Chapter XI
Applications to Algebra

by P. C. Eklof

In contrast to the situation in first-order finitary logic, the applications of infinitary logic to algebra are so scattered throughout the literature that it is extremely difficult to discern any coherent pattern. Nevertheless, there are some interesting applications; and, in this chapter, we will survey a few of them. This survey will primarily be for the benefit of the non-specialist. That being so, proofs will not always be given in detail, since our aim is simply to present enough background to state a result, indicate its significance, and explain how infinitary logic enters into the statement of the result and/or its proof.

The separate sections are organized by algebraic subject matter and are essentially independent of each other. The first four sections involve $\mathscr{L}_{\infty\omega}$, while the fifth and sixth make use of $\mathscr{L}_{\infty\kappa}$ for arbitrary κ. The last section is simply a collection of references to other relevant literature.

The first two sections of our survey deal with applications of logic to algebra in the purest sense that results expressible in algebraic terms are proved by logical means. The first section's concern—arguably the most important application to date of infinitary model theory to algebra—is the construction by Macintyre and Shelah of non-isomorphic universal locally finite groups of the same cardinality. In the second section we examine the use by Baldwin of some profound results in the model theory of $\mathscr{L}_{\omega_1\omega}$ to count the number of subdirectly irreducible algebras in a variety. The remaining sections involve applications in which logical notions are employed in the expression as well as in the proof of a result so as to provide new insight into an algebraic notion or problem.

Sections 3, 4 and 5 make use of the notion of infinitary equivalence. In Section 3, the back-and-forth characterization of $\mathscr{L}_{\infty\omega}$ equivalence is used to formulate precisely and prove the heuristic principle in algebraic geometry known as Lefschetz's principle. Classification theorems in abelian group theory are studied in Section 4 to see what information can be gained from their proofs about the $\mathscr{L}_{\lambda\omega}$-equivalence of abelian groups. Section 5 gives a characterization of the algebras in a variety which are $\mathscr{L}_{\infty\kappa}$-equivalent to a free algebra, and the question of the existence of non-free such algebras is studied, in general, and specifically in the variety of abelian groups. Finally, Section 6 presents both Hodges' formalization of the notion of a concrete (or effective) construction and an examination of his use of it in proving that certain algebraic constructions are not concrete.

I am deeply appreciative to the authors of the other chapters of this volume for their help. And, to Wilfred Hodges, Carol Jacoby, David Kueker, and Alan Mekler, I extend my special thanks.

1. Universal Locally Finite Groups

Using the model-theory of $\mathscr{L}_{\omega_1\omega}$ Macintyre and Shelah [1976] (to be denoted hereafter simply as [M–S]) answered questions raised by Kegel and Wehrfritz [1973] about the groups in the title.

Recall that a group is *locally finite* if every finitely-generated subgroup is finite. The following class of groups is precisely the class of existentially closed locally finite groups and was first studied by Hall [1959].

1.1 Definition. A group G is *universal locally finite* (or $G \in$ ULF) if it is locally finite and:

(i) every finite group G can be embedded into G; and
(ii) any isomorphism between finite subgroups of G is induced by an inner automorphism of G (see Kegel–Wehrfritz [1973, pp. 177 f]).

Hall has shown that any infinite locally finite group can be embedded in a ULF group of the same cardinality, and that any two countable ULF groups are isomorphic. In fact, the latter result is easily proved by a back-and-forth argument which will show that any two ULF groups are $\mathscr{L}_{\infty\omega}$-equivalent and that if $G \subseteq H$ belong to ULF, then $G \prec_{\infty\omega} H$.

Kegel and Wehrfitz [1973, Chapter 6] posed the following questions:

1.2 Questions. (a) *Are any two* ULF *groups of the same cardinality isomorphic? If not, are there* 2^κ ULF *groups of cardinality* κ?

(b) *Does every* ULF *group of cardinality* $\kappa \geq \aleph_1$ *contain an isomorphic copy of every locally finite group of cardinality* $\leq \kappa$?

The key to the results of Macintyre and Shelah is the observation that ULF is elementary in $\mathscr{L}_{\omega_1\omega}$. Indeed, for each $m \geq 1$ let $\{\varphi_{m,n}(v_1, \ldots, v_m): n \in \omega\}$ be an enumeration of all formulas of $\mathscr{L}_{\omega_1\omega}$ which describe the multiplication table of a set of m generators of a finite group. Then a group G belongs to ULF iff $G \models \sigma$ where σ is the conjunction of the following sentences:

$$(0) \qquad \bigwedge_m \forall v_1 \ldots v_m \bigvee_n \varphi_{m,n}(v_1 \ldots v_m)$$

[that is, G is locally finite];

$$(1) \qquad \bigwedge_m \bigwedge_n \exists v_1 \ldots v_m \varphi_{m,n}(v_1 \ldots v_m)$$

[that is, G satisfies Definition 1.1(i)]; and

$$(2) \qquad \bigwedge_m \bigwedge_n \forall v_1 \ldots v_m \forall u_1 \ldots u_m \left[(\varphi_{m,n}(v_1 \ldots v_m) \wedge \varphi_{m,n}(u_1 \ldots u_m)) \rightarrow \left(\exists x \bigwedge_{i=1}^{m} x^{-1} v_i x = u_i \right) \right]$$

[that is, G satisfies Definition 1.1(ii)].

Since σ has models of all infinite cardinalities, the method of indiscernibles (see Keisler [1971a, Section 13]) implies the following:

1.3 Theorem. *For every $\kappa \geq \aleph_1$ there is a model G_κ of σ of cardinality κ such that for every countable $A \subseteq G_\kappa$, G_κ has only countably many A-types.* ∎

We will now use the following simple group-theoretic observation.

1.4 Lemma. *For every infinite cardinal κ, there is a locally finite group H_κ of cardinality κ^+ and a subset A_κ of H_κ of cardinality κ such that H_κ realizes at least κ^+ quantifier-free A_κ types.*

Proof. Let G be a finite group with two elements α and β such that $\alpha\beta \neq \beta\alpha$. Then G^κ, the direct product of κ copies of G, is locally finite (the reader is referred to [M–S, Lemma 1(b)]). Let H_κ be the subgroup of G generated by $A_\kappa \cup Y$, where A_κ consists of all functions $f_\nu \in G^\kappa$ $(\nu \in \kappa)$, where

$$f_\nu(\mu) = \begin{cases} e, & \text{if } \mu \neq \nu, \\ \alpha, & \text{if } \mu = \nu, \end{cases}$$

and Y is a subset of $\{e, \beta\}^\kappa$ of cardinality κ^+. Then any two distinct elements of Y have different quantifier-free types over A_κ; in fact, if $g, h \in Y, g(\nu) = e$, and $h(\nu) = \beta$, then g satisfies $xf_\nu = f_\nu x$, but h does not. ∎

We can now proceed to prove (Refer to Questions 1.2)

1.5 Theorem (Macintyre and Shelah [1976]). (i) *For any $\kappa \geq \aleph_1$ there are 2^κ groups in* ULF *of cardinality κ.*

(ii) *For any $\kappa \geq \aleph_1$ there is a locally finite group H_κ of cardinality κ and a group $G \in$* ULF *of cardinality κ such that H_κ is not embeddable in G_κ.*

Proof. The H_κ of Lemma 1.4 is clearly not embeddable in the G_κ of Theorem 1.3, so (ii) holds. Since H_κ is embeddable in some ULF group of cardinality κ, there are clearly at least two non-isomorphic ULF groups of cardinality κ. In order to obtain 2^κ different ULF groups, we appeal to a theorem of Shelah [1972a, Theorem 2.6] which says that if a sentence σ of $\mathscr{L}_{\lambda^+\omega}$ has for every cardinal κ a model $\mathfrak{B}$ with a subset A of cardinality κ such that $\mathfrak{B}$ realizes more than κ quantifier-free A-types, then for all $\kappa > \lambda$ σ has 2^κ models of cardinality κ. (The reader is also referred to Hodges [1984] for a proof of (i) in a more general context). ∎

These results give rise to other questions which have been posed in [M–S].

1.2. Questions (continued) (c) *Which locally finite groups H can be embedded in all* ULF *groups of cardinality $\geq |H|$?* (Such groups are called *inevitable*).

(d) *For $\kappa \geq \aleph_1$ is there a universal* ULF *group of cardinality κ? That is, is there one into which can be embedded every locally finite group of cardinality $\leq \kappa$?*

Hickin [1978] proved that no locally finite group of cardinality $\aleph_1$ is inevitable.

In fact, he constructed a family of $2^{\aleph_1}$ ULF groups of cardinality $\aleph_1$ such that no uncountable subgroup is embeddable in any two of them. Giorgetta and Shelah [1984] obtained the same result with $\aleph_1$ replaced by any κ such that $\aleph_0 < \kappa \leq 2^{\aleph_0}$. Question (d) was answered in the negative (by Grossberg–Shelah [1983]) for $\kappa = 2^{\aleph_0}$; and, assuming GCH, for all κ of uncountable cofinality. The proofs of the results mentioned above do not, however, use infinitary logic.

Problems similar to Questions 1.2(a) and (b) have been studied for algebraically closed groups and for skew fields. Here the statements of some of the results use the notion of $\mathcal{L}_{\infty\omega}$-equivalence, although the proofs themselves use specific algebraic constructions. For example, we have

1.6 Theorem (Shelah–Ziegler [1979]). *Let A be a countable algebraically closed group. Let κ be an uncountable cardinal.*

(i) *There are 2^κ algebraically closed groups of cardinality κ which are $\mathcal{L}_{\infty\omega}$-equivalent to A.*
(ii) *There is an algebraically closed group of cardinality κ which is $\mathcal{L}_{\infty\omega}$-equivalent to A and which contains no uncountable commutative subgroup.* □

See also Macintyre [1976], Ziegler [1980], and Giorgetta–Shelah [1984].

2. *Subdirectly Irreducible Algebras*

Baldwin [1980] observed that some general theorems of the model theory of $\mathcal{L}_{\omega_1\omega}$ have applications to counting the number of subdirectly irreducible algebras in a residually small variety.

Recall that a variety is a class V of algebras (all structures for the same vocabulary τ, consisting only of function symbols) which is closed under the formation of products, subalgebras and homomorphic images. A fundamental theorem of Birkhoff says that V is a variety if and only if it is the class of models of a set of equations, Σ. In the following discussion we will assume that the vocabulary τ of V is countable.

2.1 Definition. An algebra $\mathfrak{A}$ is called *subdirectly irreducible* if whenever $\mathfrak{A}$ is embeddable in a product of algebras, it is also embeddable in one of the factors. This, of course, is equivalent to requiring that every family $\mathcal{F}$ of homomorphisms on $\mathfrak{A}$ which separates points of $\mathfrak{A}$—that is, for all $a \neq b$ in $\mathfrak{A}\, \exists f \in \mathcal{F}$ such that $f(a) \neq f(b)$—contains a one–one homomorphism. A variety V is *residually small* if the class of subdirectly irreducible algebras in V forms a set, or, equivalently, if there is an upper bound to the size of subdirectly irreducible algebras in V. V is *residually countable* if every subdirectly irreducible algebra in V is countable.

Taylor [1972] has shown that if a variety V is residually small then every subdirectly irreducible algebra in V has cardinality $< (2^{\aleph_0})^+$.

2.2 Definition. A *congruence* on $\mathfrak{A}$ is a subset $\theta \subseteq A \times A$ such that there is a homomorphism f on $\mathfrak{A}$ such that $\theta = \{(a, b) \in A \times A : f(a) = f(b)\}$. θ is non-trivial, if $\theta \neq$ the diagonal on A. If $(c, d) \in A \times A$, the *principal congruence generated by* (c, d), which we denote $\theta(c, d)$, is the smallest congruence containing (c, d).

Note that (a, b) belongs to $\theta(c, d)$ if and only if for every homomorphism f on $\mathfrak{A}$ such that $f(c) = f(d)$ we have $f(a) = f(b)$. Thus, by the compactness theorem of finitary logic, we have:

2.3 Lemma. *For any $a, b, c, d \in \mathfrak{A}$, $(a, b) \in \theta(c, d)$ iff there is a positive (existential) formula $\varphi(x, y, z, u) \in \mathscr{L}_{\omega\omega}$ such that*

$$(*) \qquad \models \forall x, z, u[\varphi(x, x, z, u) \rightarrow z = u]$$

and $\mathfrak{A} \models \varphi[c, d, a, b]$. $\square$

Moreover, as an immediate consequence of the definitions we have:

2.4 Lemma. *An algebra $\mathfrak{A}$ is subdirectly irreducible iff there exists $a \neq b$ in A such that for every non-trivial congruence θ on $\mathfrak{A}$, $(a, b) \in \theta$ iff there exists $a \neq b$ in A such that for every $c \neq d$ in A, $(a, b) \in \theta(c, d)$.* $\square$

Using these results, we can now establish

2.5 Proposition. *For any variety V, there is a sentence $\sigma \in \mathscr{L}_{\omega_1\omega}$ such that $\mathfrak{A} \models \sigma$ iff $\mathfrak{A}$ is a subdirectly irreducible algebra in V.*

Proof. Let Φ be the set of all positive existential formulas of $\mathscr{L}_{\omega\omega}$ satisfying $(*)$ in Lemma 2.3. Let σ be the conjunction of the defining equations of V and the following sentence:

$$\exists z, u\, \forall x, y\left[z \neq u \wedge \left(x \neq y \rightarrow \bigvee_{\varphi \in \Phi} \varphi(x, y, z, u)\right)\right].$$

By Lemmas 2.3 and 2.4, σ has the desired property. $\square$

We can now apply the model-theory of $\mathscr{L}_{\omega_1\omega}$.

2.6 Theorem (Harnik–Makkai [1977]). *If σ is a sentence of $\mathscr{L}_{\omega_1\omega}$ and σ has at least $\aleph_1$ and fewer than $2^{\aleph_0}$ countable models, then σ has a model of power $\aleph_1$.* $\square$

2.7 Corollary (Baldwin [1980]). *If V is residually countable, then V has either $\leq \aleph_0$ or exactly $2^{\aleph_0}$ subdirectly irreducible algebras.*

Proof. This result follows immediately from Proposition 2.5 and Theorem 2.6. Baldwin [1980] has noted that all the possibilities for the number of subdirectly irreducible varieties do occur. $\square$

The following theorem was proven by Shelah [1975c] under the assumption that $V = L$ and more recently (Shelah [1983a, b]) assuming only GCH.

2.8 Theorem (G.C.H.). *If σ is a sentence of $\mathscr{L}_{\omega_1\omega}$ which has at least one but fewer than $2^{\aleph_1}$ models of power $\aleph_1$ then it has a model of power $\aleph_2$.* □

2.9 Corollary (Baldwin [1980]) (G.C.H.). *If V is residually small, and it has a subdirectly irreducible algebra of power $\aleph_1$ then it has $2^{\aleph_1}$ subdirectly irreducible algebras of power $\aleph_1$.*

Proof. As we remarked after the statement of Definition 2.1, Taylor has shown that a residually small variety has no subdirectly irreducible algebra of power $(2^{\aleph_0})^+ = \aleph_2$. □

Remarks. (i) Theorems 2.6 and 2.8 can also be used in an analogous way to count the number of simple algebras in certain varieties, because the simple algebras are axiomatized by the following sentence of $\mathscr{L}_{\omega_1\omega}$ (where Φ is as in the proof of Proposition 2.5):

$$\forall x, y\, \forall z, u\left[(x \neq y) \rightarrow \bigvee_{\varphi \in \Phi} \varphi(x, y, z, u)\right].$$

(ii) Mekler [1980b] uses the idea of Lemma 2.3 to prove that the class, $\mathscr{R}$, of residually finite groups is axiomatizable in $\mathscr{L}_{\omega_1\omega}$. It follows immediately from the downward Löwenheim–Skolem theorem for $\mathscr{L}_{\omega_1\omega}$ that $\mathscr{R}$ is of countable character. That is, a group belongs to $\mathscr{R}$ iff every countable subgroup does. (This result was first proved by B. H. Neumann.)

3. Lefschetz's Principle

Using notions from category theory and the model theory of $\mathscr{L}_{\infty\omega}$, Eklof [1973] gave a simple and yet comprehensive formalization of Lefschetz's principle from algebraic geometry. The key idea was inspired by the work of Feferman [1972] and basically asserts that certain simply characterized functors preserve $\mathscr{L}_{\infty\omega}$-equivalence.

Following Weil [1962], we will call K a *universal domain* if K is an algebraically closed field of infinite transcendence degree over its prime field. We recall that the prime field of K is the smallest field contained in K and that it is isomorphic to Q (respectively the field with p elements) if char K, the characteristic of K, is 0 (respectively the prime p).

In his foundational work, Weil [1962, p. 306] gave the following explanation of the heuristic principle attributed to S. Lefschetz:

> "For a given value of the characteristic p[= zero or a prime], every result involving only a finite number of points and varieties, which has been proved for some choice of the universal domain remains valid without restriction; there is but one algebraic geometry of characteristic p for each value of p, not one algebraic geometry for each universal domain."

Seidenberg [1958] has rightly pointed out that Lefschetz had in mind a stronger principle: That algebraic geometry is the same for any two algebraically closed ground fields—not necessarily of infinite transcendence degree—having the same characteristic. We will not deal with this stronger principle at all. The reader should consult Barwise–Eklof [1969, Section 3] for historical remarks on formalizations of Lefschefz's principle.

Notice that two universal domains are $\mathscr{L}_{\infty\omega}$-equivalent if and only if they have the same characteristic. Let $\mathscr{U}$ be the category of universal domains. The nature of the formalization of Lefschetz's principle will be that certain functors on $\mathscr{U}$ into a category $\mathscr{C}$ of algebras preserve $\mathscr{L}_{\infty\omega}$-equivalence; any particular instance of Lefschetz's principle will then follow by checking that the algebraic-geometric result in question is a statement in $\mathscr{L}_{\infty\omega}$ about structures constructed by an appropriate functor.

We shall fix a vocabulary τ consisting of a countable set of function symbols but no relation symbols. Let Alg[τ] be the category of all τ-structures and all τ-homomorphisms.

3.1 Definition. A subcategory $\mathscr{C}$ of Alg[τ] will be called a *quasivariety* if it is a full subcategory (that is, if it contains all τ-homomorphisms between objects in $\mathscr{C}$) and the class of objects of $\mathscr{C}$ is axiomatizable by a set of strict universal Horn sentences, that is, a set of sentences of the form

$$\forall x_1 \ldots \forall x_n[\theta_0 \wedge \cdots \wedge \theta_{n-1} \rightarrow \theta_m],$$

where each θ_i is atomic.

Thus defined, the class of objects of $\mathscr{C}$ is closed under products and under substructures. Clearly any variety is a quasivariety. In order to characterize the quasivarieties, we recall an important notion from category theory.

3.2 Definition. Let $\kappa \geq \omega$. $D = (I, \geq)$ is a *κ-directed set* if it is a partially ordered set such that for every subset $X \subseteq D$ of cardinality $< \kappa$, there exists $j \in I$ such that $i \leq j$ for all $i \in X$. A *diagram* $\mathfrak{D}$ over D (in Alg[τ]) is a family of τ-algebras $\mathfrak{A}_i$ for each $i \in I$ and τ-homomorphisms φ_{ij}: $\mathfrak{A}_i \rightarrow \mathfrak{A}_j$ for each $i \leq j$ in I such that $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ if $i \leq j \leq k$. The *κ-direct limit* of a diagram $\mathfrak{D}$ over a κ-directed set D is a structure $\mathfrak{A}$ together with morphisms ψ_i: $\mathfrak{A}_i \rightarrow \mathfrak{A}$ for each $i \in I$ such that given any $\mathfrak{B}$ in Alg[τ] and any family of morphisms θ_i: $\mathfrak{A}_i \rightarrow \mathfrak{B}(i \in I)$ such that for all $i \leq j$, $\theta_j \circ \varphi_{ij} = \theta_i$, there is exactly one morphism θ: $\mathfrak{A} \rightarrow \mathfrak{B}$ such that for all $i \in I$, $\theta \circ \psi_i = \theta_i$.

If κ is ω, we omit the reference to it, and simply say direct limit instead of ω-direct limit. It is a standard result of category theory that the direct limit is unique up to isomorphism and that in Alg[τ] it may be constructed as the disjoint union of the $\mathfrak{A}_i$ modulo the equivalence relation generated by all identities of the form $y = \varphi_{ij}(x)$. Observe that for the latter result, it is necessary that D be directed. We shall always use the term direct limit in this sense of "colimit over a directed set" (see Mitchell [1965, pp. 44–49]).

Mal'cev [1973, Section 11] characterized the quasivarieties $\mathscr{C}$ in Alg[τ] as the full subcategories which are closed under isomorphism, substructure, and direct limits.

We will be interested in functors which preserve direct limits. The following result gives a large class of such functors (see Feferman [1972, Lemma 4]).

3.3 Lemma. *Let $F: \mathscr{C}_0 \to \mathscr{C}_1$ be a functor, where $\mathscr{C}_0$ and $\mathscr{C}_1$ are quasivarieties. Suppose that:*

(i) *F preserves monomorphisms; and*
(ii) *for every $\mathfrak{A} \in \mathscr{C}_0$ and every finite subset $X \subseteq F(\mathfrak{A})$ there exists a finitely generated substructure $\mathfrak{A}_1$ of $\mathfrak{A}$ such that $X \subseteq F(e)[\mathfrak{A}_1]$, where $e: \mathfrak{A}_1 \to \mathfrak{A}$ is the inclusion morphism.*

Then F preserves direct limits. □

Feferman proved that functors satisfying properties (i) and (ii) of Lemma 3.3—he called them *ω-local functors*—preserve $\mathscr{L}_{\infty\omega}$-equivalence and noted that this (and its generalizations for cardinals $\kappa > \omega$) imply various preservation results for algebraic constructions (see Chapter IX, Sections 4.5.2 and 4.5.3). G. Sabbagh suggested means for obtaining some other preservation results by weakening the hypotheses given in (i) and (ii) above (see Eklof [1975a, Section 3]).

We can now state

3.4 Lefschetz's Principle (Formalized). *Let $\mathscr{C}$ be a quasivariety and $F: \mathscr{U} \to \mathscr{C}$ a functor which preserves direct limits. For any universal domains K_1 and K_2, if* char $K_1 =$ char K_2, *then $F(K_1) \equiv_{\infty\omega} F(K_2)$.*

Proof. K_l is the direct limit of the family $\mathscr{S}_l$ of all of its algebraically closed subfields of finite transcendence degree (the morphisms are inclusions between subfields). Thus, $F(K_l)$ is the direct limit of the $F(k)$, $k \in \mathscr{S}_l$, relative to certain morphisms $\psi_k: F(k) \to F(K_l)$. Let $\tilde{F}(k)$ denote the image of $F(k)$ under ψ_k. It is a subalgebra of $F(K_l)$. If $f: k_1 \to k_2$ is an isomorphism between $k_1 \in \mathscr{S}_1$ and $k_2 \in \mathscr{S}_2$, we will show that the isomorphism $F(f): F(k_1) \to F(k_2)$ induces an isomorphism $\tilde{f}$ between $\tilde{F}(k_1)$ and $\tilde{F}(k_2)$ by means of the rule $\tilde{f}(\psi_{k_1}(x)) = \psi_{k_2}(F(f)(x))$, for $x \in F(k_1)$. It suffices to verify that if $\psi_{k_1}(x) = 0$, then $\psi_{k_2}(F(f)(x)) = 0$. But $\psi_{k_1}(x) = 0$ iff there is a $k_1' \supseteq k_1$ in $\mathscr{S}_1$ such that if $e_1: k_1 \to k_1'$ is the inclusion map, $F(e_1)(x) = 0$. In that case, there is a $k_2' \supseteq k_2$ and an isomorphism $f': k_1' \to k_2'$ extending f such that if $e_2: k_2 \to k_2'$ is inclusion, $f' \circ e_1 = e_2 \circ f$. Hence, $0 = F(f')F(e_1)(x) = F(e_2)F(f)(x)$, and so $\psi_{k_2}(F(f)(x)) = 0$.

Now we appeal to the back-and-forth criterion for $\mathscr{L}_{\infty\omega}$-equivalence (Refer to Chapter IX, Theorem 4.3.1 or to Chapter XIII, Theorem 2.1.1). Indeed, the family, **I**, of all $\tilde{f}$, as f ranges over all isomorphisms from an element of $\mathscr{S}_1$ to an element of $\mathscr{S}_2$, is a family of partial isomorphisms such that $\mathbf{I}: F(K_1) \simeq_\omega^p F(K_2)$. □

Let us now consider as an example of a use of Lefschetz's principle, the paper of Murthy–Swan [1976]. In this study, Lefschetz's principle is used to carry over a result on uncountable universal domains to the case of countable universal domains. It is striking that the methods used in this paper to justify the appeal to Lefschetz's principle closely mirror the considerations of our general theorem. (In fact, the authors specifically noted this; see pp. 141 f). Murthy and Swan proved that the key constructions they were studying are functors on $\mathscr{U}$ (into Ab, the category of abelian groups, or into Sets, the category of sets) which preserve direct limits (Murthy–Swan [1976, Lemma 5.8]). They then used this result to show that certain properties of the objects constructed by these functors are independent of the choice of universal domain (the reader is referred to Murthy–Swan [1976, pp. 142–143]). For example, one of the properties that concerned them is that a certain abelian group $SA_0(X\ )$—the value at $K \in \mathscr{U}$ of a functor on $\mathscr{U}$ which preserves direct limits—is a divisible group of infinite rank. They make an *ad hoc* argument, using the limit preserving property of the functor, to show that if $SA_0(X_{K_1})$ has this property for some (uncountable) K_1 in $\mathscr{U}$, then $SA_0(X_{K_2})$ has the property for all (including countable) K_2 on $\mathscr{U}$ of the same characteristic. From our point of view, the property of being a divisible abelian group of infinite rank is expressible in $\mathscr{L}_{\infty\omega}$, so by Theorem 3.4, char K_1 = char K_2 implies that $SA_0(X_{K_1}) \equiv_{\infty\omega} SA_0(X_{K_2})$. And hence it follows that $SA_0(X_{K_1})$ is divisible of infinite rank iff $SA_0(X_{K_2})$ is.

Another example of Lefschetz's principle, given by Weil [1962], is worked out in detail in Eklof [1973].

4. *Abelian Groups*

Classification theorems in abelian group theory, due to Ulm and Warfield, were generalized by Barwise–Eklof [1970] and Jacoby [1980], respectively, to classify a larger class of groups up to $\mathscr{L}_{\infty\omega}$-equivalence. This suggests that the notion of potential isomorphism, which has an algebraic formulation in terms of partial isomorphisms, is a natural one to employ in the study of abelian groups.

For simplicity of exposition—especially in the case of mixed groups—we will restrict attention to the local case. That is, we will fix a prime p and consider abelian groups A which are $\mathbb{Z}_p$-modules, where $\mathbb{Z}_p$ is the ring of rationals with denominators prime to p. This means that every element of A is uniquely divisible by every prime different from p. From now on, we will use the word "module" to mean $\mathbb{Z}_p$-module. A torsion module is then a p-group (i.e., an abelian group A such that for all $a \in A$, there exists $n \in \omega$ such that $p^n a = 0$).

For any module A and ordinal α, define $p^\alpha A$ by induction as follows: $p^0 A = A$; $p^{\alpha+1} = p(p^\alpha A) = \{px: x \in p^\alpha A\}$; $p^\sigma A = \bigcap_{\alpha<\sigma} p^\alpha A$, if σ is a limit ordinal. For any $a \in A$, the *height*, $h(a)$, of a is the unique α such that $a \in p^\alpha A - p^{\alpha+1}A$, if it exists, or $h(a) = \infty$, otherwise. It is easy to see that there exists $\sigma < |A|^+$ such that $p^\sigma A = p^{\sigma+1}A = p^\tau A$, for all $\tau > \sigma$. Then $p^\sigma A$, denoted A_d, is a divisible module and a direct summand of A. The structure of a divisible module is easily explicated: it is a direct sum of copies of $\mathbb{Q}$, the rationals, and of $Z(p^\infty)$, the p-torsion component of $\mathbb{Q}/Z$. Thus, the classification problem easily reduces to the problem of classifying *reduced* modules; that is, modules A such that $A_d = \{0\}$.

Define $p^\alpha A[p] = \{x \in p^\alpha A: px = 0\}$; this is a vector space over $GF(p)$, the field of order p. The dimension of the quotient space $p^\alpha A[p]/p^{\alpha+1}A[p]$ is called the α^{th} Ulm *invariant of A* and is denoted by $f(\alpha, A)$. Let $\hat{f}(\alpha, A) = f(\alpha, A)$ if $f(\alpha, A)$ is finite, and $\hat{f}(\alpha, A) = \infty$ otherwise.

Ulm's theorem asserts that two countable reduced torsion modules A and B are isomorphic iff $f(\alpha, A) = f(\alpha, B)$ for all $\alpha < \omega_1$. This result is not true for arbitrary uncountable torsion modules, although the largest class of torsion modules for which the theorem holds—the class of *totally projective* modules—has been given a number of interesting characterizations (see for example, Fuchs [1973, Chapter XII]). However, the back-and-forth method of proof (see Chapter IX, Theorem 4.3.3) does yield a classification of arbitrary torsion modules up to $\mathscr{L}_{\infty\omega}$-equivalence. More precisely, we have

4.1 Theorem (Barwise–Eklof [1970]). *For any cardinal κ and any reduced torsion modules A and B, A is $\mathscr{L}_{\kappa\omega}$-equivalent to B iff $\hat{f}(\alpha, A) = \hat{f}(\alpha, B)$ for all $\alpha < \kappa$.* □

For an exposition of the proof of Theorem 4.1 the reader should see Barwise [1973b]. The proof shows that every torsion module is $\mathscr{L}_{\infty\omega}$-equivalent to a totally projective module. Barwise–Eklof [1970] also uses the back-and-forth method to classify equivalence with respect to certain subclasses of sentences of $\mathscr{L}_{\kappa\omega}$. For instance, if we let $\hat{r}(B) =$ the rank of B, if finite and $\hat{r}(B) = \infty$, otherwise, we then have

4.2 Theorem (Barwise–Eklof). *If A and B are reduced torsion modules then every existential sentence of $\mathscr{L}_{\kappa\omega}$ true in A is true in B iff for all $\alpha < \kappa$, $\hat{r}(p^\alpha A) \le \hat{r}(p^\alpha B)$.* □

For countable groups (and, even more generally, by a simple argument, for direct sums of countable groups) this yielded the following result—a result which was apparently not previously known.

4.3 Corollary. *If A and B are countable torsion modules then A is embeddable in B iff* $\text{rank}(p^\alpha A) \le \text{rank}(p^\alpha B)$ *for all $\alpha < \omega_1$.* □

This result was later extended, by different means, to all totally-projective modules by May–Toubassi [1977].

Remark. The Barwise–Eklof method is employed in Eklof [1977c, Theorem 1.6] to give an $\mathscr{L}_{\infty\omega}$-extension of a theorem of Kaplansky characterizing fully invariant subgroups of a countable p-group; the extended theorem characterizes definable subgroups of arbitrary p-groups.

Warfield [1981] defined a class of modules whose torsion members were precisely the totally projective modules and which included many non-trivial mixed modules, these latter being modules that are not a direct sum of a torsion and a torsion-free module. The modules in this class have come to be called *Warfield modules* and are characterized by the property of being summands of simply presented modules, where a *simply presented* module is a module that can be generated by a set of elements subject only to defining relations of the form $p^n x = 0$ or $p^m x = y$.

A Warfield module M has a *decomposition basis*, such a basis being a linearly independent subset X such that, if $[X]$ denotes the submodule generated by X, $M/[X]$ is torsion, and for all

$$x_0, \ldots, x_n \in X, r_0, \ldots, r_n \in Z_p, \qquad h\left(\sum_{i=0}^{n} r_i x_i\right) = \min\{h(r_i x_i): i \leq n\}.$$

In fact, a countable module is a Warfield module if and only if it has a decomposition basis. For uncountable modules, this is not the case, although the Warfield modules can be characterized as those which have a certain kind of decomposition basis X called *nice*, such that $M/[X]$ is a totally projective torsion module (the reader is referred to Hunter–Richman–Walker [1977]).

Warfield classified the Warfield modules by use of new invariants $g(e, M)$ defined as follow. If $x \in M$, the *Ulm sequence* of x, denoted $U(x)$ is the sequence $(h(p^i x))_{i \in \omega}$. Two Ulm sequences $(\alpha_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$ are called *equivalent* if there are positive integers n and m such that for all $i \in \omega$, $\alpha_{i+n} = \beta_{i+m}$. Thus, $U(x)$ and $U(y)$ are equivalent if there exists $r, s \in \mathbb{Z}_p$ such that $rx = sy$. If e is an equivalence class of Ulm sequences, and M is a module with a decomposition basis X, define $g(e, M) =$ cardinality of $\{x \in X: U(x) \in e\}$. Warfield showed that this is an invariant of M and that two reduced Warfield modules M and N are isomorphic iff for all ordinals α and all classes $e, f(\alpha, M) = f(\alpha, N)$ and $g(e, M) = g(e, N)$.

Jacoby [1980] extended Warfield's methods to give a classification result for $\mathscr{L}_{\infty\omega}$-equivalence. Let $\hat{g}(e, M) = g(e, M)$ if finite, and equal to ∞, otherwise.

4.4 Theorem (Jacoby). *If M and N are reduced modules with decomposition bases, then $M \equiv_{\infty\omega} N$ iff for all α and all $e, \hat{f}(\alpha, M) = \hat{f}(\alpha, N)$ and $\hat{g}(e, M) = \hat{g}(e, N)$.* ☐

Now the class of (non-reduced as well as reduced) modules classified (up to $\mathscr{L}_{\infty\omega}$-equivalence) using Theorem 4.1 is an elementary class in $\mathscr{L}_{\infty\omega}$: It is precisely the class of all torsion modules. But the class of all modules with decomposition bases is not even closed under $\mathscr{L}_{\infty\omega}$-equivalence. Jacoby [1980] defined in a natural algebraic way a larger class of modules closed under $\mathscr{L}_{\infty\omega}$-equivalence (but not $EC_{\infty\omega}$) which can be classified up to $\mathscr{L}_{\infty\omega}$-equivalence using Theorem 4.4. But, surprisingly enough, she was able to show that no class of modules that generalizes the class of modules with decomposition bases in any reasonable way is an elementary class in $\mathscr{L}_{\infty\omega}$. The proof uses her classification theorem for modules with decomposition bases. (Jacoby [1980] contains the proof in the global case).

4.5 Theorem. *Let $\mathscr{C}$ be a class of modules satisfying:*

(i) *every Warfield module is in $\mathscr{C}$; and*
(ii) *if $A \in \mathscr{C}$, then every pair of elements of A is contained in a submodule of A which has a decomposition basis.*

Then $\mathscr{C}$ is not an elementary class in $\mathscr{L}_{\infty\omega}$. ☐

4.6 Corollary. *The class, $\mathscr{C}$, of all modules which are $\mathscr{L}_{\infty\omega}$-equivalent to a module with a decomposition basis is not an elementary class in $\mathscr{L}_{\infty\omega}$.*

Proof. Clearly $\mathscr{C}$ satisfies (i) of Theorem 4.5. Moreover, since every module with a decomposition basis obviously satisfies (ii) of the Theorem 4.5, we can use the back-and-forth method to show that every module in $\mathscr{C}$ satisfies (ii) of Theorem 4.5. ☐

5. *Almost-Free Algebras*

Algebras which are $\mathscr{L}_{\infty\kappa}$-equivalent to a free algebra in an arbitrary variety have been studied by Kueker, Shelah, Mekler, and Eklof among others.

Fix a variety V in a countable vocabulary (see Section 2). We will say that $A \in V$ is *V-free* (on X) if there is a subset $X \subseteq A$ such that for any $B \in V$ and any set map $f: X \to B$, there is one and only one homomorphism $\hat{f}: A \to B$ such that $\hat{f} \upharpoonright X = f$. X is said to be a *set of free generators for A*. Since V will be fixed, we will simply say *free* instead of V-free.

If B is a subalgebra of C, we say B is a *free factor* of C (written $B|C$) if B and C have sets of free generators, X and Y, respectively, such that $X \subseteq Y$. In this case, *every* set of free generators of B extends to a set of free generators of C. If $B|C$, we say C has *infinite rank over B*, if there are X, Y as above such that in addition $Y - X$ is finite.

It follows easily from the back-and-forth criterion (see Chapter IX, Theorem 4.3.3) that if $\kappa \geq \omega_1$, any two free algebras of cardinality $\geq \kappa$ are $\mathscr{L}_{\infty\kappa}$-equivalent. Define A to be *$\mathscr{L}_{\infty\kappa}$-free*, if A is $\mathscr{L}_{\infty\kappa}$-equivalent to a free algebra. The back-and-forth criterion implies that A is $\mathscr{L}_{\infty\kappa^+}$-free iff A is the κ^+-direct limit of a set S of free subalgebras of cardinality κ, where the maps are inclusions, such that S is ω-directed under $|$(see Definition 3.2). The latter condition means that if $G_0, \ldots, G_n \in S$, then there is an $H \in S$ such that for all $i \leq n$, $G_i | H$.

Surprisingly enough, Kueker [1973] has shown that $\mathscr{L}_{\infty\kappa^+}$-free algebras satisfy the following stronger condition. The proof uses game-theoretic methods (see Kueker [1981]).

5.1 Theorem (Kueker). *A is $\mathscr{L}_{\infty\kappa^+}$-free iff A is the κ^+- direct limit of a set S of free subalgebras of cardinality κ such that S is κ^+-directed under $|$. In particular, if*

$|A| = \kappa^+$, A is $\mathscr{L}_{\infty\kappa^+}$-free iff $A = \cup_{\nu<\kappa^+} A_\nu$ where each A_ν is a free subalgebra of cardinality κ and for all $\mu < \nu < \kappa^+$, $A_\mu | A_\nu$.

Proof. If Y is a subset of A of cardinality κ, the *Y-Shelah game* on A is the game of length ω, where player I (respectively II) chooses X_n, a subalgebra of A of cardinality κ, when n is even (respectively odd), and II wins if for all k, $X_{2k} \subset X_{2k+1}$ and $Y | X_{2k+1} | X_{2k+3}$. Let $S(A) = \{Y\colon$ player II has a winning strategy in the Y-Shelah game on $A\}$. Observe that if F is the free algebra on κ^+ generators, $Y \in S(A)$ iff for some $B | F$, $(A, Y) \equiv_{\infty\kappa^+} (F, B)$. Hence, $S(A)$ is $\mathscr{L}_{\infty\kappa^+}$-definable (see Chang [1968c, Proposition 7]). Now $S(F)$ is clearly κ^+-directed under $|$ and F is the κ^+-direct limit of $S(F)$. Thus, since these facts are expressible in $L_{\infty\kappa^+}$, the same holds when F is replaced by A. □

5.2 Corollary (Kueker). *If A is $\mathscr{L}_{\infty\kappa^+}$-free then there is a free algebra F on a set of free generators of cardinality κ^+ such that $F \prec_{\infty\kappa} A$.* □

It follows from the back-and-forth criterion that for any uncountable λ, A is $\mathscr{L}_{\infty\lambda}$-free iff A is $\mathscr{L}_{\infty\kappa^+}$-free, for every $\kappa < \lambda$ (see Shelah [1975a, Theorem 2.6(c)]).

A natural question is whether or not there are non-free $\mathscr{L}_{\infty\kappa}$-free algebras. The following profoundly interesting result is due to Shelah (Shelah [1975a, Theorem 2.6(d)]).

5.3 Theorem (Shelah). *If λ is singular and A is $\mathscr{L}_{\infty\lambda}$-free and of cardinality λ then A is free.* □

Remarks. (i) Hodges [1981] gives a very clear exposition of Shelah's "singular compactness theorem" in a general form. For those familiar with Hodges [1981], we now indicate how to derive Theorem 5.3 from the results in that paper. It suffices to prove that for every $\kappa < \lambda$, player II has a winning strategy in the κ^+-Shelah game on A (see Hodges [1981, p. 207]). If S is a set of free subalgebras of cardinality κ such that A is the κ^+-direct limit of S and S is ω-directed under $|$, then player II can win by always choosing his subalgebra B_i (i is odd) to be an element of S such that $B_{i-2} | B_i$.

(ii) Mekler [1980a, Theorem 1.6] proved that if κ is a regular cardinal and A is a κ^+-free group (that is, every subgroup of A of cardinality $<\kappa^+$ is free), then A is $\mathscr{L}_{\infty\kappa}$-free. For varieties in which it is not the case that a subalgebra of a free algebra is always free, a different definition of κ^+-free is needed; one (weak) notion of κ^+-free is that A is to satisfy

$$\neg(\forall x_{2i}\, \exists x_{2i+1})_{i<\kappa} \text{“}\langle x_i\colon i < \kappa\rangle \text{ is not free”}$$

(that is, it is not the case that almost every subalgebra of cardinality κ is non-free. The reader should consult Kueker [1977]). It follows from an argument similar to that in Lemma 3.1 of Hodges [1981] that (for regular κ) if A is a κ^+-free algebra in this sense, then A is $\mathscr{L}_{\infty\kappa}$-free (the reader should compare this result to that in Shelah [1975a, Theorem 2.6(b)]). If A is not E^κ_κ-non-free, then A is κ^+-free (in the above sense). Whether or not an $\mathscr{L}_{\infty\kappa^+}$-free algebra is always κ^+-free remains an open question. (It is true under certain hypotheses on V.)

A general theorem of Shelah (see Chapter IX, Theorem 4.3.7) implies that, assuming $V = L$, if κ is regular and not weakly-compact, then there are either 1 or 2^κ $\mathscr{L}_{\infty\kappa}$-free algebras of cardinality κ. Eklof–Mekler [1982] recently proved the following general result about the existence of non-free $\mathscr{L}_{\infty\kappa}$-free algebras.

5.4 Theorem (Eklof–Mekler). (1) ($V = L$). *If there is a non-free $\mathscr{L}_{\infty\omega_1}$-free algebra of cardinality ω_1, then for every regular non-weakly-compact κ there is a non-free $\mathscr{L}_{\infty\kappa}$-free algebra of cardinality κ.*

(2) *If every $\mathscr{L}_{\infty\omega_1}$-free algebra of cardinality ω_1 is free, then for every κ, every $\mathscr{L}_{\infty\kappa}$-free algebra of cardinality κ is free.* □

Moreover, under certain general conditions on the variety V, the hypothesis given in (2) holds if and only if the class of free algebras is definable in $\mathscr{L}_{\omega_1\omega}$ (see also Kueker [1980]).

Much work has been done on the problem of constructing non-free $\mathscr{L}_{\infty\kappa}$-free algebras for $V =$ the variety of groups or abelian groups. Kueker proved that a group (or abelian group) is $\mathscr{L}_{\infty\omega}$-free iff it is ω_1-free. Higman constructed a non-free ω_1-free group. Mekler, as well as Kueker, constructed a non-free $\mathscr{L}_{\infty\omega_1}$-free group of cardinality ω_1. See Mekler [1980a] for more results on groups. Pope [1982] deals with other varieties of groups and rings.

The $\mathscr{L}_{\infty\kappa}$-free abelian groups (for uncountable κ) are characterized by the property that A is κ-free and every subset of A of cardinality $< \kappa$ is contained in a subgroup B of cardinality $< \kappa$, such that A/B is κ-free (see Eklof [1974]). These groups had arisen naturally in the study of Whitehead's problem: by Chase [1963], CH implies that every Whitehead group is $\mathscr{L}_{\infty\omega_1}$-free. But by Shelah [1979b] MA + $\neg$CH implies that there are Whitehead groups which are not $\mathscr{L}_{\infty\omega_1}$-free. The following theorem sums up the main results about the existence of $\mathscr{L}_{\infty\kappa}$-free abelian groups. (See Eklof [1977c] for more details and references).

5.5 Theorem. (i) (Eklof [1975b]) *For all $n \in \omega$ there is a non-free $\mathscr{L}_{\infty\omega_{n+1}}$-free abelian group of cardinality ω_{n+1}.*

(ii) (Shelah [1979a]). GCH $\Rightarrow$ *for all $\kappa < \aleph_{\omega^2}$ there is a non-free $\mathscr{L}_{\infty\kappa^+}$-free abelian group of cardinality κ^+.*

(iii) (Magidor–Shelah [1983]). (*Assuming the consistency of the existence of many supercompact cardinals*). *It is consistent with* GCH *that if $\kappa = \aleph_{\omega^2}$ every κ^+-free abelian group of cardinality κ^+ is free.*

(iv) *If κ is weakly compact, every κ-free abelian group of cardinality κ is free.*

(v) *If κ is strongly compact, every κ-free abelian group is free.*

(vi) (Gregory). $V = L \Rightarrow$ *for every non-weakly-compact regular κ there exists a non-free $\mathscr{L}_{\infty\kappa}$-free abelian group of cardinality κ.*

(vii) *If there is no inner model with a measurable cardinal, then there exist arbitrarily large κ such that there is a non-free $\mathscr{L}_{\infty\kappa^+}$-free abelian group of cardinality κ^+.* □

5.6 Corollary. *If there is a strongly compact cardinal, then the class of free abelian groups is definable in $\mathscr{L}_{\infty\infty}$. If the class of free abelian groups is definable in $\mathscr{L}_{\infty\infty}$ then there is an inner model with a measurable cardinal.*

Proof. If κ is strongly compact, then by (v) the class of free abelian groups is defined by the sentence of $\mathscr{L}_{\kappa\kappa}$ which says that the group is κ-free. Conversely, we prove a little more: if there is no inner model with a measurable cardinal, then the class of free abelian groups is not definable in any $\mathscr{L}$ which has the following downward Löwenheim–Skolem property: there is a cardinal λ such that for every sentence θ of $\mathscr{L}$, if $\mathfrak{A} \models \theta$, then $\mathfrak{B} \models \theta$, for some substructure $\mathfrak{B}$ of $\mathfrak{A}$ of cardinality $\leq \lambda$. Suppose there is a sentence θ of $\mathscr{L}$ which is true in a group G iff G is a free abelian group, and let λ be as above. By (vii), there is a non-free $\mathscr{L}_{\infty\kappa^+}$-free abelian group A of cardinality κ^+ for some $\kappa \geq \lambda$. So $A \models \neg\theta$. But every subgroup of A of cardinality $\leq \lambda$ is free and hence satisfies θ—a contradiction. ☐

For Theorem 5.5(vii) and Corollary 5.6, the crucial fact from the theory of the core model is that if there is a largest κ such that $\square_{\kappa^+}$ holds, then there is an inner model with many measurable cardinals.

6. Concrete Algebraic Constructions

Using notions from $\mathscr{L}_{\infty\kappa}$, Hodges gave a formalization of the intuitive idea of an effective algebraic construction and used it in conjunction with set-theoretic methods to give a negative answer to Taylor's question (Taylor [1971]) as to whether or not there is a concrete construction of the pure injective hull of every abelian group.

6.1 Notation. Let $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ be vocabularies consisting of function symbols (possibly of infinite arity), and let Alg[$\boldsymbol{\sigma}$] (respectively Alg[$\boldsymbol{\tau}$]) be the category of σ-structures (respectively $\boldsymbol{\tau}$-structures). Let $\mathscr{B}$ (respectively $\mathscr{C}$) be a quasivariety in Alg[$\boldsymbol{\tau}$] (respectively in Alg[$\boldsymbol{\sigma}$]) (see Definition 3.1). The following definition is a formalization of the intuitive idea of a construction which is uniformly definable by generators and relations (see Hodges [1975]).

6.2 Definition. Let $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$, $\mathscr{B}$, $\mathscr{C}$ be as in Notation 6.1, and let κ be a regular cardinal. A function F from objects of $\mathscr{B}$ to objects of $\mathscr{C}$ is a *κ-word-construction* if there is a vocabulary $\boldsymbol{\sigma}'$ extending $\boldsymbol{\sigma}$, a set T of terms of $\mathscr{L}[\boldsymbol{\sigma}']$, a set A of atomic formulas of $\mathscr{L}[\sigma']$, and a function $\Gamma\colon T \cup A \to \mathscr{L}_{\infty\kappa}[\boldsymbol{\tau}]$ such that, for all $\alpha \in T \cup A$, $\Gamma(\alpha)$ has free variables among those in α; and, for all $\mathfrak{B}$ in $\mathscr{B}$, $F(\mathfrak{B})$ is isomorphic to $df\langle X, \Phi\rangle$, the structure given by the presentation $\langle X, \Phi\rangle$ where $X = X^{\mathfrak{B}}$, the set of generators, is $\{t(\vec{b})\colon t \in T,\ \mathfrak{B} \models \Gamma(t)[\vec{b}]\}$ and $\Phi = \Phi^{\mathfrak{B}}$, the set of relations, is $\{\varphi(\vec{b})\colon \varphi \in A,\ \mathfrak{B} \models \Gamma(\varphi)[\vec{b}]\}$. (Let $\vec{b}$ run over all sequences of elments of $\mathfrak{B}$ of length $<\kappa$). More precisely, $df\langle X, \Phi\rangle$ is the structure whose universe is the closure $\bar{X}$ of X under the function symbols of σ, modulo the equivalence relation $\sim$ on $\bar{X}$ defined by

$$x_1 \sim x_2 \quad \text{iff} \quad \Phi \models x_1 = x_2;$$

if $\tilde{x}$ denotes the equivalence class of $x \in \bar{X}$, the operations on $df\langle X, \Phi\rangle$ are given by: if f is a n-ary function symbol of σ,

$$f(\tilde{x}_1, \ldots, \tilde{x}_n) = f(x_1, \ldots, x_n)^{\sim}$$

F is a *word-construction* if it is a κ-word-construction for some κ.

6.3 Examples. (1) Let $\mathscr{B} = \mathscr{C} =$ the variety of rings in the vocabulary $\boldsymbol{\sigma} = \boldsymbol{\tau} = \{+, \cdot\}$. We will now show that the function F which takes a ring B to the formal power series ring $B[[Y]]$ is an ω_1-word construction. Let $\boldsymbol{\sigma}'$ add to $\boldsymbol{\sigma}$ the extra ω-ary function symbol p. Let $T = \{p(v_0, v_1, v_2, \ldots)\}$ and let $\Gamma(p(v_0, v_1, v_2, \ldots))$ be $\forall x(x = x)$. Hence, $X = \{p(\vec{b}) : \vec{b} \in B^\omega\}$. Let $A = \{\varphi_1, \varphi_2\}$, where φ_1 is $p(v_0, v_1, v_2, \ldots) + p(u_0, u_1, u_2, \ldots) = p(w_0, w_1, w_2, \ldots)$, and φ_2 is $p(v_0, v_1, v_2, \ldots) \cdot p(u_0, u_1, u_2, \ldots) = p(w_0, w_1, w_2, \ldots)$. Let $\Gamma(\varphi_1)$ be $\bigwedge_{i \in \omega}(v_i + u_i = w_i)$, and let $\Gamma(\varphi_2)$ be $\bigwedge_{j \in \omega}(w_j = \sum_{l+k=j} v_l + u_k)$. Observe that these are formulas of $\mathscr{L}_{\infty\omega_1}$ but not of $\mathscr{L}_{\infty\omega}$, since they have infinitely many free variables. Now it is easy to check that $df\langle X^B, \Phi^B\rangle \cong B[[Y]]$.

(2) Let $\mathscr{B} =$ the variety of sets; that is, $\mathscr{B} = \mathrm{Alg}[\boldsymbol{\tau}]$, where $\boldsymbol{\tau} = \varnothing$ and $\mathscr{C} =$ the variety of groups ($\subseteq \mathrm{Alg}[\boldsymbol{\sigma}]$, where $\boldsymbol{\sigma} = \{\cdot\}$). We shall show that the function F which takes a set B to the free group on B is an ω-word construction. Let $\boldsymbol{\sigma}'$ be obtained by adding to $\boldsymbol{\sigma}$ a unary function symbol i, and a 0-ary function symbol (constant) e. Let T be the set of all terms in $\mathscr{L}[\boldsymbol{\sigma}']$ and let $\Gamma(t)$ be $\forall x(x = x)$, for all $t \in T$. Let $A = \{v \cdot i(v) = e,\ i(v) \cdot v = e,\ e \cdot v = v,\ v \cdot e = v,\ v_1 \cdot (v_2 \cdot v_3) = (v_1 \cdot v_2) \cdot v_3\}$; and, for each $\varphi \in A$, let $\Gamma(\varphi)$ be $\forall x(x = x)$. Then $df(X^B, \Phi^B)$ is the free group on B.

Other examples of word-constructions are the following—the first three being ω-word-constructions, and the last an ω_1-word-construction.

(3) An integral domain to its quotient field (the example is worked out in Hodges [1975, Example 6]).

(4) An ordered field to its real closure (see Hodges [1976, Theorem 2.1]).

(5) A valued field to its Henselization (see Hodges [1976, Theorem 2.4]).

(6) A rank 1 valued field to its completion (see Hodges [1976, Theorem 2.6]).

By using several sorts, the word construction can be defined so that (for example, in Example (4)) it gives the embedding $F \to \tilde{\tilde{F}}$ of F in its real closure.

Hodges [1975] advances the thesis that every effective—or, synonymously, concrete—construction occurring naturally in algebra can be put into the form of a word-construction. That word-constructions are effective is given by the following result. Let $\mathscr{P}_{<\kappa}$ denote the function given by $\mathscr{P}_{<\kappa}(X) = \{Y: Y \subseteq X$ and $\mathrm{Card}(Y) < \kappa\}$.

6.4 Theorem (Hodges [1975]). *If $F: \mathscr{B} \to \mathscr{C}$ is a κ-word-construction, then F is provably $\Sigma_1(\mathscr{P}_{<\kappa})$. That is, there is a formula $\theta(x, y)$ in the language of set theory including the symbol $\mathscr{P}_{<\kappa}$ (possibly with parameters) which has all universal quantifiers bounded and which satisfies*

$$\mathrm{ZF} + \textit{definition of } \mathscr{P}_{<\kappa} \vdash \forall x \exists !\, y\theta(x, y)$$

and for all $\mathfrak{B} \in \mathscr{B}$,

$$\mathrm{ZF} + \textit{definition of } \mathscr{P}_{<\kappa} \vdash \theta(\mathfrak{B}, F(\mathscr{L})). \quad \square$$

Hodges [1975] also proves that κ-word-constructions preserve $\mathscr{L}_{\infty\kappa}$-equivalence, and discusses connections with Feferman [1972] Eklof [1973, 1975a] and Gaifman [1974].

The following result provides a useful algebraic method of proving that certain constructions are word-constructions (see, Hodges [1980a, Lemma]).

6.5 Lemma. *Let $\mathscr{B}$, $\mathscr{C}$, $\boldsymbol{\sigma}$, $\boldsymbol{\tau}$ be as in Notation 6.1 and let κ be a regular cardinal. If $F: \mathscr{B} \to \mathscr{C}$ is a functor which preserves κ-direct limits (see Definition 3.2) then F is a κ-word-construction.*

Proof. Any structure $\mathfrak{B}$ is the κ-direct limit of $\mathscr{D}(\mathfrak{B})$, the κ-directed diagram of the κ-generated—that is, is generated by fewer than κ elements—substructures of $\mathfrak{B}$, where the maps between substructures are inclusions. So it suffices to define a word construction which sends every $\mathfrak{B}$ to the κ-direct limit of $F(\mathscr{D}(\mathfrak{B}))$. To do this, let $\{\mathfrak{B}_\nu: \nu \in \lambda\}$ be the set of all κ-generated substructures of $\mathfrak{B}$, and for each $\mathfrak{B}_\nu$ let $f_\nu: \rho_\nu \to \mathfrak{B}_\nu$ be a function ($\rho_\nu < \kappa$) whose image is a set of generators of $\mathfrak{B}_\nu$. Then we extend $\boldsymbol{\sigma}$ to $\boldsymbol{\sigma}'$ by adding a set T of function symbols $\zeta_{\nu,c}$ where c ranges over all elements of $F(\mathfrak{B}_\nu)$ and the arity of $\zeta_{\nu,c}$ is ρ_ν. Let A be the set of all atomic formulas of $\mathscr{L}[\boldsymbol{\sigma}']$. We claim that, for all $\mathfrak{B}$ in $\mathscr{B}$, the κ-direct limit of $F(\mathfrak{D}(\mathfrak{B}))$ is $df\langle X, \Phi\rangle$, where X is the set of all $\zeta_{\nu,c}(\vec{b})$ and where the map $f_\nu(i) \mapsto b_i$, for $i < \rho$, induces an isomorphism of $\mathfrak{B}_\nu$ to the substructure $\langle \vec{b}\rangle$ of $\mathfrak{B}$ generated by $\vec{b}$; and where, furthermore, Φ consists of atomic formulas which are of the form $\varphi(\zeta_{\nu,c_1}(\vec{b}), \ldots, \zeta_{\nu,c_n}(\vec{b}))$, where the $\zeta_{\nu,c_i}(\vec{b})$ are in X and $F(\mathfrak{B}_\nu) \models \varphi[c_1, \ldots, c_n]$, or are of the form $\zeta_{\nu,c}(\vec{b}) = \zeta_{\mu,e}(\vec{d})$ (both terms in X), where there is an inclusion $\iota: \langle \vec{b}\rangle \to \langle \vec{d}\rangle$ and $F(\iota)c = e$; or are logical consequences in the quasivariety $\mathscr{C}$ of formulas of these forms. We leave it to the reader to verify that there is a function $\Gamma: T \cup A \to \mathscr{L}_{\infty\kappa}[\tau]$ which determines X and Φ as in Definition 6.2 (see Hodges [1975, pp. 457 ff] for details). $\square$

Note that, in fact, we need only that F preserve κ-direct limits over diagrams whose map are monomorphisms. For example, for any right R-module M, the functor which takes a left R-module N to the abelian group $M \otimes_R N$ preserves ω-direct limits and is thus an ω-word construction. Observe that this functor is not ω-local, but does preserve $\mathscr{L}_{\infty\omega}$-equivalence (see Section 3).

Let Div be the functor which takes an abelian group A to the push-out diagram illustrated below

$$\begin{array}{ccc} \bigoplus_{a \in A} \mathbb{Q}c_a & \longrightarrow & \mathrm{Div}'(A) \\ {\scriptstyle i}\uparrow & & \uparrow \\ \bigoplus_{a \in A} \mathbb{Z}c_a & \underset{g}{\longrightarrow} & A \end{array}$$

where i is inclusion and g takes c_a to a. It is not hard to check that Div preserves ω-direct-limits. Hence, by Lemma 6.5 there is a concrete construction which takes every abelian group A to an embedding of A into a divisible group, $\mathrm{Div}'(A)$, containing A. On the other hand, we have

6.6 Theorem (Hodges [1980a, Corollary 5]). *There is no word-construction F on the variety of abelian groups such that for all A, $F(A)$ is an embedding of A in a divisible hull of A.*

Sketch of Proof. Suppose to the contrary, that there is such an F. Then, using the definition of a word-construction, it is easy to see that F induces an embedding of the automorphism group of A into the automorphism group of $F(A)$. One obtains a contradiction by taking $A = \mathbb{Z}_5 \oplus \mathbb{Z}_5$, the direct sum of two copies of the cyclic group of order 5, and by showing—via a direct computation—that the automorphism of A, given by the matrix,

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

has order 5 but has no extension to the divisible hull—or even to $\mathbb{Z}_{25} \oplus \mathbb{Z}_{25}$—which has order 5. (This argument—though different from the one in Hodges [1980a]—is also due to Hodges). $\square$

As a corollary, Hodges [1980a, Theorem 6] also gives a negative answer to Taylor's question: if there were a word-construction sending an abelian group A to an embedding of A in a pure-injective hull of A, then, using some constructions satisfying the hypothesis of Lemma 6.5 one could define a word-construction of divisible hulls.

Other negative results are given in Hodges [1976], such as, for example, that there is no word-construction sending a field to its algebraic closure, or a formally real field to its real closure (the reader should compare this to Example 6.3(4)).

7. *Miscellany*

Here we mention a few other examples of the interaction of infinitary logic and algebra.

One important application is Shelah's construction of arbitrarily large rigid real closed fields. This result uses infinitary logic in a general construction that has a variety of other uses (see Shelah [1983d]).

The model theory of $\mathscr{L}_{\omega_1\omega}$ has also been applied to group theory by Kopperman–Mathias [1968]. There use was made of the downward Löwenheim–Skolem notions for $\mathscr{L}_{\omega_1\omega}$ to give new proofs of results of Hall, results showing that certain classes of groups are bountiful, where a class $\mathscr{C}$ of groups is called *bountiful* if whenever $G \subseteq H$ and $H \in \mathscr{C}$, then there exists $H' \in \mathscr{C}$ such that $G \subseteq H'$ and $|H'| = |G| + \aleph_0$.

Dickman has analyzed the Erdös–Gillman–Henriksen isomorphism theorem for real closed fields from the point of view of the back-and-forth method (see Chapter IX, Theorems 4.5.8 and 4.5.9).

Using the model theory of $\mathscr{L}_{\infty\omega}$, Eklof [1977b] contains a new proof of a result of Hill characterizing the classes of abelian groups closed under substructures and direct limits.

Eklof and Sabbagh [1971] discuss $\mathscr{L}_{\infty\omega}$-equivalence and $\mathscr{L}_{\infty\omega}$-definability for various classes of modules and rings. For example, it is proved that the class of Noetherian rings is not definable in $\mathscr{L}_{\infty\omega}$. But it is definable in $\mathscr{L}_{\omega_1\omega_1}$ (see Kopperman [1969]). An algebraic result of Gordon and Robson [1973, Theorem 9.8] implies that the class $\mathscr{C}$ of *commutative* Noetherian rings is not definable in $\mathscr{L}_{\infty\omega}$. (The argument in Eklof–Sabbagh [1971, p. 644] immediately implies that $\mathscr{C}$ is not definable in $\mathscr{L}_{\omega_1\omega}$, but an argument found by Hodges—an argument which uses the ordinal rank of prime ideals—yields the stronger conclusion).