarrow \exists z \exists w \exists t [(z)_0 = y \wedge \models_{\langle J_\alpha, a\rangle} [\theta(\mathring{w}, \mathring{z}, \mathring{t}) \wedge \bar{\varphi}_i((\mathring{z})_0, \mathring{x}, (\mathring{z})_1) \\ \wedge (\forall v \in \mathring{w}) \neg \bar{\varphi}_i((v)_0, \mathring{x}, (v)_1)]].$$

Let $\theta_i(u, y, x)$ be the Σ_0 $\mathscr{L}$-formula:

$$[((u)_0)_0 = y \wedge \theta((u)_1, (u)_0, (u)_2) \wedge \bar{\varphi}_i(((u)_0)_0, x, ((u)_0)_1) \\ \wedge (\forall v \in (u)_1) \neg \bar{\varphi}_i((v)_0, x, (v)_1)].$$

(More precisely, let θ_i be the canonical rendering of this formula in true Σ_0 form.)

Then θ_i is independent of the choice of α, A. But clearly, for any $\langle J_\alpha, A\rangle$,

$$y = h_{\alpha,A}(i, x) \leftrightarrow (\exists u \in J_\alpha) [\models_{\langle J_\alpha, A\rangle} \theta_i(\mathring{u}, \mathring{y}, \mathring{x})].$$

We establish several important facts concerning the functions $h_{\alpha,A}$.

3.5 Lemma. *The sequence $(\theta_i \mid i < \omega)$ is $\Delta_1^{J_1}$.*

Proof. Since the sequence $(\bar{\varphi}_i \mid i < \omega)$ is $\Delta_1^{J_1}$. □

3.6 Lemma. *Let $1 < \bar{\alpha} < \alpha$, $A \subseteq J_\alpha$. If $y = h_{\bar{\alpha}, A \cap J_{\bar{\alpha}}}(i, x)$, then $y = h_{\alpha,A}(i, x)$.*

Proof. By Σ_0-absoluteness (I.9.14). □

Notice that we have not so far required that the structure $\langle J_\alpha, A\rangle$ is amenable. As we shall show presently, in the case where we do have amenability, the function $h_{\alpha,A}$ is Σ_1-definable over $\langle J_\alpha, A\rangle$. In such cases, it is possible to deduce our next three lemmas from II.6.1–II.6.3. We do not do it this way because we shall need these results in cases where amenability is not available.

3.7 Lemma. *Let $A \subseteq J_\alpha$, $x \in J_\alpha$. Then*

$$x \in h^*_{\alpha,A}(\{x\}) \prec_1 \langle J_\alpha, A\rangle.$$

Proof. Set $h = h_{\alpha,A}$, $N = h^*(\{x\})$. Let $P \in \Sigma_1^{\langle J_\alpha, A\rangle}(N) \cap \mathscr{P}(J_\alpha)$. We show that if $P \neq \emptyset$ then $P \cap N \neq \emptyset$.

Let P be $\Sigma_1^{\langle J_\alpha, A\rangle}(\{x_1, \ldots, x_n\})$, where $x_1, \ldots, x_n \in N$. Pick $i_1, \ldots, i_n \in \omega$ so that $x_1 = h(i_1, x), \ldots, x_n = h(i_n, x)$. For each $k = 1, \ldots, n$, x_k is the unique y in J_α such that $\models_{\langle J_\alpha, A\rangle} \exists z \theta_{i_k}(z, \mathring{y}, \mathring{x})$. Hence for each such k, x_k is Σ_1-definable from x in $\langle J_\alpha, A\rangle$. Hence P is in fact $\Sigma_1^{\langle J_\alpha, A\rangle}(\{x\})$. Thus for some $i \in \omega$,

$$P(y) \leftrightarrow \models_{\langle J_\alpha, A\rangle} \varphi_i(\mathring{y}, \mathring{x}).$$

Since $P \neq \emptyset$, let y be the $<_J$-least element of P. Then clearly, $y = h(i, x)$. Hence $y \in N$, proving that $P \cap N \neq \emptyset$. $\square$

By modifying the proof of the above lemma along the lines of II.6.2 and II.6.3, we obtain:

3.8 Lemma. *Let $A \subseteq J_\alpha, p \in J_\alpha$, $X \subseteq J_\alpha$. If X is closed under ordered pairs, then*

$$X \cup \{p\} \subseteq h^*_{\alpha,A}(X \times \{p\}) \prec_1 \langle J_\alpha, A\rangle. \quad \square$$

3.9 Lemma. *Let $A \subseteq J_\alpha$, $X \subseteq J_\alpha$. If $h^*_{\alpha,A}(X)$ is closed under ordered pairs, then*

$$X \subseteq h^*_{\alpha,A}(x) \prec_1 \langle J_\alpha, A\rangle. \quad \square$$

3.10 Lemma. *If $\langle J_\alpha, A\rangle$ is amenable, the function $h_{\alpha,A}$ is (uniformly) $\Sigma_1^{\langle J_\alpha, A\rangle}$.*

Proof. We have

$$y = h_{\alpha,A}(i, x) \leftrightarrow \models_{\langle J_\alpha, A\rangle} \exists u \theta_i(u, \mathring{y}, \mathring{x}).$$

By 3.5 and 3.4, the result follows immediately. $\square$

Let $H_{\alpha,A}$ denote the uniformly $\Sigma_0^{\langle J_\alpha, A\rangle}$ predicate such that for amenable $\langle J_\alpha, A\rangle$,

$$y = h_{\alpha,A}(i, x) \leftrightarrow (\exists z \in J_\alpha)\, H_{\alpha,A}(z, y, i, x).$$

As an immediate corollary to the above result we have:

3.11 Lemma.

(i) *The function $h_{\alpha,A}$ is a (uniformly Σ_1) Σ_1 skolem function for amenable $\langle J_\alpha, A\rangle$ with $\alpha > 1$.*

(ii) *The function $h_{\alpha,\emptyset}$ is a (uniformly Σ_1) Σ_1 skolem function for J_α for each $\alpha > 1$.* $\square$

We often write h_α for $h_{\alpha,\emptyset}$. The notation $h_{\alpha,A}$, h_α, θ_i, $H_{\alpha,A}$ $H_\alpha(= H_{\alpha,\emptyset})$ is fixed for the rest of this book.

As an illustration of the use of the skolem functions h_α, we shall prove an analogue of II.6.8 for the Jensen hierarchy, showing that for any ordinal α there is a $\Sigma_1(J_\alpha)$ map from $\omega\alpha$ onto J_α. This will require some preliminary lemmas, but before we give them we introduce an important notion which should throw some light upon our construction of the Σ_1 skolem function.

A function r is said to *uniformise* a relation R iff $\mathrm{dom}(r) = \mathrm{dom}(R)$ and for all x,

$$\exists y R(y, \vec{x}) \leftrightarrow R(r(\vec{x}), \vec{x}).$$

We say a structure of the form $\mathbf{M} = \langle M, (A_i)_{i<\omega}\rangle$ is Σ_n-*uniformisable* iff every $\Sigma_n(\mathbf{M})$ relation on M is uniformised by a $\Sigma_n(\mathbf{M})$ function.

In general, Σ_n-uniformisability is a very strong condition to demand of a structure. Indeed, the existence of *any* uniformising function definable over the structure concerned is quite a strong property, let alone the existence of one whose definition is no more complex than that of the relation it is uniformising. It is thus perhaps rather surprising to learn that for all $\alpha > 1$ and all $n \geqslant 1$, J_α is Σ_n-uniformisable. In the general case the proof is rather tricky, and will be given in the next section, where Σ_n-uniformisation will play an important role in our study of the Σ_n-projectum. But the case $n = 1$ is quite straightforward, and we shall consider this case here, using it to obtain an analogue of II.6.7 for the Jensen hierarchy. (In the proof of II.6.7 we did in fact make implicit use of the fact that for limit $\alpha > \omega$, L_α is Σ_1-uniformisable, but we did not dwell upon this point there.)

First let us see how Σ_n-uniformisability affects the existence of Σ_n-skolem functions.

3.12 Lemma. *Let $n \geqslant 1$, $\alpha > 1$. If J_α is Σ_n-uniformisable, then it has a Σ_n-skolem function.*

Proof. Let $(\varphi_i \mid i < \omega)$ be a $\Delta_1^{J_1}$ enumeration of all Σ_n-formulas of $\mathscr{L}$ with free variables v_0, v_1. By 3.4, the relation

$$\{(y, i, x) \mid \models_{J_\alpha} \varphi_i(\mathring{y}, \mathring{x})\}$$

is $\Sigma_n^{J_\alpha}$. Let r be a $\Sigma_n(J_\alpha)$ function uniformising this relation. Pick $p \in J_\alpha$ so that r is $\Sigma_n^{J_\alpha}(\{p\})$. Set

$$h(i, x) \simeq r(i, (x, p)) \qquad (x \in J_\alpha).$$

It is easily seen that h is a Σ_n skolem function for J_α and that p is a good parameter for h. $\square$

We note that the converse to the above lemma is trivially true.

For the case $n = 1$ we now prove:

3.13 Lemma. *Let $\alpha > 1$. Then J_α is Σ_1-uniformisable.*

Proof. Let $R(y, \vec{x})$ be a $\Sigma_1(J_\alpha)$ relation on J_α. Let S be a $\Sigma_0(J_\alpha)$ relation such that

$$R(y, \vec{x}) \leftrightarrow (\exists z \in J_\alpha)\, S(z, y, \vec{x}).$$

Define g on J_α by

$$g(\vec{x}) \simeq \text{the } <_J\text{-least } w \text{ such that } S((w)_0, (w)_1, \vec{x}).$$

The function g is $\Sigma_1(J_\alpha)$. For it has the definition

$$\begin{aligned} w = g(\vec{x}) \leftrightarrow\ & S((w)_0, (w)_1, \vec{x}) \\ & \wedge \exists u[u = \{w' \mid w' <_J w\} \wedge (\forall w' \in u) \neg S((w')_0, (w')_1, \vec{x})], \end{aligned}$$

which is $\Sigma_1(J_\alpha)$ by 2.8(v). Now set

$$r(\vec{x}) \simeq (g(\vec{x}))_1.$$

Then r is $\Sigma_1(J_\alpha)$, and r clearly uniformises R. $\square$

At this point the reader might like to see what goes wrong when we try to generalise the above argument to the case $n > 1$. (As we shall see in the next section, proving Σ_n-uniformisability of J_α for $n > 1$ is by no means a simple matter, though it is achieved by somehow pushing through an argument such as the above.)

Now for our analogue of II.6.8. As in II.6.6, let

$$\Phi: \mathrm{On} \times \mathrm{On} \leftrightarrow \mathrm{On}$$

be Gödel's pairing function. By the same argument as in II.6.6, we have:

3.14 Lemma. *$\Phi^{-1} \upharpoonright \omega\alpha$ is uniformly $\Sigma_1^{J_\alpha}$ for all α.* $\square$

Analogous to II.6.7 we have:

3.15 Lemma. *There is a $\Sigma_1(J_\alpha)$ map from $\omega\alpha$ onto $\omega\alpha \times \omega\alpha$.*

Proof. Set

$$Q = \{\alpha \mid \Phi: \alpha \times \alpha \leftrightarrow \alpha\},$$

a closed unbounded class of ordinals. It is easily seen that $\omega\alpha = \alpha$ for any ordinal α such that $\omega\alpha \in Q$. Moreover,

$$Q = \{\alpha \mid \Phi(0, \alpha) = \alpha\}.$$

We prove the lemma by induction on α. For $\alpha = 0$ the result is trivial, so we assume $\alpha > 0$ now and that the lemma holds for all $\beta < \alpha$. There are three cases to consider.

Case 1. $\omega\alpha \in Q$.

In this case, $\Phi^{-1} \upharpoonright \omega\alpha$ suffices.

Case 2. $\alpha = \beta + 1$.

If $\beta = 0$ here, then $\omega\alpha = \omega \in Q$ and we are done by Case 1. So we may assume that $\beta > 0$. Define $j: \omega\alpha \leftrightarrow \omega\beta$ by

$$j(\xi) = \begin{cases} 2\xi, & \text{if } \xi < \omega \\ \xi, & \text{if } \omega \leqslant \xi < \omega\beta \\ 2n+1, & \text{if } \omega\beta + n. \end{cases}$$

Clearly, j is $\Sigma_1^{J_\alpha}(\{\omega, \omega\beta\})$.

By induction hypothesis, there is a $\Sigma_1(J_\beta)$ map g from $\omega\beta$ onto $\omega\beta \times \omega\beta$. Let

$$G = \{(v, x) \mid g(v) = x\},$$

a $\Sigma_1(J_\beta)$ relation on J_β. Let $\bar{g}$ be a $\Sigma_1(J_\beta)$ function uniformising G. Clearly, $\bar{g}$ maps $\omega\beta \times \omega\beta$ one-one into $\omega\beta$. Now, $\bar{g} \in \text{rud}(J_\beta) = J_\alpha$ (since $\text{rud}(J_\beta) \cap \mathscr{P}(J_\beta) = Def(J_\beta)$), so f is a $\Sigma_1(J_\alpha)$ map from $\omega\alpha \times \omega\alpha$ one-one into $\omega\beta$, where we define f by

$$f((v, \tau)) = \bar{g}((j(v), j(\tau))).$$

Now, j is onto $\omega\beta$, so $\text{ran}(f) = \text{ran}(\bar{g}) \in J_\alpha$. Hence h is a $\Sigma_1(J_\alpha)$ map from $\omega\alpha$ onto $\omega\alpha \times \omega\alpha$, where we define h by

$$h(v) = \begin{cases} f^{-1}(v), & \text{if } v \in \text{ran}(f) \\ (0, 0), & \text{otherwise.} \end{cases}$$

The map h is as required.

Case 3. $\omega\alpha \notin Q$ and $\lim(\alpha)$.

Set $(v, \tau) = \Phi^{-1}(\omega\alpha)$. Since $\omega\alpha \notin Q$, we have $v, \tau < \omega\alpha$. Let $<^*$ be the well-ordering of On $\times$ On used to define Φ (see II.6.6), and set

$$c = \{z \mid z <^* (v, \tau)\}.$$

Then $c \in J_\alpha$, and moreover, $\Phi \upharpoonright c$ is a $\Sigma_1(J_\alpha)$ bijection from c onto $\omega\alpha$. Pick $\gamma < \alpha$ such that $v, \tau < \omega\gamma$. (Possible since $\lim(\alpha)$.) Then $\Phi^{-1} \upharpoonright \omega\gamma$ is a $\Sigma_1(J_\alpha)$ map from $\omega\alpha$ one-one into $\omega\gamma$. Also, arguing as in Case 2, the induction hypothesis implies the existence of a map $\bar{g} \in J_\alpha$ one-one from $\omega\gamma \times \omega\gamma$ into $\omega\gamma$. Then f is a $\Sigma_1(J_\alpha)$ bijection from $\omega\alpha \times \omega\alpha$ onto $d = \bar{g}''[\bar{g}''c \times \bar{g}''c]$, where we define f by

$$f((\xi, \zeta)) = \bar{g}((\bar{g}(\Phi^{-1}(\xi)), \bar{g}(\Phi^{-1}(\zeta)))).$$

But $d \in J_\alpha$, so h is a $\Sigma_1(J_\alpha)$ map from $\omega\alpha$ onto $\omega\alpha \times \omega\alpha$, where we define h by

$$h(\xi) = \begin{cases} f^{-1}(\xi), & \text{if } \xi \in d, \\ (0, 0), & \text{otherwise.} \end{cases}$$

Then h is as required. The proof is complete. □

We may now prove our analogue of II.6.8.

3.16 Lemma. *Let $\alpha > 1$. There is a $\Sigma_1(J_\alpha)$ map from $\omega\alpha$ onto J_α.*

Proof. Let f be a $\Sigma_1^{J_\alpha}(\{p\})$ map from $\omega\alpha$ onto $\omega\alpha \times \omega\alpha$, where $p \in J_\alpha$ is the $<_J$-least for which such an f exists. Define f^0, f^1 by

$$f(v) = (f^0(v), f^1(v)) \qquad (v \in \omega\alpha).$$

By induction, define f_n from $\omega\alpha$ onto $(\omega\alpha)^n$ by:

$$f_1 = id \upharpoonright \omega\alpha,$$
$$f_{n+1}(v) = (f^0(v), f_n \circ f^1(v)).$$

Notice that each f_n is $\Sigma_1^{J_\alpha}(\{p\})$.

Let $h = h_\alpha$, $H = H_\alpha$, and set $X = h^*(\omega\alpha \times \{p\})$.

Claim 1. X is closed under ordered pairs.

To see this, let $x_1, x_2 \in X$, say $x_i = h(j_i, (v_i, p))$. Let $(v_1, v_2) = f_2(\tau)$. Then $\{(x_1, x_2)\}$ is a $\Sigma_1^{J_\alpha}(\{\tau, p\})$ predicate on J_α. So by definition of h, $(x_1, x_2) \in X$, as claimed.

By claim 1 and 3.9, $X \prec_1 J_\alpha$. Let $\pi: X \cong J_\beta$, where $\beta \leqslant \alpha$, by the Condensation Lemma. Clearly, $\omega\alpha \subseteq X$, so we must have $\beta = \alpha$ here.

Claim 2. For all $i \in \omega$, $x \in X$,

$$\pi(h(i, x)) \simeq h(i, \pi(x)).$$

Let $i \in \omega$, $x \in X$. Suppose first that $y = h(i, x)$ is defined. Note that as $x \in X \prec_1 J_\alpha$, we have $y \in X$. Now (with $(\theta_i \mid i < \omega)$ as defined in the definition of the Σ_1 skolem function),

$$\models_{J_\alpha} \exists z\, \theta_i(z, \mathring{y}, \mathring{x}).$$

So as $x, y \in X \prec_1 J_\alpha$,

$$\models_X \exists z\, \theta_i(z, \mathring{y}, \mathring{x}).$$

Pick $z \in X$ such that

$$\models_X \theta_i(\mathring{z}, \mathring{y}, \mathring{x}).$$

Applying $\pi: X \cong J_\alpha$,

$$\models_{J_\alpha} \theta_i(\pi(z)^\circ, \pi(y)^\circ, \pi(x)^\circ).$$

Thus

$$\models_{J_\alpha} \exists z\, \theta_i(z, \pi(y)^\circ, \pi(x)^\circ).$$

Thus $\pi(y) = h(i, \pi(x))$.

Conversely, suppose $h(i, \pi(x))$ is defined. Then $h(i, \pi(x)) \in J_\alpha = \pi'' X$, so for some $y \in X$, $h(i, \pi(x)) = \pi(y)$, and we can reverse the above steps to obtain $y = h(i, x)$. This proves claim 2.

Now, $f: \omega\alpha \to \omega\alpha \times \omega\alpha$, so as $\pi \restriction \omega\alpha = id \restriction \omega\alpha$, $\pi'' f = f$. And by isomorphism, $\pi'' f$ is $\Sigma_1^{J_\alpha}(\{\pi(p)\})$. So by choice of p, $p \leqslant_J \pi(p)$. But by 2.9(iii), $\pi(p) \leqslant_J p$. Hence $\pi(p) = p$.

By claim 2 now, for $i \in \omega$, $v \in \omega\alpha$, we have

$$\pi(h(i, (v, p))) \simeq h(i, \pi((v, p))) \simeq h(i, (v, p)).$$

Thus $\pi \restriction X = id \restriction X$. Thus $X = J_\alpha$. It follows at once that if we set

$$r(v) \simeq h((f(v))_0, ((f(v))_1, p)),$$

then r is a $\Sigma_1(J_\alpha)$ map such that $r''\omega\alpha = J_\alpha$. However, we are not yet done, since the map r just defined is not total on $\omega\alpha$. To achieve this, define $g\colon \omega\alpha \times \omega\alpha \times \omega\alpha \to J_\alpha$ by:

$$g(i, \nu, \tau) = \begin{cases} y, & \text{if } (\exists z \in S_\tau)\, H(z, y, i, (\nu, p)) \\ \emptyset, & \text{otherwise.} \end{cases}$$

Then g is $\Sigma_1(J_\alpha)$. And clearly,

$$g''(\omega\alpha \times \omega\alpha \times \omega\alpha) = h^*(\omega\alpha \times \{p\}) = X = J_\alpha.$$

Thus $g \circ f_3$ satisfies the lemma. $\square$

By examining the proofs of 3.15 and 3.16, we see that in the case where $\alpha \in Q$, no parameters are required in the functions we defined. Hence, noting that $\omega\alpha = \alpha$ whenever $\alpha \in Q$, we have:

3.17 Lemma. *If α is closed under the Gödel pairing function, there is a (uniform) $\Sigma_1^{J_\alpha}$ map from $\omega\alpha$ onto J_α.* $\square$

4. The Σ_n-Projectum

As we indicated in IV.4, the Σ_n-projectum of an ordinal plays an important role in the reduction of Σ_n predicates to Σ_1 predicates, the main idea behind the fine structure theory. Indeed, if ϱ is the Σ_n-projectum of α, then it is as a Σ_1 predicate on $\langle J_\varrho, A\rangle$ for some set A that we shall code a given Σ_n predicate on J_α.

Let $n > 0$, $\alpha > 0$. The *Σ_n-projectum* of α, ϱ_α^n, is the least ordinal $\varrho \leqslant \alpha$ such that there is a $\Sigma_n(J_\alpha)$ function f over J_α such that $f''J_\varrho = J_\alpha$.

By 3.16, it is easily seen that ϱ_α^n is the least $\varrho \leqslant \alpha$ such that there is a $\Sigma_n(J_\alpha)$ map f for which $f''\omega\varrho = \omega\alpha$.

Clearly, $0 < m < n \to \varrho_\alpha^n \leqslant \varrho_\alpha^m$. So it is natural to define $\varrho_\alpha^0 = \alpha$ for each ordinal α.

4.1 Lemma. *If $\varrho_\alpha^n > 1$, then* $\lim(\varrho_\alpha^n)$.

Proof. Suppose that $\varrho = \varrho_\alpha^n = \gamma + 1$, where $\gamma > 0$. Let f be a $\Sigma_n(J_\alpha)$ function such that $f''\omega\varrho = \omega\alpha$. Define $g\colon \omega\gamma \to \omega\varrho$ by

$$g(\nu) = \begin{cases} m, & \text{if } \nu = 2m < \omega, \\ \omega\gamma + m, & \text{if } \nu = 2m + 1 < \omega, \\ \nu, & \text{if } \omega \leqslant \nu < \omega\gamma. \end{cases}$$

Clearly, g is $\Sigma_1(J_\alpha)$. Thus $f \circ g$ is $\Sigma_n(J_\alpha)$. But $(f \circ g)''\omega\gamma = \omega\alpha$, so this contradicts the choice of ϱ. $\square$

In order to obtain more information about the Σ_n-projectum we shall prove that for all $\alpha > 1$ and all $n > 0$, J_α is Σ_n-uniformisable. The proof is fairly intricate, and requires several preliminary lemmas. Before we begin, we outline the general strategy.

We begin by examining the proof of Σ_1-uniformisation given in 3.13. This reduced to proving that every Σ_0 relation is uniformised by a Σ_1 function. (In 3.13, what we really did was to uniformise the Σ_0 relation S, obtaining the uniformisation of the Σ_1 relation R as a simple consequence.) This worked in the case $n = 1$ because, if $S(y, \vec{x})$ is Σ_0, then so too is $(\forall z \in y) \neg S(z, \vec{x})$. But consider now the analogous situation for $n > 1$. We seek a Σ_n uniformisation of a Π_{n-1} relation S. Now, if $S(y, \vec{x})$ is Π_{n-1}, then $(\forall z \in y) \neg S(z, x)$ is in general Σ_{n+1}, not Σ_n. Roughly speaking, we overcome this difficulty as follows. We reduce the predicate S on J_α to a predicate on $J_{\varrho_\alpha^{n-1}}$. The structure $\mathbf{J}_{\varrho_\alpha^{n-1}}$ is sufficiently suited to handling $\Sigma_{n-1}(J_\alpha)$ predicates on it that the canonical uniformisation procedure applied to the reduced predicate turns out to be $\Sigma_n(J_\alpha)$, thereby providing us with the desired Σ_n uniformisation of S. The precise property of the projectum which we need in order to make this work is described below.

Let $P(y, \vec{x})$ be any predicate on J_α. For $\varrho \leqslant \alpha$, we say that $P(y, \vec{x})$ is *$\Sigma_n(J_\alpha)$ on J_ϱ* iff there is a Σ_n formula $\varphi(y, \vec{x})$ of $\mathscr{L}_{J_\alpha}$ such that

$$(\forall y \in J_\varrho)(\forall \vec{x} \in J_\alpha)[P(y, \vec{x}) \leftrightarrow \models_{J_\alpha} \varphi(\mathring{y}, \mathring{\vec{x}})].$$

Similarly for *$\Pi_n(J_\alpha)$ on J_ϱ*.

For any predicate $R(y, \vec{x})$, we denote by $R^\forall(y, \vec{x})$ the predicate

$$\{(y, \vec{x}) \mid (\forall z \in y)\, R(z, \vec{x})\},$$

and by $R^\exists(y, \vec{x})$ the predicate

$$\{(y, \vec{x}) \mid (\exists z \in y)\, R(z, \vec{x})\}.$$

Let $\alpha > 1$, $n > 0$, $0 < \varrho \leqslant \alpha$. We denote by $\Gamma(\alpha, n, \varrho)$ the following property: whenever $R(y, \vec{x})$ is $\Sigma_n(J_\alpha)$, then $R^\forall(y, \vec{x})$ is $\Sigma_{n+1}(J_\alpha)$ on J_ϱ.

We shall prove that for any $\alpha > 1$, $n > 0$, $\Gamma(\alpha, n, \varrho_\alpha^n)$ is valid. Using $\Gamma(\alpha, n, \varrho_\alpha^n)$ we shall be able to prove that J_α is Σ_{n+1}-uniformisable, the proof being a variation of the proof for the Σ_1 case (3.13) as outlined above. (In fact the proof of $\Gamma(\alpha, n, \varrho_\alpha^n)$ and that of Σ_{n+1}-uniformisability proceeds by a simultaneous induction on n.) But first we need some preliminary results.

4.2 Lemma. *Let $\alpha > 1$, $n > 0$, $\varrho > 0$. Assume $\Gamma(\alpha, n, \varrho)$. Then:*

(i) *if $R(y, \vec{x})$ is $\Pi_n(J_\alpha)$, then $R^\exists(y, \vec{x})$ is $\Pi_{n+1}(J_\alpha)$ on J_ϱ;*

(ii) *if $R(y, \vec{x})$ is $\Sigma_n(J_\alpha)$, then $Q(y, \vec{x})$ is $\Sigma_{n+1}(J_\alpha)$ on J_ϱ, where $Q = \{(y, \vec{x}) \mid (\forall z <_J y)\, R(z, \vec{x})\}$.*

Proof. (i) This follows from $\Gamma(\alpha, n, \varrho)$ by taking negations.

(ii) For $y, \vec{x} \in J_\varrho$, we have

$$Q(y, \vec{x}) \leftrightarrow (\exists u, w, v \in J_\varrho)\,[y \in S_v \wedge w = <_v^S \wedge (\forall z)(z \in u \leftrightarrow (z, y) \in w) \wedge (\forall z \in u)\, R(z, \vec{x})].$$

Using $\Gamma(\alpha, n, \varrho)$, this is easily seen to be $\Sigma_{n+1}(J_\alpha)$. (In case $\varrho < \alpha$, we must use J_ϱ as a parameter to ensure that $u \in J_\varrho$. If $\varrho = \alpha$ there is no need to mention ϱ at all, of course.) $\square$

4.3 Lemma. *Let $\alpha > 1$, $n > 0$, and set $\varrho = \varrho_\alpha^n$. Suppose that J_α is Σ_n-uniformisable. Then $\langle J_\varrho, A\rangle$ is amenable for all $A \in \Sigma_n(J_\alpha) \cap \mathscr{P}(J_\varrho)$.*

Proof. Let $A \in \Sigma_n(J_\alpha) \cap \mathscr{P}(J_\varrho)$. We show that $\langle J_\varrho, A\rangle$ is amenable. If $\varrho = 1$, then $J_\varrho = H_\omega$, so this is immediate. Now assume $\varrho > 1$. Thus by 4.1, $\lim(\varrho)$. So it suffices to show that $\gamma < \varrho$ implies $A \cap J_\gamma \in J_\varrho$.

Let $\gamma < \varrho$ be given. Set $B = A \cap J_\gamma$. Thus B is $\Sigma_n(J_\alpha)$. Let B be $\Sigma_n^{J_\alpha}(\{p\})$. Let $\varphi(v_0, v_1)$ be a Σ_n-formula such that

$$(*) \qquad x \in B \quad \text{iff } \models_{J_\alpha} \varphi(\mathring{x}, \mathring{p}).$$

By assumption, J_α is Σ_n-uniformisable, so by 3.12, J_α has a Σ_n skolem function, h. Set $X = h^*(J_\gamma \times \{p\})$. By 3.2, $X \prec_n J_\alpha$. Let $\pi: X \cong J_{\bar\alpha}$. Set $\bar p = \pi(p)$, $\bar h = \pi''(h \cap (X \times \omega \times X))$. Since $B \subseteq J_\gamma$, $\pi'' B = B$. So by $(*)$

$$(**) \qquad x \in B \quad \text{iff } \models_{J_{\bar\alpha}} \varphi(\mathring{x}, \mathring{\bar p}).$$

Thus B is $\Sigma_n^{J_{\bar\alpha}}(\{\bar p\})$. Hence $B \in J_{\bar\alpha+1}$. If $\bar\alpha < \varrho$ then this means that $B \in J_\varrho$ and we are done. So we are reduced to proving that $\bar\alpha < \varrho$.

Suppose, on the contary, that $\bar\alpha \geqslant \varrho$. By definition of X, $J_{\bar\alpha} = \bar h^*(J_\gamma \times \{\bar p\})$. So, as $\bar h$ is $\Sigma_n(J_{\bar\alpha})$, there is a $\Sigma_n(J_{\bar\alpha})$ function f such that $f'' J_\gamma = J_{\bar\alpha}$. Let g be a $\Sigma_n(J_{\bar\alpha})$ map such that $g'' J_\varrho = J_\alpha$. Since $\varrho \leqslant \bar\alpha$, $g \circ f$ is a $\Sigma_n(J_\alpha)$ map such that $g \circ f'' J_\gamma = J_\alpha$, contrary to $\gamma < \varrho$. The lemma is proved. $\square$

Our proof of Σ_n uniformisability will be by induction on n. The key to the induction is provided by the following lemma.

4.4 Lemma. *Let $\alpha > 1$, $n > 0$, and assume $\Gamma(\alpha, n, \varrho_\alpha^n)$. If J_α is Σ_n-uniformisable, then it is Σ_{n+1}-uniformisable.*

Proof. The procedure is not unlike that adopted in proving Σ_1-uniformisability, except that we reduce the predicate to one on $J_{\varrho_\alpha^n}$ before we commence.

Let $R(y, \vec{x})$ be $\Sigma_{n+1}(J_\alpha)$, and let S be $\Pi_n(J_\alpha)$ such that

$$R(y, \vec{x}) \leftrightarrow (\exists z \in J_\alpha)\, S(z, y, \vec{x}).$$

Let $\varrho = \varrho_\alpha^n$, and let f be a $\Sigma_n(J_\alpha)$ function such that $f'' J_\varrho = J_\alpha$. We shall consider the case where $\varrho < \alpha$. The case where $\varrho = \alpha$ is a little simpler, since there is no need to mention ϱ at all. Set

$$r(\vec{x}) \simeq \text{the } <_J\text{-least } z \text{ such that } S((f(z))_0, (f(z))_1, \vec{x}),$$

$$\bar r(\vec{x}) \simeq (f \circ r(\vec{x}))_1.$$

Clearly, $\bar r$ uniformises R. If r is $\Sigma_{n+1}(J_\alpha)$, so too is $\bar r$, so what we must do is prove that r is indeed $\Sigma_{n+1}(J_\alpha)$. We have, by definition,

$$y = r(\vec{x}) \leftrightarrow [y \in \mathrm{dom}(f)] \wedge [\forall z(z = f(y) \rightarrow S((z)_0, (z)_1, \vec{x})]$$
$$\wedge [(\forall y' <_J y)(y' \in \mathrm{dom}(f) \rightarrow \neg S((f(y'))_0, (f(y'))_1, \vec{x})].$$

The first conjunct here is $\Sigma_n(J_\alpha)$ and the second is $\Pi_n(J_\alpha)$. Also, $\mathrm{dom}(f)$ is $\Sigma_n(J_\alpha)$, and for $y \in J_\varrho$, $\{y' \mid y' <_J y\} \in J_\varrho$, so by 4.3,

$$\mathrm{dom}(f) \cap \{y' \mid y' <_J y\} \in J_\varrho$$

for each $y \in J_\varrho$. Hence the third conjunct reduces to

$$(\exists u \in J_\varrho)\,[(\forall y' \in u)\,(y' <_J y \wedge y' \in \mathrm{dom}(f))$$
$$\wedge (\forall y')\,(y' <_J y \wedge y' \in \mathrm{dom}(f) \rightarrow y' \in u)$$
$$\wedge (\forall y' \in u)\,(\exists z)\,(z = f(y') \wedge \neg S((z)_0, (z)_1, \vec{x}))].$$

This is of the form

$$(\exists u \in J_\varrho)\,[(\forall y' \in u)\,(\Sigma_n(J_\alpha)) \wedge (\forall y')\,(\Pi_n(J_\alpha)) \wedge (\forall y' \in u)\,(\Sigma_n(J_\alpha))].$$

Using $\Gamma(\alpha, n, \varrho)$, we see that it is in fact of the form

$$(\exists u \in J_\varrho)\,[\Sigma_{n+1}(J_\alpha) \wedge \Pi_n(J_\alpha) \wedge \Sigma_{n+1}(J_\alpha)].$$

Hence r is $\Sigma_{n+1}(J_\alpha)$, as required. □

4.5 Theorem (Uniformisation Theorem). *Let $\alpha > 1, n > 0$. Then J_α is Σ_n-uniformisable.*

Proof. By 3.13 we are done if $n = 1$. By 4.4, the result follows by induction if we can establish $\Gamma(\alpha, n, \varrho_\alpha^n)$ for all $n > 0$. We do this by induction on n as well.

Let $n \geqslant 1$, and in case $n > 1$ assume $\Gamma(\alpha, 1, \varrho_\alpha^1), \ldots, \Gamma(\alpha, n-1, \varrho_\alpha^{n-1})$. We prove that $\Gamma(\alpha, n, \varrho_\alpha^n)$. Note that by 4.4, J_α is Σ_m-uniformisable for all $m \leqslant n$, $m \geqslant 1$.

Set $\varrho = \varrho_\alpha^n$, $\eta = \varrho_\alpha^{n-1}$. Notice that $\varrho \leqslant \eta \leqslant \alpha$. There are two cases to consider.

Case 1. There is no $\Sigma_n(J_\alpha)$ map from any $\gamma < \omega\varrho$ cofinally into $\omega\eta$.

In this case we commence by proving a sort of Σ_n-Collection Axiom.

Claim. If $R(y, x)$ is $\Sigma_n(J_\alpha)$ and $u \in J_\varrho$, then

$$(\forall x \in u)\,(\exists y \in J_\eta)\,R(y, x) \rightarrow (\exists v \in J_\eta)\,(\forall x \in u)\,(\exists y \in v)\,R(y, x).$$

Proof of claim. If $\varrho = 1$ the claim is trivial, so assume $\varrho > 1$. Hence $\lim(\varrho)$, and we can pick $\gamma < \varrho$ so that $u \in J_\gamma$. Let $j\colon \omega\gamma \xrightarrow{\text{onto}} J_\gamma$ be $\Sigma_1(J_\gamma)$. Let r be a $\Sigma_n(J_\alpha)$ function uniformising R. Define $f\colon \omega\gamma \rightarrow \omega\eta$ by

$$f(v) = \begin{cases} \text{the least } \tau < \omega\eta \text{ such that } r \circ j(v) \in S_\tau, & \text{if } j(v) \in u, \\ 0, \quad \text{otherwise}. \end{cases}$$

Thus:

$$\tau = f(v) \leftrightarrow \models_{J_\alpha} [(j(v) \in u) \wedge \exists z \exists f [z = r \circ j(v) \wedge f = (S_\xi \mid \xi \leqslant \tau) \\ \wedge z \in f(\tau) \wedge (\forall \xi \in \tau)(z \notin f(\xi))]] \vee [(j(v) \notin u) \wedge (\tau = 0)].$$

Thus f is $\Sigma_n(J_\alpha)$. So, by assumption there is a $\delta < \omega\eta$ such that $f''\omega\gamma \subseteq \delta$. Then

$$(\forall x \in u)(\exists y \in S_\delta) R(y, x),$$

which proves the claim.

We must now consider two subcases.

Case 1.1. $n = 1$.

Let $R(y, \vec{x})$ be $\Sigma_1(J_\alpha)$. Let S be $\Sigma_0(J_\alpha)$ with

$$R(y, \vec{x}) \leftrightarrow (\exists t \in J_\alpha) S(t, y, \vec{x}).$$

Let $y \in J_\varrho$, $\vec{x} \in J_\alpha$. Since $\eta = \varrho_\alpha^0 = \alpha$, the claim gives

$$(\forall z \in y) R(z, \vec{x}) \leftrightarrow (\forall z \in y)(\exists t \in J_\eta) S(t, z, \vec{x}) \\ \leftrightarrow (\exists v \in J_\eta)(\forall z \in y)(\exists t \in v) S(t, z, \vec{x}),$$

which is $\Sigma_1(J_\alpha)$. Thus $R^\forall$ is $\Sigma_1(J_\alpha)$ on J_ϱ, proving $\Gamma(\alpha, 1, \varrho)$.

Case 1.2. $n > 1$.

Let $R(y, \vec{x})$ be $\Sigma_n(J_\alpha)$, and let S be $\Pi_{n-1}(J_\alpha)$ with

$$R(y, \vec{x}) \leftrightarrow (\exists t \in J_\alpha) S(t, y, \vec{x}).$$

Let f be a $\Sigma_{n-1}(J_\alpha)$ function such that $f''J_\eta = J_\alpha$. Let $y \in J_\varrho$, $\vec{x} \in J_\alpha$. By the claim,

$$(\forall z \in y) R(z, \vec{x}) \leftrightarrow (\forall z \in y)(\exists t \in J_\eta) S(f(t), z, \vec{x}) \\ \leftrightarrow (\exists v \in J_\eta)(\forall z \in y)(\exists t \in v) S(f(t), z, \vec{x}).$$

Now, J_α is Σ_{n-1}-uniformisable and $\mathrm{dom}(f)$ is $\Sigma_{n-1}(J_\alpha)$, so by 4.3,

$$v \in J_\eta \to \mathrm{dom}(f) \cap v \in J_\eta.$$

Hence

$$R^\forall(y, \vec{x}) \leftrightarrow (\exists v \in J_\eta) [(\forall x \in v)(x \in \mathrm{dom}(f)) \\ \wedge (\forall z \in y)(\exists t \in v)(\forall w)[w = f(t) \to S(w, z, \vec{x})]].$$

This is of the form

$$(\exists v \in J_\eta) [\Pi_n(J_\alpha) \wedge (\forall z \in y)(\exists t \in v)(\Pi_{n-1}(J_\alpha))].$$

Using $\Gamma(\alpha, n-1, \eta)$, together with 4.2(i), this is in fact of the form

$$(\exists v \in J_\eta) [\Pi_n(J_\alpha) \wedge (\forall z \in y)(\Pi_n(J_\alpha))],$$

which is the same as

$$(\exists v \in J_\eta)\,(\Pi_n(J_\alpha)),$$

which is $\Sigma_{n+1}(J_\alpha)$, as required.

Case 2. Otherwise.

Let $\gamma < \omega\varrho$ be least such that there is a $\Sigma_n(J_\alpha)$ map g from γ cofinally into $\omega\eta$. Let $R(y, \vec{x})$ be $\Sigma_n(J_\alpha)$. We commence by proving:

Claim. There is a $\Delta_n(J_\alpha)$ predicate $Q(v, y, \vec{x})$ such that for any $y \in J_\varrho$, $\vec{x} \in J_\alpha$,

$$R(y, \vec{x}) \leftrightarrow (\exists v \in \gamma)\, Q(v, y, \vec{x}).$$

Proof of claim. Let f be a $\Sigma_{n-1}(J_\alpha)$ function such that $f''J_\eta = J_\alpha$. (If $n = 1$, then $\eta = \alpha$, so take $f = id \restriction J_\alpha$.) Let S be $\Pi_{n-1}(J_\alpha)$ with

$$R(y, \vec{x}) \leftrightarrow (\exists t \in J_\alpha)\, S(t, y, \vec{x}).$$

Define Q by

$$Q(v, y, \vec{x}) \leftrightarrow (v \in \gamma) \wedge (\exists t \in S_{g(v)})\, S(f(t), y, \vec{x}).$$

Since g is cofinal in $\omega\eta$ and $f''J_\eta = J_\alpha$, we have

$$R(y, \vec{x}) \leftrightarrow (\exists v \in \gamma)\, Q(v, y, \vec{x}).$$

We show that Q is $\Delta_n(J_\alpha)$. It is clearly $\Sigma_n(J_\alpha)$.

Define $\tilde{Q}$ by

$$\tilde{Q}(u, y, \vec{x}) \leftrightarrow (\exists t \in u)\, S(f(t), y, \vec{x}).$$

Thus:

$$\begin{aligned} Q(v, y, \vec{x}) &\leftrightarrow (v \in \gamma) \wedge \tilde{Q}(S_{g(v)}, y, \vec{x}) \\ &\leftrightarrow (v \in \gamma) \wedge \forall w \forall \tau [\tau = g(v) \wedge w = S_\tau \rightarrow \tilde{Q}(w, y, \vec{x})]. \end{aligned}$$

So it suffices to show that $\tilde{Q}$ is $\Pi_n(J_\alpha)$.

Well, if $n = 1$, then $f = \mathrm{id} \restriction J_\alpha$, so

$$\tilde{Q}(u, y, \vec{x}) \leftrightarrow (\exists t \in u)\, S(t, y, \vec{x}),$$

which is in fact $\Sigma_0(J_\alpha)$. So suppose $n > 1$. Then

$$\tilde{Q}(u, y, \vec{x}) \leftrightarrow (\exists t \in u \cap \mathrm{dom}(f))\, (\forall w)\, [w = f(t) \rightarrow S(w, y, \vec{x})].$$

Define T by

$$T(t, y, \vec{x}) \leftrightarrow (\forall w)\, [w = f(t) \rightarrow S(w, y, \vec{x})].$$

Then T is $\Pi_{n-1}(J_\alpha)$, and by the above

$$\tilde{Q}(u, y, \vec{x}) \leftrightarrow (\exists t \in u \cap \mathrm{dom}(f))\ T(t, y, \vec{x}).$$

Now, J_α is Σ_{n-1}-uniformisable and $\mathrm{dom}(f)$ is $\Sigma_{n-1}(J_\alpha)$, so by 4.3,

$$u \in J_\eta \rightarrow u \cap \mathrm{dom}(f) \in J_\eta.$$

Thus

$$\tilde{Q}(u, y, \vec{x}) \leftrightarrow (\forall v \in J_\eta)\ [v = u \cap \mathrm{dom}(f) \rightarrow (\exists t \in v)\ T(t, y, \vec{x})].$$

But we have

$$v = u \cap \mathrm{dom}(f) \leftrightarrow (\forall x \in v)\ (x \in u \wedge x \in \mathrm{dom}(f)) \\ \wedge (\forall x \in u)\ (x \in \mathrm{dom}(f) \rightarrow x \in v).$$

This is of the form

$$(\forall x \in v)\ (\Sigma_{n-1}(J_\alpha)) \wedge (\forall x \in u)\ (\Pi_{n-1}(J_\alpha)).$$

Using $\Gamma(\alpha, n-1, \eta)$, as we may since $v \in J_\eta$, we see that this is of the form

$$\Sigma_n(J_\alpha) \wedge \Pi_{n-1}(J_\alpha),$$

and is thus $\Sigma_n(J_\alpha)$. Hence $\tilde{Q}(u, y, \vec{x})$ is of the form

$$\tilde{Q}(u, y, \vec{x}) \leftrightarrow (\forall v \in J_\eta)\ [\Sigma_n(J_\alpha) \rightarrow (\exists t \in v)\ (\Pi_{n-1}(J_\alpha))].$$

Using $\Gamma(\alpha, n-1, \eta)$ again, this is of the form

$$(\forall v \in J_\eta)\ [\Sigma_n(J_\alpha) \rightarrow \Pi_n(J_\alpha)],$$

which is $\Pi_n(J_\alpha)$. That completes the proof of the claim.

By the claim, we have, for $y \in J_\varrho$, $\vec{x} \in J_\alpha$,

$$R^\forall(y, \vec{x}) \leftrightarrow (\forall z \in y)\ (\exists v \in \gamma)\ Q(v, z, \vec{x}).$$

For each $\vec{x} \in J_\alpha$, we define

$$G(\vec{x}) = \{(v, z) \mid v \in \gamma \wedge z \in J_\varrho \wedge Q(v, z, \vec{x})\}.$$

Thus,

$$R^\forall(y, \vec{x}) \leftrightarrow (\forall z \in y)\ (\exists v \in \gamma)\ [(v, z) \in G(\vec{x})] \\ \leftrightarrow \models_{\langle J_\rho, G(x) \rangle} \varphi(\mathring{y}, \mathring{\gamma}),$$

where φ is the Σ_0 formula

$$\varphi(v_0, v_1)\colon (\forall v_2 \in v_0)\ (\exists v_3 \in v_1)\ \mathring{A}((v_3, v_2)),$$

in the language $\mathscr{L}(A)$.

Now, $\lim(\varrho)$, so as φ is Σ_0, by Σ_0 absoluteness we have (cf. the proof of II.6.3)

$$\begin{aligned} R^{\forall}(y, \vec{x}) &\leftrightarrow (\exists w \in J_\varrho)\,[(w \text{ is transitive}) \wedge (y, \gamma \in w) \\ &\qquad \wedge (\models_{\langle w, G(\vec{x}) \cap w\rangle} \varphi(\mathring{y}, \mathring{\gamma}))] \\ &\leftrightarrow (\exists w \in J_\varrho)\,[(\forall u \in w)\,(\forall v \in u)\,(v \in w) \wedge (y, \gamma \in w) \\ &\qquad \wedge Sat^A(w, G(\vec{x}) \cap w, \varphi(\mathring{y}, \mathring{\gamma}))]. \end{aligned}$$

Now, Q is $\Sigma_n(J_\alpha)$, so for each $\vec{x} \in J_\alpha$, $G(\vec{x})$ is a $\Sigma_n(J_\alpha)$ subset of J_ϱ. Moreover, J_α is Σ_n-uniformisable. So by 4.3, for each $\vec{x} \in J_\alpha$ we have

$$w \in J_\varrho \to G(\vec{x}) \cap w \in J_\varrho.$$

Hence,

$$\begin{aligned} R^{\forall}(y, \vec{x}) &\leftrightarrow (\exists w \in J_\varrho)\,(\exists a \in J_\varrho)\,[(\forall u \in w)\,(\forall v \in u)\,(v \in w) \wedge (y, \gamma \in w) \\ &\qquad \wedge (a = G(\vec{x}) \cap w) \wedge Sat^A(w, a, \varphi(\mathring{y}, \mathring{\gamma}))]. \end{aligned}$$

So in order to show that $R^{\forall}(y, \vec{x})$ is $\Sigma_{n+1}(J_\alpha)$ on J_ϱ it suffices to show that the function $a(w, \vec{x}) = G(\vec{x}) \cap w$ is $\Sigma_{n+1}(J_\alpha)$.

Well, we have

$$a = a(w, \vec{x}) \leftrightarrow \forall z[z \in a \leftrightarrow z \in w \wedge (z)_0 \in \gamma \wedge (z)_1 \in J_\varrho \wedge Q((z)_0, (z)_1, \vec{x})].$$

So, as Q is $\Delta_n(J_\alpha)$, the function $a(w, \vec{x})$ is in fact $\Pi_n(J_\alpha)$. The proof is complete. $\square$

With the aid of the Uniformisation Theorem, we are now able to provide some useful information about the Σ_n projectum.

4.6 Theorem. *Let $\alpha > 1$, $n > 0$. Then ϱ_α^n is equal to the largest ordinal δ such that $\langle J_\delta, A\rangle$ is amenable for all $A \in \Sigma_n(J_\alpha) \cap \mathscr{P}(J_\delta)$.*

Proof. By 4.5 and 4.3, $\langle J_\varrho, A\rangle$ is amenable for all $A \in \Sigma_n(J_\alpha) \cap \mathscr{P}(J_\varrho)$, where we have set $\varrho = \varrho_\alpha^n$ for convenience. Suppose δ were a larger ordinal with this property. Let f be a $\Sigma_n(J_\alpha)$ function such that $f'' J_\varrho = J_\alpha$. Set

$$A = \{u \in J_\varrho \mid u \notin f(u)\}.$$

A is $\Sigma_n(J_\alpha)$ and $A \subseteq J_\varrho$, so $\langle J_\delta, A\rangle$ is amenable. But then

$$A = A \cap J_\varrho \in J_\delta \subseteq J_\alpha,$$

so for some $u \in J_\varrho$, we have $A = f(u)$, which leads to the contradiction

$$u \in f(u) \leftrightarrow u \in A \leftrightarrow u \notin f(u).$$

This proves the theorem. $\square$

4.7 Theorem. *Let $\alpha > 1$, $n > 0$. Then ϱ_α^n is equal to the smallest ordinal η such that $\mathscr{P}(\omega\eta) \cap \Sigma_n(J_\alpha) \nsubseteq J_\alpha$.*

Proof. Let $\varrho = \varrho_\alpha^n$, and let f be a $\Sigma_n(J_\alpha)$ function such that $f''J_\varrho = J_\alpha$. Let j be a $\Sigma_1(J_\varrho)$ map from $\omega\varrho$ onto J_ϱ. Set

$$A = \{v \in \omega\varrho \mid v \notin f \circ j(v)\}.$$

A is a $\Sigma_n(J_\alpha)$ subset of $\omega\varrho$. If $A \in J_\alpha$, then $A = f \circ j(v)$ for some $v < \omega\varrho$, and we get the contradiction

$$v \in A \leftrightarrow v \notin f \circ j(v) \leftrightarrow v \notin A.$$

Hence $\mathscr{P}(\omega\varrho) \cap \Sigma_n(J_\alpha) \nsubseteq J_\alpha$. But if $\eta < \varrho$ and $B \in \mathscr{P}(\omega\eta) \cap \Sigma_n(J_\alpha)$, then by 4.6, $\langle J_\varrho, B\rangle$ is amenable, so $B = B \cap J_\eta \in J_\varrho \subseteq J_\alpha$. Thus $\mathscr{P}(\omega\eta) \cap \Sigma_n(J_\alpha) \subseteq J_\alpha$. The theorem is proved. $\square$

To complete this section, we state the following key fact that was used in our proof of the Uniformisation Theorem.

4.8 Lemma. *Let $\alpha > 1, n > 0, \varrho = \varrho_\alpha^n$. If $R(y, \vec{x})$ is $\Sigma_n(J_\alpha)$, then $R^\forall(y, \vec{x})$ is $\Sigma_{n+1}(J_\alpha)$ on J_ϱ. That is, there is a $\Sigma_{n+1}(J_\alpha)$ predicate $Q(y, \vec{x})$ such that*

$$(\forall y \in J_\varrho)(\forall \vec{x} \in J_\alpha)[(\forall z \in y) R(z, \vec{x}) \leftrightarrow Q(y, \vec{x})]. \quad \square$$

5. Standard Codes

Let $\alpha > 0$, $n > 0$. A Σ_n *code* for J_α is a set $A \subseteq J_{\varrho_\alpha^n}$. $A \in \Sigma_n(J_\alpha)$, such that for any $m \geqslant 1$,

$$\Sigma_{n+m}(J_\alpha) \cap \mathscr{P}(J_{\varrho_\alpha^n}) = \Sigma_m(\langle J_{\varrho_\alpha^n}, A\rangle).$$

In this section we show that not only does each J_α have a Σ_n code for each n, but there are particularly nice codes which are preserved under condensation arguments.

We begin by recalling the following result (V.5.9).

5.1 Lemma. *Let $\pi: J_{\bar{\alpha}} \prec_0 J_\alpha$. Then for any $v < \omega\bar{\alpha}$, $\pi(S_v) = S_{\pi(v)}$.* $\square$

Using 5.1, we prove:

5.2 Lemma. *Let $\pi: \langle J_{\bar{\alpha}}, \bar{A}\rangle \prec_0 \langle J_\alpha, A\rangle$ and suppose that $\pi''\omega\bar{\alpha}$ is cofinal in $\omega\alpha$. Then in fact $\pi: \langle J_{\bar{\alpha}}, \bar{A}\rangle \prec_1 \langle J_\alpha, A\rangle$.*

Proof. Let φ be a Σ_0 formula of $\mathscr{L}$ such that

$$\models_{\langle J_\alpha, A\rangle} \exists z\, \varphi(z, \pi(\vec{x})).$$

Since $\pi''\omega\bar{\alpha}$ is cofinal in $\omega\alpha$, we can find a $v < \omega\bar{\alpha}$ such that

$$\models_{\langle J_\alpha, A\rangle} (\exists z \in S_{\pi(v)})\, \varphi(z, \pi(\vec{x})).$$

By 5.1, this can be written as

$$\models_{\langle J_\alpha, A\rangle} (\exists z \in \pi(S_\nu))\, \varphi(z, \pi(\vec{x})).$$

So as π is Σ_0-elementary, this gives,

$$\models_{\langle J_{\bar{\alpha}}, \bar{A}\rangle} (\exists z \in S_\nu)\, \varphi(z, \vec{x}).$$

So, as required,

$$\models_{\langle J_{\bar{\alpha}}, \bar{A}\rangle} \exists z\, \varphi(z, \vec{x}). \quad \square$$

Let $\alpha > 0$. The *standard codes*, A_α^n, and the *standard parameters*, p_α^n, are defined by recursion on n.

To commence, set

$$A_\alpha^0 = \emptyset, \quad p_\alpha^0 = \emptyset.$$

Now let $n \geqslant 0$ and assume that A_α^n and p_α^n are defined, and that if $n \geqslant 1$, A_α^n is a Σ_n code for J_α. We define A_α^{n+1} and p_α^{n+1}. By definition of ϱ_α^{n+1} there is a $\Sigma_{n+1}(J_\alpha)$ map $f \subseteq J_\alpha \times J_{\varrho_\alpha^{n+1}}$ such that $f''J_{\varrho_\alpha^{n+1}} = J_\alpha$. Let $\bar{f} = f \cap (J_{\varrho_\alpha^n} \times J_{\varrho_\alpha^{n+1}})$. Then $\bar{f}$ is also a $\Sigma_{n+1}(J_\alpha)$ map, and $\bar{f}''J_{\varrho_\alpha^{n+1}} = J_{\varrho_\alpha^n}$. But $\bar{f} \subseteq J_{\varrho_\alpha^n}$, so as A_α^n is a Σ_n code for J_α, $\bar{f}$ is in fact $\Sigma_1(\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle)$. Hence we may define

p_α^{n+1} = the $<_J$-least $p \in J_{\varrho_\alpha^n}$ such that every $x \in J_{\varrho_\alpha^n}$ is Σ_1-definable in $\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle$ from parameters in $J_{\varrho_\alpha^{n+1}} \cup \{p\}$.

As in section 3, $(\varphi_i \mid i < \omega)$ is a fixed $\Delta_1^{J_1}$ enumeration of all the Σ_1 formulas of $\mathscr{L}(A)$ of the form

$$\varphi_i(v_0, v_1) \equiv \exists v_2\, \bar{\varphi}_i(v_0, v_1, v_2),$$

where $\bar{\varphi}_i$ is Σ_0. Set

$$A_\alpha^{n+1} = \{(i, x) \mid i \in \omega \wedge x \in J_{\varrho_\alpha^{n+1}} \wedge \models_{\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle} \varphi_i(\mathring{x}, \mathring{p}_\alpha^{n+1})\}.$$

5.3 Lemma. *A_α^{n+1} is a Σ_{n+1} code for J_α.*

Proof. By assumption, A_α^n is a $\Sigma_n(J_\alpha)$ set. So by 4.6, $\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle$ is amenable. So by 3.4, A_α^{n+1} is $\Sigma_1(\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle)$. Hence as A_α^n is a Σ_n code for J_α, A_α^{n+1} is $\Sigma_{n+1}(J_\alpha)$. We must show that for $m \geqslant 1$,

$$\Sigma_{n+1+m}(J_\alpha) \cap \mathscr{P}(J_{\varrho_\alpha^{n+1}}) = \Sigma_m(\langle J_{\varrho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle).$$

Suppose first that $R \in \Sigma_0(\langle J_{\varrho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle)$. Let φ be a Σ_0 formula of $\mathscr{L}$ and q an element from $J_{\varrho_\alpha^{n+1}}$ such that

$$R(x) \leftrightarrow \models_{\langle J_{\varrho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle} \varphi(\mathring{x}, \mathring{q}).$$

Since $\langle J_{\varrho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle$ is amenable, we have, by Σ_0-absoluteness:

$$R(x) \leftrightarrow (\exists u \in J_{\varrho_\alpha^{n+1}})(\exists a \in J_{\varrho_\alpha^{n+1}})[u \text{ is transitive} \wedge x \in u \wedge q \in u \\ \wedge a = A_\alpha^{n+1} \cap u \wedge \models_{\langle u,a\rangle} \varphi(\mathring{x}, \mathring{q})].$$

Consider the function $a = A_\alpha^{n+1} \cap u$. Since A_α^{n+1} is J_α-definable, so is this function (as a function on $J_{\varrho_\alpha^{n+1}}$). Indeed, it has the definition

$$a = A_\alpha^{n+1} \cap u \leftrightarrow (\forall v \in a)(v \in u \wedge v \in A_\alpha^{n+1}) \wedge (\forall v \in u)(v \in A_\alpha^{n+1} \rightarrow v \in a).$$

This is of the form

$$a = A_\alpha^{n+1} \cap u \leftrightarrow (\forall v \in a)(\Sigma_{n+1}(J_\alpha)) \wedge (\forall v \in u)(\Pi_{n+1}(J_\alpha)).$$

By 4.8, for $a \in J_{\varrho_\alpha^{n+1}}$, this is of the form

$$\Sigma_{n+2}(J_\alpha) \wedge \Pi_{n+1}(J_\alpha),$$

and hence is $\Sigma_{n+2}(J_\alpha)$.

It follows at once from our above definition that R is $\Sigma_{n+2}(J_\alpha)$. Hence

$$\Sigma_0(\langle J_{\varrho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle) \subseteq \Sigma_{n+2}(J_\alpha).$$

It follows immediately that

$$\Sigma_1(\langle J_{\varrho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle) \subseteq \Sigma_{n+2}(J_\alpha).$$

By a simple induction on m, we get, for $m \geqslant 1$,

$$\Sigma_m(\langle J_{\varrho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle) \subseteq \Sigma_{n+1+m}(J_\alpha).$$

It remains to prove that for every $m \geqslant 1$,

$$\Sigma_{n+1+m}(J_\alpha) \cap \mathscr{P}(J_{\varrho_\alpha^{n+1}}) \subseteq \Sigma_m(\langle J_{\varrho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle).$$

Since A_α^n is a Σ_n code for J_α, it suffices to prove that

$$\Sigma_{m+1}(\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle) \cap \mathscr{P}(J_{\varrho_\alpha^{n+1}}) \subseteq \Sigma_m(\langle J_{\varrho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle).$$

Let f be a $\Sigma_{n+1}(J_\alpha)$ function such that $f'' J_{\varrho_\alpha^{n+1}} = J_\alpha$. Set $\bar{f} = f \cap (J_{\varrho_\alpha^n} \times J_{\varrho_\alpha^{n+1}})$. Then $\bar{f}$ is $\Sigma_{n+1}(J_\alpha)$ and $\bar{f}'' J_{\varrho_\alpha^{n+1}} = J_{\varrho_\alpha^n}$. Moreover, $\bar{f} \subseteq J_{\varrho_\alpha^n}$, so $\bar{f}$ is $\Sigma_1(\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle)$.

Let $R \in \Sigma_{m+1}(\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle) \cap \mathscr{P}(J_{\varrho_\alpha^{n+1}})$. Assume for the sake of argument that m is even. Let P be a $\Sigma_1(\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle)$ relation such that for $x \in J_{\varrho_\alpha^{n+1}}$,

$$R(x) \leftrightarrow (\exists y_1 \in J_{\varrho_\alpha^n})(\forall y_2 \in J_{\varrho_\alpha^n}) \ldots (\exists y_{m-1} \in J_{\varrho_\alpha^n})(\forall y_m \in J_{\varrho_\alpha^n}) P(\vec{y}, x).$$

Define a relation $\tilde{P}$ by

$$\tilde{P}(\vec{z}, x) \leftrightarrow [(\vec{z}, x \in J_{\varrho_\alpha^{n+1}}) \wedge \exists y[y = f(\vec{z}) \wedge P(y, x)]].$$

Now, there are $p, q \in J_{\varrho_\alpha^n}$ such that $\bar{f}$ is $\Sigma_1^{\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle}(\{p\})$ and P is $\Sigma_1^{\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle}(\{q\})$. By choice of p_α^{n+1}, the pair (p, q) is Σ_1-definable from elements of $J_{\varrho_\alpha^{n+1}} \cup \{p_\alpha^{n+1}\}$ in $\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle$. Hence both $\bar{f}$ and P are $\Sigma_1^{\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle}(\{u, p_\alpha^{n+1}\})$ for some $u \in J_{\varrho_\alpha^{n+1}}$. Thus $\tilde{P}$ is $\Sigma_1^{\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle}(\{u, p_\alpha^{n+1}\})$. (In case $\varrho_\alpha^{n+1} < \varrho_\alpha^n$, we may assume that ϱ_α^{n+1} is Σ_1-definable from u and p_α^{n+1} in $\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle$ as well.) So for some $i \in \omega$,

$$(*) \qquad \begin{aligned} P(\vec{z}, x) &\leftrightarrow [(\vec{z}, x \in J_{\varrho_\alpha^n}) \wedge \models_{\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle} \varphi_i((\vec{z}, x, u)^\circ, \mathring{p}_\alpha^{n+1})] \\ &\leftrightarrow [(\vec{z}, x \in J_{\varrho_\alpha^n}) \wedge (i, (\vec{z}, x, u)) \in A_\alpha^{n+1}]. \end{aligned}$$

Similarly, if we define D by

$$D(z) \leftrightarrow z \in \operatorname{dom}(\bar{f}),$$

then D is $\Sigma_1(\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle)$ and there is a $v \in J_{\varrho_\alpha^{n+1}}$ and a $j \in \omega$ such that

$$(**) \qquad D(z) \leftrightarrow [(z \in J_{\varrho_\alpha^n}) \wedge (j, (z, v)) \in A_\alpha^{n+1}].$$

Now, by definition of $\tilde{P}$ we have, for $x \in J_{\varrho_\alpha^{n+1}}$,

$$\begin{aligned} R(x) \leftrightarrow\ & (\exists z_1 \in J_{\varrho_\alpha^{n+1}})(\forall z_2 \in J_{\varrho_\alpha^{n+1}}) \dots (\exists z_{m-1} \in J_{\varrho_\alpha^{n+1}})(\forall z_m \in J_{\varrho_\alpha^{n+1}}) \\ & \cdot [(D(z_1) \wedge D(z_3) \wedge \dots \wedge D(z_{m-1})) \wedge (D(z_2) \wedge D(z_4) \wedge \dots \\ & \wedge D(z_m) \rightarrow \tilde{P}(\vec{z}, x))]. \end{aligned}$$

By $(*)$ and $(**)$, this is $\Sigma_m(\langle J_{\varrho_\alpha^{n+1}}, A_\alpha^{n+1}\rangle)$, as required. □

Let $\langle J_\alpha, A\rangle$ be amenable. The *Σ_n-projectum* of the structure $\langle J_\alpha, A\rangle$ is defined to be the largest ordinal $\varrho \leqslant \alpha$ such that $\langle J_\varrho, B\rangle$ is amenable for all $B \in \Sigma_n(\langle J_\alpha, A\rangle) \cap \mathscr{P}(J_\varrho)$, and is denoted by $\varrho_{\alpha,A}^n$. Note that this *definition* is not just a generalisation of the definition of the Σ_n-projectum of an ordinal. Though by 4.6, the *notion* is a generalisation of that of a Σ_n-projectum of an ordinal. Indeed, we can say more, as the next lemma indicates:

5.4 Lemma. *Let $\alpha > 1$, $n \geqslant 0$. Then $\varrho_\alpha^{n+1} = \varrho_{\varrho_\alpha^n, A_\alpha^n}^1$.*

Proof. By 4.6, ϱ_α^{n+1} is the largest $\varrho \leqslant \alpha$ such that $\langle J_\varrho, A\rangle$ is amenable for all $A \in \Sigma_{n+1}(J_\alpha) \cap \mathscr{P}(J_\varrho)$. Set $\eta = \varrho_{\varrho_\alpha^n, A_\alpha^n}^1$.

Suppose that $A \in \Sigma_1(\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle) \cap \mathscr{P}(J_{\varrho_\alpha^{n+1}})$. Then, as A_α^{n+1} is a Σ_{n+1} code for J_α, $A \in \Sigma_{n+1}(J_\alpha) \cap \mathscr{P}(J_{\varrho_\alpha^{n+1}})$. Thus by our above remark $\langle J_{\varrho_\alpha^{n+1}}, A\rangle$ is amenable. Thus by definition of η, $\varrho_\alpha^{n+1} \leqslant \eta$.

Now let $A \in \Sigma_{n+1}(J_\alpha) \cap \mathscr{P}(J_\eta)$. By choice of η, we have (trivially) $\eta \leqslant \varrho_\alpha^n$. Thus $A \in \Sigma_{n+1}(J_\alpha) \cap \mathscr{P}(J_{\varrho_\alpha^n})$. Hence $A \in \Sigma_1(\langle J_{\varrho_\alpha^n}, A_\alpha^n\rangle)$. Thus $\langle J_\eta, A\rangle$ is amenable. So, by definition, $\eta \leqslant \varrho_\alpha^{n+1}$. □

Again, let $\langle J_\alpha, A\rangle$ be amenable, and set $\varrho = \varrho^1_{\alpha,A}$. Suppose that every $x \in J_\alpha$ is Σ_1-definable in $\langle J_\alpha, A\rangle$ from parameters in $J_\varrho \cup \{p\}$ for some $p \in J_\alpha$. Then we define $p^1_{\alpha,A}$ to be the $<_J$-least such p, and set

$$A^1_{\alpha,A} = \{(i, x) \mid i \in \omega \wedge x \in J_\varrho \wedge \models_{\langle J_\alpha, A\rangle} \varphi_i(\mathring{x}, \mathring{p}^1_{\alpha,A})\}.$$

5.5 Lemma. *Let* $\alpha > 1$, $n \geqslant 0$. *Then*:

(i) $p^{n+1}_\alpha = p^1_{\varrho^n_\alpha, A^n_\alpha}$;

(ii) $A^{n+1}_\alpha = A^1_{\varrho^n_\alpha, A^n_\alpha}$.

Proof. (i) By definition,

$p^{n+1}_\alpha =$ the $<_J$-least $p \in J_{\varrho^n_\alpha}$ such that every $x \in J_{\varrho^n_\alpha}$ is Σ_1-definable in $\langle J_{\varrho^n_\alpha}, A^n_\alpha\rangle$ from parameters in $J_{\varrho^{n+1}_\alpha} \cup \{p\}$.

By 5.4, $\varrho^n_\alpha = \varrho^1_{\varrho^n_\alpha, A^n_\alpha}$. So by definition, $p^1_{\varrho^n_\alpha, A^n_\alpha} = p^{n+1}_\alpha$.

(ii) Likewise, by virtue of 5.4 and (i) above, the definitions of A^{n+1}_α and $A^1_{\varrho^n_\alpha, A^n_\alpha}$ coincide. □

It is the following result which will enable us to carry out condensation type arguments with structures of the form $\langle J_{\varrho^n_\alpha}, A^n_\alpha\rangle$, thereby enabling us to handle Σ_n predicates on the J_α's as coded Σ_1 predicates.

5.6 Theorem ("Condensation Lemma"). *Let* $\alpha > 1$, $n \geqslant 0$, $m \geqslant 0$. *Let* $\langle J_{\bar\varrho}, \bar A\rangle$ *be amenable, and let*

$$\pi\colon \langle J_{\bar\varrho}, \bar A\rangle \prec_m \langle J_{\varrho^n_\alpha}, A^n_\alpha\rangle.$$

Then:

(I) *There is a unique* $\bar\alpha \geqslant \bar\varrho$ *such that* $\bar\varrho = \varrho^n_{\bar\alpha}$, $A = A^n_{\bar\alpha}$.

(II) *There is a unique* $\tilde\pi \supseteq \pi$ *such that*:

 (i) $\tilde\pi\colon J_{\bar\alpha} \prec_{m+n} J_\alpha$, *and*

 (ii) *for* $i = 1, \ldots, n$, $\tilde\pi(p^i_{\bar\alpha}) = p^i_\alpha$.

(III) *For* $i = 1, \ldots, n$,

$$(\tilde\pi \restriction J_{\varrho^i_{\bar\alpha}})\colon \langle J_{\varrho^i_{\bar\alpha}}, A^i_{\bar\alpha}\rangle \prec_{m+n-i} \langle J_{\varrho^i_\alpha}, A^i_\alpha\rangle. \quad \square$$

The proof of 5.6 is quite long. Before we commence we make a few remarks. Firstly, notice that the result is indeed a condensation lemma. In many applications, the embedding π will simply be the inverse of the collapsing map obtained from some Σ_1 elementary submodel of $\langle J_{\varrho^n_\alpha}, A^n_\alpha\rangle$. Secondly, note that we allow for the case where $m = 0$. We will require this case in applications. Notice that this is the only case where we need to explicitly demand that $\langle J_{\bar\varrho}, \bar A\rangle$ be amenable. In all other cases this is automatic by the elementarity of π. Finally, some nomenclature. The embedding $\tilde\pi\colon J_{\bar\alpha} \to J_\alpha$ is called the *canonical extension* of $\pi\colon \langle J_{\bar\varrho}, \bar A\rangle \to \langle J_{\varrho^n_\alpha}, A^n_\alpha\rangle$.

Now for the proof of 5.6. This proceeds by induction on n. For $n = 0$ the theorem reduces to a triviality, so we are at once left with the proof that if the theorem holds for $n - 1$, then it holds for n, where $n > 0$. To simplify the notation, let us write $\varrho = \varrho_\alpha^n$, $A = A_\alpha^n$. So we are given an amenable structure $\langle J_{\bar{\varrho}}, \bar{A} \rangle$ and an embedding

$$\pi: \langle J_{\bar{\varrho}}, \bar{A} \rangle \prec_m \langle J_\varrho, A \rangle.$$

We shall show that there is a unique structure $\langle J_{\bar{\beta}}, \bar{B} \rangle$ such that $\bar{\varrho} = \varrho^1_{\bar{\beta}, \bar{B}}$ and $\bar{A} = A^1_{\bar{\beta}, \bar{B}}$, and a unique $\tilde{\pi} \supseteq \pi$ such that, setting $\beta = \varrho_\alpha^{n-1}$, $B = A_\alpha^{n-1}$, $p = p_\alpha^n$:

(i) $\tilde{\pi}: \langle J_{\bar{\beta}}, \bar{B} \rangle \prec_{m+1} \langle J_\beta, B \rangle$;

(ii) $\tilde{\pi}(p^1_{\bar{\beta}, \bar{B}}) = p$.

The induction step, and hence the theorem, follow directly from this. For by induction hypothesis there is a unique $\bar{\alpha}$ such that $\bar{\beta} = \varrho_{\bar{\alpha}}^{n-1}$, $\bar{B} = A_{\bar{\alpha}}^{n-1}$, etc., and we have, by 5.4 and 5.5, $p = p^1_{\beta, B}$, $\bar{\varrho} = \varrho^1_{\bar{\beta}, \bar{B}} = \varrho^n_{\bar{\alpha}}$, $\bar{A} = A^1_{\bar{\beta}, \bar{B}} = A^n_{\bar{\alpha}}$.

The function $\tilde{\pi}$ will be the inverse to a certain collapsing isomorphism. The set which $\tilde{\pi}^{-1}$ collapses is defined thus:

$$X = \{x \in J_\beta \mid x \text{ is } \Sigma_1\text{-definable in } \langle J_\beta, B \rangle \text{ from parameters in } \operatorname{ran}(\pi) \cup \{p\}\}.$$

Since $X \prec_1 \langle J_\beta, B \rangle$, there is an isomorphism

$$\tilde{\pi}: \langle J_{\bar{\beta}}, \bar{B} \rangle \cong \langle X, B \cap X \rangle,$$

for some unique $\bar{\beta}$, $\bar{B}$. Thus

$$\tilde{\pi}: \langle J_{\bar{\beta}}, \bar{B} \rangle \prec_1 \langle J_\beta, B \rangle.$$

Define $\tilde{\varrho} \leqslant \varrho$ by

$$\omega\tilde{\varrho} = \sup(\pi''\omega\bar{\varrho}).$$

Set

$$\tilde{A} = A \cap J_{\tilde{\varrho}}.$$

Then

$$\pi: \langle J_{\bar{\varrho}}, \bar{A} \rangle \prec_0 \langle J_{\tilde{\varrho}}, \tilde{A} \rangle.$$

But $\pi''\omega\bar{\varrho}$ is cofinal in $\omega\tilde{\varrho}$. So by 5.2,

$$\pi: \langle J_{\bar{\varrho}}, \bar{A} \rangle \prec_1 \langle J_{\tilde{\varrho}}, \tilde{A} \rangle.$$

5.7 Lemma. $\operatorname{ran}(\pi) = X \cap J_{\tilde{\varrho}}$.

Proof. Clearly, $\operatorname{ran}(\pi) \subseteq X \cap J_{\tilde{\varrho}}$. To prove the opposite inclusion, let $y \in X \cap J_{\tilde{\varrho}}$. Then for some $i \in \omega$ and some $x \in \operatorname{ran}(\pi)$,

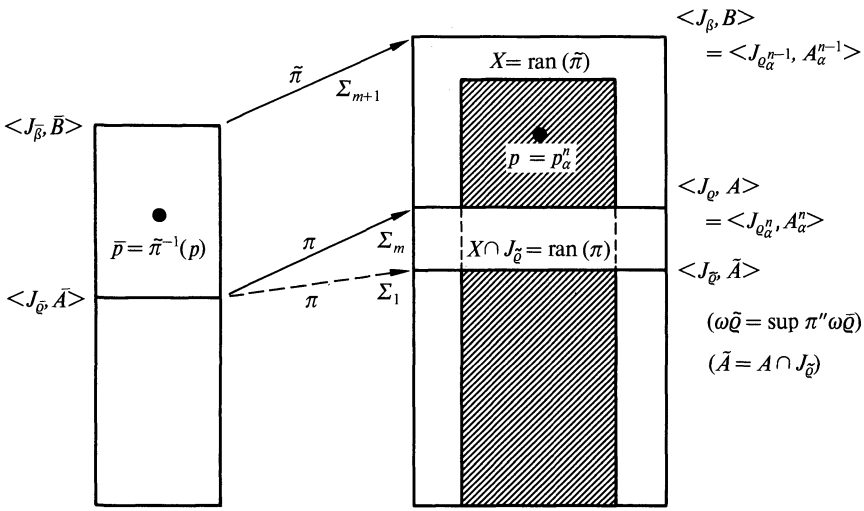
$$y = \text{the unique } x \in J_\beta \text{ such that } \models_{\langle J_\beta, B \rangle} \varphi_i((\mathring{y}, \mathring{x}), \mathring{p}).$$

Thus by definition of $A = A^n_\alpha$,

$$y = \text{the unique } y \in J_{\bar\beta} \text{ such that } \tilde{A}(i, (y, x)).$$

But $x \in \operatorname{ran}(\pi)$ and $\pi: \langle J_{\bar\varrho}, \bar{A}\rangle \prec_1 \langle J_{\tilde\varrho}, \tilde{A}\rangle$, so we conclude that $y \in \operatorname{ran}(\pi)$. That proves the lemma. $\square$

By 5.7, π^{-1} is the unique collapsing isomorphism for $X \cap J_{\tilde\varrho}$. But $X \cap J_{\tilde\varrho}$ is an $\in$-initial segment of X and $\tilde\pi^{-1}$ is the unique collapsing isomorphism for X, so $\tilde\pi^{-1} \restriction (X \cap J_{\tilde\varrho})$ is the unique collapsing isomorphism for $X \cap J_{\tilde\varrho}$. Thus $\tilde\pi^{-1} \restriction (X \cap J_{\tilde\varrho}) = \pi^{-1}$. Thus $\pi = \tilde\pi \restriction J_{\bar\varrho}$ and $\pi \subseteq \tilde\pi$. (Fig. 1 sums up the situation now.)



Shaded part $= X = \{x \in J_\beta \mid x$ is Σ_1-definable in $\langle J_\beta, B\rangle$ from parameters in $\operatorname{ran}(\pi) \cup \{p\}\}$.

Fig. 1

5.8 Lemma. $\tilde\pi: \langle J_{\bar\beta}, \bar{B}\rangle \prec_{m+1} \langle J_\beta, B\rangle$.

Proof. If $m = 0$ there is nothing to prove. So assume $m > 0$.

Let y be Σ_{m+1}-definable in $\langle J_\beta, B\rangle$ from parameters in $\operatorname{ran}(\tilde\pi)$. We must show that $y \in \operatorname{ran}(\tilde\pi)$. Now, by the definition of $\operatorname{ran}(\tilde\pi) = X$, y is Σ_{m+1}-definable in $\langle J_\beta, B\rangle$ from parameters in $\operatorname{ran}(\pi) \cup \{p\}$. Let φ be a Σ_{m+1}-formula of $\mathscr{L}(B)$ such that y is the unique $y \in J_\beta$ for which $\models_{\langle J_\beta, B\rangle} \varphi(\mathring{y}, \vec{\mathring{x}}, \mathring{p})$, where $\vec{x} \in \operatorname{ran}(\pi)$. Then we have

$$\varphi(u, \vec{v}, w) = \exists z_1 \forall z_2 \exists z_3 \ldots - z_m \psi(\vec{z}, u, \vec{v}, w),$$

where ψ is Σ_1 if m is even and Π_1 if m is odd.

Suppose first that $\varrho = \beta$. Now, y is the unique y such that

$$(*) \qquad (\exists z_1 \in J_\beta)(\forall z_2 \in J_\beta) \ldots (- z_m \in J_\beta)\,[\models_{\langle J_\beta, B\rangle} \psi(\vec{\mathring{z}}, \mathring{y}, \vec{\mathring{x}}, \mathring{p})].$$

But $\beta = \varrho = \varrho_\alpha^n = \varrho_{\beta,B}^1, p = p_\alpha^n = p_{\beta,B}^1$, and $A = A_\alpha^n = A_{\beta,B}^1$. So as ψ is Σ_1 or Π_1, (*) is a $\Sigma_m^{\langle J_\rho, A\rangle}(\{\vec{x}\})$ predicate of y. But $\vec{x} \in \operatorname{ran}(\pi) \prec_m \langle J_\varrho, A\rangle$. Thus $y \in \operatorname{ran}(\pi) \subseteq \operatorname{ran}(\tilde{\pi})$, and we are done.

Now suppose that $\varrho < \beta$. Let $h = h_{\beta,B}$, and set

$$\tilde{h}((i, x)) \simeq h(i, (x, p)).$$

Let $D = \operatorname{dom}(\tilde{h}) \cap J_\varrho$. For $u \in D$, $\tilde{h}(u)$ is Σ_1-definable in $\langle J_\beta, B\rangle$ from u, p, so if $u \in X$, then since $p \in X \prec_1 \langle J_\beta, B\rangle$, we have $\tilde{h}(u) \in X$. Thus in order to show that $y \in X$ it suffices to show that for some $u \in D \cap X$, we have

$$\models_{\langle J_\beta, B\rangle} \varphi(\tilde{h}(\mathring{u}), \mathring{\vec{x}}, \mathring{p}).$$

(For then by uniqueness, $y = \tilde{h}(u) \in X$.) Now, $\varrho = \varrho_\alpha^n$, so by definition of $p = p_\alpha^n$, every $x \in J_\beta$ is Σ_1-definable in $\langle J_\beta, B\rangle$ from parameters in $J_\varrho \cup \{p\}$. So in particular, $\tilde{h}'' J_\varrho = J_\beta$, i.e. $\tilde{h}'' D = J_\beta$. Thus it suffices to show that for some $u \in D \cap X$ we have

$$(**) \qquad (\exists z_1 \in D)(\forall z_2 \in D)\ldots(- z_m \in D)\,[\models_{\langle J_\beta, B\rangle} \psi(\tilde{h}(\mathring{z}_1), \ldots, \tilde{h}(\mathring{z}_m), \tilde{h}(\mathring{u}), \mathring{\vec{x}}, \mathring{p})].$$

If we can show that (**) is a $\Sigma_m^{\langle J_\rho, A\rangle}(\{\vec{x}\})$ predicate of u we shall be done, since $\vec{x} \in \operatorname{ran}(\pi) \prec_m \langle J_\varrho, A\rangle$ and $\operatorname{ran}(\pi) \subseteq X$.

Let us assume that m is even. (We deal with the similar case m odd later.) There is an $i_0 < \omega$ such that for any $z \in J_\varrho$,

$$\begin{aligned} z \in D &\leftrightarrow \exists y[y = \tilde{h}(z)] \\ &\leftrightarrow \exists y[y = h((z)_0, ((z)_1, p))] \\ &\leftrightarrow \models_{\langle J_\beta, B\rangle} \varphi_{i_0}(\mathring{z}, \mathring{p}) \\ &\leftrightarrow (i_0, z) \in A, \end{aligned}$$

where the last equivalence follows from the definition of $A = A_\alpha^n$. Similarly, as ψ is Σ_1 (for m even) there is a $j_0 < \omega$ such that for any $z_1, \ldots, z_m, u \in D$,

$$\begin{aligned} &\models_{\langle J_\beta, B\rangle} \psi(\tilde{h}(\mathring{z}_1), \ldots, \tilde{h}(\mathring{z}_m), \tilde{h}(\mathring{u}), \mathring{\vec{x}}, \mathring{p}) \\ &\quad\leftrightarrow \models_{\langle J_\beta, B\rangle} \varphi_{j_0}((\mathring{z}_1, \ldots, \mathring{z}_m, \mathring{u}, \mathring{\vec{x}}), \mathring{p}) \\ &\quad\leftrightarrow (j_0, (z_1, \ldots, z_m, u, \vec{x})) \in A. \end{aligned}$$

Hence (**) is equivalent to the following (for any $u \in J_\varrho$)

$$\begin{aligned} &[(i_0, u) \in A] \wedge (\exists z_1 \in J_\varrho)(\forall z_2 \in J_\varrho)(\exists z_3 \in J_\varrho)(\forall z_4 \in J_\varrho)\ldots \\ &\qquad (\exists z_{m-1} \in J_\varrho)(\forall z_m \in J_\varrho)[((i_0, z_1) \in A \\ &\qquad \wedge (i_0, z_3) \in A \wedge \ldots \wedge (i_0, z_{m-1}) \in A) \wedge ((i_0, z_2) \in A \\ &\qquad \wedge (i_0, z_4) \in A \wedge \ldots \wedge (i_0, z_m) \in A \rightarrow (j_0, (z_1, \ldots, z_m, u, \vec{x})) \in A)]. \end{aligned}$$

But this is $\Sigma_m^{\langle J_\varrho, A\rangle}(\{\vec{x}\})$, so we are done.

The case m odd is fairly similar. The only difference is that we rewrite $(**)$ as

$$(\exists z_1 \in D)(\forall z_2 \in D)\ldots(\exists z_m \in D)\neg\,[\models_{\langle J_\beta, B\rangle}\neg\,\psi(\mathring{\tilde{h}}(z_1),\ldots,\mathring{\tilde{h}}(z_m), \mathring{\tilde{h}}(u), \vec{\mathring{x}}, \mathring{p})],$$

so that $(\neg\,\psi)$ is Σ_1. The rest of the proof is modified accordingly.

That completes the proof of the lemma. $\square$

Now let $\bar{p} = \tilde{\pi}^{-1}(p)$. We must prove that $\bar{\varrho} = \varrho^1_{\bar{\beta},\bar{B}}$, $\bar{A} = A^1_{\bar{\beta},\bar{B}}$, $\bar{p} = p^1_{\bar{\beta},\bar{B}}$.

5.9 Lemma. $\bar{A} = \{(i, x) \mid i \in \omega \wedge x \in J_{\bar{\varrho}} \wedge \models_{\langle J_{\bar{\beta}}, B\rangle} \varphi_i(\mathring{x}, \mathring{\bar{p}})\}$.

Proof. Since $\pi: \langle J_{\bar{\varrho}}, \bar{A}\rangle \prec_m \langle J_\varrho, A\rangle$, we have, for $x \in J_{\bar{\varrho}}$,

$$\bar{A}((i, x)) \leftrightarrow A((i, \pi(x))).$$

And since $\beta = \varrho_\alpha^{n-1}$, $B = A_\alpha^{n-1}$, $A = A_\alpha^n$, we have

$$A((i, \pi(x))) \leftrightarrow \models_{\langle J_\beta, B\rangle} \varphi_i(\mathring{\pi}(x), \mathring{p}).$$

Finally, since $\tilde{\pi}: \langle J_{\bar{\beta}}, \bar{B}\rangle \prec_1 \langle J_\beta, B\rangle$ and $\pi(x), p \in \operatorname{ran}(\tilde{\pi})$, we have

$$\models_{\langle J_\beta, B\rangle} \varphi_i(\mathring{\pi}(x), \mathring{p}) \leftrightarrow \models_{\langle J_\beta, B\rangle} \varphi_i(\mathring{x}, \mathring{\bar{p}}).$$

The above three equivalences yield the lemma. $\square$

5.10 Lemma. $\bar{\varrho} = \varrho^1_{\bar{\beta},\bar{B}}$.

Proof. Since $J_{\bar{\beta}}$ is the collapse of X, every $x \in J_{\bar{\beta}}$ is Σ_1-definiable in $\langle J_{\bar{\beta}}, \bar{B}\rangle$ from parameters in $J_{\bar{\varrho}} \cup \{\bar{p}\}$. Thus if $\bar{h} = h_{\bar{\beta},\bar{B}}$, we have

$$J_{\bar{\beta}} = \tilde{h}^*(J_{\bar{\varrho}} \times \{\bar{p}\}).$$

Hence there is a $\Sigma_1(\langle J_{\bar{\beta}}, \bar{B}\rangle)$ map f from a subset of $\omega\bar{\varrho}$ onto $J_{\bar{\beta}}$. It follows that $\varrho^1_{\bar{\beta},\bar{B}} \leqslant \bar{\varrho}$. For suppose, on the contary, that $\bar{\varrho} < \varrho^1_{\bar{\beta},\bar{B}}$. Let $E = \{\xi \in \omega\bar{\varrho} \mid \xi \notin f(\xi)\}$. Then E is a $\Sigma_1(\langle J_{\bar{\beta}}, \bar{B}\rangle)$ subset of $\omega\bar{\varrho}$. By definition of $\varrho^1_{\bar{\beta},\bar{B}}$, $\langle J_{\varrho^1_{\bar{\beta},\bar{B}}}, E\rangle$ must be amenable. Thus $E = E \cap \omega\bar{\varrho} \in J_{\varrho^1_{\bar{\beta},\bar{B}}} \subseteq J_{\bar{\beta}}$. So for some $\xi \in \omega\bar{\varrho}$, $E = f(\xi)$. But then we get

$$\xi \in f(\xi) \leftrightarrow \xi \in E \leftrightarrow \xi \notin f(\xi),$$

a contradiction. Thus, as claimed, $\varrho^1_{\bar{\beta},\bar{B}} \leqslant \bar{\varrho}$. We now prove the opposite inequality.

Let $C \in \Sigma_1(\langle J_{\bar{\beta}}, \bar{B}\rangle) \cap \mathscr{P}(J_{\bar{\varrho}})$. Since every member of $J_{\bar{\beta}}$ is Σ_1-definable from parameters in $J_{\bar{\varrho}} \cup \{\bar{p}\}$ in $\langle J_{\bar{\beta}}, \bar{B}\rangle$, $C \in \Sigma_1^{\langle J_{\bar{\beta}}, \bar{B}\rangle}(J_{\bar{\varrho}} \cup \{\bar{p}\})$. So for some $i \in \omega$ and some $y \in J_{\bar{\varrho}}$ we have, for $x \in J_{\bar{\varrho}}$,

$$x \in C \quad \text{iff } \models_{\langle J_{\bar{\beta}}, \bar{B}\rangle} \varphi_i((\mathring{x}, \mathring{y}), \mathring{\bar{p}}).$$

So by 5.9, we have, for $x \in J_{\bar{\varrho}}$,

$$x \in C \quad \text{iff } (i, (x, y)) \in \bar{A}.$$

Let $u \in J_{\bar{\varrho}}$, and set

$$v = \{(i, (x, y)) \mid x \in u\}.$$

Note that $v \in J_{\bar{\varrho}}$. Since $\langle J_{\bar{\varrho}}, \bar{A} \rangle$ is amenable, $\bar{A} \cap v \in J_{\bar{\varrho}}$. But look,

$$x \in C \cap u \quad \text{iff } (i, (x, y)) \in \bar{A} \cap v.$$

So as $J_{\bar{\varrho}}$ is rud closed, $C \cap u \in J_{\bar{\varrho}}$. Thus $\langle J_{\bar{\varrho}}, C \rangle$ is amenable. Thus, by definition, $\bar{\varrho} \leqslant \varrho^1_{\bar{\beta}, \bar{B}}$, and the lemma is proved. $\square$

5.11 Lemma. $\bar{p} = p^1_{\bar{\beta}, \bar{B}}$.

Proof. Since every $x \in J_{\bar{\beta}}$ is Σ_1-definable from parameters in $J_{\bar{\varrho}} \cup \{\bar{p}\}$ in $\langle J_{\bar{\beta}}, \bar{B} \rangle$ and $\bar{\varrho} = \varrho^1_{\bar{\beta}, \bar{B}}$, it suffices to show that $\bar{p}$ is $<_J$-least with this property. Well suppose not, and let $\bar{p}' <_J \bar{p}$ have the same property. For some $i \in \omega$ and some $x \in J_{\bar{\varrho}}$, we have $\bar{p} = \bar{h}(i, \bar{p}'))$. Since $\tilde{\pi}: \langle J_{\bar{\beta}}, \bar{B} \rangle \prec_1 \langle J_\beta, B \rangle$ and $\bar{h} = h_{\bar{\beta}, \bar{B}}$, $h = h_{\beta, B}$, setting $p' = \tilde{\pi}(\bar{p}')$ and applying $\tilde{\pi}$ gives $p = h(i, (\tilde{\pi}(x), p'))$. Now, $\tilde{\pi}(x) = \pi(x) \in X \cap J_\varrho$. So, as every $y \in J_\beta$ is Σ_1-definable from parameters in $J_\varrho \cup \{p\}$ in $\langle J_\beta, B \rangle$, it follows that every $y \in J_\beta$ is Σ_1-definable from parameters in $J_\varrho \cup \{p'\}$ in $\langle J_\beta, B \rangle$. But $p' <_J p = p^n_\alpha$, so this contradicts the definition of p^n_α. The lemma is proved. $\square$

Since $\bar{\varrho} = \varrho^1_{\bar{\beta}, \bar{B}}$ and $\bar{p} = p^1_{\bar{\beta}, \bar{B}}$, 5.9 implies immediately that $\bar{A} = A^1_{\bar{\beta}, \bar{B}}$. That proves the existence part of 5.6. We turn to the question of uniqueness.

Suppose that $\langle J_{\bar{\beta}_0}, \bar{B}_0 \rangle$ and $\langle J_{\bar{\beta}_1}, \bar{B}_1 \rangle$ are such that $\bar{\varrho} = \varrho^1_{\bar{\beta}_i, \bar{B}_i}$ and $\bar{A} = A^1_{\bar{\beta}_i, \bar{B}_i}$, $i = 0, 1$. Set $\bar{p}_i = p^1_{\bar{\beta}_i, \bar{B}_i}$. For each $j \in \omega$ and each $\vec{x} \in J_{\bar{\varrho}}$ we have

$$\models_{\langle J_{\bar{\beta}_0}, \bar{B}_0 \rangle} \varphi_j((\mathring{\vec{x}}), \mathring{\bar{p}}_0) \quad \text{iff } \bar{A}((j, (\vec{x}))) \quad \text{iff } \models_{\langle J_{\bar{\beta}_1}, \bar{B}_1 \rangle} \varphi_j((\mathring{\vec{x}}), \mathring{\bar{p}}_1).$$

Since $(\varphi_j \mid j < \omega)$ enumerates all the Σ_1 formulas of $\mathscr{L}(A)$ with free variables v_0, v_1, we have, for all x, y in $J_{\bar{\varrho}}$ and all $j, k \in \omega$,

(a) $\quad h_{\bar{\beta}_0, \bar{B}_0}(j, (x, p_0)) = h_{\bar{\beta}_0, \bar{B}_0}(k, (y, p_0)) \quad \text{iff } h_{\bar{\beta}_1, \bar{B}_1}(j, (x, p_1)) = h_{\bar{\beta}_1, \bar{B}_1}(k, (y, p_1))$;

(b) $\quad h_{\bar{\beta}_0, \bar{B}_0}(j, (x, p_0)) \in h_{\bar{\beta}_0, \bar{B}_0}(k, (y, p_0)) \quad \text{iff } h_{\bar{\beta}_1, \bar{B}_1}(j, (x, p_1)) \in h_{\bar{\beta}_1, \bar{B}_1}(k, (y, p_1))$;

(c) $\quad h_{\bar{\beta}_0, \bar{B}_0}(j, (x, p_0)) \in \bar{B}_0 \quad \text{iff } h_{\bar{\beta}_1, \bar{B}_1}(j, (x, p_1)) \in \bar{B}_1$.

But

$$h^*_{\bar{\beta}_i, \bar{B}_i}(J_{\bar{\varrho}} \times \{p_i\}) = J_{\bar{\beta}_i}$$

for $i = 0, 1$, so by (a)–(c) we have

$$\sigma: \langle J_{\bar{\beta}_0}, \bar{B}_0 \rangle \cong \langle J_{\bar{\beta}_1}, \bar{B}_1 \rangle,$$

where for $x \in J_{\bar{\varrho}}$, $j \in \omega$, we set

$$\sigma(h_{\bar{\beta}_0, \bar{B}_0}(j, (x, p_0)) \simeq h_{\bar{\beta}_1, \bar{B}_1}(j, (x, p_1)).$$

This means that $\bar{\beta}_0 = \bar{\beta}_1$ and that $\sigma = \mathrm{id} \restriction J_{\bar{\beta}_0}$, so $\bar{B}_0 = \bar{B}_1$ as well. Hence $\bar{\beta}$, $\bar{B}$ are unique and it remains only to show that $\tilde{\pi}$ is unique.

Let $\tilde{\pi}_i \supseteq \pi$, $\tilde{\pi}_i \colon \langle J_{\bar{\beta}}, \bar{B}\rangle \prec_{m+1} \langle J_\beta, B\rangle$, $\tilde{\pi}_i(\bar{p}) = p$, for $i = 0, 1$. Let $y \in J_{\bar{\beta}}$. For some $j \in \omega$ and some $x \in J_{\bar{\varrho}}$, we have $y = h_{\bar{\beta},\bar{B}}(j, (x, \bar{p}))$. Then

$$\begin{aligned}\tilde{\pi}_0(x) &= \tilde{\pi}_0 \circ h_{\bar{\beta},\bar{B}}(j, (x, \bar{p})) = h_{\beta,B}(j, (\tilde{\pi}_0(x), \tilde{\pi}_0(\bar{p}))) = h_{\beta,B}(j, \pi(x), p)) \\ &= h_{\beta,B}(j, (\tilde{\pi}_1(x), \tilde{\pi}_1(\bar{p}))) = \tilde{\pi}_1 \circ h_{\bar{\beta},\bar{B}}(j, (x, \bar{p})) = \tilde{\pi}_1(y).\end{aligned}$$

Hence $\tilde{\pi}_0 = \tilde{\pi}_1$, and the proof of 5.6 is complete.

6. An Application: A Global $\Box$-Principle

S Let S denote the class of all singular limit ordinals. Given any class E of limit ordinals, we shall denote the following principle by $\Box(E)$: there is a sequence $(C_\alpha \mid \alpha \in S)$ such that:

(i) C_α is a club subset of α;
(ii) $\mathrm{otp}(C_\alpha) < \alpha$;
(iii) if $\bar{\alpha} < \alpha$ is a limit point of C_α, then $\bar{\alpha} \in S$, $\bar{\alpha} \notin E$, and $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$.

Using our fine structure theory we shall prove the following theorem (which will be utilised in the next chapter):

6.1 Theorem. *Assume $V = L$. Then there is a class E of limit ordinals such that:*

(i) $\alpha \in E \to \mathrm{cf}(\alpha) = \omega$;
(ii) *if $\kappa > \omega$ is regular, then $E \cap \kappa$ is a stationary subset of κ;*
(iii) *$\Box(E)$ is valid.* $\Box$

In fact by a slightly different argument, it is possible to prove the following more general result.

6.1′ Theorem. *Assume $V = L$. Let A be a class of limit ordinals. Then there is a class $E \subseteq A$ such that:*

(i) *if $\kappa > \omega$ is regular and $A \cap \kappa$ is stationary in κ, then $E \cap \kappa$ is stationary in κ;*
(ii) $\Box(E)$. $\Box$

This more general result is proved in detail in Chapter IX, using Silver machines instead of the Fine Structure theory. It is also possible to adapt the proof given in this chapter using the fine structure (see Exercise 4), but in order to avoid making an already complicated proof look even worse, we prove the more spelised version (which in any case is enough for our needs here). As will be seen, the advantage with the specialised version is that the existence and behaviour of the set E can be relegated to a special case of the construction, and thus may be ignored for most of the proof. (This advantage does not arise with the machine proof, which does not involve a number of separate cases.)

Before we commence the proof, let us see how this relates to the principles $\Box_\kappa$ considered in Chapter IV. Let $\Box$ denote the principle $\Box(\emptyset)$. Clearly, if $F \subseteq E$, then $\Box(E)$ implies $\Box(F)$, so $\Box$ is the weakest of the global $\Box$-principles of the above kind.

6.2 Theorem. *Assume $\Box$. Then $\Box_\kappa$ holds for any uncountable cardinal κ.*

Proof. Recall that $\Box_\kappa$ asserts the existence of a sequence $(C_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ such that:

(i) C_α is a club subset of α;

(ii) $\mathrm{cf}(\alpha) < \kappa \rightarrow |C_\alpha| < \kappa$;

(iii) if $\bar{\alpha}$ is a limit point of C_α, then $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$.

We shall denote by $\Box_\kappa^S$ the following principle: there is a sequence $(C_\alpha \mid \alpha \in S \cap \kappa^+)$ such that:

(i) C_α is a club subset of α;

(ii) $\mathrm{cf}(\alpha) < \kappa \rightarrow |C_\alpha| < \kappa$;

(iii) if $\bar{\alpha}$ is a limit point of C_α, then $\bar{\alpha} \in S \cap \kappa^+$ and $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$.

We shall prove the implications $\Box \rightarrow \Box_\kappa^S \rightarrow \Box_\kappa$. We deal with the second implication first. Let $(C_\alpha \mid \alpha \in S \cap \kappa^+)$ be as in $\Box_\kappa^S$. Define a $\Box_\kappa$-sequence $(\tilde{C}_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ as follows.

Suppose first that κ is regular. Then we define $\tilde{C}_\alpha = C_\alpha - \kappa$ for $\kappa < \alpha < \kappa^+$, $\lim(\alpha)$, and $\tilde{C}_\alpha = \alpha$ for $\alpha \leqslant \kappa$, $\lim(\alpha)$. If $\kappa < \alpha < \kappa^+$, $\lim(\alpha)$, then $\alpha \in S \cap \kappa^+$, so C_α is defined. Hence $\tilde{C}_\alpha$ is defined for all limit ordinals $\alpha < \kappa^+$. Clearly, $(\tilde{C}_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ is a $\Box_\kappa$-sequence.

Now suppose κ is singular. In this case the above method will not work, since in order to satisfy $\Box_\kappa$ we shall require $|\tilde{C}_\kappa| < \kappa$, which prevents us from defining $\tilde{C}_\kappa = \kappa$. So we proceed as follows. Let $\theta = \mathrm{cf}(\kappa) < \kappa$. Let $\tilde{C}_\kappa$ be a club subset of κ of order-type θ with $\min(\tilde{C}_\kappa) = 0$. If $\alpha < \kappa$ is a limit point of $\tilde{C}_\kappa$, set $\tilde{C}_\alpha = \alpha \cap C_\kappa$. If $\alpha < \kappa$ is a limit ordinal but is not a limit point of $\tilde{C}_\kappa$, then there is a largest element $\nu \in \tilde{C}_\kappa$ such that $\nu < \alpha$, and we set $\tilde{C}_\alpha = \alpha - \nu$. Finally, in case $\kappa < \alpha < \kappa^+$, $\lim(\alpha)$, we set $\tilde{C}_\alpha = C_\alpha - \kappa$. It is easily seen that $(\tilde{C}_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ is a $\Box_\kappa$-sequence.

We turn now to the considerably less simple problem of deducing $\Box_\kappa^S$ from $\Box$. We start with a $\Box$-sequence $(C_\alpha^0 \mid \alpha \in S)$. For each $\alpha \in S \cap \kappa^+$, we set $C_\alpha^1 = C_\alpha^0 - \kappa$ in case $\alpha > \kappa$ and $C_\alpha^1 = C_\alpha^0$ in case $\alpha \leqslant \kappa$. It is clear that the sequence $(C_\alpha^1 \mid \alpha \in S \cap \kappa^+)$ satisfies the following conditions:

1(i) C_α^1 is a club subset of α;

1(ii) $\mathrm{otp}(C_\alpha^1) < \alpha$;

1(iii) if $\bar{\alpha}$ is a limit point of C_α^1, then $\bar{\alpha} \in S \cap \kappa^+$ and $C_{\bar{\alpha}}^1 = \bar{\alpha} \cap C_\alpha^1$;

1(iv) if $\alpha \in S \cap \kappa^+$, $\alpha > \kappa$, then $C_\alpha^1 \cap \kappa = \emptyset$.

We next define a sequence $(C_\alpha^2 \mid \alpha \in S \cap \kappa^+)$ such that:

2(i) $C_\alpha^2 \subseteq C_\alpha^1$;
2(ii) C_α^2 is a club subset of α;
2(iii) $\operatorname{otp}(C_\alpha^2) \leqslant \kappa$;
2(iv) if $\bar\alpha$ is a limit point of C_α^2, then $\bar\alpha \in S \cap \kappa^+$ and $C_{\bar\alpha}^2 = \bar\alpha \cap C_\alpha^2$.

For $\alpha \in S \cap \kappa^+$, let $\theta_\alpha = \operatorname{otp}(C_\alpha^1)$ and let $f_\alpha: \theta_\alpha \to C_\alpha^1$ be the monotone enumeration of C_α^1. We define C_α^2 by recursion on α.

For $\alpha \leqslant \kappa$, set $C_\alpha^2 = C_\alpha^1$. This part of the C^2-sequence clearly satisfies 2(i)–2(iv). And by 1(iv), the remaining case ($\alpha > \kappa$) will not affect the situation below κ, so we shall not beed to worry about any clashes when we come to check 2(iv) for the rest of the C^2-sequence.

Now suppose $\alpha > \kappa$ and we have defined $C_{\bar\alpha}^2$ for $\bar\alpha \in S \cap \alpha$. If $\theta_\alpha \leqslant \kappa$, we set $C_\alpha^2 = C_\alpha^1$. It is immediate that 2(i)–2(iii) are satisfied in this case. We check 2(iv). Let $\bar\alpha$ be a limit point of C_α^2. Then $\bar\alpha$ is a limit point of C_α^1, so by 1(iii), $\bar\alpha \in S$ and $C_{\bar\alpha}^1 = \bar\alpha \cap C_\alpha^1$. Thus $\theta_{\bar\alpha} = \operatorname{otp}(C_{\bar\alpha}^1) \leqslant \operatorname{otp}(C_\alpha^1) = \theta_\alpha \leqslant \kappa$. But by 1(iv), $\bar\alpha > \kappa$. Thus $C_{\bar\alpha}^2 = C_{\bar\alpha}^1 = \bar\alpha \cap C_\alpha^1 = \bar\alpha \cap C_\alpha^2$.

We are left with the case where $\theta_\alpha > \kappa$. In this case, θ_α is singular, since $\operatorname{cf}(\theta_\alpha) = \operatorname{cf}(\alpha) \leqslant \kappa < \theta_\alpha$. Hence $\theta_\alpha \in S \cap \kappa^+$. By 1(ii), $\theta_\alpha < \alpha$, so $C_{\theta_\alpha}^2$ is defined already. Set $C_\alpha^2 = f_\alpha'' C_{\theta_\alpha}^2$. Using the induction hypothesis, it is immediate that 2(i)–2(iii) are satisfied. We check 2(iv). Let $\bar\alpha < \alpha$ be a limit point of C_α^2. Then $\operatorname{cf}(\bar\alpha) < \operatorname{otp}(C_\alpha^2) \leqslant \kappa$. But by 1(iv), $\bar\alpha > \kappa$. Thus $\bar\alpha \in S$. Now, $\bar\alpha$ is a limit point of C_α^1, so $C_{\bar\alpha}^1 = \bar\alpha \cap C_\alpha^1$. Hence $\theta_{\bar\alpha} = \operatorname{otp}(C_{\bar\alpha}^1) < \operatorname{otp}(C_\alpha^1) = \theta_\alpha$ and $f_{\bar\alpha} = f_\alpha \restriction \theta_{\bar\alpha}$. Clearly, $f_\alpha(\theta_{\bar\alpha}) = \bar\alpha$. So as $\bar\alpha$ is a limit point of C_α^2 and $C_\alpha^2 = f_\alpha'' C_{\theta_\alpha}^2$, $\theta_{\bar\alpha}$ must be a limit point of $C_{\theta_\alpha}^2$. Thus $\theta_{\bar\alpha} \in S$ and $C_{\theta_{\bar\alpha}}^2 = \theta_{\bar\alpha} \cap C_{\theta_\alpha}^2$. But $C_{\theta_\alpha}^2 \subseteq C_{\theta_\alpha}^1$, so $\theta_{\bar\alpha}$ is a limit point of $C_{\theta_\alpha}^1$, so by 1(iv), $\theta_{\bar\alpha} > \kappa$. This means that $C_{\bar\alpha}^2 = f_{\bar\alpha}'' C_{\theta_{\bar\alpha}}^2$, and we have (since $f_\alpha(\theta_{\bar\alpha}) = \bar\alpha$ and $f_\alpha'' C_{\theta_\alpha}^2 = C_\alpha^2$ and $\bar\alpha$ is a limit point of C_α^2):

$$C_{\bar\alpha}^2 = f_{\bar\alpha}'' C_{\theta_{\bar\alpha}}^2 = f_{\bar\alpha}''(\theta_{\bar\alpha} \cap C_{\theta_\alpha}^2) = f_\alpha''(\theta_{\bar\alpha} \cap C_{\theta_\alpha}^2) = \bar\alpha \cap C_\alpha^2.$$

That completes the definition of $(C_\alpha^2 \mid \alpha \in \kappa^+)$. If κ is regular, then $(C_\alpha^2 \mid \alpha \in S \cap \kappa^+)$ clearly satisfies $\Box_\kappa^S$, and we are done. If κ is singular, we extract from $(C_\alpha^2 \mid \alpha \in S \cap \kappa^+)$ a $\Box_\kappa^S$-sequence $(C_\alpha^3 \mid \alpha \in S \cap \kappa^+)$ in the same way as in the proof of IV.5.1 (at the very end). The proof of 6.2 is complete. $\Box$

Notice that in the above proof of 6.2 we commenced with a $\Box$-sequence $(C_\alpha \mid \alpha \in S)$ and constructed a $\Box_\kappa$-sequence $(\tilde C_\alpha \mid \alpha < \kappa^+ \wedge \lim(\alpha))$ such that, in particular, $\tilde C_\alpha \subseteq C_\alpha$ for $\kappa < \alpha < \kappa^+$. Thus the same argument establishes the following more general result:

6.2′ Theorem. *Assume $\Box(E)$. Then for any uncountable cardinal κ, $\Box_\kappa(F)$ holds, where $F = (E \cap \kappa^+) - (\kappa + 1)$. (So if $E \cap \kappa^+$ is stationary in κ^+, F is stationary in κ^+.)* $\Box$

This relates to 6.1′, of course.

We turn now to the proof of 6.1. We assume $V = L$ from now on.

Define a class E of limit ordinals as follows. E is the class of all limit ordinals E
α such that for some $\beta > \alpha$:

(i) α is regular over J_β; and
(ii) for some $p \in J_\beta$, if $p \in X \prec J_\beta$ and $X \cap \alpha$ is transitive, then $X = J_\beta$.

6.3 Lemma. *If $\kappa > \omega$ is regular, then $E \cap \kappa$ is stationary in κ.*

Proof. Let $C \subseteq \kappa$ be club in κ. We prove that $E \cap C \neq \emptyset$. Let N be the smallest $N \prec J_{\kappa^+}$ such that $C \in N$ and $N \cap \kappa$ is transitive. Since κ is regular, $N \cap \kappa \in \kappa$. Let $\alpha = N \cap \kappa$.

Let $\pi: J_\beta \cong N$. Then $\pi \upharpoonright \alpha = \mathrm{id} \upharpoonright \alpha$ and $\pi(\alpha) = \kappa$. Since $C \in N$, we have $C \cap \alpha \in J_\beta$ and $\pi(c \cap \alpha) = C$. Since C is club in κ, by absoluteness we have

$$\models_{J_{\kappa^+}} \text{“}C \text{ is club in } \kappa\text{”}.$$

So, as $\pi: J_\beta \prec J_{\kappa^+}$,

$$\models_{J_\beta} \text{“}C \cap \alpha \text{ is club in } \alpha\text{”}.$$

Thus by absoluteness again, $C \cap \alpha$ is indeed club in α. But C is closed in κ. Hence $\alpha \in C$. We show that $\alpha \in E$ as well.

Suppose that there were a J_β-definable map from a bounded subset of α cofinally into α. Then by applying $\pi: J_\beta \prec J_{\kappa^+}$ we would obtain a J_{κ^+}-definable map from a bounded subset of κ cofinally into κ, which is impossible. Hence α is regular over J_β.

Now suppose that $C \cap \alpha \in X \prec J_\beta$ and that $X \cap \alpha$ is transitive. Applying $\pi: J_\beta \cong N \prec J_{\kappa^+}$ we get $C \in (\pi'' X) \prec N \prec J_{\kappa^+}$. But $\pi(\alpha) = \kappa$, so $(\pi'' X) \cap \kappa = \pi''(X \cap \alpha) = X \cap \alpha$, which is transitive. So by the choice of N we must have $(\pi'' X) = N$. Thus $X = J_\beta$.

Thus β and $C \cap \alpha$ testify that $\alpha \in E$. The proof is complete. □

6.4 Lemma. *Let $\alpha \in E$, and let $\beta > \alpha$ be as in the definition of E. Then* $\mathrm{cf}(\alpha) = \omega$ *and there is a $\Sigma_1(J_{\beta+1})$ map from ω cofinally into α.*

Proof. Let $p \in J_\beta$ be such that whenever $p \in X \prec J_\beta$ and $X \cap \alpha$ is transitive, then $X = J_\beta$. Let $h = h_{\beta+1}$, the canonical Σ_1 skolem function for $J_{\beta+1}$, and let $H = H_{\beta+1}$ be the uniformly $\Sigma_0^{J_{\beta+1}}$ predicate such that

$$y = h(i, x) \quad \text{iff } (\exists z \in J_{\beta+1})\, H(z, y, i, x).$$

For $n < \omega$, define partial functions h_n by

$$y = h_n(i, x) \quad \text{iff } x, y \in S_{\omega\beta+n} \wedge (\exists z \in S_{\omega\beta+n})\, H(z, y, i, x).$$

Since $J_{\beta+1}$ is amenable (and hence closed under Σ_0 subset formation), $h_n \in J_{\beta+1}$. And clearly, the sequence $(h_n \mid n < \omega)$ is $\Sigma_1(J_{\beta+1})$.

Define a sequence of sets $(X_n \mid n < \omega)$ and a sequence of ordinals $(\alpha_n \mid n < \omega)$ as follows.

$$\alpha_0 = 1;$$
$$X_n = h_n^*(J_{\alpha_n} \times \{(p, J_\beta)\});$$
$$\alpha_{n+1} = \sup(X_n \cap \alpha).$$

Let $X = \bigcup_{n<\omega} X_n$, and set $\alpha_\omega = \bigcup_{n<\omega} \alpha_n$. Then clearly, $X = h^*(J_{\alpha_\omega} \times \{(p, J_\beta)\})$ and $X \cap \alpha = \alpha_\omega$.

Let $Y = X \cap J_\beta$. Since $J_\beta \in X$ and $X \prec_1 J_{\beta+1}$, we clearly have $Y \prec J_\beta$. But $p \in Y$ and $Y \cap \alpha = X \cap \alpha = \alpha_\omega$. So by choice of p, $Y = J_\beta$. Thus $\alpha_\omega = Y \cap \alpha = \alpha$. This shows that $\alpha = \bigcup_{n<\omega} \alpha_n$. Since $(\alpha_n \mid n < \omega)$ is easily seen to be $\Sigma_1(J_{\beta+1})$, we shall be done if we can show that $\alpha_n < \alpha$ for all $n < \omega$.

For each $n < \omega$, let j_n be a J_{α_n}-definable map from $\omega\alpha_n$ onto J_{α_n}. For $\nu < \omega\alpha_n$, $i < \omega$, set

$$f_n(\nu, i) = \begin{cases} h_n(i, (j_n(\nu), (p, J_\beta))), & \text{if this is defined and is an element of } \alpha; \\ \text{undefined}, & \text{in all other cases.} \end{cases}$$

Since $h_n \in J_{\beta+1}$ and $J_{\beta+1}$ is closed under Σ_0 subset formation, $f_n \in J_{\beta+1}$. But $f_n \subseteq J_\beta$. So $f_n \in \mathrm{Def}(J_\beta)$, i.e. f_n is J_β-definable. Since α is regular over J_β, it follows that for each $\nu < \omega\alpha_n$, $\sup_{i<\omega} f_n(\nu, i) < \alpha$. Likewise, it then follows that if $\omega\alpha_n < \alpha$, then $\sup_{\nu<\omega\alpha_n} \sup_{i<\omega} f_n(\nu, i) < \alpha$. But clearly, $\sup_{\nu<\omega\alpha_n} \sup_{i<\omega} f_n(\nu, i) = \alpha_{n+1}$. Thus $\omega\alpha_n < \alpha$ implies $\alpha_{n+1} < \alpha$. But α is regular over J_β, so if $\alpha_{n+1} < \alpha$ then $\omega\alpha_{n+1} < \alpha$. Thus by induction on n we obtain $\alpha_n < \alpha$ for all $n < \omega$. The proof is complete. $\square$

By 6.3 and 6.4, E is a class of limit ordinals, each cofinal with ω, such that $E \cap \kappa$ is stationary in κ for every regular $\kappa > \omega$. We complete the proof of 6.1 by showing that $\square(E)$ holds: that is, there is a sequence $(C_\alpha \mid \alpha \in S)$ such that:

(i) C_α is a club subset of α;

(ii) $\mathrm{otp}(C_\alpha) < \alpha$;

(iii) if $\bar{\alpha} < \alpha$ is a limit point of C_α, then $\bar{\alpha} \in S$, $\bar{\alpha} \notin E$, and $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$.

In the definition of C_α there are several cases to consider.

Case 1. $\alpha < \omega_1$.

In this case, let C_α be any ω-sequence cofinal in α. There is nothing to check in this case.

In order to describe the next case we make use of the Gödel Pairing Function, Φ (see II.8.6). Set

Q $$Q = \{\alpha \mid \Phi''(\alpha \times \alpha) \subseteq \alpha\}.$$

By the properties of Φ,

$$Q = \{\alpha \mid (\Phi \restriction \alpha \times \alpha) : \alpha \times \alpha \leftrightarrow \alpha\}.$$

Q is clearly a club class. And it is an elementary exercise to verify that if $\alpha \in Q$, the next element of Q beyond α is α^ω.

Case 2. $\alpha > \omega_1$ and $\alpha \notin Q$.

Let β be the largest element of Q below α. Thus $\beta < \alpha < \beta^\omega$. Hence we can find a unique integer $n > 0$ and unique ordinals $\xi_0, \xi_1, \ldots, \xi_n, \xi_n \neq 0, 0 \leqslant \xi_i < \beta$, such that

$$\alpha = \xi_n \beta^n + \xi_{n-1} \beta^{n-1} + \ldots + \xi_1 \beta + \xi_0.$$

Let m be the least integer such that $\xi_m \neq 0$.

Suppose first that $\xi_m = \zeta_m + 1$. Since $\lim(\alpha)$ we must have $m > 0$. Set

$$C_\alpha = \{(\xi_n \beta^n + \xi_{n-1} \beta^{n-1} + \ldots + \xi_{m+1} \beta^{m+1} + \zeta_m \beta^m + \xi \beta^{m-1}) \mid 1 \leqslant \xi < \beta\}.$$

It is easily seen that C_α is club in α and of order-type $\beta < \alpha$.

Now suppose that $\lim(\xi_m)$. Then set

$$C_\alpha = \{(\xi_n \beta^n + \xi_{n-1} \beta^{n-1} + \ldots + \xi_{m+1} \beta^{m+1} + \xi \beta^m) \mid 1 \leqslant \xi < \xi_m\}.$$

Again C_α is club in α. And C_α has order-type $\xi_m < \beta < \alpha$.

In either case now, if $\bar{\alpha} < \alpha$ is a limit point of C_α, then with β as above we have $\beta < \bar{\alpha} < \beta^\omega$ and $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$. (This is elementary.) Moreover, it is clear that $E \subseteq Q$, so we have $\bar{\alpha} \notin E$.

Case 3. $\alpha > \omega_1$ and $\alpha \in Q$ and $\sup(Q \cap \alpha) < \alpha$.

Let $\beta = \sup(Q \cap \alpha)$. Then α is the successor of β in Q. Hence $\alpha = \beta^\omega$, and we may set

$$C_\alpha = \{\beta^n \mid n < \omega\}.$$

There is nothing to check in this case.

From now on we shall assume that α does not fall under Cases 1–3. Thus, $\alpha > \omega_1$ and α is a limit point of Q. Notice that, in particular, $\omega\alpha = \alpha$. Let

$\beta = \beta(\alpha) =$ the least β such that α is singular over J_β; $\beta(\alpha), \beta$

$n = n(\alpha) =$ the least n such that α is Σ_n-singular over J_β. $n(\alpha), n$

Case 4. $n = 1$ and β is a successor ordinal.

By IV.5.2, $\mathrm{cf}(\alpha) = \omega$, so may let C_α be any ω-sequence cofinal in α. There is nothing to check in this case.

Notice that by 6.4, every element of E falls under Case 1 or Case 4.

Case 5. $n > 1$ or $\lim(\beta)$.

This is the only remaining case, and is by far the most difficult one. To commence, set

$\varrho = \varrho(\alpha) = \varrho_\beta^{n-1}, \quad A = A(\alpha) = A_\beta^{n-1}.$ $\varrho(\alpha), \varrho$ $A(\alpha), A$

Notice that we must have $\lim(\varrho)$ here.

By definition of ϱ_β^{n-1}, there is a $\Sigma_{n-1}(J_\beta)$ map from a subset of $\omega\varrho$ onto β. Hence there is a $\Sigma_{n-1}(J_\beta)$ map from a subset of $\omega\varrho$ onto α. But α is Σ_{n-1}-regular over J_β. Thus $\alpha \leqslant \omega\varrho$. Hence as $\omega\alpha = \alpha$, we have $\alpha \leqslant \varrho$. Again, there is a $\Sigma_n(J_\beta)$ map from a bounded subset of α cofinally into α. Since $\models_{J_\beta}$ "α is regular", this map cannot lie in J_β. Hence $\mathscr{P}(\alpha \times \alpha) \cap \Sigma_n(J_\beta) \nsubseteq J_\beta$. So, utilising Gödel's pairing function on $\alpha \times \alpha$, we see that $\mathscr{P}(\alpha) \cap \Sigma_n(J_\beta) \nsubseteq J_\beta$. Thus $\varrho_\beta^n \leqslant \alpha$. Hence we have proved that

$$\varrho_\beta^n \leqslant \alpha \leqslant \varrho.$$

$p(\alpha), p$ By virtue of the first of the above inequalities, we may define $p = p(\alpha) =$ the $<_J$-least $p \in J_\varrho$ such that every $x \in J_\varrho$ is Σ_1-definable from elements of $\alpha \cup \{p\}$ in $\langle J_\varrho, A\rangle$. (Thus $p \leqslant_J p_\beta^n$.)

h, H Let $h = h_{\varrho, A}$, the canonical Σ_1 skolem function for $\langle J_\varrho, A\rangle$, and let $H = H_{\varrho, A}$ be the uniformly $\Sigma_0^{\langle J_\rho, A\rangle}$ predicate such that

$$y = h(i, x) \leftrightarrow (\exists z \in J_\varrho)\, H(z, y, i, x).$$

6.5 Lemma. *There is a $\gamma < \alpha$ such that $h^*(\gamma \times \{p\}) \cap \alpha$ is unbounded in α.*

Proof. By choice of β there is a $\tau < \alpha$ and a $\Sigma_n(J_\beta)$ function f such that $f''\tau$ is cofinal in α. Since $\alpha \leqslant \varrho$, $f \subseteq J_\varrho$. But $\varrho = \varrho_\beta^{n-1}$, $A = A_\beta^{n-1}$. Thus f is $\Sigma_1(\langle J_\varrho, A\rangle)$. By choice of p, f will in fact be $\Sigma^{\langle J_\rho, A\rangle}(\{v, p\})$ for some $v < \alpha$. Since α is a limit point of Q, we can pick a $\gamma \in Q$ such that $v, \tau < \gamma < \alpha$. We show that $h^*(\gamma \times \{p\}) \cap \alpha$ is unbounded in α. It suffices to show that $f''\tau \subseteq h^*(\gamma \times \{p\})$.

Let $X = h^*(\gamma \times \{p\})$. We show that X is closed under the formation of orderdered pairs. Let $x_0, x_1 \in X$, say $x_k = h(i_k, (\xi_k, p))$. Let $\xi = \Phi(\xi_0, \xi_1)$. Since $\gamma \in Q$, $\xi < \gamma$. Moreover, by the nature of Φ, ξ_0 and ξ_1 are Σ_1-definable from ξ in J_ϱ. Hence (x_0, x_1) is Σ_1-definable from ξ, p in $\langle J_\varrho, A\rangle$. So for some $i \in \omega$,

$$(x_0, x_1) = h(i, (\xi, p)) \in X.$$

Since X is closed under ordered pairs, 3.3 tells us that $X \prec_1 \langle J_\varrho, A\rangle$. But $\gamma \cup \{p\} \subseteq X$ and $\tau \subseteq \gamma$. So as f is $\Sigma_1^{\langle J_\rho, A\rangle}(\gamma \cup \{p\})$, we have $f''\tau \subseteq X$, as required. $\square$

h_τ For $\tau < \varrho$, we shall write h_τ for $h_{\tau, A \cap J_\tau}$. Thus:

$$y = h_\tau(i, x) \quad \text{iff } (x, y \in J_\tau) \wedge (\exists z \in J_\tau)\, H(z, y, i, x).$$

$g^{(\alpha)}, g$ Define a map $g = g^{(\alpha)}$ from a subset of α onto J_ϱ by

$$g(\omega v + i) \simeq h(i, (v, p)).$$

G Thus g is $\Sigma_1^{\langle J_\rho, A\rangle}(\{p\})$. Let G be the canonical $\Sigma_0^{\langle J_\rho, A\rangle}(\{p\})$ predicate such that

$$g(v) = x \quad \text{iff } (\exists z \in J_\varrho)\, G(z, x, v).$$

Let γ be the smallest ordinal such that $\alpha \cap g''\gamma$ is unbounded in α. By 6.5, $\gamma < \alpha$. And it is clear that γ must be a limit ordinal. For $\gamma \leqslant \tau < \alpha$ we have $\bigcup(\alpha \cap g''\tau) = \alpha > \tau$. Hence there is a maximal $\kappa = \kappa^{(\alpha)} < \alpha$ such that $\bigcup(\alpha \cap g''\kappa) \leqslant \kappa$, and moreover $\kappa < \gamma$. We fix γ, κ for the rest of the proof. Note that $\bigcup(\alpha \cap g''\tau) > \tau$ whenever $\kappa < \tau < \gamma$. γ $\kappa^{(\alpha)}, \kappa$

6.6 Lemma. *If* $(\kappa, p) \in X \prec_1 \langle J_\varrho, A\rangle$ *and* $X \cap \alpha$ *is transitive, then* $X \cap \alpha = \alpha$.

Proof. Let X be as above, and set $\bar{\alpha} = X \cap \alpha$. Since $\kappa \in X$, $\bar{\alpha} > \kappa$. Thus if it were the case that $\bar{\alpha} < \alpha$, we should have $\sup(\alpha \cap g''\bar{\alpha}) > \bar{\alpha}$. So for some $\nu < \bar{\alpha}$, $\bar{\alpha} < g(\nu) < \alpha$. But $g(\nu) = h(i, (\tau, p))$, where $\nu = \omega\tau + i$, so as $p \in X$ and $\tau \leqslant \nu \in \bar{\alpha} \subseteq X$ and $X \prec_1 \langle J_\varrho, A\rangle$, we have $g(\nu) \in X$. Then $g(\nu) \in \bar{\alpha}$, a contradiction. Hence $\bar{\alpha} = \alpha$. □

We define, by recursion, functions $k\colon \theta \to \gamma$, $m\colon \theta \to \varrho$, and sequences $(X_\nu \mid \nu < \theta)$, $(\alpha_\nu \mid \nu < \theta)$, for some $\theta \leqslant \gamma$, as follows. (The exact order in which the definition proceeds is described after we have stated all of the clauses.)

$k(\nu) =$ the least $\tau \in \mathrm{dom}(g) - \kappa$ such that: $k(\nu)$

(i) $\tau \geqslant \bigcup(k''\nu)$;

(ii) $g(\tau) \in \alpha$ and $g(\tau) > \alpha_\nu$;

(iii) $m(\nu) \in h^*(g(\tau) \times \{p\})$.

$m(0) =$ the least $\eta > \kappa$ such that $p \in J_\eta$;

$m(\nu + 1) =$ the least $\eta > m(\nu)$ such that: $m(\nu)$

(i) $\eta > k(\nu), g \circ k(\nu)$;

(ii) $A \cap J_{m(\nu)} \in J_\eta$;

(iii) $m(\nu) \in h^*(g \circ k(\nu) \times \{p\})$;

(iv) $(\exists z \in J_\eta)\, G(z, g \circ k(\nu), k(\nu))$;

$m(\lambda) = \sup_{\nu<\lambda} m(\nu)$, if $\lim(\lambda)$ and $\sup_{\nu<\lambda} m(\nu) < \varrho$

(otherwise undefined).

$X_\nu = h^*_{m(\nu)}(J_\eta \times \{p\})$, where $\eta = \max(\bigcup[k''\nu], \bigcup[g \circ k''\nu])$. X_ν

$\alpha_\nu = \sup(X_\nu \cap \alpha)$. α_ν

We stop the construction when an ordinal θ is reached such that $k''\theta$ is cofinal in γ, unless the construction breaks down earlier. (We shall prove that this is not the case.) θ

Let us see how the construction proceeds. The definition of $m(0)$ is unproblematical. Now suppose that $m(\nu)$ is defined. This presupposes that we have not yet reached θ, so $\bigcup(k''\nu) < \gamma < \alpha$. Since $\bigcup(k''\nu) < \gamma$, the choice of γ implies that $\alpha \cap g''(\bigcup k''\nu)$ is bounded in α, so $\alpha \cap g \circ k''\nu$ will be bounded in α (because $g \circ k''\nu \subseteq g''(\bigcup k''\nu)$). Hence the η in the definition of X_ν satisfies $\eta < \alpha$. There is no difficulty in defining X_ν and α_ν of course. Since $m(\nu) < \varrho$ and $\langle J_\varrho, A\rangle$ is amenable, we have $h_{m(\nu)} \in J_\varrho \subseteq J_\beta$. So as α is a regular cardinal inside J_β and $\eta < \alpha$, we have $\alpha_\nu < \alpha$. By the choice of p, $h^*(\alpha \times \{p\}) = J_\varrho$, so there is now no problem in defining $k(\nu)$. Then we define $m(\nu + 1)$. This causes no difficulty as far as clauses (i), (ii) and (iv) are concerned, but what about clause (iii)? Well, by definition of $k(\nu)$ we have $m(\nu) \in h^*(g \circ k(\nu) \times \{p\})$. So as $\lim(\varrho)$ there is an $\eta < \varrho$ such that $m(\nu) \in h^*_\eta(g \circ k(\nu) \times \{p\})$. Thus we can easily satisfy clause (iii) as well.

Now suppose that λ is a limit ordinal and that $k \restriction \lambda$, $m \restriction \lambda$, $(X_\nu \mid \nu < \lambda)$, $(\alpha_\nu \mid \nu < \lambda)$ are defined and that $\sup k''\lambda < \gamma$. Then by choice of γ, $\eta = \sup(g \circ k''\lambda) < \alpha$. Suppose it were not possible to define $m(\lambda)$. Thus it must be the case that $\sup_{\nu<\lambda} m(\nu) = \varrho$. Let $X = \bigcup_{\nu<\lambda} X_\nu$. Clearly, in this case, $X = h^*(J_\eta \times \{p\})$ and $X \cap \alpha = \sup_{\nu<\lambda}\alpha_\nu$. Now for all $\nu < \lambda$, by the definition of k we have $g \circ k(\nu) > \alpha_\nu$, and by the definition of $m(\nu + 1)$ (clause (iv)), $g \circ k(\nu) \in X_{\nu+1}$, so $g \circ k(\nu) < \alpha_{\nu+1}$. Thus $\alpha_\nu < g \circ k(\nu) < \alpha_{\nu+1}$ for all $\nu < \lambda$. Hence $X \cap \alpha = \sup_{\nu<\lambda} g \circ k(\nu) = \eta < \alpha$. But $(\kappa, p) \in X_0 \subseteq X \prec_1 \langle J_\varrho, A\rangle$, so this contradicts 6.6. Hence $m(\lambda)$ can be defined. We may now define $X_\lambda, \alpha_\lambda, k(\lambda)$ without trouble, just as before.

Thus the construction proceeds until an ordinal θ is reached for which $\sup k''\theta = \gamma$. Clearly, θ must be a limit ordinal. Since k is monotone increasing from θ into γ, we have $\theta \leqslant \gamma$. Note also that, as we observed above, $\alpha_\nu < g \circ k(\nu) < \alpha_{\nu+1}$ for all $\nu < \theta$.

6.7 Lemma.

(i) $\sup_{\nu<\theta}\alpha_\nu = \alpha$.

(ii) $\sup_{\nu<\theta} m(\nu) = \varrho$.

(iii) $\bigcup_{\nu<\theta} X_\nu = J_\varrho$.

Proof. (i) By our last observation above,

$$\alpha_\nu < g \circ k(\nu) < \alpha_{\nu+1}$$

for all $\nu < \theta$. Hence

$$(*) \qquad \sup_{\nu<\theta}\alpha_\nu = \sup_{\nu<\theta} g \circ k(\nu).$$

Suppose now that (i) were false, and that

$$\eta = \sup_{\nu<\theta}\alpha_\nu = \sup_{\nu<\theta} g \circ k(\nu) < \alpha.$$

By choice of γ, $\alpha \cap g''\gamma$ is unbounded in α. So let $\tau_0 \in \mathrm{dom}(g)$ be least such that $\kappa, \eta < g(\tau_0) < \alpha$. By definition of κ, $\tau_0 \in \mathrm{dom}(g) - \kappa$. As τ_0 is minimal, by the choice of γ we must have $\tau_0 < \gamma$. So there is a least $\nu < \theta$ such that $k(\nu) > \tau_0$. Consider the definition of $k(\nu)$: namely, the least $\tau \in \mathrm{dom}(g) - \kappa$ such that $\tau \geqslant \bigcup(k''\nu)$, $\alpha_\nu < g(\tau) < \alpha$, and $m(\nu) \in h^*(g(\tau) \times \{p\})$. Now look at τ_0. We have already observed that $\tau_0 \in \mathrm{dom}(g) - \kappa$. By the minimality of ν, we have $k''\nu \subseteq \tau_0$, so $\tau_0 \geqslant \bigcup(k''\nu)$. By the choice of τ_0, we have $\alpha_\nu < \eta < g(\tau_0) < \alpha$. Finally, since $g \circ k(\nu) < \eta < g(\tau_0)$, we have (by the definition of $k(\nu)$) $m(\nu) \in h^*(g(\tau_0) \times \{p\})$. Thus τ_0 is a candidate in the choice of $k(\nu)$. Hence $k(\nu) \leqslant \tau_0$. But we chose ν so that $k(\nu) > \tau_0$. This contradiction proves (i).

(ii) Let $\bar{\varrho} = \sup_{\nu<\theta} m(\nu)$. Then for all $\nu < \theta$ we can find a $z \in J_{\bar{\varrho}}$ such that $G(z, g \circ k(\nu), k(\nu))$. Thus as $\sup_{\nu<\theta} g \circ k(\nu) = \sup_{\nu<\theta}\alpha_\nu = \alpha$ (by (i) and (*)), if we define f from a subset of γ into α by the $\Sigma_1(\langle J_{\bar{\varrho}}, A \cap J_{\bar{\varrho}}\rangle)$ definition

$$\zeta = f(\xi) \leftrightarrow (\exists z \in J_{\bar{\varrho}})\, G(z, \zeta, \xi),$$

then $f''\gamma$ is unbounded in α. But if $\bar{\varrho} < \varrho$, then as $\langle J_\varrho, A\rangle$ is amenable, $f \in J_\varrho \subseteq J_\beta$, so α is not regular inside J_β. Contradiction! Hence $\bar{\varrho} = \varrho$.

(iii) By (i), (ii) and (*), we have

$$\bigcup_{\nu<\theta} X_\nu = h^*(J_\alpha \times \{p\}).$$

So by choice of p,

$$\bigcup_{\nu<\theta} X_\nu = J_\varrho. \quad \square$$

For each $\tau < \theta$, define a map g_τ from a subset of α_τ into $J_{m(\tau)}$ by g_τ

$$g_\tau(\xi) = x \leftrightarrow (\exists z \in J_{m(\tau)})\, G(z, x, \xi).$$

By definition of m, if $\lim(\tau)$, then $\langle J_{m(\tau)}, A \cap J_{m(\tau)}\rangle$ is amenable, and in this case g_τ is $\Sigma^{\langle J_{m(\tau)}, A \cap J_{m(\tau)}\rangle}(\{p\})$.

We define κ_τ from g_τ in the same way that κ was defined from g: that is, we let κ_τ be the largest $\kappa_\tau \leqslant \alpha_\tau$ such that $\bigcup(\alpha_\tau \cap g''_\tau \kappa_\tau) \leqslant \kappa_\tau$. κ_τ

6.8 Lemma. *For sufficiently large ordinals $\tau < \theta$, $\kappa_\tau = \kappa$.*

Proof. Clearly, if $\nu < \tau < \theta$, then $g_\nu \subseteq g_\tau$. Moreover, $\bigcup_{\tau<\theta} g_\tau = g$. Thus for any $\tau < \theta$, $\bigcup(\alpha_\tau \cap g''_\tau \kappa) \leqslant \bigcup(\alpha \cap g''\kappa) \leqslant \kappa$. Thus $\kappa_\tau \geqslant \kappa$. Similarly, $\nu < \tau < \theta$ implies that $\kappa_\tau \leqslant \kappa_\nu$. So for some $\nu < \theta$ we must have $\kappa_\tau = \kappa_\nu \geqslant \kappa$ for all $\tau > \nu$. Suppose that $\kappa_\nu > \kappa$. Then $\bigcup(\alpha \cap g''\kappa_\nu) > \kappa_\nu$. So for some $\tau < \theta$, $\bigcup(\alpha \cap g''_\tau \kappa_\nu) > \kappa_\nu$. But we may assume that $\tau > \nu$ and that, in fact, $\bigcup(\alpha_\tau \cap g''\kappa_\nu) > \kappa_\nu$. Then $\kappa_\tau = \kappa_\nu$ and so $\bigcup(\alpha_\tau \cap g''_\tau \kappa_\tau) > \kappa_\tau$. Contradiction! That proves the lemma. $\square$

By recursion, we define a strictly increasing, continuous function $t\colon \tilde{\theta} \to \theta$, for some $\tilde{\theta} \leqslant \theta$. First of all we let $t(0)$ be the least ν such that $(\nu \leqslant \tau < \theta) \to (\kappa_\tau = \kappa)$ and $\alpha_\nu > \omega_1$. t

In case $n = 1$, when $t(\iota)$ is defined we let $t(\iota + 1)$ be the least $\nu < t(\iota)$ such that $\Phi''(\alpha_{t(\iota)} \times \alpha_{\tau(\iota)}) \subseteq \alpha_\nu$. Since α is a limit point of Q, $t(\iota + 1) < \theta$ is always defined.

In case $n > 1$ and $t(\iota)$ is defined, we let $t(\iota + 1)$ be the least $\nu > t(\iota)$ such that $\Phi''(\alpha_{t(\iota)} \times \alpha_{t(\iota)}) \subseteq \alpha_\nu$ and

$$J_\alpha \cap h^*_{\varrho^{n-2}_\beta, A^{n-2}_\beta}(X_{t(\iota)} \times \{p^{n+1}_\beta\}) \subseteq X_\nu.$$

We must check that $t(\iota + 1) < \theta$ is well-defined.

Let

$$Y = J_\alpha \cap h^*_{\varrho^{n-2}_\beta, A^{n-2}_\beta}(X_{t(\iota)} \times \{p^{n-1}_\beta\}).$$

We must show that $Y \subseteq X_\xi$ for some $\xi < \theta$. Since $Y \subseteq J_\alpha$, it suffices to show that $Y \subseteq J_\tau$ for some $\tau < \alpha$; for if $\tau < \alpha$, then $\alpha_\xi > \tau$ for some $\xi < \theta$, and we have $g \circ k(\xi) > \alpha_\xi$, so by definition, $J_\tau \subseteq X_{\xi+1}$. Now, for some $\eta < \alpha$, we have $X_{t(\iota)} = h^*_{m(t(\iota))}(J_\eta \times \{p\})$. Since $\langle J_\varrho, A\rangle$ is amenable, $h_{m(t(\iota))} \in J_\varrho \subseteq J_\beta$. Thus J_β contains a function mapping $\omega\eta$ onto $\omega \times (X_{t(\iota)} \times \{p^{n-1}_\beta\})$. Again, by the definition of Y, Y is the image of a $\Sigma_1(\langle J_{\varrho^{n-2}_\beta}, A^{n-2}_\beta\rangle)$ function defined on a subset of $\omega \times (X_{t(\iota)} \times \{p^{n-1}_\beta\})$. By the properties of the standard code A^{n-2}_β, this function is $\Sigma_{n-1}(J_\beta)$. Combining these two functions gives us a $\Sigma_{n-1}(J_\beta)$ function f such that $f''\omega\eta = Y$. Since f is $\Sigma_{n-1}(J_\beta)$, so too is $\bar{f}\colon \omega\eta \to \alpha$, defined by letting $\bar{f}(\nu)$ be the least τ such that $f(\nu) \in J_\tau$. Since α is Σ_{n-1}-regular over J_β, $\bar{f}''\omega\eta \subseteq \tau$ for some $\tau < \alpha$. Then $Y \subseteq J_\tau$, as required.

Finally, if $\lim(\lambda)$ and $t \restriction \lambda$ is defined, we let $t(\lambda) = \sup_{\iota<\lambda} t(\iota)$, if this is less than θ, with $t(\lambda)$ undefined otherwise.

$\tilde{\theta}$ Thus for some limit ordinal $\tilde{\theta} \leqslant \theta$ we shall have $\sup_{\iota<\tilde{\theta}} t(\iota) = \theta$, at which point the definition of t is complete.

We define

C_α $$C_\alpha = \{\alpha_{t(\nu)} \mid \nu < \tilde{\theta}\}.$$

Thus C_α is a club subset of α of order-type $\tilde{\theta} \leqslant \theta \leqslant \gamma < \alpha$. To complete the proof
$\bar{\alpha}$ of $\square$ we must show that if $\bar{\alpha} < \alpha$ is a limit point of C_α, then $\bar{\alpha} \in S$, $\bar{\alpha} \notin E$, and
λ $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$. Let $\bar{\alpha} = \alpha_\lambda$, where $\lim(\lambda)$.

6.9 Lemma. *$\bar{\alpha} > \omega_1$ and $\bar{\alpha} \in Q$. Moreover, if $n > 1$ and f is a $\Sigma_1^{\langle J_{\varrho^{n-2}_\beta}, A^{n-2}_\beta\rangle}(X_\lambda \cup \{p^{n-1}_\beta\})$ function from a bounded subset of $\bar{\alpha}$ into $\bar{\alpha}$, then f is bounded in $\bar{\alpha}$.*

Proof. That $\bar{\alpha} > \omega_1$ and $\bar{\alpha} \in Q$ is an immediate consequence of the definition of t. Now let $n > 1$, and let f be as above. Since the function m is continuous, so too is the sequence $(X_\nu \mid \nu < \theta)$. Thus $X_\lambda = \bigcup_{\nu<\lambda} X_\nu$, and the finitely many parameters in the definition of f will all lie in X_ν for some $\nu < \lambda$. We may choose ν here so that $\mathrm{dom}(f) \subseteq \alpha_\nu$. Let $\iota < \tilde{\theta}$ be least such that $t(\iota) > \nu$. Since $\bar{\alpha} = \alpha_\lambda$ is a limit point of C_α, λ is a limit point of t and so $t(\iota)$, $t(\iota+1) < \lambda$. But f is $\Sigma_1^{\langle J_{\varrho^{n-2}_\beta}, A^{n-2}_\beta\rangle}(X_{t(\iota)} \times \{p^{n-1}_\beta\})$ and $\mathrm{dom}(f) \subseteq \alpha_{t(\iota)} \subseteq X_{t(\iota)}$. So by definition of $t(\iota+1)$, $\mathrm{ran}(f) \subseteq X_{t(\iota+1)}$. Thus $\mathrm{ran}(f) \subseteq \alpha_{t(\iota+1)} < \alpha_\lambda = \bar{\alpha}$. $\square$

Let

$\pi, \bar{\varrho}, \bar{A}$ $$\pi\colon \langle J_{\bar{\varrho}}, \bar{A}\rangle \cong \langle X_\lambda, A \cap X_\lambda\rangle.$$

Thus

$$\pi\colon \langle J_{\bar{\varrho}}, \bar{A}\rangle \prec_1 \langle J_{m(\lambda)}, A \cap J_{m(\lambda)}\rangle.$$

But by Σ_0-absoluteness,

$$\langle J_{m(\lambda)}, A \cap J_{m(\lambda)}\rangle \prec_0 \langle J_\varrho, A\rangle.$$

Thus

$$\pi\colon \langle J_{\bar{\varrho}}, \bar{A}\rangle \prec_0 \langle J_\varrho, A\rangle.$$

So by 5.6 there are unique $\bar{\beta}$, $\tilde{\pi}$ such that $\bar{\varrho} = \varrho_{\bar{\beta}}^{n-1}$, $A = A_{\bar{\beta}}^{n-1}$, $\tilde{\pi}\colon J_{\bar{\beta}} \prec_{n-1} J_{\beta}$, $\pi \subseteq \tilde{\pi}$, $\tilde{\pi}(p_{\bar{\beta}}^{n-1}) = p_{\beta}^{n-1}$. Note that by definition of k, $g \circ k(v) > \alpha_v$ for all $v < \lambda$, so by definition of X_{v+1}, $\alpha_v \subseteq X_{v+1}$ for all $v < \lambda$. Thus $\bar{\alpha} \subseteq X_\lambda$ and in fact $\bar{\alpha} = X_\lambda \cap \alpha$. So we have $\pi \upharpoonright \bar{\alpha} = \mathrm{id} \upharpoonright \bar{\alpha}$, and in case $\bar{\alpha} < \bar{\beta}$, $\tilde{\pi}(\bar{\alpha}) \geqslant \alpha$. $\bar{\beta}, \tilde{\pi}$

Let $\bar{h} = h_{\bar{\varrho}, \bar{A}}$, $\bar{H} = H_{\bar{\varrho}, \bar{A}}$. Set $\bar{p} = \pi^{-1}(p)$. $\bar{h}, \bar{H}, \bar{p}$

6.10 Lemma. *$\bar{p}$ = the $<_J$-least element of $J_{\bar{\varrho}}$ such that every $x \in J_{\bar{\varrho}}$ is Σ_1-definable from parameters in $\bar{\alpha} \cup \{\bar{p}\}$ in $\langle J_{\bar{\varrho}}, \bar{A}\rangle$.*

Proof. By definition,

$$X_\lambda = h^*_{m(\lambda)}(J_\eta \times \{p\}),$$

where $\eta = \max(\kappa + 1, \sup[g \circ k'' \lambda])$. But $\alpha_v < g \circ k(v) < \alpha_{v+1}$ for all $v < \theta$. So as $\bar{\alpha} = \alpha_\lambda$ and $\lim(\lambda)$, $\eta = \bar{\alpha}$. Thus

$$X_\lambda = h^*_{m(h)}(J_{\bar{\alpha}} \times \{p\}).$$

Applying π^{-1}, we get

$$J_{\bar{\varrho}} = \bar{h}^*(J_{\bar{\alpha}} \times \{\bar{p}\}).$$

But by definition of t, we have $\bar{\alpha} \in Q$, so by 3.19 there is a $\Sigma_1^{J_\alpha}$ map from $\bar{\alpha}$ onto $J_{\bar{\alpha}}$. Hence

$$J_{\bar{\alpha}} = h^*_{\bar{\alpha}, \emptyset}(\bar{\alpha}) \subseteq \bar{h}^*(\bar{\alpha}).$$

Thus

$$J_{\bar{\varrho}} = \bar{h}^*(\bar{\alpha} \times \{\bar{p}\}).$$

This shows that every element of $J_{\bar{\varrho}}$ is Σ_1-definable from members of $\bar{\alpha} \cup \{\bar{p}\}$ in $\langle J_{\bar{\varrho}}, \bar{A}\rangle$. We must now show that $\bar{p}$ is the $<_J$-least such member of $J_{\bar{\varrho}}$. Suppose, on the contrary, that $\bar{p}' <_J \bar{p}$ also has this property. Then, in particular, for some $i < \omega$ and some $v < \bar{\alpha}$, we have $\bar{p} = \bar{h}(i, (v, \bar{p}'))$. Applying $\pi\colon \langle J_{\bar{\varrho}}, \bar{A}\rangle \prec_1 \langle J_{m(\lambda)}, A \cap J_{m(\lambda)}\rangle$, we get $p = h_{m(\lambda)}(i,(v, p'))$, where $p' = \pi(\bar{p}')$. Thus $p = h(i, (v, p'))$. Hence by choice of p, every element of J_ϱ will be Σ_1-definable from parameters in $\alpha \cup \{p'\}$ in $\langle J_\varrho, A\rangle$. But $\bar{p}' <_J \bar{p}$, so $p' <_J p$, and so we have contradicted the choice of p. □

Now define $\bar{g}$ from $\bar{h}, \bar{\alpha}, \bar{p}$ just as g was defined from h, α, p. Thus, we define $\bar{g}$ from a subset of $\bar{\alpha}$ into $J_{\bar{\varrho}}$ by $\bar{g}$

$$\bar{g}(\omega v + i) \simeq \bar{h}(i, (v, \bar{p})).$$

Let $\bar{G}$ be the canonical $\Sigma^{\langle J_{\bar{\rho}}, A\rangle}(\{\bar{p}\})$ predicate such that $\bar{G}$

$$\bar{g}(v) = x \quad \text{iff } (\exists z \in J_{\bar{\varrho}})\, \bar{G}(z, x, v).$$

Note that the Σ_0 formula which defines $\bar{G}$ from $\bar{p}$ in $\langle J_{\bar{\varrho}}, \bar{A}\rangle$ will be the same as that which defines G from p in $\langle J_\varrho, A\rangle$. But

$$\pi\colon \langle J_{\bar{\varrho}}, \bar{A}\rangle \prec_1 < J_{m(\lambda)}, A \cap J_{m(\lambda)}\rangle,$$

$\pi \upharpoonright \bar{\alpha} = \mathrm{id} \upharpoonright \bar{\alpha}$, and $\pi(\bar{p}) = p$. Thus for $\nu, \tau \in \bar{\alpha}$,

$$\begin{aligned} \bar{g}(\nu) = \tau \quad & \text{iff } (\exists z \in J_{\bar{\varrho}})\, \bar{G}(z, \tau, \nu) \\ & \text{iff } (\exists z \in J_{m(\lambda)})\, G(z, \tau, \nu) \\ & \text{iff } g_\lambda(\nu) = \tau. \end{aligned}$$

Hence

(1) $\qquad \bar{g} \cap (\bar{\alpha} \times \bar{\alpha}) = g_\lambda \cap (\bar{\alpha} \times \bar{\alpha}).$

$\bar{\kappa}$ Next we define $\bar{\kappa}$ from $\bar{g}$, $\bar{\alpha}$ just as κ was defined from g, α. That is, let $\bar{\kappa}$ be the largest $\bar{\kappa} \leqslant \bar{\alpha}$ such that $\bigcup(\bar{\alpha} \cap \bar{g}''\bar{\kappa}) \leqslant \bar{\kappa}$. By (1) and the fact that $\bar{\alpha} = \alpha_\lambda$, this is the same as the definition of κ_λ, so $\bar{\kappa} = \kappa_\lambda$. But by the definition of $t(0)$, $\kappa_\lambda = \kappa$. Thus $\bar{\kappa} = \kappa$.

η Let $\eta = \bigcup k''\lambda$. By definition of $X_{\nu+1}$, we have $k(\nu) \in X_{\nu+1}$, so $k(\nu) < \alpha_{\nu+1}$ for all $\nu < \lambda$. Thus $\eta \leqslant \bar{\alpha}$.

Since $\alpha_\nu < g \circ k(\nu) < \alpha_{\nu+1}$ for all $\nu < \lambda$, we have

(2) $\qquad \bar{\alpha} = \bigcup_{\nu < \lambda} g \circ k(\nu).$

Now by clause (iv) in the definition of $m(\nu + 1)$, $g_\lambda \upharpoonright k''\lambda = g \upharpoonright k''\lambda$. Thus by (2), we have

(3) $\qquad \bar{\alpha} = \bigcup_{\nu < \lambda} g_\lambda \circ k(\nu).$

Since k is monotone increasing, we have $k''\nu \subseteq k(\nu)$ for all $\nu < \lambda$. Thus $g''_\lambda(k''\nu) \subseteq g''_\lambda k(\nu)$ for all $\nu < \lambda$, i.e. $g_\lambda \circ k''\nu \subseteq g''_\lambda k(\nu)$ for all $\nu < \lambda$. So from (3) we have

(4) $\qquad \bar{\alpha} = \bigcup_{\nu < \lambda} (\bar{\alpha} \cap g''_\lambda k(\nu)).$

This is the same as

(5) $\qquad \bar{\alpha} = \bigcup(\bar{\alpha} \cap g''_\lambda \eta).$

So by (1) and (5) (noting that $\eta \leqslant \bar{\alpha}$) we have

(6) $\qquad \bar{\alpha} = \bigcup(\bar{\alpha} \cap \bar{g}''\eta).$

Now by definition of k we have $k(0) > \kappa$, so $\eta = \bigcup k''\lambda > \kappa$. So as $\bar{\kappa} = \kappa$ we have $\kappa < \eta \leqslant \bar{\alpha}$. So by choice of $\bar{\kappa}$ we have $\bigcup(\bar{\alpha} \cap \bar{g}''\eta) > \eta$. Thus by (6) we have $\bar{\alpha} > \eta$. But (6) also tells us that $\bar{g}$ maps a subset of η cofinally into $\bar{\alpha}$. Thus, in particular, $\bar{\alpha} \in S$.

6.11 Lemma. $\bar{\beta} = \beta(\bar{\alpha})$.

Proof. By definition, $\bar{g}$ is $\Sigma_1(\langle J_{\bar{\varrho}}, \bar{A}\rangle)$. So $\bar{g} \cap (\bar{\alpha} \times \bar{\alpha})$ is $\Sigma_1(\langle J_{\bar{\varrho}}, \bar{A}\rangle)$. But $\bar{\varrho} = \varrho_{\bar{\beta}}^{n-1}$, $\bar{A} = A_{\bar{\beta}}^{n-1}$. Thus $\bar{g} \cap (\bar{\alpha} \times \bar{\alpha})$ is $\Sigma_n(J_{\bar{\beta}})$. By (6) above, $\bar{g} \cap (\bar{\alpha} \times \bar{\alpha})$ maps a subset of $\eta < \bar{\alpha}$ cofinally into $\bar{\alpha}$. Hence $\bar{\alpha}$ is Σ_n-singular over $J_{\bar{\beta}}$. Thus $\beta(\bar{\alpha}) \leqslant \bar{\beta}$.

Suppose that $\beta(\bar{\alpha}) < \bar{\beta}$. Then there is an $f \in J_{\bar{\beta}}$ and a $\delta < \bar{\alpha}$ such that f maps δ cofinally into $\bar{\alpha}$. Now, $\tilde{\pi} \upharpoonright \bar{\alpha} = \mathrm{id} \upharpoonright \bar{\alpha}$, so we have $\tilde{\pi}(\delta) = \delta$ and $f \subseteq \tilde{\pi}(f)$. But $\models_{J_\beta}$"$\mathrm{dom}(f) = \delta$", so applying $\tilde{\pi}: J_{\bar{\beta}} \prec_{n-1} J_\beta$ we have $\models_{J_\beta}$"$\mathrm{dom}(\tilde{\pi}(f)) = \delta$". Thus we must have $\tilde{\pi}(f) = f$. But $\models_{J_{\bar{\beta}}}$"$\bigcup f''\delta = \bar{\alpha}$", so applying $\tilde{\pi}$, $\models_{J_\beta}$"$\bigcup f''\delta = \tilde{\pi}(\bar{\alpha})$". Since $\tilde{\pi}(\bar{\alpha}) \geqslant \alpha > \bar{\alpha}$, this is impossible. Hence $\beta(\bar{\alpha}) = \bar{\beta}$. □

6.12 Lemma. $n = n(\bar{\alpha})$.

Proof. By the properties of $\bar{g} \cap (\bar{\alpha} \times \bar{\alpha})$ mentioned above we have $n(\bar{\alpha}) \leqslant n$. So if $n = 1$ we are done. Assume that $n > 1$.

Let $\bar{f}$ be a $\Sigma_{n-1}(J_{\bar{\beta}})$ function from a bounded subset of $\bar{\alpha}$ into $\bar{\alpha}$. We shall show $\bar{f}$
that $\bar{f}''\bar{\alpha}$ is bounded in $\bar{\alpha}$, thereby proving that $n(\bar{\alpha}) = n$. Let $u = \mathrm{dom}(\bar{f})$. Let u
$\pi = \tilde{\pi} \upharpoonright J_{\varrho_{\bar{\beta}}^{n-2}}$. By 5.6 we know that

$$\pi: \langle J_{\varrho_{\bar{\beta}}^{n-2}}, A_{\bar{\beta}}^{n-2}\rangle \prec_1 \langle J_{\varrho_\beta^{n-2}}, A_\beta^{n-2}\rangle$$

and

$$\pi(p_{\bar{\beta}}^{n-1}) = p_\beta^{n-1}.$$

Since $\bar{\varrho} = \varrho_{\bar{\beta}}^{n-1}$, we can find an $x \in J_{\bar{\varrho}}$ such that $\bar{f}$ is $\Sigma_1^{\langle J_{\varrho_{\bar{\beta}}^{n-2}}, A_{\bar{\beta}}^{n-2}\rangle}(\{x, p_{\bar{\beta}}^{n-1}\})$. Let x
f be defined over $\langle J_{\varrho_\beta^{n-2}}, A_\beta^{n-2}\rangle$ by means of the same Σ_1 definition in parameters f
$\pi(x)$, p_β^{n-1}.

Since $\bar{f} \subseteq \bar{\alpha} \times \bar{\alpha}$ and $\pi \upharpoonright \bar{\alpha} = \mathrm{id} \upharpoonright \bar{\alpha}$, we have $\bar{f} \subseteq f$. Again, u is a $\Sigma_{n-1}(J_{\bar{\beta}})$ subset of $\bar{\alpha} \leqslant \bar{\varrho} = \varrho_{\bar{\beta}}^{n-1}$, so by 4.6, $\langle J_{\bar{\varrho}}, u\rangle$ is amenable. But u is bounded in $\bar{\alpha}$. Hence $u \in J_{\bar{\varrho}}$. Thus $\pi(u)$ is defined. Since u is a bounded subset of $\bar{\alpha}$ and $\pi \upharpoonright \bar{\alpha} = \mathrm{id} \upharpoonright \bar{\alpha}$, we have $\pi(u) = u$. But the statements

"$\bar{f}$ is a function" and "$\mathrm{dom}(\bar{f}) \subseteq u$"

are $\Pi_1^{\langle J_{\varrho_{\bar{\beta}}^{n-2}}, A_{\bar{\beta}}^{n-2}\rangle}(\{x, p_{\bar{\beta}}^{n-1}, u\})$. Hence as π is Σ_1-elementary, f is a function and $\mathrm{dom}(f) \subseteq u$. Thus $f = \bar{f}$.

This shows that $\bar{f}$ is $\Sigma_1^{\langle J_{\varrho_\beta^{n-2}}, A_\beta^{n-2}\rangle}(\{\pi(x), p_\beta^{n-1}\})$. But $\pi(x) \in X_\lambda$. So by 6.9, f is bounded in $\bar{\alpha}$, and we are done. □

6.13 Lemma. $\bar{\varrho} = \varrho(\bar{\alpha})$ *and* $\bar{A} = A(\bar{\alpha})$.

Proof. Directly from 6.11 and 6.12. □

6.14 Lemma. $\bar{p} = p(\bar{\alpha})$.

Proof. Directly from 6.13 and 6.10. □

6.15 Lemma. $\bar{g} \cap (\bar{\alpha} \times \bar{\alpha}) = g_\lambda \cap (\bar{\alpha} \times \bar{\alpha}) = g^{(\bar{\alpha})} \cap (\bar{\alpha} \times \bar{\alpha})$ *and* $\kappa^{(\bar{\alpha})} = \kappa^{(\alpha)} = \kappa$.

Proof. By our previous results. □

6.16 Lemma. $\bar{\alpha}$ falls under Case 5 in the definition of $C_{\bar{\alpha}}$.

Proof. Since $\bar{\alpha} > \omega_1$, $\bar{\alpha}$ does not fall under Case 1. Since $\bar{\alpha} \in Q$, $\bar{\alpha}$ does not fall under Case 2. Since $\bar{\alpha}$ is a limit point of Q (by definition of the function t) $\bar{\alpha}$ does not fall under Case 3. If $n > 1$, then by 6.12, $\bar{\alpha}$ does not fall under Case 4. And if $n = 1$, then $\bar{\beta} = \bar{\varrho}$, so as $\pi: J_{\bar{\varrho}} \prec_1 J_{m(\lambda)}$ and $\lim(\lambda)$, $\bar{\beta}$ is a limit ordinal, so by 6.11, $\bar{\alpha}$ still does not fall under Case 4. Hence $\bar{\alpha}$ must fall under Case 5. □

6.17 Corollary. $\bar{\alpha} \notin E$.

Proof. Since all members of E fall under Case 1 or Case 4. □

6.18 Lemma. $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$.

Proof. Define $\bar{k}: \bar{\theta} \to \bar{\gamma}$, $\bar{m}: \bar{\theta} \to \bar{\varrho}$, $(\bar{X}_\nu \mid \nu < \bar{\theta})$, $(\bar{\alpha}_\nu \mid \nu < \bar{\theta})$ from $\bar{\alpha}$ just as k, m, $(X_\nu \mid \nu < \theta)$, $(\alpha_\nu \mid \nu < \theta)$ were defined from α. Since $\bar{\alpha}$ is a limit point of C_α, we clearly have $\bar{\theta} = \lambda$ here. And a straightforward induction proof shows that for $\nu < \lambda$, $\bar{k}(\nu) = k(\nu)$, $\pi(\bar{m}(\nu)) = m(\nu)$, $\pi'' \bar{X}_\nu = X_\nu$, $\bar{\alpha}_\nu = \alpha_\nu$.

Now define $\bar{t}$ from $\bar{\alpha}$ as t was defined from α. For some $\bar{\lambda}$, we will have $\lambda = t(\bar{\lambda})$. By induction on $\nu < \bar{\lambda}$, we get $\bar{t}(\nu) = t(\nu)$. Hence

$$C_{\bar{\alpha}} = \{\bar{\alpha}_{\bar{t}(\nu)} \mid \nu < \bar{\lambda}\} = \{\alpha_{t(\nu)} \mid \nu < \bar{\lambda}\} = \bar{\alpha} \cap C_\alpha. \quad \square$$

The proof of 6.1 is finally complete.

Exercises

1. Strong Embeddings

This exercise is concerned with establishing a sort of "dual" to theorem 5.6. This result says that if there is an embedding

$$\sigma: \langle J_{\bar{\varrho}}, \bar{A} \rangle \prec_1 \langle J_{\varrho^n_\beta}, A^n_\beta \rangle,$$

then $\langle J_{\bar{\varrho}}, \bar{A} \rangle$ must have the form $\bar{\varrho} = \varrho^n_{\bar{\beta}}$, $\bar{A} = A^n_{\bar{\beta}}$, and the embedding σ can be extended to an embedding

$$\tilde{\sigma}: J_{\bar{\beta}} \prec_{n+1} J_\beta.$$

In the result proved below, the roles of $\langle J_{\bar{\varrho}}, \bar{A} \rangle$ and $\langle J_{\varrho^n_\beta}, A^n_\beta \rangle$ in the above are interchanged.

Let $\langle J_{\bar{\varrho}}, \bar{A} \rangle$, $\langle J_\varrho, A \rangle$ be amenable structures. We say that an embedding

$$\sigma: \langle J_{\bar{\varrho}}, \bar{A} \rangle \prec_1 \langle J_\varrho, A \rangle$$

is *strong* iff, whenever $\varphi(x, y)$ is a Σ_0 formula of $\mathscr{L}(A)$, if

$$\{(x, y) \in J_{\bar{\varrho}} \mid \models_{\langle J_{\bar{\rho}}, \bar{A} \rangle} \varphi(\mathring{x}, \mathring{y})\}$$

is well-founded, then

$$\{(x, y) \in J_\varrho \mid \models_{\langle J_\rho, A \rangle} \varphi(\mathring{x}, \mathring{y})\}$$

is well-founded. (Notice that in describing this property as an attribute of σ, we are really using the fact that in order to specify a mapping it is necessary to specify the domain and the range. The actual behaviour of σ plays no part in the definition of strongness.)

We shall prove that, for any $n > 0$, if $\langle J_\varrho, A\rangle$ is amenable and

$$\sigma\colon \langle J_{\varrho^n_{\bar\beta}}, A^n_{\bar\beta}\rangle \prec_1 \langle J_\varrho, A\rangle$$

is strong, then there is a unique ordinal β such that $\varrho = \varrho^n_\beta$, $A = A^n_\beta$, and a (strong) embedding

$$\tilde\sigma\colon J_{\bar\beta} \prec_{n+1} J_\beta$$

such that $\sigma \subseteq \tilde\sigma$.

It suffices to prove the following: Let $n, i > 0$, and suppose that

$$\sigma\colon \langle J_{\varrho^n_{\bar\beta}}, A^n_{\bar\beta}\rangle \prec_i \langle J_\varrho, A\rangle$$

is strong, where $\langle J_\varrho, A\rangle$ is amenable. Then there are η, B, $\tilde\sigma$, such that $\sigma \subseteq \tilde\sigma$ and

(i) $\varrho = \varrho^1_{\eta,B}$, $A = A^1_{\eta,B}$, $\tilde\sigma(p^{n-1}_{\bar\beta}) = p^1_{\eta,B}$;
(ii) $\tilde\sigma\colon \langle J_{\varrho^{n-1}_{\bar\beta}}, A^{n-1}_{\bar\beta}\rangle \prec_{i+1} \langle J_\eta, B\rangle$ is strong.

Set: $\bar\varrho = \varrho^n_{\bar\beta}$, $\bar A = A^n_{\bar\beta}$, $\bar\eta = \varrho^{n-1}_{\bar\beta}$, $\bar B = A^{n-1}_{\bar\beta}$, $\bar p = p^{n-1}_{\bar\beta}$.

Note that: $J_{\bar\eta} = h^*_{\bar\eta,\bar B}(J_{\bar\varrho} \times \{\bar p\})$.

Define: $\bar h((i, x)) \simeq h_{\bar\eta,\bar B}(i, (x, p)) \quad (x \in J_{\bar\varrho})$.

Define relations $\bar D$, $\bar E$, $\bar I$, $\bar B'$ on $J_{\bar\varrho}$ by:

$$\bar D = \mathrm{dom}(\bar h);$$
$$\bar E = \{(x, y) \in \bar D^2 \mid \bar h(x) \in \bar h(y)\};$$
$$\bar I = \{(x, y) \in \bar D^2 \mid \bar h(x) = \bar h(y)\};$$
$$\bar B' = \{x \in \bar D \mid \bar h(x) \in \bar B\}.$$

Since $\bar D$, $\bar E$, $\bar I$, $\bar B'$ are $\Sigma_1^{\langle J_{\bar\eta}, \bar B\rangle}(\{\bar p\})$, they are $\Sigma_0^{\langle J_{\bar\varrho}, A\rangle}$. Let D, E, I, B' have the same Σ_0 definitions over $\langle J_\varrho, A\rangle$. Since σ is strong, E is well-founded. Let

$$\bar M = \langle \bar D, \bar I, \bar E, \bar B'\rangle,$$
$$M = \langle D, I, E, B'\rangle.$$

Let $\bar T$ be the Σ_1 satisfaction relation for the structure $\bar M$. Then

$$\bar T(\varphi, (\vec x)) \leftrightarrow \models^{\Sigma_1}_{\langle J_{\bar\eta}, B\rangle} \varphi(\bar h(\vec x)^\circ).$$

Since $\bar T$ is $\Sigma_1^{\langle J_{\bar\eta}, B\rangle}(\{\bar p\})$, it is $\Sigma_0^{\langle J_{\bar\varrho}, A\rangle}$. Let T have the same Σ_0 definition over $\langle J_\varrho, A\rangle$.

1 A. Prove that T is the Σ_1 satisfaction relation for the structure M.

Since the satisfaction relations $\bar{T}$, T are Σ_0 in $\langle J_{\bar{\varrho}}, \bar{A}\rangle$, $\langle J_\varrho, A\rangle$, respectively, by the same definition, and σ is Σ_i-elementary, we have

$$(\sigma \restriction \bar{D})\colon \bar{M} \prec_{i+1} M.$$

Thus M satisfies the identity axioms (for I) and the Axiom of Extensionality. So we may define the factor models

$$\bar{M}^* = \bar{M}/\bar{I} = \langle \bar{D}^*, \bar{E}^*, \bar{B}^*\rangle$$
$$M^* = M/I = \langle D^*, E^*, B^*\rangle.$$

Let $\bar{k}\colon \bar{M} \to \bar{M}^*$ and $k\colon M \to M^*$ be the natural projections. Since $\bar{M}^*$, M^* are well-founded and extensional, let $\bar{l}$, l be their transitivisation isomorphisms, respectively. Clearly,

$$\bar{l}\colon \bar{M}^* \cong \langle J_{\bar{\eta}}, \bar{B}\rangle, \qquad \bar{h} = \bar{l} \circ \bar{k}.$$

Let

$$l\colon M^* \cong \langle J_\eta, B\rangle,$$

and set

$$h = l \circ k.$$

Define $\sigma^*\colon \bar{M}^* \prec_{i+1} M^*$ by $\sigma^* \circ \bar{k} = k \circ \sigma$, and define

$$\tilde{\sigma}\colon \langle J_{\bar{\eta}}, \bar{B}\rangle \prec_{i+1} \langle J_\eta, B\rangle$$

by $\tilde{\sigma} \circ \bar{h} = \bar{h} \circ \sigma$. We have the following commutative diagram of the situation.

$$\begin{tikzcd}
M \arrow[r, "k"] \arrow[rr, bend left, "h"] & M^* \arrow[r, "l"] & \langle J_\eta, B\rangle \\
\bar{M} \arrow[u, "\sigma \restriction D"] \arrow[r, "\bar{k}"] \arrow[rr, bend right, "\bar{h}"'] & \bar{M}^* \arrow[u, "\sigma^*"] \arrow[r, "\bar{l}"] & \langle J_{\bar{\eta}}, \bar{B}\rangle \arrow[u, "\tilde{\sigma}"']
\end{tikzcd}$$

1 B. Prove that $\tilde{\sigma} \restriction J_{\bar{\varrho}} = \sigma$.

Set $p = \tilde{\sigma}(\bar{p})$.

1 C. Prove that

$$(i, x) \in D \to h((i, x)) = h_{\eta, B}(i, (x, p)).$$

1D. Prove that

$$A = \{(i, x) \mid x \in J_\varrho \wedge \models_{\langle J_\eta, B\rangle} \varphi_i(\mathring{x}, \mathring{p})\},$$

where $(\varphi_i \mid i < \omega)$ is as usual.

1E. Prove that $\varrho = \varrho^1_{\eta, B}$.

1F. Prove that $p = p^1_{\eta, B}$.

1G. Conclude that $A = A^1_{\eta, B}$.

1H. Prove that $\tilde{\sigma}$ is strong. (Hint. Pull back to $\bar{D}$ and D, and use the fact that σ is strong.)

That completes the proof.

The result just proved may be used to give a proof of the Covering Lemma (Chapter V) different from the one given in this book. This alternative proof may be found in *Devlin and Jensen* (1975).

2. The Combinatorial Principle $\Box^\kappa(E)$

For each infinite cardinal κ, let

$$S_\kappa = \{\alpha \in S \mid \mathrm{cf}(\alpha) \leqslant \kappa\}.$$

Let $\Box^\kappa(E)$ denote the following assertion. There is a sequence $(C_\alpha \mid \alpha \in S_\kappa)$ such that:

(i) C_α is a club subset of α;
(ii) if $\mathrm{cf}(\alpha) < \kappa$, then $\mathrm{otp}(C_\alpha) < \kappa$;
(iii) if $\bar{\alpha} < \alpha$ is a limit point of C_α, then $\bar{\alpha} \in S_\kappa$, $\bar{\alpha} \notin E$, and $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$.

2A. Prove that $\Box^\kappa(E)$ implies $\Box_\kappa(F)$, where $F = E \cap (\kappa^+ - \kappa)$. (Hint: Let $(C_\alpha \mid \alpha \in S_\kappa)$ be as in $\Box^\kappa(E)$. For $\kappa < \alpha < \kappa^+$, let $C'_\alpha = C_\alpha \cap (\kappa^+ - \kappa)$. For $\alpha \leqslant \kappa$, define C'_α in two cases. If κ is regular, let $C'_\alpha = \alpha$. If κ is singular, and if $\delta = \mathrm{cf}(\kappa)$, let C'_κ be a club subset of κ of type δ. If $\alpha < \kappa$ is a limit point of C'_κ, let $C'_\alpha = \alpha \cap C'_\kappa$. If $\alpha < \kappa$ is such that $\mu < \alpha \leqslant \nu$, where $\mu, \nu \in C'_\kappa$ are such that ν is the least element of C'_κ above μ, let $C'_\alpha = \alpha - \mu$. If $\alpha < \min(C'_\kappa)$, let $C'_\alpha = \alpha$. Then $(C'_\alpha \mid \alpha < \kappa^+ \;\&\; \lim(\alpha))$ is a $\Box_\kappa(F)$-sequence.)

2B. Prove that $\Box(E)$ implies that $\Box^\kappa(E)$ holds for any infinite cardinal κ. (Hint: Since the case $\kappa = \omega$ is trivial, assume $\kappa > \omega$. First define $(C'_\alpha \mid \alpha \in S_\kappa)$ to satisfy:

(i) C'_α is a club subset of α;
(ii) $\mathrm{otp}(C'_\alpha) \leqslant \kappa$;
(iii) if $\bar{\alpha} < \alpha$ is a limit point of C'_α, then $\bar{\alpha} \in S_\kappa$, $\bar{\alpha} \notin E$, and $C'_{\bar{\alpha}} = \bar{\alpha} \cap C'_\alpha$.

This is done as follows. Let $(C_\alpha \mid \alpha \in S)$ satisfy $\Box(E)$ with the additional assumption $C_\alpha \subseteq \alpha - \kappa$ for $\alpha > \kappa$. (For a fixed κ this is trivially arranged.) For α singular, set $\xi_\alpha = \mathrm{otp}(C_\alpha)$, and let $f_\alpha\colon \xi_\alpha \to C_\alpha$ be the monotone enumeration of C_α. Define C'_α by recursion on α. For $\alpha \in S_\kappa$ such that $\xi_\alpha \leqslant \kappa$, let $C'_\alpha = C_\alpha$. Now suppose $\alpha \in S_\kappa$ and we wish to define C'_α. Thus $\xi_\alpha > \kappa$. Since $\mathrm{cf}(\xi_\alpha) = \mathrm{cf}(\alpha) \leqslant \kappa < \kappa < \xi_\alpha$, ξ_α is singular, so $\xi_\alpha \in S_\kappa$. By (ii) of $\square(E)$, $\xi_\alpha < \alpha$, so C'_{ξ_α} is defined. Set $C'_\alpha = f''_\alpha C'_{\xi_\alpha}$. Then $(C'_\alpha \mid \alpha \in S_\kappa)$ satisfies (i)–(iii) above. If κ is regular, $(C'_\alpha \mid \alpha \in S_\kappa)$ satisfies $\square^\kappa(E)$ already. Suppose κ is singular and let $\delta = \mathrm{cf}(\kappa)$. Let $(\delta_\nu \mid \nu < \delta)$ be a normal sequence cofinal in κ with $\delta_0 = 0$. Define $(\tilde{C}_\alpha \mid \alpha \in S_\kappa)$ as follows. Let $g_\alpha : \theta_\alpha \to C'_\alpha$ be the monotone enumeration of C'_α. If $\delta_\nu < \theta_\alpha \leqslant \delta_{\nu+1}$, set $\tilde{C}_\alpha = g''_\alpha(\theta_\alpha - (\delta_\nu + 1))$. If $\theta_\alpha = \sup\{\delta_\nu \mid \delta_\nu < \theta_\alpha\}$, set $\tilde{C}_\alpha = g''_\alpha\{\delta_\nu \mid \delta_\nu < \theta_\alpha\}$. Then $(\tilde{C}_\alpha \mid \alpha \in S_\kappa)$ is as required.)

2C. Prove that if $V = L$, then for any uncountable regular cardinal κ, there is a sequence $(X_\xi \mid \xi < \kappa^+)$ of classes such that for each closed set $X \subseteq \mathrm{On}$ of order-type κ:

(i) for all $\xi < \kappa^+$, $X \cap X_\xi$ is stationary in X;
(ii) if $\xi < \eta < \kappa^+$, then $X \cap X_\xi \cap X_\eta$ is not stationary in X.

(Hint: First use $\diamondsuit_\kappa$ to show that there are stationary sets $Y_\xi \subseteq \kappa$, $\xi < \kappa^+$, such that $Y_\xi \cap Y_\eta$ is not stationary whenever $\xi < \eta < \kappa^+$. Now let $(C_\alpha \mid \alpha \in S_\kappa)$ be as in $\square^\kappa(\emptyset)$. Let $(\varrho^\alpha_\xi \mid \xi < \eta_\alpha)$ be the monotone enumeration of C_α. Let

$$X_\delta = Y_\delta \cup \{\alpha \in \bigcup_{\nu<\kappa} S_\nu - \kappa \mid (\exists \xi \in Y_\delta)(\exists \beta \in S_\kappa)[\lim(\xi) \wedge \alpha = \varrho^\beta_\xi]\}.)$$

2*D*. Prove that if $V = L$, then for any uncountable regular cardinal κ there is a sequence $(X_\xi \mid \xi < \kappa)$ of pairwise disjoint classes such that for any closed set $X \subseteq \mathrm{On}$ of order-type κ, $X \cap X_\xi$ is stationary in X for every $\xi < \kappa$. (Hint: Use 2C.)

Deduce that, if $V = L$, then for each cardinal κ there is a set $A \subseteq \kappa$ such that neither A nor $\kappa - A$ contains a closed set of order-type ω_1. (See also the Notes on this chapter.)

3. The Failure of $\square_\kappa$ and Large Cardinals

Show that if κ^+ is not Mahlo in L, then $\square_\kappa$ holds. (Hint: Let $C \in L$ be a club subset of κ^+ consisting of singular cardinals in L. By 6.1, $\square$ holds in L, so there is a "$\square$-sequence" on C. Using the ideas from the proof of 6.2, modify this sequence to a $\square_\kappa$-sequence.)

Deduce that if $\square_\kappa$ fails, then κ^+ is Mahlo in L.

Notice that the above result provides an alternative solution to Exercise IV.5.

4. The Principles $\square(E)$

Prove Theorem VI.6.1′. (Use the argument of IX.2 as a starting point.)