Chapter X

Jumps of Minimal Degrees

Jump inversion theorems are used to characterize the range of the jump operator on various classes of degrees. In Chap. III, we proved two such theorems. The Friedberg Jump Inversion Theorem classified $\mathbf{J}[\mathbf{0}, \infty)$, the range of the jump operator on $\mathbf{D}[\mathbf{0}, \infty)$, as $\mathbf{D}[\mathbf{0}', \infty)$. And the Shoenfield Jump Inversion Theorem classified $\mathbf{J}[\mathbf{0}, \mathbf{0}']$, the range of the jump operator on $\mathbf{D}[\mathbf{0}, \mathbf{0}']$, as $\mathbf{D}[\mathbf{0}', \mathbf{0}^{(2)}] \cap \{\mathbf{d} : \mathbf{d}$ is recursively enumerable in $\mathbf{0}'\}$. This chapter is devoted to a proof of the Cooper Jump Inversion Theorem which classifies $\mathbf{J}(\mathbf{M})$, the range of the jump operator on the class of minimal degrees, as $\mathbf{D}[\mathbf{0}', \infty)$. This result contrasts sharply with the classification problem for $\mathbf{J}(\mathbf{M}[\mathbf{0}, \mathbf{0}'])$, the range of the jump operator on the class of minimal degrees below $\mathbf{0}'$, a problem which is still unsolved. The natural analogy would be to guess that $\mathbf{J}(\mathbf{M}[\mathbf{0}, \mathbf{0}']) = \mathbf{D}[\mathbf{0}', \mathbf{0}^{(2)}] \cap \{\mathbf{d} : \mathbf{d}$ is recursively enumerable in $\mathbf{0}'\}$. However, by IV.3.6, if $\mathbf{d} \in \mathbf{J}(\mathbf{M}[\mathbf{0}, \mathbf{0}'])$ then $\mathbf{d}' = \mathbf{0}^{(2)}$, so this guess is incorrect. Jockusch has conjectured that $\mathbf{J}(\mathbf{M}[\mathbf{0}, \mathbf{0}']) = \{\mathbf{d} : \mathbf{d} \geqslant \mathbf{0}' \;\&\; \mathbf{d}' = \mathbf{0}^{(2)} \;\&\; \mathbf{d}$ is recursively enumerable in $\mathbf{0}'\}$.

1. Targets

The strategy for proving the Cooper Jump Inversion Theorem is to combine the construction of a minimal degree using partial trees with the ideas introduced in the proof of the Friedberg Jump Inversion Theorem (III.4.2), making certain important modifications. One of these modifications involves defining a *jump target function*, which we do in this section. The proof of Cooper's theorem is presented in Sect. 2.

Given $C \subseteq N$, we build a set A such that $A' \equiv_T C \oplus \emptyset'$ as the union of a sequence $\{\alpha_s : s \in N\}$ of binary strings, through the use of an oracle of degree $\mathbf{0}'$. At stage e of the construction, we try to resolve whether or not $\Phi_e^A(e)\downarrow$. As, for $s > e$, α_s will be constrained to lie on certain partial recursive trees, we will not be able to ask an oracle of degree $\mathbf{0}'$ the same question as we asked in the proof of the Friedberg Jump Inversion Theorem. For with most reasonable recursively defined conditions, the search for a string which satisfies these conditions and which is not terminal on a given partial recursive tree requires an appeal to an oracle of degree $\mathbf{0}^{(2)}$. We therefore ask a different question, and insure that the answer to the new question at stage e of the construction will be the same as the answer to the question "$\Phi_e^A(e)\downarrow$?". We ask if we can find an extension α of α_e for which $\Phi_e^\alpha(e)\downarrow$ and which is *potentially* on a certain tree T. Potentially will mean that we are looking at a recursive approximation $\{T_s : s \in N\}$ to T and we seek $s \in N$ and α of length $\leqslant s$ which is either on T_s or extends a terminal branch σ of T_s. In order to insure that the answers to the two questions are the same, we require that if $\sigma \subset A$ then $\alpha \subset A$. Thus we define a *target function* for T which, when given σ and e, outputs the e-target α towards which any extension of σ on T must head at stages $\geqslant e$. The use of targets is similar to that in III.5.6.

Given an index i for a partial recursive tree T_i, a recursive approximation $\{T_{i,s}: s \in N\}$ to T_i is generated in a natural way, where $T_i = \cup\{T_{i,s}: s \in N\}$, $T_{i,s+1}$ extends $T_{i,s}$ and $T_{i,s}$ is finite for all $s \in N$. Similarly, $\lambda \in \mathscr{S}$ (the set of strings) can be thought of as coding recursive approximations to $\{T_i: i < \mathrm{lh}(\lambda)\}$ where $\lambda(i)$ is an index for the partial recursive tree T_i. Each tree used in this chapter is specified through a particular recursive enumeration. Hence no confusion should arise when we identify a partial recursive tree with an index for one of its enumerations, or if we identify a finite sequence of partial recursive trees with $\lambda \in \mathscr{S}$ coding indices for the trees in the sequence.

The following definition will be useful in defining the jump target function.

1.1 Definition. Let T be a tree and let $\beta \in \mathscr{S}_2$ be given. Then β is *compatible* with T if either $\beta \subseteq T(\xi)$ for some $\xi \in \mathscr{S}_2$ such that $T(\xi)\downarrow$, or $\beta \supset \sigma$ for some terminal $\sigma \subset T$.

Target functions point the way to leave a tree T if A must leave T. However, T may be a subtree of another partial tree T^* which A may also be forced to leave, so T^* must have its own target function. In order to successfully combine the use of partial trees with target functions in this setting, these target functions will have to be mutually consistent. Thus a target function for T cannot consider T in isolation. Rather, it will depend on a finite sequence of trees $\mathrm{Id}_2 = T_0 \supseteq T_1 \supseteq \cdots \supseteq T_k = T$. We will need to specify an index for computing $T_0 = \mathrm{Id}_2$ in order to begin the construction. Thus we specify the particular recursive approximation $\{\mathrm{Id}_{2,s}: s \in N\}$ to Id_2 defined by

$$\mathrm{Id}_{2,s}(\sigma) = \begin{cases} \sigma & \text{if } \mathrm{lh}(\sigma) \leqslant s \\ \uparrow & \text{otherwise.} \end{cases}$$

The target function used to prove the Cooper Jump Inversion Theorem is now introduced. Recall that for all $\lambda \in \mathscr{S}$ for which $\mathrm{lh}(\lambda) > 0$, $\lambda^- = \lambda \upharpoonright (\mathrm{lh}(\lambda) - 1)$. We will also use λ_j to denote $\lambda \upharpoonright j + 1$ for $j < \mathrm{lh}(\lambda)$. Thus if λ codes the sequence of trees $T_0, T_1, \ldots, T_m$ and $j \leqslant m$, then λ_j codes $T_0, T_1, \ldots, T_j$.

1.2 Definition. The *jump target function* $f: \mathscr{S}_2 \times \mathscr{S} \times N^2 \to \mathscr{S}_2$ is defined by induction on $\mathrm{lh}(\lambda) - 1$ for those $\lambda \in \mathscr{S}$ coding sequences of trees $\{T_i: i \leqslant m = \mathrm{lh}(\lambda) - 1\}$ such that $T_{i+1,s} \subseteq T_{i,s}$ for all $s \in N$ and $i < m$, and then by subinduction on $\{s: s \in N\}$. $f(\sigma, \lambda, n, s)$ produces the n-target for σ at stage s with respect to the sequence of trees coded by λ. This n-target will specify a string τ which *forces n into the jump*, i.e., $\Phi_n^\tau(n)\downarrow$. Fix a recursive one-one correspondence $\{\sigma_i: i \in N\}$ of $\mathscr{S}_2$ with N such that for all $i, j \in N$, if $\mathrm{lh}(\sigma_i) < \mathrm{lh}(\sigma_j)$ then $i < j$.

Stage 0. $\mathrm{lh}(\lambda) = 1$ and λ codes $\{\mathrm{Id}_{2,s}: s \in N\}$. We proceed by induction on s, defining $f(\sigma, \lambda, n, s) = f(\sigma, \lambda, n, s-1)$ if $s > 0$ and $f(\sigma, \lambda, n, s-1)\downarrow$. Otherwise, we search for the least i such that $\sigma_i \subset \mathrm{Id}_{2,s}$, $\sigma \subseteq \sigma_i$ and $\Phi_n^{\sigma_i}(n)\downarrow$, and set $f(\sigma, \lambda, n, s) = \sigma_i$. If no such i exists, then $f(\sigma, \lambda, n, s)\uparrow$. Thus at stage 0, we have defined targets for all strings of length $\leqslant s$ for which a potential target of length $\leqslant s$ exists, as $T_{0,s} = \mathrm{Id}_{2,s}$. By our compatibility constraints, we will always choose targets at subsequent stages from the list of targets already found, and so will only have to worry about making our choices consistently at later stages.

Stage $m > 0$. $\mathrm{lh}(\lambda) = m + 1$ and λ codes $\{T_i: i \leqslant m\}$. If there is a $\xi \in \mathscr{S}_2$ such that for all t, if $T_{m,t} \neq \emptyset$ then $T_{m,t} = \mathrm{PExt}_2(T_{m-1,t}, \xi)$, let $f(\sigma, \lambda, n, s) = f(\sigma, \lambda^-, n, s)$ for all σ, n and s such that $\sigma \subset T_{m,s}$ and $f(\sigma, \lambda^-, n, s)\downarrow$. Otherwise, we assume by induction that $f(\sigma, \lambda^-, n, s)$ has been defined for all $\sigma \in \mathscr{S}_2$ and $n, s \in N$ such that $f(\sigma, \lambda^-, n, s)\downarrow$.

Substage s. Proceed by induction on $\mathrm{lh}(\sigma)$. If $\sigma \not\subset T_{m,s}$ (in which case a definition is irrelevant) or either $n > s$ or $\mathrm{lh}(\sigma) > s$ (and so a definition is premature), then $f(\sigma, \lambda, n, s)\uparrow$. Thus fix $\sigma \subset T_{m,s}$ such that $\mathrm{lh}(\sigma) \leqslant s$ and $n \in N$ such that $n \leqslant s$. We proceed by cases.

Case 1. $f(\sigma, \lambda, n, s-1)\downarrow$ and is compatible with $T_{j,s}$ for all $j \leqslant m$. Let $f(\sigma, \lambda, n, s) = f(\sigma, \lambda, n, s-1)$. In this case, we preserve the previous definition, which remains compatible with all trees of sufficiently high priority.

Case 2. Case 1 is not followed, $\mathrm{lh}(\sigma) > 0$, and there is a $\delta \subset \sigma$ such that $f(\delta, \lambda, n, s)\downarrow \supseteq \sigma$. Fix such a δ of shortest length, and let $f(\sigma, \lambda, n, s) = f(\delta, \lambda, n, s)$. In this case, we define the n-target of σ at stage s to be the same as the n-target specified by some $\delta \subset \sigma$.

Case 3. Neither Case 1 nor Case 2 is followed and there is a $\tau \supset \sigma$ such that $f(\tau, \lambda, n, s-1)\downarrow$ and is compatible with $T_{j,s}$ for all $j \leqslant m$. Fix the least such τ and let $f(\sigma, \lambda, n, s) = f(\tau, \lambda, n, s-1)$. In this case, we will later define $f(\tau, \lambda, n, s) = f(\tau, \lambda, n, s-1)$ through Case 1, and will want σ and τ to have the same n-target.

Case 4. None of the first three cases is followed and there is a $\gamma \subset T_{m-1,s}$ such that $f(\gamma, \lambda^-, n, s)\downarrow \supseteq \sigma$ and is compatible with $T_{m,s}$. Fix the least such γ and let $f(\sigma, \lambda, n, s) = f(\gamma, \lambda^-, n, s)$. In this case, we choose a suitable n-target from the n-targets of the previous tree.

Case 5. Otherwise. Then $f(\sigma, \lambda, n, s)\uparrow$. No definition is possible here if the n-target of σ is to force the jump on n and be compatible with T_j for all $j \leqslant m$.

For the remainder of this chapter, we fix f as the jump target function introduced in Definition 1.2.

The next lemma summarizes the important properties of the jump target function.

1.3 Lemma. *The jump target function f is a partial recursive function with recursive domain. Fix $\sigma \in \mathscr{S}_2, n, s \in N$ and $\lambda \in \mathscr{S}$ such that λ codes $\{T_i: i \leqslant m\}$. If $f(\sigma, \lambda, n, s)\downarrow$, then*:

(i) $\Phi_n^{f(\sigma,\lambda,n,s)}(n)\downarrow$ & $n \leqslant s$ & $\mathrm{lh}(\sigma) \leqslant s$ & $\sigma \subset T_{m,s}$ & $\sigma \subseteq f(\sigma, \lambda, n, s)$ & $f(\sigma, \lambda_0, n, s) \subset T_{0,s}$. (*This condition places an effective bound on the domain of f, and stipulates that the n-target of a string always extends that string and is on T_0.*)

(ii) $\mathrm{lh}(\lambda) > 1 \to \exists \gamma \in \mathscr{S}_2(f(\sigma, \lambda, n, s) = f(\gamma, \lambda^-, n, s))$. (*Thus the range of f on $T_{m,s}$ is contained in the range of f on $T_{m-1,s}$.*)

(iii) $\forall j \leqslant m(f(\sigma, \lambda, n, s)$ *is compatible with* $T_{j,s})$. (*Thus the n-target for σ for a particular tree is eligible to be placed on all previous trees in the sequence of trees.*)

(iv) $f(\tau, \lambda, n, s-1)\downarrow$ *and is compatible with* $T_{j,s}$ *for all* $j \leqslant m \to f(\tau, \lambda, n, s)\downarrow = f(\tau, \lambda, n, s-1)$. (*Thus once defined, n-targets do not change unless an incompatibility with a previous tree in the sequence of trees is discovered.*)

(v) $\forall \delta \in \mathscr{S}_2(\delta \subseteq \sigma \,\&\, \delta \subset T_{m,s} \to f(\delta, \lambda, n, s)\downarrow)$. (*Thus the property of having an n-target is closed under inclusion for strings on a given tree. Warning: It is possible for* $\delta \subset \sigma \subset T_{m,s}$ *and yet* $\delta \not\subset T_{m,s}$. *For* $\delta \subset T_{m,s}$ *says that* δ *is in the range of* $T_{m,s}$, *but* $\delta \subset \sigma \subset T_{m,s}$ *says that* δ *is compatible with the range of* $T_{m,s}$.)

(vi) $\forall \tau \subset T_{m,s}(\Phi_n^\tau(n)\downarrow \to f(\tau, \lambda, n, s)\downarrow = \tau)$. (*Thus if τ forces the jump on n, then τ has itself as an n-target.*)

(vii) $\forall \tau \subset T_{m,s}(\sigma \subseteq \tau \subseteq f(\sigma, \lambda, n, s) \to f(\tau, \lambda, n, s)\downarrow = f(\sigma, \lambda, n, s))$. (*This condition stipulates that if τ is contained in the n-target of σ, then τ has the same n-target as σ.*)

(viii) $m > 0 \to f(\sigma, \lambda^-, n, s)\downarrow$. (*Thus the assignment of n-targets to strings must proceed tree by tree in the sequence of trees.*)

(ix) $\mathrm{lh}(\lambda) > 1 \,\&\, \sigma$ *terminal on* $T_{m,s} \to f(\sigma, \lambda, n, s) = f(\sigma, \lambda^-, n, s)$. (*Thus we can specify γ in* (ii) *when σ is terminal on* $T_{m,s}$.)

(x) $\exists \xi \in \mathscr{S}_2 \,\forall t(T_{m,t} \neq \emptyset \to T_{m,t} = \mathrm{PExt}_2(T_{m-1,t}, \xi)) \to f(\sigma, \lambda, n, s) = f(\sigma, \lambda^-, n, s)$.

Also, if $m > 0$ *and* $f(\sigma, \lambda^-, n, s)\downarrow$ *and* $\delta \subseteq \sigma$ *and* $\delta \subset T_{m,s}$, *then*:

(xi) *If* $f(\sigma, \lambda^-, n, s)$ *is compatible with* $T_{m,s}$ *then* $f(\delta, \lambda, n, s)\downarrow$. (*This condition asserts that if there is an n-target of a string on* T_{m-1} *which is a suitable choice for the n-target of δ, then such an n-target from* T_{m-1} *is chosen as the n-target for δ. There may be many possible choices for the n-target for δ coming from* T_{m-1}, *so we cannot specify this n-target.*)

Proof. The proof is by induction, first on $\mathrm{lh}(\lambda)$, then on s, and finally on $\mathrm{lh}(\sigma)$. Fix m, σ, n, s, λ and $\{\lambda_j : j \leqslant m\}$ as in the hypothesis of the lemma. If $f(\sigma, \lambda, n, s)\downarrow$, then $\Phi_n^{f(\sigma,\lambda,n,s)}(n)\downarrow$ by induction and the definition of f at stage 0.

(i)–(xi) are easily verified if for some $\xi \in \mathscr{S}_2$ and all $t \in N$, if $T_{m,t} \neq \emptyset$ then $T_{m,t} = \mathrm{PExt}_2(T_{m-1,t}, \xi)$. Assume this not to be the case, and assume that $f(\sigma, \lambda, n, s)\downarrow$.

(i) Immediate from Definition 1.2 and induction. It thus also follows that f is partial recursive with recursive domain.

(ii) Assume that $\mathrm{lh}(\lambda) \geqslant 1$. If $f(\sigma, \lambda, n, s)$ is defined through Case 1 or Case 3 of Definition 1.2, then there is a $\tau \supseteq \sigma$ such that $f(\sigma, \lambda, n, s) = f(\tau, \lambda, n, s-1)$. By induction on s applied to (ii), there is a $\gamma \subset T_{m-1,s-1}$ such that $f(\tau, \lambda, n, s-1) = f(\gamma, \lambda^-, n, s-1)$. By Cases 1 and 3, $f(\gamma, \lambda^-, n, s-1) = f(\tau, \lambda, n, s)$ is compatible with $T_{j,s}$ for all $j \leqslant m$, hence applying (iv) by induction on λ, $f(\gamma, \lambda^-, n, s) = f(\gamma, \lambda^-, n, s-1) = f(\sigma, \lambda, n, s)$. If $f(\sigma, \lambda, n, s)$ is defined through Case 2 of Definition 1.2, then there is a $\delta \subset \sigma$ such that $f(\sigma, \lambda, n, s) = f(\delta, \lambda, n, s)$. By induction, $f(\delta, \lambda, n, s) = f(\gamma, \lambda^-, n, s)$ for some $\gamma \subset T_{m-1,s}$. Finally, (ii) is immediate if $f(\sigma, \lambda, n, s)$ is defined through Case 4 of Definition 1.2.

(iii) If we perform a case by case analysis of Definition 1.2, it will follow by induction that $f(\sigma, \lambda, n, s)$ is compatible with $T_{j,s}$ for all $j \leqslant m$.

(iv) Immediate from Case 1 of Definition 1.2.

(v) Clear if $\mathrm{lh}(\lambda) = 1$. Otherwise, we note that by (i), (ii) and (iii), there is a $\gamma \subset T_{m-1,s}$ such that $\delta \subseteq \sigma \subseteq f(\sigma, \lambda, n, s) = f(\gamma, \lambda^-, n, s)$ and $f(\gamma, \lambda^-, n, s)$ is compatible with $T_{m,s}$. Hence $f(\sigma, \lambda, n, s)$ will be defined through Case 4 of Definition 1.2 if it is not defined through an earlier case.

(vii) Let $\tau \subset T_{m,s}$ be given such that $\sigma \subset \tau \subseteq f(\sigma, \lambda, n, s)$. We note that $f(\tau, \lambda, n, s)$ will be defined through Case 2 of Definition 1.2 if it is not defined through Case 1 of Definition 1.2, and, for Case 2, there will be a $\delta \subset T_{m,s}$ such that $\delta \subseteq \tau$ and $f(\tau, \lambda, n, s) = f(\delta, \lambda, n, s)$. Since $\delta \subseteq \sigma \subseteq \tau \subseteq f(\delta, \lambda, n, s) = f(\tau, \lambda, n, s)$, applying (vii) by induction we see that $f(\sigma, \lambda, n, s) = f(\delta, \lambda, n, s) = f(\tau, \lambda, n, s)$. It thus remains to consider the case where $f(\tau, \lambda, n, s)$ is defined through Case 1 of Definition 1.2, and so $f(\tau, \lambda, n, s) = f(\tau, \lambda, n, s-1)$. We must now consider the case which was used to define $f(\sigma, \lambda, n, s)$.

Suppose that $f(\sigma, \lambda, n, s)$ was defined through Case 1 of Definition 1.2. Then $f(\sigma, \lambda, n, s) = f(\sigma, \lambda, n, s-1)$. Hence $\sigma \subset \tau \subseteq f(\sigma, \lambda, n, s) = f(\sigma, \lambda, n, s-1)$. By induction on s, $f(\tau, \lambda, n, s-1) = f(\sigma, \lambda, n, s-1)$. Hence $f(\tau, \lambda, n, s) = f(\tau, \lambda, n, s-1) = f(\sigma, \lambda, n, s-1) = f(\sigma, \lambda, n, s)$.

Suppose that $f(\sigma, \lambda, n, s)$ was defined through Case 2 of Definition 1.2. Then there is a $\delta \subset \sigma$ of shortest length such that $\delta \subset T_{m,s}$ and $f(\sigma, \lambda, n, s) = f(\delta, \lambda, n, s)$. Thus $\delta \subset \sigma \subset \tau \subseteq f(\sigma, \lambda, n, s) = f(\delta, \lambda, n, s)$, so by induction, we apply (vii) to δ and τ to obtain $f(\tau, \lambda, n, s) = f(\delta, \lambda, n, s) = f(\sigma, \lambda, n, s)$.

Otherwise, we note that since $f(\tau, \lambda, n, s)$ is defined through Case 1 of Definition 1.2, τ can be used to define $f(\sigma, \lambda, n, s)$ through Case 3 of Definition 1.2. Hence $f(\sigma, \lambda, n, s)$ will be defined through Case 3 of Definition 1.2, and $f(\sigma, \lambda, n, s) = f(\tau', \lambda, n, s-1)$ for some least $\tau' \supset \sigma$ such that $f(\tau', \lambda, n, s-1)\downarrow$. By (i) and the hypothesis for this case, $\tau', \tau \subseteq f(\sigma, \lambda, n, s) = f(\tau', \lambda, n, s-1)$, so $\tau \subseteq \tau'$ or $\tau' \subseteq \tau$. Since τ' was chosen to define $f(\sigma, \lambda, n, s)$, $\tau' \subseteq \tau$. Hence $\tau' \subseteq \tau \subseteq f(\sigma, \lambda, n, s) = f(\tau', \lambda, n, s-1)$, so applying (vii) by induction on s, $f(\sigma, \lambda, n, s) = f(\tau', \lambda, n, s-1) = f(\tau, \lambda, n, s-1) = f(\tau, \lambda, n, s)$.

(vi) Suppose that $\Phi_n^\tau(n)\downarrow$. Fix the least t such that $\tau \subset T_{m,t}$. It suffices to show that $f(\tau, \lambda, n, t) = \tau$; for if $r > t$ and $f(\tau, \lambda, n, r-1) = \tau$, then by (i), $\tau \subset T_{m,r-1} \subseteq T_{m,r}$, so τ is compatible with $T_{j,r}$ for all $j \leqslant m$. Thus by (iv), $f(\tau, \lambda, n, r)$ is defined in Case 1 of Definition 1.2 and $f(\tau, \lambda, n, r) = f(\tau, \lambda, n, r-1) = \tau$.

(vi) follows easily if $\mathrm{lh}(\lambda) = 1$. Assume that $\mathrm{lh}(\lambda) > 1$. Fix the least $r \leqslant t$ such that for some $\delta \subset T_{m,r}$, $f(\tau, \lambda, n, t) = f(\delta, \lambda, n, r)$, and the least $\delta \subset T_{m,r}$ such that $\delta \subseteq \tau$ and $f(\delta, \lambda, n, r) = f(\tau, \lambda, n, t)$. We verify (vi) by showing that $f(\delta, \lambda, n, r) = \tau$. By choice of r, $f(\delta, \lambda, n, r)$ cannot be defined through Cases 1 or 3 of Definition 1.2. By choice of δ, $f(\delta, \lambda, n, r)$ cannot be defined through Case 2 of Definition 1.2. Hence $f(\delta, \lambda, n, r)$ is defined through Case 4 of Definition 1.2, and $f(\delta, \lambda, n, r) = f(\gamma, \lambda^-, n, r)$ for the least γ for which $f(\gamma, \lambda^-, n, r) \supseteq \delta$. Since $f(\tau, \lambda, n, t) = f(\delta, \lambda, n, r) = f(\gamma, \lambda^-, n, r) \supseteq \gamma, \tau$ by (i), γ and τ are comparable. If $\tau \subset \gamma$, then since $\gamma \subset T_{m-1,t}$ and $\tau \subset T_{m,t} \subseteq T_{m-1,t}$, we conclude that $\tau \subset T_{m-1,r}$. By induction, $f(\tau, \lambda^-, n, r) = \tau$. By the minimality of γ and since $\delta \subseteq \tau$, it follows that $\gamma \subseteq \tau$, a contradiction. Hence $\gamma \subseteq \tau$. But then by (i), $\gamma \subseteq \tau \subseteq f(\tau, \lambda, n, t) = f(\gamma, \lambda^-, n, r)$ so applying (vii), $f(\gamma, \lambda^-, n, r) = f(\tau, \lambda^-, n, r) = \tau$. Hence $f(\tau, \lambda, n, t) = f(\gamma, \lambda^-, n, r) = \tau$.

(viii) Assume that $\mathrm{lh}(\lambda) > 1$. By (ii), there is a $\gamma \subset T_{m-1,s}$ such that $f(\sigma,\lambda,n,s) = f(\gamma,\lambda^-,n,s)$. By (i), $\sigma,\gamma \subseteq f(\sigma,\lambda,n,s) = f(\gamma,\lambda^-,n,s)$ so σ and γ are comparable. If $\sigma \subseteq \gamma$, then since $f(\gamma,\lambda^-,n,s)\downarrow$, it follows from (v) that $f(\sigma,\lambda^-,n,s)\downarrow$. Otherwise, $\gamma \subseteq \sigma \subseteq f(\gamma,\lambda^-,n,s)$, so by (vii), $f(\sigma,\lambda^-,n,s)\downarrow = f(\gamma,\lambda^-,n,s)$.

Before proving (ix), we prove the following fact:

(1) σ terminal on $T_{m,s} \Rightarrow \forall r \leqslant s \forall \delta \subset T_{m,r} \forall \tau \subset T_{m-1,r}(\delta \subseteq \sigma\ \&$ $\sigma \subseteq f(\delta,\lambda,n,r)\ \&\ \tau$ is least such that $f(\delta,\lambda,n,r) = f(\tau,\lambda^-,n,r) \rightarrow \tau \subseteq \sigma)$.

Fix r, δ, and τ as in (1). If $f(\delta,\lambda,n,r)$ is defined through Case 1 of Definition 1.2, then there is a $\tau' \subset T_{m-1,r-1}$ such that $f(\delta,\lambda,n,r) = f(\delta,\lambda,n,r-1) = f(\tau',\lambda^-,n,r-1)$. By (iii) applied to $f(\delta,\lambda,n,r)$, $f(\delta,\lambda,n,r) = f(\tau',\lambda^-,n,r-1)$ is compatible with $T_{j,r}$ for all $j \leqslant m$, so we will have defined $f(\tau',\lambda^-,n,r) = f(\tau',\lambda^-,n,r-1)$ through Case 1 of Definition 1.2. If $f(\tau,\lambda^-,n,r) = f(\delta,\lambda,n,r) = f(\tau',\lambda^-,n,r)$, then by (i), $\tau,\tau' \subseteq f(\tau,\lambda^-,n,r) = f(\tau',\lambda^-,n,r)$ so τ and τ' are comparable. Hence by the minimality of τ, $\tau \subseteq \tau'$. Since $\tau' \subseteq \sigma$ by induction on $\mathrm{lh}(\lambda)$, $\tau \subseteq \sigma$.

We now proceed by induction on $\mathrm{lh}(\delta)$, assuming (1) for all $\delta' \subset \delta$ in place of δ and also all δ such that $f(\delta,\lambda,n,r)$ is defined through Case 1 of Definition 1.2.

Suppose that $f(\delta,\lambda,n,r)$ is defined through Case 2 of Defintion 1.2. Then $f(\delta,\lambda,n,r) = f(\delta',\lambda,n,r)$ for some $\delta' \subset \delta$. (1) for δ now follows from (1) for δ' since $\delta' \subseteq \delta \subseteq \sigma \subseteq f(\sigma,\lambda,n,r) = f(\delta',\lambda,n,r)$.

Suppose that $f(\delta,\lambda,n,r)$ is defined through Case 3 of Definition 1.2. Then $f(\delta,\lambda,n,r) = f(\beta,\lambda,n,r-1)$ for some $\beta \supset \delta$ such that $f(\beta,\lambda,n,r) = f(\beta,\lambda,n,r-1)$ is defined through Case 1 of Definition 1.2. By (i) and the hypothesis of (1), $\beta,\sigma \subseteq f(\delta,\lambda,n,r) = f(\beta,\lambda,n,r)$ so β and σ are comparable. It also follows from (i) and since $f(\beta,\lambda,n,r)\downarrow$ that $\beta \subset T_{m,r}$. Since $T_{m,r} \subseteq T_{m,s}$ and σ is terminal on $T_{m,s}$, we cannot have $\beta \supset \sigma$; hence $\beta \subseteq \sigma$. Thus $\beta \subseteq \sigma \subseteq f(\delta,\lambda,n,r) = f(\beta,\lambda,n,r)$, so (1) for δ follows by induction from (1) for β.

Suppose that $f(\delta,\lambda,n,r)$ is defined through Case 4 of Definition 1.2. Since $\delta \subseteq \sigma$ and σ is terminal on $T_{m,s}$, if $\sigma \subset T_{m,r}$ then by (i), if $f(\sigma,\lambda^-,n,r)\downarrow$, then it is compatible with $T_{m,r}$. Hence by the minimality of τ in Case 4 and (v), we cannot have $\tau \supset \sigma$. By (i) and the hypothesis of (1), $\tau,\sigma \subseteq f(\delta,\lambda,n,r) = f(\tau,\lambda^-,n,r)$ so τ and σ are comparable. Hence $\tau \subseteq \sigma$. And if $\sigma \not\subset T_{m,r}$ then since τ and σ are comparable, $\tau \subset T_{m,r}$ and $\sigma \subset T_{m,t}$, $\tau \subset \sigma$.

(ix) Suppose that $\mathrm{lh}(\lambda) > 1$ and σ is terminal on $T_{m,s}$. By (ii), $f(\sigma,\lambda,n,s) = f(\gamma,\lambda^-,n,s)$ for some least $\gamma \subset T_{m-1,s}$. Applying (1) to $\delta = \sigma$, we see that $\gamma \subseteq \sigma$. Thus by (i), $\gamma \subseteq \sigma \subseteq f(\sigma,\lambda,n,s) = f(\gamma,\lambda^-,n,s)$, so by (vii), $f(\sigma,\lambda^-,n,s) = f(\gamma,\lambda^-,n,s) = f(\sigma,\lambda,n,s)$.

(x) Immediate from the construction of Definition 1.2 at stage $m > 0$.

(xi) Fix δ,σ,λ and s as in the hypothesis of the lemma, and assume that $m > 0$ and $f(\sigma,\lambda^-,n,s)\downarrow$ and is compatible with $T_{m,s}$ and $\delta \subseteq \sigma$ and $\delta \subset T_{m,s}$. Then $f(\delta,\lambda,n,s)$ will be defined through Case 4 of Definition 1.2 if it has not previously been defined. ▯

The construction given in the next section depends on sequences of trees which *respect* the jump target function. A preliminary definition is needed.

1.4 Definition. Fix $\sigma \in \mathscr{S}_2$, $n,s \in N$ and $\lambda \in \mathscr{S}$ such that λ codes $\{T_i\colon i \leqslant m\}$ and $\sigma \subset T_{m,s}$. We say that σ *is n-active on* $\{T_i\colon i \leqslant m\}$ *at stage s* if $\sigma \subset f(\sigma,\lambda,n,s)\downarrow$ and for all $e < n$, either $f(\sigma, \lambda, e, s)\uparrow$ or $\sigma = f(\sigma, \lambda, e, s)$. s is *$\langle \sigma, n\rangle$-good* on $\{T_i: i \leqslant m\}$ if for all $e < n$, σ is not e-active on $\{T_i: i \leqslant m\}$ at stage s.

The n-active strings at a given stage are those for which action can be taken at that stage to force n into the jump without ignoring similar desires for $e < n$. A stage is $\langle \sigma, n\rangle$-good if its desire to force n into the jump will not be injured by forcing e into the jump for some $e < n$. The next definition tells us that a sequence of trees *respects* the jump target function if it always acts to force the smallest possible number into the jump, while not violating the properties needed to prove a computation lemma.

1.5 Definition. Let $\lambda \in \mathscr{S}$ code $\{T_i: i \leqslant m\}$. We say that $\{T_i: i \leqslant m\}$ *respects* f if the following conditions hold:

(i) $T_0 = \mathrm{Id}_2$.

(ii) $\forall i < m \forall s \in N(T_{i+1,s} \subseteq T_{i,s} \,\&\, T_{i,s+1}$ extends $T_{i,s})$.

(iii) $\forall \sigma \in \mathscr{S}_2 \forall j, n, s \in N(j \leqslant m \,\&\, n \leqslant s \,\&\, \sigma$ is terminal on $T_{j,s} \,\&\, \sigma$ is not terminal on $T_{j,s+1} \,\&\, \sigma$ is n-active on $\{T_i: i \leqslant j\}$ at stage $s \to f(\sigma, \lambda_j, n, s)$ is compatible with $T_{i,s+1}$ for all $i \leqslant j)$. (Recall that $\lambda_j = \lambda \restriction j + 1$.)

The crucial property of Definition 1.5 is (iii). This property states that if we extend a tree in the sequence at a terminal string σ which is n-active, then we must follow the n-target of that string with our extension. It is this property which will allow us to show that $\lim_s f(\sigma, \lambda, n, s)$ exists for suitably chosen σ, λ and n.

Given a sequence of trees $\{T_i: i \leqslant m\}$ which respects f, the sequence will have to be extended in various ways to sequences $\{T_i: i \leqslant m + 1\}$ which also respect f. Extensions letting T_{m+1} be $\mathrm{PExt}_2(T_m, \xi)$ or $\mathrm{PDiff}_2(T_m, e)$ for some $\xi \in \mathscr{S}_2$ or $e \in N$ are easily obtained.

1.6 Definition. Let $\{T_{i,s}: s \in N\}$ be a recursive sequence of finite trees such that for all $s \in N$, $T_{i,s+1}$ extends $T_{i,s}$ and let $T_i = \bigcup\{T_{i,s}: s \in N\}$. We define the approximation to $\mathrm{PExt}_2(T_i, \xi)$ for $\xi \in \mathscr{S}_2$ by

$$\mathrm{PExt}_{2,s}(T_i, \xi) = \mathrm{PExt}_2(T_{i,s}, \xi).$$

1.7 Remark. Let $\{T_i: i \leqslant m\}$ be a sequence of trees which respects f. Let $T_{m+1} = \mathrm{PExt}_2(T_m, \xi)$ for some $\xi \in \mathscr{S}_2$ such that $T_m(\xi)\downarrow$ and let the approximation to T_{m+1} be given as in Definition 1.6. Then $\{T_i: i \leqslant m + 1\}$ respects f. (Note that 1.5(iii) follows from 1.3(i) and (iii).)

Since e-differentiating trees are just extension trees for which $T_{m+1}(\emptyset)$ is carefully chosen, Remark 1.7 applies also to $\mathrm{PDiff}_2(T_m, e)$. Splitting trees, however, require more delicate approximations. We now indicate how to construct such approximations for e-splitting trees.

1.8 Lemma. *Let $e \in N$ be given. Let $\{T_i: i \leqslant m\}$ be a sequence of trees which respects f. Then there is an e-splitting tree $T_{m+1} = \mathrm{PSp}_2(\{T_i: i \leqslant m\}, e, f) = \bigcup\{T_{m+1,s}: s \in N\}$ such that $\{T_i: i \leqslant m + 1\}$ respects f. The approximation $\{T_{m+1,s}: s \in N\}$ is recursive, and an index for this approximation can be obtained uniformly and recursively from a string λ which codes $\{T_i: i \leqslant m\}$.*

Proof. We proceed by induction on $\{s: s \in N\}$. If $s > 0$, define $T_{m+1,s}(\xi) = T_{m+1,s-1}(\xi)$ if $T_{m+1,s-1}(\xi)\downarrow$. Suppose that either $s = 0$ or $T_{m+1,s-1}(\xi)\uparrow$. There are three cases.

Case 1. $s \geqslant 0$ and $T_{m,s}(\emptyset)\downarrow$ and if $s > 0$ then $T_{m,s-1}(\emptyset)\uparrow$. In this case, $T_{m+1,s}(\emptyset) = T_{m,s}(\emptyset)$.

Case 2. $s > 0$ and $\mathrm{lh}(\xi) > 0$ and $T_{m+1,s-1}(\xi^-)\downarrow$ and $T_{m+1,s-1}(\xi)\uparrow$. Let $\sigma^\# = T_{m+1,s-1}(\xi^-)$. Fix the least $n < s$, if any, such that $\sigma^\#$ is n-active at stage $s - 1$ for $\{T_i: i \leqslant m + 1\}$. If such an n exists, let $\sigma^* = f(\sigma^\#, \lambda, n, s - 1)$ and if no such n exists, let $\sigma^* = \sigma^\#$. Search for the least $\langle \tau_0, \tau_1, x\rangle \in \mathscr{S}_2^2 \times N$ (under some fixed recursive one-one correspondence of $\mathscr{S}_2^2 \times N$ with N) such that $x \leqslant s$, $\mathrm{lh}(\tau_i) \leqslant s$, and $\sigma^* \subseteq \tau_i \subset T_{m,s+1}$ for $i \leqslant 1$, and $\langle \tau_0, \tau_1\rangle$ forms an e-splitting on x. If no such $\langle \tau_0, \tau_1, x\rangle$ exists, then $T_{m+1,s}(\xi)\uparrow$. Otherwise, fix $\langle \tau_0, \tau_1, x\rangle$ and let $T_{m+1,s}(\xi) = \tau_j$ where $\xi = \xi^- * j$.

Case 3. Otherwise. Then $T_{m+1,s}(\xi)\uparrow$.

The lemma is now easily verified. (The proof that 1.5(iii) holds follows by induction from the choice of $\sigma^\#$ and 1.3(i) and (ix).) ▯

It follows from the proof of the Computation Lemma (V.2.6) that for all branches g of $\mathrm{PSp}_2(\{T_i: i \leqslant m\}, e, f)$, $g \leqslant_T \Phi_e^g$. It also follows from the definition of $\mathrm{PSp}_2(\{T_i: i \leqslant m\}, e, f)$ that:

(2) If σ is terminal on $\mathrm{PSp}_2(\{T_i: i \leqslant m\}, e, f)$ and σ is e-active for $\{T_i: i \leqslant m\}$ at all sufficiently large stages and λ codes $\{T_i: i \leqslant m\}$ and $\lim_s f(\sigma, \lambda, e, s)\downarrow$, then there is no e-splitting of $\lim_s f(\sigma, \lambda, e, s)$ on T_m.

The next remark notes that if we have a sequence of trees which respects f, then every subsequence also respects f.

1.9 Remark. Let $\{T_i: i \leqslant m + 2\}$ be a sequence of trees which respects f. Let $T_i^* = T_i$ for $i \leqslant m$ and $T_{m+1}^* = T_{m+2}$. Then $\{T_i^*: i \leqslant m + 1\}$ respects f.

The final lemmas of this section will be used to show that there is a question which can be asked of an oracle of degree $\mathbf{0}'$, the answer to which will determine whether or not $\Phi_e^A(e)\downarrow$ where A is the set of minimal degree which is constructed in the next section. The following definition will be useful.

1.10 Definition. Let $\{T_i: i \leqslant m\}$ be a sequence of trees which respects f. Let $\alpha, \beta \in \mathscr{S}_2$ and $n, s \in N$ be given such that $\alpha \subset T_{m,s}$ and $\alpha \subseteq \beta$. Then β is *n-desirable for α on $\{T_i: i \leqslant m\}$ at stage s* if there are $j \leqslant m$ and $\gamma \subset T_{j,s}$ such that:

(i) $\exists \xi \in \mathscr{S}_2(\beta \subseteq T_{j,s}(\xi) \,\&\, f(\gamma, \lambda_j, n, s)\downarrow = \beta)$.

(ii) $$\exists \sigma_{j+1} \cdots \exists \sigma_m \left(\bigwedge_{i=j+1}^{m} (\sigma_i \text{ is terminal on } T_{i,s}) \,\& \right.$$
$$\left. \alpha \subseteq \sigma_m \subseteq \sigma_{m-1} \subseteq \cdots \subseteq \sigma_{j+1} \subseteq \gamma \subseteq \beta \right).$$

In the construction of Section 2, $\Phi_n^A(n)$ will converge if, and only if $f(\alpha_n, \lambda, n, t(n))\downarrow$, which will be the case if, and only if α_n has an n-desirable extension on $\{T_i: i \leqslant k(n)\}$ coded by λ (α_n and $k(n)$ are defined during the construction). The equivalence of these conditions will follow from the remaining lemmas of this section.

1.11 Lemma. *Let $\{T_i: i \leqslant m\}$ be a sequence of trees which respects f. Fix $n \in N$ and $\alpha \in \mathscr{S}_2$ such that $\alpha \subset T_{m,s}$. Let $s \in N$ and $\beta \in \mathscr{S}_2$ be given such that $f(\alpha, \lambda, n, s) = \beta$. Then β is n-desirable for α on $\{T_i: i \leqslant m\}$ at stage s. Furthermore, there are $j \leqslant m$ and $\gamma \subset T_{j,s}$ such that* 1.10(i) *is satisfied as well as*:

(i) $$\exists\sigma_j \cdots \exists\sigma_m\Bigg(\bigwedge_{i=j+1}^{m} (\sigma_i \textit{ is terminal on } T_{i,s})\ \&$$
$$\alpha \subseteq \sigma_m \subseteq \sigma_{m-1} \subseteq \cdots \subseteq \sigma_j = \gamma \subseteq \beta\ \&\ \bigwedge_{i=j}^{m} f(\sigma_i, \lambda_i, n, s) = \beta\Bigg).$$

Proof. Fix the greatest $j \leqslant m$ such that $\beta \subset T_{j,s}(\xi)$ for some $\xi \in \mathscr{S}_2$. By 1.5(i), such a j must exist. For all $i \leqslant m$, fix the longest string $\sigma_i \subset T_{i,s}$ such that $\sigma_i \subseteq \beta$. By 1.3(iii), β is compatible with $T_{i,s}$ for all $i \leqslant m$, hence σ_i must exist for all $i \leqslant m$ and if $j < i \leqslant m$ then σ_i must be terminal on $T_{i,s}$. Since $T_{j,s} \supseteq T_{j+1,s} \supseteq \cdots \supset T_{m,s}$ and $\alpha \subset T_{m,s}$, $\alpha \subseteq \sigma_m \subseteq \sigma_{m-1} \subseteq \cdots \subseteq \sigma_j \subseteq \beta$. By 1.3(vii) and (ix), $\beta = f(\alpha, \lambda, n, s) = f(\sigma_m, \lambda, n, s) = f(\sigma_m, \lambda_{m-1}, n, s) = f(\sigma_{m-1}, \lambda_{m-1}, n, s) = \cdots = f(\sigma_j, \lambda_j, n, s)$. Hence β is n-desirable for on $\{T_i: i \leqslant m\}$ at stage s as witnessed by $\gamma = \sigma_j$, and (i) holds. ▯

1.12 Lemma. *Let $\{T_i: i \leqslant m\}$ be a sequence of trees which respects f. Fix $n, s \in N$ and $\alpha \in \mathscr{S}_2$ such that $\alpha \subset T_{m,s}$. Let $\beta \in \mathscr{S}_2$ be given such that β is n-desirable for α on $\{T_i: i \leqslant m\}$ at stage s. Then $f(\alpha, \lambda, n, t)\downarrow$.*

Proof. Fix $j \leqslant m$ and $\gamma = \sigma_j, \sigma_{j+1}, \ldots, \sigma_m$ as in Definition 1.10. We proceed by induction on $\{i: j \leqslant i \leqslant m\}$, showing that $f(\sigma_i, \lambda_i, n, s)\downarrow$. Since $\gamma = \sigma_j$, it follows from 1.10(i) that $f(\sigma_j, \lambda_j, n, s)\downarrow$. Assume by induction, that $i < m$ and $f(\sigma_i, \lambda_i, n, s)\downarrow$. By 1.10(ii), $\sigma_i \supseteq \sigma_{i+1}$ so by 1.3(v) and (xi), $f(\sigma_{i+1}, \lambda_{i+1}, n, s)\downarrow$. Hence we conclude that $f(\sigma_m, \lambda_m, n, s)\downarrow$.

By 1.10(ii), $\alpha \subseteq \sigma_m$. Hence by 1.3(v), $f(\alpha, \lambda, n, s)\downarrow$. ▯

1.13 Remarks. The targets which we have used in this section are called *followers* in the literature. They were used by Cooper [1973] to prove a jump inversion theorem for minimal degrees. Cooper defined the followers, the trees, and the set of minimal degree stage by stage in a full approximation construction, and did not separate the various definitions. We have separated the various definitions, and this will enable us to give an oracle construction proof of the Cooper Jump Inversion Theorem.

2. Jumps of Minimal Degrees

We now characterize the range of the jump operator restricted to the set of minimal degrees.

2.1 Cooper Jump Inversion Theorem. *Let* $\mathbf{c} \in \mathbf{D}$ *be given such that* $\mathbf{c} \geqslant \mathbf{0}'$. *Then there is a minimal degree* $\mathbf{a}$ *such that* $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{c}$.

Proof. Recursively in $\mathbf{c}$, we will construct sequences of strings $\{\alpha_s: s \in N\}$ and $\{\alpha_s^*: s \in N\}$, functions $k, k^*, t: N \to N$, and an array of trees $\{T_i^s: s \in N \;\&\; i \leqslant k(s)\}$ by induction on $\{s: s \in N\}$. α_s^* and k^* play roles similar to those played by α_s and k in the proof of Theorem IX.2.1, except that they consider the targets of the strings rather than the strings themselves. Thus there are two steps in the construction. Given α_s and $k(s)$, the s-target of α_s determines α_s^* and $k^*(s)$ which are then used to determine α_{s+1} and $k(s+1)$.

Fix a set C of degree $\mathbf{c}$. $A = \bigcup\{\alpha_s: s \in N\}$ will be a set of minimal degree. Recall that f is the jump target function.

The following induction hypotheses will be satisfied at the end of stage s of the construction, where λ_j^s will code $\{T_i^s: i \leqslant j\}$ for all $j \leqslant k(s)$:

(1) $\quad s \geqslant 1 \to \alpha_{s-1} \subseteq \alpha_{s-1}^* \subset \alpha_s.$

(2) $\quad \alpha_s \subset T_{k(s)}^s.$

(3) $\quad \forall s \in N(T_0^s = \mathrm{Id}_2).$

(4) $\quad \forall i < k(s)(T_{i+1}^s \subseteq T_i^s).$

(5) $\quad s \geqslant 1 \to \forall i < k(s)(T_i^s = T_i^{s-1}).$

(6) $\quad \{T_i^s: i \leqslant k(s)\}$ respects f.

The construction proceeds as follows:

Stage 0. Set $k(0) = 0$, $T_0^0 = \mathrm{Id}_2$ coded by $\lambda_0^0 = \lambda$, and $\alpha_0 = \emptyset$.

Stage $s + 1$. For all $j \leqslant k(s)$, let λ_j^s code $\{T_i^s: i \leqslant j\}$. Let $t(s)$ be the least stage t such that:

(7) $\quad t \geqslant t(s-1) \quad \text{if} \quad s > 0,$

(8) $\quad \alpha_s \subset T_{k(s),t}^s,$

(9) $\quad \text{If} \quad s > 0 \quad \text{then} \quad \alpha_{s-1}^+ \subset T_{k^*(s-1),t}^{s-1},$

(10) $\quad f(\alpha_s, \lambda_{k(s)}^s, s, t)\downarrow,$

if such a stage exists, and let $t(s)$ be the least stage t satisfying (7)–(9) otherwise. Define

$$\alpha_s^* = \begin{cases} f(\alpha_s, \lambda_{k(s)}^s, s, t(s)) & \text{if} \quad f(\alpha_s, \lambda_{k(s)}^s, s, t(s))\downarrow \\ \alpha_s & \text{otherwise.} \end{cases}$$

Find the greatest $k \leqslant k(s)$ such that for some $\xi \in \mathscr{S}_2, \alpha_s^* \subseteq T_k^s(\xi)$. (By (3), k must exist.) Let $k^*(s) = k$. Let α_s^+ be the string of shortest length such that $\alpha_s^* \subseteq \alpha_s^+ \subseteq T_{k^*(s)}^s$.

Let $r(s)$ be the greatest $r \leqslant k^*(s)$ such that α_s^+ is not terminal on T_r^s. (Again by (3), r must exist.) Define $k(s+1) = r(s) + 1$, and $T_i^{s+1} = T_i^s$ for all $i < k(s+1)$.

Fix $\eta_s \in \mathscr{S}_2$ such that $T^s_{r(s)}(\eta_s) = \alpha_s^+$. Let $X^{s+1} = \mathrm{PExt}_2(T^{s+1}_{r(s)}, \eta_s)$ and let

$$Y_i^{s+1} = \begin{cases} T_i^{s+1} & \text{if } i < k(s+1) \\ \mathrm{PDiff}_2(X^{s+1}, s) & \text{if } i = k(s+1)\,\&\, \mathbf{c} = \mathbf{0}' \\ \mathrm{PExt}_2(X^{s+1}, C(s)) & \text{if } i = k(s+1)\,\&\, \mathbf{c} > \mathbf{0}'. \end{cases}$$

Let $\alpha_{s+1} = Y^{s+1}_{k(s+1)}(\emptyset)$, and define

$$T^{s+1}_{k(s+1)} = \begin{cases} Y^{s+1}_{k(s+1)} & \text{if } r(s) < k(s) \\ \mathrm{PSp}_2(\{Y_i^{s+1}: i \leqslant k(s+1)\}, r(s), f) & \text{if } r(s) = k(s). \end{cases}$$

The construction is now complete. Induction hypotheses (1)–(5) are easily verified. And (6) follows from Lemma 1.8 and Remarks 1.7 and 1.9; again we note that $\mathrm{PDiff}_2(T, e) = \mathrm{PExt}_2(T, \xi)$ for some choice of $\xi \in \mathscr{S}_2$. By (1), $A = \bigcup\{\alpha_s : s \in N\} \subseteq N$.

Given $e < s \in N$, we will show that one of the following conditions holds:

(11) $\quad \forall t \geqslant t(s)(f(\alpha_s, \lambda^s_{k(s)}, e, t)\downarrow = \alpha_s)$.

(12) $\quad \forall t \geqslant t(s)(f(\alpha_s, \lambda^s_{k(s)}, e, t)\uparrow)$.

Furthermore, we will have to show that (11) holds if, and only if (11) holds for $s+1$ in place of s. It will then follow that A forces e into the jump whenever possible, and so we will be able to compute A' from an oracle of degree $\mathbf{a} \cup \mathbf{0}'$. We will verify (11) or (12) by induction on s, introducing an intermediate step wherein one of these conditions is verified for α_s^+ in place of α_s and $r(s)$ in place of $k(s)$ for each $e \leqslant s$. The next lemma will be used to show that (11) is inherited by α_{s+1} from α_s.

2.2 Lemma. *Let λ^j code a sequence of trees $\{T_i : i \leqslant m_j\}$ which respects f for $j \leqslant 1$. Let $\alpha, \beta, \gamma \in \mathscr{S}_2$ be given such that $\alpha \subseteq \beta \subseteq \gamma$, let $e, u \in N$ be given such that $\alpha \subset T_{m_0,u}$ and $\gamma \subset T_{m_1,u}$ and assume that $\forall t \geqslant u(f(\alpha, \lambda^0, e, t)\downarrow = \beta)$. Then $\forall t \geqslant u(f(\gamma, \lambda^1, e, t)\downarrow = \gamma)$.*

Proof. Fix $t \geqslant u$. Since $f(\alpha, \lambda_0, e, t) = \beta$, it follows from 1.3(i) that $\Phi_e^\beta(e)\downarrow$. Hence by the Enumeration Theorem (I.3.1(i)), $\Phi_e^\gamma(e)\downarrow$. It now follows from 1.3(vi) that $f(\gamma, \lambda_1, e, t) = \gamma$. ▯

We now show that a condition of the same type as (12) is inherited by α_s^+ from α_s.

2.3 Lemma. *Let $e < s \in N$ be given and assume that* (12) *holds. Fix $t \geqslant t(s)$, $k \in \{r : k^*(s) \leqslant r \leqslant k(s)\}$ and $\gamma \subset T^s_{k,t}$ such that γ is the longest string on $T^s_{k,t}$ which is contained in α_s^* and assume that if $k > k^*(s)$ then γ is terminal on $T^s_{k,t}$. Let λ code $\{T_i^s : i \leqslant k\}$. Then $f(\gamma, \lambda, e, t)\uparrow$. Furthermore,*

(13) $\quad \forall u \geqslant t(s)(f(\alpha_s^+, \lambda^s_{k^*(s)}, e, u)\uparrow)$.

Proof. We first assume that $f(\gamma, \lambda, e, t)\downarrow = \delta$ and obtain a contradiction. If $f(\alpha_s, \lambda^s_{k(s)}, s, t)\uparrow$, then $\alpha_s^* = \alpha_s$ which is on $T^s_{i,t}$ for all $i \leqslant k(s)$. And if $f(\alpha_s, \lambda^s_{k(s)}, s, t)\downarrow = \alpha_s^* = \beta$, then it follows from 1.3(iii) that β is compatible with $T^s_{i,t}$ for all $i \leqslant k(s)$. Hence for all i such that $k < i \leqslant k(s)$, there is a unique $\sigma_i \subseteq \alpha_s^*$ such that σ_i is terminal on $T^s_{i,t}$. By Lemma 1.11, δ is e-desirable for γ on $\{T_i : i \leqslant k\}$, producing a sequence $\{\sigma_i : k^* \leqslant i \leqslant k\}$ satisfying 1.11(i). Since σ_i is terminal on $T^s_{i,t}$ for all i such that $k < i \leqslant k(s)$,

$$\alpha_s \subseteq \sigma_{k(s)} \subseteq \sigma_{k(s)-1} \subseteq \cdots \subseteq \sigma_{k+1} \subseteq \gamma \subseteq \sigma_k \subseteq \cdots \subseteq \sigma_{k^*},$$

so $\{\sigma_i : k^* \leqslant i \leqslant k(s)\}$ witnesses the fact that δ is e-desirable for α_s at stage t. By Lemma 1.12, $f(\alpha_s, \lambda^s_{k(s)}, e, t)\downarrow$, contradicting (12). Hence $f(\gamma, \lambda, e, t)\uparrow$.

Let $k = k^*(s)$. Then by 1.3(v), since $f(\gamma, \lambda, e, t)\uparrow$ and $\gamma \subseteq \alpha^*_s \subseteq \alpha^+_s$, $f(\alpha^+_s, \lambda, e, t)\uparrow$. Hence (13) follows. ▯

Lemma 2.3 yields no information when $e = s$. This case is covered in the next lemma, where we show that one of the following conditions holds:

(14) $\quad \forall t \geqslant t(s+1)(f(\alpha^+_s, \lambda^s_{k^*(s)}, s, t)\downarrow = \alpha^+_s)$.

(15) $\quad \forall t \geqslant t(s+1)(f(\alpha^+_s, \lambda^s_{k^*(s)}, s, t)\uparrow)$.

2.4 Lemma. *Let $s \in N$ be given, and assume that either* (11) *or* (12) *holds for each $e < s$. Then either* (14) *or* (15) *holds.*

Proof. If $\forall t \geqslant t(s)(f(\alpha_s, \lambda^s_{k(s)}, s, t)\uparrow)$, then the lemma follows immediately (note that, in this case, $\alpha^+_s = \alpha^*_s = \alpha_s$, $k^*(s) = k(s)$ and $t(s+1) \geqslant t(s)$). Otherwise, we may fix the least $t \geqslant t(s)$ such that $f(\alpha_s, \lambda^s_{k(s)}, s, t)\downarrow = \beta$. We proceed by induction, showing that for all $r \geqslant t$, $f(\alpha_s, \lambda^s_{k(s)}, s, r)\downarrow = \beta$. It follows from (7) and (10) that $t \leqslant t(s+1)$.

Assume, by induction, that $r \geqslant t$ and $f(\alpha_s, \lambda^s_{k(s)}, s, r)\downarrow = \beta$. By Lemma 1.11, there is a sequence $\{\sigma_i : k^*(s) \leqslant i \leqslant k(s)\}$ satisfying 1.11(i). By 1.11(i), $\alpha_s \subseteq \sigma_{k^*(s)} \subseteq \beta$ and for each i such that $k^*(s) < i \leqslant k(s)$, $\alpha_s \subseteq \sigma_i \subseteq \beta$, σ_i is terminal on $T^s_{i,r}$ and $f(\sigma_i, \lambda^s_i, s, t)\downarrow = \beta$. For each $e < s$ and i such that $k^*(s) < i \leqslant k(s)$, it follows from (11), (12) and Lemmas 2.2 and 2.3 that σ_i is not e-active on $\{T^s_j : j \leqslant i\}$ at stage r. Hence either $\beta = \alpha_s$ or for all i such that $k^*(s) < i \leqslant k(s)$, σ_i is s-active on $\{T^s_j : j \leqslant i\}$ at stage r. Since, for each such i, $\{T^s_j : j \leqslant i\}$ respects f, β is compatible with $T^s_{j,r+1}$ for all $j \leqslant i$, so by 1.3(iv), $f(\alpha_s, \lambda^s_{k(s)}, s, r+1)\downarrow = \beta$. Thus the induction is complete.

Since, $\forall r \geqslant t(f(\alpha_s, \lambda^s_{k(s)}, s, r)\downarrow = \beta$, we must have $\beta = \alpha^*_s \subseteq \alpha^+_s$. Hence by Lemma 2.2 and (9), (14) must hold. ▯

We now move from $T^s_{k^*(s)}$ to $T^s_{r(s)}$, showing that conditions (13) and (15) are inherited.

2.5 Lemma. *Fix $e \leqslant s \in N$ and suppose that either* (13) *or* (15) *holds. Then*

(16) $\quad \forall t \geqslant t(s+1)(f(\alpha^+_s, \lambda^s_{r(s)}, e, t)\uparrow)$.

Proof. Fix $t \geqslant t(s+1)$, and assume that $f(\alpha^+_s, \lambda^s_{r(s)}, e, t)\downarrow = \beta$ in order to obtain a contradiction. Since α^+_s is terminal on $T^s_{j,t}$ for all j such that $r(s) < j \leqslant k^*(s)$, β is compatible with $T^s_{j,t}$ for all such j. Applying 1.3(xi) repeatedly, we see that $f(\alpha^+_s, \lambda^s_{k^*(s)}, e, t)\downarrow$, contradicting (13) or (15). ▯

We now move from $T^s_{r(s)}$ to $T^{s+1}_{k(s+1)}$, showing that (16) is inherited.

2.6 Lemma. *Fix $e \leqslant s \in N$, and assume that* (16) *holds. Then* (12) *holds for $s+1$ in place of s.*

Proof. Since $k(s+1) = r(s)+1$, it follows from (5) that $T_j^s = T_j^{s+1}$ for all $j \leqslant r(s)$. Hence by (16), $\forall t \geqslant t(s+1)(f(\alpha_s^+, \lambda_{r(s)}^{s+1}, e, t)\uparrow)$. Suppose that for some $t \geqslant t(s+1)$, $f(\alpha_{s+1}, \lambda_{k(s+1)}^{s+1}, e, t)\downarrow$ for the sake of obtaining a contradiction. By 1.3(viii), $f(\alpha_{s+1}, \lambda_{r(s)}^{s+1}, e, t)\downarrow$. Since $\alpha_s^+ \subseteq \alpha_{s+1}$, it follows from 1.3(v) that $f(\alpha_s^+, \lambda_{r(s)}^{s+1}, e, t)\downarrow$, yielding the desired contradiction. ▯

The next lemma summarizes the facts proved relating to (11) and (12).

2.7 Lemma. *Fix $e \in N$. Then one of the following conditions holds*:

(i) $\forall s > e \, \forall t \geqslant t(s)(f(\alpha_s, \lambda_{k(s)}^s, e, t)\downarrow = \alpha_s)$.

(ii) $\forall s > e \, \forall t \geqslant t(s)(f(\alpha_s, \lambda_{k(s)}^s, e, t)\uparrow)$.

Proof. Fix $e \in N$. We assume, by induction, that the lemma holds for e_0 in place of e for each $e_0 < e$. We then proceed by induction on $\{s: s > e\}$.

First assume that $s = e+1$. By Lemma 2.4, either (14) or (15) holds. If (14) holds, then by Lemma 2.2 and (8), (11) holds. If (15) holds, then by Lemma 2.5, (16) holds; hence by Lemma 2.6, (12) holds for $s+1$ in place of s.

Assume that $s > e+1$. By induction, either (11) or (12) holds for $s-1$ in place of s. If (11) holds for $s-1$ in place of s, then by Lemma 2.2 and (8), (11) holds for s. Suppose that (12) holds for $s-1$ in place of s. By Lemma 2.3, (13) must hold for $s-1$ in place of s. By Lemma 2.5, (16) must hold for $s-1$ in place of s. Hence by Lemma 2.6, (12) must hold for s. This completes the induction step. ▯

We use Lemma 2.7 to show that $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{c}$. Note that $t(s)$ can be obtained from α_{s-1}^+, α_s and $\lambda_{k(s)}^s$ through the use of an oracle of degree $\mathbf{0}'$. For to determine whether a string is on a tree at a given stage is a question of degree $\leqslant \mathbf{0}'$. Thus by Lemma 2.7, once we have a stage t such that α_{s-1}^+ and α_s satisfy (7)–(9), then $t(s)$ is the least $r \geqslant t$ such that $f(\alpha_s, \lambda_{k(s)}^s, s, r)\downarrow$ if such an r exists, and $t(s) = t(s-1)$ (0 if $s=0$) otherwise. And we can determine whether $\exists r \geqslant t(f(\alpha_s, \lambda_{k(s)}^s, s, r)\downarrow)$ from an oracle of degree $\mathbf{0}'$. Hence A, k, k^*, r, t and $\{T_i^s: s \in N \,\&\, i \leqslant k(s)\}$ are recursive in a set of degree $\mathbf{c}$. Furthermore, the only use of a C oracle which cannot be replaced by an appeal to an oracle of degree $\mathbf{0}'$ is the use of $C(s)$ in defining $Y_{k(s+1)}^{s+1}$.

We show that $\mathbf{c} \leqslant \mathbf{a} \cup \mathbf{0}' \leqslant \mathbf{a}' \leqslant \mathbf{c}$. C can be computed from A and an oracle K of degree $\mathbf{0}'$. If $\mathbf{c} = \mathbf{0}'$, then there is nothing to show. Assume that $\mathbf{c} > \mathbf{0}'$. Suppose that for all $n < e$, α_n, $k(n)$, $\{T_i^n: i \leqslant k(n)\}$ and $t(n)$ have been computed. Then α_e^*, α_e^+ and X_e can be computed from K. Now $C(e) = i$ if, and only if, $X_e(i) \subset A$, so $C(e)$ is determined by A. To complete the induction, α_e, $k(e)$, $\{T_i^e: i \leqslant k(e)\}$ and $t(e)$ are now determined by an appeal to the oracle K. Since this procedure is uniform in e, $\mathbf{c} \leqslant \mathbf{a} \cup \mathbf{0}'$.

By III.2.3(iv) and (v), and since $\mathbf{c} \geqslant \mathbf{0}'$, $\mathbf{a}, \mathbf{0}' \leqslant \mathbf{a}'$ so $\mathbf{a} \cup \mathbf{0}' \leqslant \mathbf{a}'$.

It has already been noted that α_n, $k(n)$, $\{T_i^n: i \leqslant k(n)\}$, $k^*(n)$ and $t(n)$ can be computed recursively in C uniformly in n. Thus in order to show that $\mathbf{a}' \leqslant \mathbf{c}$, it suffices to verify the following fact:

(17) $\Phi_e^A(e)\downarrow \leftrightarrow \Phi_e^{\alpha_{e+1}}(e)\downarrow$.

Since $\Phi_e^A(e)\downarrow$ if, and only if $\Phi_e^{\alpha_s}(e)\downarrow$ for some $s \in N$, (17) follows from Lemma 2.7.

The next lemma is the heart of the proof that A is a set of minimal degree.

2.8 Lemma. *Let $e, s \in N$ be given such that $k(s) = e$ and $k(t) > e$ for all $t > s$. Then*

(i) $\forall t \geqslant s(\alpha_t \subset T_e^s)$.

Furthermore, $k(s+1) = e+1$ and

(ii) T_{e+1}^{s+1} *is an e-splitting tree.*

If there is a $u > s+1$ such that $k(u) = e+1$, then

(iii) *there is a terminal $\gamma \subset T_{e+1}^{u-1}$ such that either*

$$f(\gamma, \lambda, u-1, t(u-1))\uparrow \qquad or \qquad f(\gamma, \lambda, u-1, t(u-1))\downarrow \subseteq \alpha_u$$

where λ codes $\{T_i^{u-1}: i \leqslant e+1\}$, and

(iv) $\forall t > u(k(t) > e+1)$.

Sketch of proof. With the exception of (iii), the proof of Lemma 2.8 is similar to the proof of Lemma IX.2.2. The proof of (iii) follows from 1.3(vii). We leave the proof of Lemma 2.8 to the reader. ▯

It follows from Lemma 2.8(iv) that $\lim_t k(t) = \infty$. Hence by 2.8(i)–(iii) and condition (2) of the previous section, for all $e \in N$, if Φ_e^A is total then either Φ_e^A is recursive or $A \equiv_T \Phi_e^A$. Also, if $\mathbf{c} = \mathbf{0}'$, then it follows from the definition of α_{e+1} that $A \neq \Phi_e$, so A is not recursive. This completes the proof of Theorem 2.1. ▯

We now characterize the jumps of minimal degrees. The characterization follows from Lemma III.2.3(v) and Theorem 2.1.

2.9 Corollary. *The following are equivalent:*

(i) $\mathbf{c}$ *is the jump of a minimal degree.*

(ii) $\mathbf{c} \geqslant \mathbf{0}'$.

The following corollary follows immediately from Theorem 2.1.

2.10 Corollary. *There is a minimal degree $\mathbf{a} \in \mathbf{L}_1$. Also, there are continuum many minimal degrees in $\mathbf{GL}_1 - \mathbf{L}_1$.*

2.11 Remarks. Yates [1970a] constructed a minimal degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{0}'$ as a corollary to his theorem that every recursively enumerable degree bounds a minimal degree. Yates' result is proved in the next chapter. Cooper [1973] proved Theorem 2.1. Both Yates' and Cooper's proofs are full approximation style proofs in that they do not use an oracle during the construction (which is recursively performed).

2.12 Exercise. Let $\mathbf{c}, \mathbf{d} \in \mathbf{D}$ be given such that $\mathbf{c} \geqslant \mathbf{d}'$. Show that there is a minimal cover $\mathbf{a}$ of $\mathbf{d}$ such that $\mathbf{a}' = \mathbf{a} \cup \mathbf{d}' = \mathbf{c}$.