Chapter X

Freeness and Isolation

In this chapter we investigate the relation between the notion of isolation (analogous to dependence) and the notion of freeness (analogous to independence). Since in fact we study only one notion of freeness (nonforking) but several notions of isolation, we could consider this study as the further investigation of properties of isolation relations. In fact, one of the notions of isolation, namely **S**-isolation, will be seen to play a rather privileged role. Of the dependence relations we consider, it is the most natural counterpart to the independence relation of nonforking. Unfortunately, study of **S**-isolation gives direct information about only the **S**-saturated structures. By making more restrictive assumptions on the theory we are able to widen the class of models treated. In particular, for ω-stable theories $\mathbf{AT}_{\aleph_0}$ will be covered by our discussion. In Section 1 we develop a set of axioms relating the notions of isolation and independence. Section 2 explores the notion of a 'powerful' isolation relation. In Section 3 we prove the uniqueness of prime models. Section 4 discusses the posssible sizes of indiscernible sequences over a set A which lie in prime models over A.

WARNING: While we have attempted to isolate in Chapter II all relevant features of nonforking, in Chapter IX the relevant features of isolation relations and here the properties which govern their interaction, attempts to apply these results to other isolation relations should be undertaken with caution. In particular, we have not carefully surveyed the situation for **F**-isolation. Our major purpose in this axiomatization has been to systematize the exposition of the properties of $\mathbf{AT}_\lambda$, $\mathbf{SET}_\lambda$, and $\mathbf{S}_\lambda$ for regular λ. We also include some material on **L**-isolation for countable languages.

1. Axioms Relating Freeness and Isolation

In this section we discuss four principles from which we derive all the information relating forking and the formation of prime models (subject to the caveat in the introduction to this chapter). The first of these makes the rather obvious assertion that a type which is isolated over a set A does not fork over A. The second is a transfer principle which asserts that if an element is independent from a set then it is independent from all points whose types are isolated over that set. This property has immediate algebraic consequences. The third principle is the topologically oriented open mapping theorem while the fourth is a rather technical consequence of the finite character of forking.

We also introduce in this section the notion of a relatively **I**-saturated substructure. This concept clarifies some of the arguments here and extends to prove similar results for independent trees in Chapter XVII. The following notations will simplify much of the discussion.

1.1 Notation. Let M be **I**-saturated and A a set; $M[A]$ denotes a strictly **I**-prime model over $M \cup A$. We also frequently write $M[\bar{a}]$, ignoring the distinction between the sequence $\bar{a}$ and its range. If $p \in S(M)$ then $M[p]$ denotes $M[\bar{a}]$ for some $\bar{a}$ realizing p.

There are several potential ambiguities in this notation. A minor ambiguity is that we suppress mention of **I**; the choice of **I** will always be clear from context. A more serious ambiguity is the abuse of functional notation to denote an operation which is unique only up to isomorphism. We use the notation as a shorthand since in most cases it causes no problems. However, sometimes the *specific* structure which is strictly prime over $M \cup A$ is important. In such situations one must be very careful not to be misled by the notation. This difficulty is discussed in detail in connection with Theorem 1.28.

This problem becomes more serious if **I** is taken as **L**. For, as we observed in Section IX.4, there is no uniqueness theorem for **L**-constructible models. In general, we try to avoid the $M[a]$ notation in this context. It occasionally appears because theorems which are proved with the other **I** in mind apply to **L** if $M[\bar{a}]$ is taken as a meaning, 'an arbitrary **L**-constructible model containing $M \cup \bar{a}$.'

A lesser problem is the distinction between $M[p]$ and $M[\bar{a}]$ when $\bar{a}$ is a realization of p. The next two exercises illustrate this distinction.

1.2 Exercise. Deduce from the uniqueness of strictly **I**-prime models that i) $M[\bar{a}]$ is unique up to isomorphism over $M \cup \bar{a}$, while ii) $M[p]$ is unique only up to isomorphism over M.

1.3 Exercise. Let T be $\mathrm{Th}(Z_4^{\aleph_0})$. Let M be a countable model of T and let p, q be the nonforking extensions to M of the types of an element of order four and an element of order two respectively. Let a realize p and b realize q. Show that there are many copies of $M[b]$ (all isomorphic over $M \cup b$) but that within $\mathcal{M}$, $M[a]$ is unique, period, not just up to isomorphism.

Recall that if p is a type over A then p does not fork over A. We strengthen this requirement by demanding that if p is isolated over A then p does not fork over A.

1.4 Axiom $\mathbf{FI_0}$. If $A \subseteq B$ and $p \in S(B)$ is isolated by A then p does not fork over A.

For $\mathbf{AT}_\lambda$, L, and $\mathbf{SET}_\lambda$ the verification of this axiom is immediate since we have a type $q \subseteq p$ such that $q \vdash p$ and q is over A. But for $\mathbf{S}_\lambda$, the result is equally clear, noting only that if q is almost over A then q does not fork over A.

Consider the following situation: $(\bar{a}\downarrow\bar{b}; M)$ and $t(\bar{c}; M \cup \bar{b})$ is **I**-isolated by $D \cup \bar{b}$. What can be said about the relation of $\bar{a}$ and $\bar{c}$? Perhaps the most intuitive idea is that if $\bar{a}$ is independent from $\bar{b}$ then $\bar{a}$ is independent from any element whose type over $\bar{b}$ is **I**-isolated. Before stating the axiom we introduce two further concepts which will simplify its verification. The crucial relation between $M \cup \bar{a}$ and $M \cup \{\bar{a}, \bar{b}\}$ is crystalized in the following definition.

1.5 Definition. The set A is *relatively* **I**-*saturated* in B if each **I**-formula q over A which is satisfied in B is satisfied in A. We write $A \leq_{\mathbf{I}} B$.

Shelah has defined several special cases of this notion. When **I** is **AT** we have Shelah's notion of the *Tarski-Vaught property.* This concept also encompasses Shelah's *strong elementary submodel* [Shelah 198?]. Thus, his notions of $\subseteq_g$ and $\subseteq_h$ correspond to relative $\mathbf{SET}_{\aleph_0}$ and $\mathbf{S}_{\aleph_0}$ saturation respectively.

1.6 Exercise. Show that $\leq_{\mathbf{I}}$ is transitive.

1.7 Exercise. If M is an **I**-saturated model of T then for any A with $M \subseteq A$, $M \leq_{\mathbf{I}} A$.

The next two exercises were suggested by Leo Harrington.

1.8 Exercise. Show that if $A \leq_{AT} B$ then any formula almost over A which is satisfied in B is satisfied in A.

1.9 Exercise. Prove that if $A \leq_{\mathbf{AT}} B$ and $c \downarrow_A B$ then $t(B; A \cup c)$ is finitely satisfiable in A. Find an example where $t(c; B)$ is not finitely satisfied in A. (Hint: For the proof use the fact that $t(c; B)$ is definable almost over A and the previous exercise.)

The last exercise shows that although forking is symmetric when T is stable, the motivating notion of finite satisfiability may not be.

We now extend the concept of strong saturation discussed in Section III.2 to a more general context. We will verify that **I**-saturation implies strong **I**-saturation and derive several extremely useful corollaries from this result.

1.10 Definition. The structure M is *strongly* **I**-*saturated* if for any set A with $|A| < \lambda(\mathbf{I})$ and any **I**-formula, q, over $M \cup A$, if there is a $\bar{b}$ with $\lg(\bar{b}) < \lambda(\mathbf{I})$, realizing q such that $\bar{b} \downarrow_M A$, then there is a $\bar{b}' \in M$ which realizes q.

1.11 Theorem. *Let T be a stable theory. If* **I** *is* $\mathbf{AT}_\lambda$, $\mathbf{S}_\lambda$ *or* **L** *every* **I**-*saturated model is strongly* **I**-*saturated. If $\lambda > |T|$, any* $\mathbf{SET}_\lambda$-*saturated model is strongly* $\mathbf{SET}_\lambda$-*saturated.*

Proof. For the **AT** and **L**, the result follows immediately from the definition of coheir. For **S** the result is obvious. For the other isolation notions slight variations of the proof of Lemma III.2.29 yield the result.

We can rephrase strong **I**-saturation in terms of relative **I**-saturation.

1.12 Lemma. *If the strongly* **I***-saturated model M is contained in $A \cap B$ then $A \downarrow_M B$ if and only if $A \leq_{\mathbf{I}} A \cup B$.*

Proof. Let q be an **I**-formula over A which is realized in B. Since the conjunction of any finite subset of q is realized in M, strong **I**-saturation guarantees that q is realized in M.

The following stronger version will be applied both here and in Chapter XII (to aid the discussion of regular types). For **L** and **AT** the Corollary does not really add anything to the theorem.

1.13 Corollary. *Suppose the strongly* **I***-saturated model M is contained in $A \cap B$ and $A \downarrow_M B$. Let the* **I***-formula $q(\overline{x}, \overline{a}, \overline{b})$ be consistent (where $\lg(\overline{a}), \lg(\overline{b}) < \lambda(\mathbf{I})$). Then, there is a $\overline{b}' \in M$ such that $q(\overline{x}, \overline{a}, \overline{b}')$ is consistent.*

Proof. The type $\{(\exists \overline{x})[\phi(\overline{x}, \overline{a}, \overline{y})] : \phi(\overline{x}, \overline{a}, \overline{x}) \in q\}$ is an **I**-formula. By the definition of strong **I**-saturation it must be realized in M by some $\overline{b}'$. Now the required type over $\overline{a} \cup \overline{b}'$ is easily seen to be consistent since we may assume q is closed under finite conjunction.

1.14 Exercise. Show that if **I** contains $\mathbf{AT}_{\aleph_0}$ then any structure which is algebraically closed and strongly **I**-saturated must be a model.

The following axiom establishes the relation between $M[\overline{a}]$ and $\overline{b}$ when $\overline{a} \downarrow_M \overline{b}$.

1.15 Axiom $\mathbf{FI_1}$. Let $M \models T$ be strongly **I**-saturated. If $\overline{a} \downarrow_M \overline{b}$ and $t(\overline{c}; M \cup \overline{b})$ is **I**-isolated then $t(\overline{c}; M \cup \overline{b}) \vdash t(\overline{c}; M \cup \{\overline{a}, \overline{b}\})$.

This axiom clearly implies the following more precise formulation. Let $M \models T$ be **I**-saturated. If $\overline{a} \downarrow_M \overline{b}$ and $t(\overline{c}; M \cup \overline{b})$ is **I**-isolated by $D \cup \overline{b}$ with $D \subseteq M$ then $t(\overline{c}; M \cup \{\overline{a}, \overline{b}\})$ is **I**-isolated by $D \cup \overline{b}$.

It is easy to see that this axiom requires some hypothesis on the set M.

1.16 Exercise. Let T be the theory of an infinite set and let $a, b, c,$ be elements of a model of T. Show $t(c; a)$ is isolated by a and $a \downarrow_\emptyset b$, but $t(c; \{a, b\})$ is not isolated over a.

We derive Axiom FI_1 from a stronger result which will be applied in Chapter XVII. The stronger result arises by generalizing the context, replacing $M \cup \overline{a}$ and $M \cup \{\overline{a}, \overline{b}\}$ by any A, B with $A \leq_{\mathbf{I}} B$. For our standard isolation relations we can deduce the theorem from Corollary 1.13. Because of the weakness of **L**-saturation, this proof does not suffice in the **L** case but a minor variant does.

1.17 Theorem. *Suppose $A \leq_{\mathbf{I}} B$ and $t(\overline{c}; A)$ is* **I***-isolated over A. Then $t(\overline{c}; A) \vdash t(\overline{c}; B)$.*

Proof. First, let $\mathbf{I}$ be any of the isolation relations except $\mathbf{L}$ and choose an $\mathbf{I}$ formula $q(\overline{x};\overline{a})$ such that $q(\overline{x};\overline{a}) \vdash t(\overline{c};A)$. Suppose there is a formula $\phi(\overline{x};\overline{b}) \in F(B)$ with both $q(\overline{x};\overline{a}) \cup \{\phi(\overline{x};\overline{b})\}$ and $q(\overline{x};\overline{a}) \cup \{\neg\phi(\overline{x};\overline{b})\}$ consistent. Then, by Corollary 1.13 there is a $\overline{b}' \in A$ such that both $q(\overline{x};\overline{a}) \cup \{\phi(\overline{x};\overline{b}')\}$ and $q(\overline{x};\overline{a}) \cup \{\neg\phi(\overline{x};\overline{b}')\}$ are consistent. But this contradicts the choice of q.

Now suppose $\mathbf{I}$ is $\mathbf{L}$ and let p denote $t(\overline{c};A)$. Fix $\phi(\overline{x};\overline{b}) \in F(B)$ so that both $p \cup \{\phi(\overline{x};\overline{b})\}$ and $p \cup \{\neg\phi(\overline{x};\overline{b})\}$ are consistent. Now for $\phi(\overline{x};\overline{y})$, choose $\psi_\phi(\overline{x};\overline{a})$ such that $\psi_\phi(\overline{x};\overline{a}) \vdash p_\phi$. Then both $\psi_\phi(\overline{x};\overline{a}) \cup \{\phi(\overline{x};\overline{b})\}$ and $\psi_\phi(\overline{x};\overline{a}) \cup \{\neg\phi(\overline{x};\overline{b})\}$ are consistent. Since $A \leq_L B$ there is a $\overline{b}' \in A$ with both $\psi_\phi(\overline{x};\overline{a}) \cup \{\phi(\overline{x};\overline{b}')\}$ and $\psi_\phi(\overline{x};\overline{a}) \cup \{\neg\phi(\overline{x};\overline{b}')\}$ consistent. This contradicts the choice of ψ_ϕ.

From Lemma 1.12 and Theorem 1.17 we deduce that FI_1 holds for each $\mathbf{I}$. The following version of Theorem 1.17 is easier to apply. As in Theorem 1.17, for ii) the case $\mathbf{I}$ is $\mathbf{L}$ requires a slightly different proof which we omit this time.

1.18 Corollary. *If $A \leq_{\mathbf{I}} B$ and C is $\mathbf{I}$-atomic over A then*

i) $C \downarrow_A B$.
ii) $A \cup C \leq_{\mathbf{I}} B \cup C$.

Proof. Statement i) is immediate from Theorem 1.17. For ii), suppose that $\phi(\overline{a};\overline{b};\overline{c})$ holds and that the $\mathbf{I}$-formula $q(\overline{a}';\overline{x})$ implies $t(\overline{c};A)$. By Lemma 1.14, and the completeness theorem there is a formula $\theta(\overline{x};\overline{a}_1)$ (almost) over A such that $\models (\forall\overline{x})\theta(\overline{x};\overline{a}_1) \to \phi(\overline{a},\overline{b};\overline{x})$. Since $A \leq_{\mathbf{I}} B$, there exists $\overline{b}' \in A$ such that $\models (\forall\overline{x})\theta(\overline{x};\overline{a}_1) \to \phi(\overline{a},\overline{b}';\overline{x})$. Since $\models \theta(\overline{c};\overline{a}_1)$, we can deduce $\models \phi(\overline{a};\overline{b}';\overline{c})$ and finish.

1.19 Exercise. Prove Corollary 1.18 when $\mathbf{I}$ is $\mathbf{L}$.

The following result is now immediate.

1.20 Corollary. *Suppose* FI_0 *and* FI_1 *hold, M is strongly $\mathbf{I}$-saturated and $\lambda(\mathbf{I})$ is regular. Then, $\overline{b} \rhd_M M[\overline{b}]$.*

1.21 Exercise. Suppose FI_0 and FI_1 hold, M is strongly $\mathbf{I}$-saturated and $\lambda(\mathbf{I})$ is regular. Prove that for any $\overline{b}$ if $\overline{c} \in M[\overline{b}] - M$ then $\overline{c} \not\downarrow_M \overline{b}$.

The following theorem is proved for the isolation relations other than $\mathbf{L}$. The proof for $\mathbf{L}$ is a minor variant but we don't rely on that case. The full significance of this result will only become apparent with the discussion of strongly regular types in Section XII.1.

1.22 Corollary. *If M is strongly $\mathbf{I}$-saturated, $\overline{a} \downarrow_M D$ but $(\overline{b} \not\downarrow \overline{a}; D \cup M)$, and N is any $\mathbf{I}$-saturated model containing $M \cup \overline{a}$. Then for any $\mathbf{I}$-formula q with* $\mathrm{dom}^* q$ *contained in M such that $\models q(\overline{b})$, there exists $\overline{b}'$ such that the following hold.*

i) $\overline{a} \not\downarrow_M \overline{b}'$

ii) $\bar b' \in q(N)$
iii) $\bar b' \notin M$.

Proof. (Fig. 1). Let A denote dom* q. Without loss of generality (since we assume $\lambda(\mathbf{I}) \geq \kappa(T)$), $\bar a \downarrow_A M$. Thus $(\bar a \not\downarrow \bar d^\frown \bar b; A)$. By the local character of forking there is a formula $\theta(\bar x; \bar y^\frown \bar z)$ such that $\models \theta(\bar a; \bar b^\frown \bar d)$ for some $\bar d \in D$ and for any $\bar b', \bar d'$, if $\models \theta(\bar a; \bar b', \bar d')$ then $(\bar a \not\downarrow \bar b'^\frown \bar d'; A)$. Now by Corollary 1.13, there exist $\bar b', \bar d'$, with $\bar d' \in M$ such that $\models \theta(\bar a; \bar b', \bar d')$. Since $\bar a \downarrow_A M$, this implies $\bar a \not\downarrow_M \bar b'$ so $\bar b' \notin M$. Since the requirements on $\bar b'$ were expressed by an $\mathbf{I}$-formula over $A \cup \bar a$, without loss of generality $\bar b' \in M[\bar a]$.

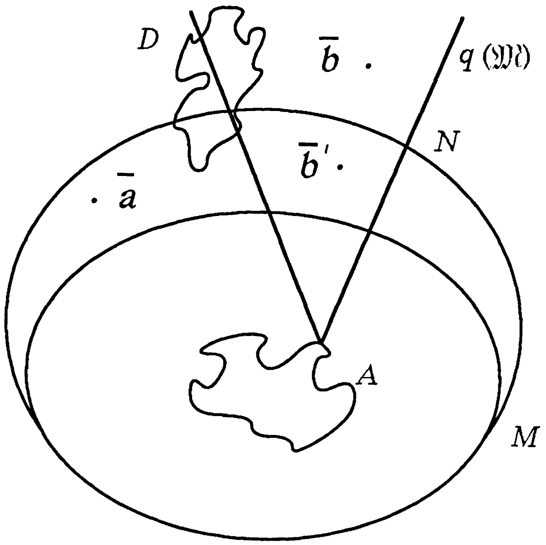

Fig. 1. Corollary X.1.22

The following exercises provide further rephrasings of axiom FI_1 and Corollary 1.20. In all cases M is to be assumed strongly $\mathbf{I}$-saturated.

1.23 Exercise. Show that if p is realized in $M[q]$ then $q \rhd_M p$.

1.24 Exercise. Show that if X is an independent set over M then X is a maximal independent (over M) subset of $M[X]$.

The following exercise emphasizes that dependence is not a simple negation of nonforking by showing $\bar b$ can depend on $\bar a$ over M without being in $M[\bar a]$.

1.25 Exercise. Using the theory of a single equivalence relation with infinitely many infinite classes show the following assertion is false for $\mathbf{AT}_{\aleph_0}$. If $b \notin M[a]$ then $b \downarrow_M M[a]$.

This phenomenon can be made even sharper.

1.26 Exercise. Show using the theory of two refining equivalence relations that there exist M, a, and b such that $b \notin M[a]$ but $b \not\downarrow_{M \cup a} M[a]$.

1.27 Exercise. Prove that $p \perp q$ implies $M[p] \downarrow_M M[q]$. Indeed, $p \perp q$ implies $t(M[p]; M) \perp t(M[q]; M)$.

What is the relation between $M[\overline{a}, \overline{b}]$ and $M[\overline{a}][\overline{b}]$? It is easy to see that there are embeddings $M[\overline{a}] \mapsto M[\overline{a}, \overline{b}] \mapsto M[\overline{a}][\overline{b}]$. There are several variations on the question of whether this last map has an inverse. These variations depend on such subtleties as whether we are working in the category of constructions or the category of models and embeddings. The next theorem gives the strongest possible positive answer when $\overline{a} \downarrow_M \overline{b}$.

1.28 Theorem. i) *Let M be strongly* **I**-*saturated and $A = \{\overline{a}_0, \ldots, \overline{a}_n\}$ be an independent set over M. Suppose $M_{i+1} = M_i[\overline{a}_i]$. Then M_n is* **I**-*constructible over $M \cup A$.*

ii) *More precisely, if E_1 is an* **I**-*construction of N_1 over $M \cup \overline{a}$, E is an* **I**-*construction of N over $N_1 \cup \overline{b}$, and $\overline{a} \downarrow_M \overline{b}$ then N is* **I**-*constructible over $M \cup \{\overline{a}, \overline{b}\}$.*

Proof. The proof of ii) is included in the proof of i). We show i) by induction on $|A|$. Suppose $A = B \cup \overline{a}$, $|B| = m$ and, by induction, that M_m is **I**-constructible over $M \cup B$. Note that by FI_1 the construction of M_m over $M \cup B$ is, in fact, a construction of M_m over $M \cup A$. Now, appending the construction of M_{m+1} over $M_m \cup \overline{a}$ yields a construction of M_{m+1} over $M \cup A$ as required.

1.29 Exercise. Extend Theorem 1.28 by induction to infinite indiscernible sequences.

1.30 Exercise. Show that if $\lambda(\mathbf{I})$ is regular, FI_0 and FI_1 hold and $\overline{a} \downarrow_M \overline{b}$ then $t(M[\overline{b}]; M \cup \overline{b}) \vdash t(M[\overline{b}]; M \cup \overline{a} \cup \overline{b})$.

One important fact about **AT** is the following variant of the open mapping theorem: If $t(\overline{a}; A \cup \overline{b})$ is **AT**-isolated and $\overline{a} \downarrow_A \overline{b}$ then $t(\overline{a}; A)$ is **AT**-isolated. For the topological explanation of this nomenclature see Section III.3. We designate this property as the third basic relation between freedom and isolation. Both this property and FI_3 which follows are closely related to the finite character of forking, so there may be some more elegant way to unify these two notions.

1.31 Axiom $\mathbf{FI_2}$ (The Open Mapping Theorem). If both ($\overline{a}\downarrow\overline{b}$;A) and $t(\overline{a}; A \cup \overline{b})$ is **I**-isolated over $A \cup \overline{b}$ then $t(\overline{a}; A)$ is **I**-isolated.

We have already seen (III.3.24) that FI_2 holds for $\mathbf{AT}_{\aleph_0}$. We now verify it for **S**-isolation.

1.32 Lemma. *If $\overline{a} \downarrow_A B$ and $t(\overline{a}; B)$ is* **S**-*isolated then $t(\overline{a}; A)$ is* **S**-*isolated.*

Proof. Choose $C \subseteq B$ such that $stp(\overline{a}; C) \vdash t(\overline{a}; B)$, and $D \subseteq A$ such that $\overline{a} \downarrow_D A$, $|D| < \kappa(T)$ and D contains $C \cap A$. We claim that $stp(\overline{a}; D) \vdash t(\overline{a}; A)$. This is a property of A, D, and an arbitrary realization $\overline{a}'$ of $stp(\overline{a}; D)$. Thus, we are completely free to choose the type of B over $A \cup \overline{a}'$. So without loss of generality we may assume that $A^\frown\overline{a}' \downarrow_D B$. Then since $stp(\overline{a}; D)$ is stationary, $stp(\overline{a}'; C) = stp(\overline{a}; C)$. But $stp(\overline{a}; C) \vdash t(\overline{a}; B)$ so $t(\overline{a}'; B) = t(\overline{a}; B)$ which is more than is required.

An alternative exposition of this proof would involve a new copy B' of B rather than relying on 'without loss of generality'. Although we concluded that $t(\overline{a}; B) = t(\overline{a}'; B)$, easy examples show that this phenomenon depended on our auspicious choice of B.

1.33 Exercise. Verify that FI_2 holds for **L**-isolation. (Hint: This is easily deduced from the original Open Mapping Theorem.)

There is a final axiom connecting isolation and nonforking which holds in all the classes we consider. In order to describe this principle we need some additional notation.

1.34 Notation. Let $A \subseteq \mathcal{M}$ and $p \in S(A)$. Suppose p is an n-type. We let $p^*(\mathcal{M})$ denote $\{c \in \mathcal{M} : \text{for some } \overline{c} \text{ realizing } p, c \in \mathrm{rng}(\overline{c})\}$.

Thus $p(\mathcal{M})$ denotes a set of n-tuples while $p^*(\mathcal{M})$ denotes a subset of $\mathcal{M}$.

1.35 Axiom $\mathbf{FI_3}$ (The Local Nature of Implication). Suppose that M is **I**-saturated and p is an **I**-formula with $\mathrm{dom}^*\, p = B \subseteq M$. If q is any extension of p to $S(M)$ then $q|(B \cup p^*(M)) \vdash q$.

1.36 Exercise. Deduce from FI_3 that if $p \in S(A)$, $(\overline{c} \not\downarrow M; A)$ and $\overline{c}$ realizes p then $(\overline{c} \not\downarrow p^*(M); A)$.

If $p \in S(A)$, keeping in mind that if $\overline{a}^\frown\overline{b} \in p(\mathcal{M})$ and $q = t(\overline{a}^\frown\overline{b}; A)$ then $q^*(\mathcal{M}) \subseteq p^*(\mathcal{M})$, we easily see

1.37 Proposition. *Suppose* FI_3 *holds. Let* M *be* **I**-*saturated. For* $\overline{a}, \overline{b}$ *realizing* p, *if* $q = t(\overline{a}^\frown\overline{b}; M)$ *then, letting* $B = \mathrm{dom}^*\, p$, $q|(B \cup p^*(M)) \vdash q$.

The following result, which easily yields that the set of models of an ω-stable (indeed an arbitrary stable theory) satisfies FI_3 for both **AT** and **L**-isolation, was first enunciated in this form by Lascar and Bouscaren in [Bouscaren & Lascar 1983].

1.38 Theorem. *Let* M *be a model of a stable theory,* $p \in S(M)$, $\phi(\overline{x}, \overline{a}) \in p$ *and set* $B = \phi(M, \overline{a}) \cup \overline{a}$. *Then,* $p|B \vdash p$.

Proof. Assume for contradiction that there are $\psi(\overline{x}; \overline{m}) \in F(M)$ and $\overline{c}$, $\overline{c}'$ in $\mathcal{M}$ such that $\overline{c}$ realizes p and $t(\overline{c}; B) = t(\overline{c}'; B)$ but $\models \psi(\overline{c}; \overline{m}) \wedge \neg\psi(\overline{c}'; \overline{m})$. Let d define $t(\ \overline{m}; B)$. In particular, $d\psi(\overline{y}) \in F(B)$ and for any $\overline{b} \in B$, $\models d\psi(\overline{b})$ iff $\models \psi(\overline{b};\ \overline{m})$. Since $t(\overline{c}; B) = t(\overline{c}'; B)$ and $\models \phi(\overline{c})$, we have $\models \phi(\overline{c}')$. Suppose $\models d\psi(\overline{c})$ and thus $\models d\psi(\overline{c}')$. (If $\models \neg d\psi(\overline{c})$ we obtain a similar contradiction.) Since M is an elementary submodel of $\mathcal{M}$, there is a $\overline{c}'' \in M$ such that

$\models \phi(\overline{c}'') \wedge d\psi(\overline{c}'')$ and $\models \neg\psi(\overline{c}''; \overline{m})$. But then $\overline{c}'' \in B$ and this contradicts the choice of d.

With a similar diagonal argument we extend this result to saturated models. This imposes a stricter requirement on the model M but makes the subset on which the type depends smaller. We can easily conclude that FI_3 holds for $\mathbf{SET}_\lambda$ and $\mathbf{S}_\lambda$.

1.39 Theorem. *If T is stable, M is a λ-saturated model of T, $B \subseteq M$ with $|B| < \lambda$, and $p \in S(M)$ with $p|B = p_0$ then $p|(B \cup p_0^*(M)) \vdash p$.*

Proof. Let $A = B \cup p_0^*(M)$. For each $\overline{n} \in M$ and each formula $\phi(\overline{x}; \overline{n})$, the definition d of $t(\overline{n}; A)$ yields a $d\phi$ such that:

$$\text{For all } \overline{a} \in A, M \models \phi(\overline{a}; \overline{n}) \text{ iff } M \models d\phi(\overline{a}). \qquad (*)$$

But if $p|B \cup d\phi(\overline{x}) \cup \neg\phi(\overline{x}; \overline{n})$ is consistent then it is realized by some $\overline{a}' \in M$ contradicting $(*)$.

1.40 Exercise. Verify that for each $i < 4$ and each regular $\lambda \geq \kappa(T)$ the axiom FI_i holds for $\mathbf{SET}_\lambda$ and $\mathbf{S}_\lambda$.

1.41 Exercise. Suppose FI_3 holds and M is $\mathbf{I}$-saturated. For $\overline{a}$, $\overline{b}$ realizing p, if $\overline{a}^\frown\overline{b} \not\downarrow_A M$ then there exists an $X \subseteq p^*(M)$ such that $\overline{a}^\frown\overline{b} \not\downarrow_A X$

1.42 Historical Notes. This section unifies various arguments scattered in [Shelah 1978], [Lascar & Poizat 1979], [Bouscaren & Lascar 1983] and [Makkai 1984]. The fact that Corollary 1.20 holds for local isolation is new here. The weaker Exercise 1.21 was proved (for $\mathbf{L}$) earlier by Pillay and Steinhorn [Pillay & Steinhorn 1985]. The origins of Theorem 1.33 are a little murky. Although, first given explicitly in this form in [Bouscaren & Lascar 1983],there are variations in both [Lascar & Poizat 1979] and [Shelah 1978] (V.1.11).

2. Powerful Isolation Relations

We introduce here an important strengthening of the isolation concept. Most of the notions we have considered, but not $\mathbf{AT}$, satisfy this additional condition. These results show that under certain conditions the concepts of isolation and domination are very closely related. Corollary 2.5 records another crucial property of $\mathbf{S}$-isolation.

2.1 Definition. The isolation relation $\mathbf{I}$ is *powerful* if for any finite sequence $\overline{a}$ and any set B with $|B| < \lambda(\mathbf{I})$, $t(\overline{a}; B)$ is $\mathbf{I}$-isolated.

Of course, if $t(\overline{a}; B)$ is an $\mathbf{I}$-formula, it is $\mathbf{I}$-isolated, so for any λ, $\mathbf{SET}_\lambda$ and $\mathbf{S}_\lambda$ are powerful isolation relations. It is equally easy to see that even for ω-stable T, $\mathbf{AT}_{\aleph_0}$ is not always a powerful isolation relation (see Exercise 2.4).

2.2 Exercise. Show that if T is a countable, ω-stable, $\aleph_0$-categorical theory then $\mathbf{AT}_{\aleph_0}$ is a powerful isolation relation.

The following theorem is one of the main reasons for introducing this notion.

2.3 Theorem. *Let M be $\mathbf{I}$-saturated for a powerful isolation relation $\mathbf{I}$. Suppose also that if M is $\mathbf{I}$-saturated, then for any $\overline{a}$, $t(\overline{a}; M)$ is strongly based on a subset of M with fewer than $\lambda(\mathbf{I})$ elements. Then for any $\overline{a}$ and $\overline{b}$, $\overline{a} \rhd_M \overline{a}^\frown\overline{b}$ if and only if there exists a structure N which is $\mathbf{I}$-strictly prime over $M \cup \overline{a}$ and contains $\overline{b}$.*

Proof. Suppose $\overline{a} \rhd_M \overline{a}^\frown\overline{b}$; let $p = t(\overline{a}; M)$ and $r = t(\overline{b}; M \cup \overline{a})$. Choose $B \subseteq M$ such that $|B| < \lambda(\mathbf{I})$, $\overline{a}^\frown\overline{b} \downarrow_B M$, and $t(\overline{a}^\frown\overline{b}; M)$ is strongly based on B. Then, by Theorem VI.3.12, $t(\overline{a}; B) \rhd_B t(\overline{a}^\frown\overline{b}'; B)$ for any $\overline{b}'$ realizing $r_0 = r|(B \cup \overline{a})$. Since $\mathbf{I}$ is powerful, r_0 is $\mathbf{I}$-isolated. We now show r_0 $\mathbf{I}$-isolates $t(b; M \cup \overline{a})$. If $\overline{b}'$ realizes r_0, $\overline{a} \downarrow_B M$ implies $\overline{a}^\frown\overline{b}' \downarrow_B M$. Thus $t(\overline{a}^\frown\overline{b}'; M) = t(\overline{a}^\frown\overline{b}; M)$ and, in particular, $t(\overline{b}; M \cup \overline{a}) = t(\overline{b}'; M \cup \overline{a})$ as required. Thus, a strictly $\mathbf{I}$-prime model can be constructed over $M \cup \overline{a}$ beginning with $\overline{b}$. This is more than is required. The converse is an easy consequence of Corollary 1.20.

2.4 Exercise. Show that Theorem 2.3 fails for $\mathbf{AT}_{\aleph_0}$. (Hint: Consider the theory of an equivalence relation with infinitely many infinite classes and a model of Th(Z, S) in each class.)

Note that as phrased the result applies to $\mathbf{SET}_\lambda$ for $\lambda \geq |T|^+$. If T is a countable ω-stable theory, Theorem 2.3 applies to $\mathbf{SET}_{\aleph_0}$ but not if T is strictly superstable.

2.5 Corollary. *Let M be $\mathbf{I}$-saturated and good, $\mathbf{I}$ a powerful isolation relation, and p, $q \in S(M)$. Then $p \rhd q$ if and only if q is realized in $M[p]$.*

Proof. Let $\overline{a}$, $\overline{b}$ realize p and q and witness $p \rhd q$. Choose $B \subseteq M$ with $|B| < \lambda(\mathbf{I})$, p, q strongly based on B, and $\overline{a}^\frown\overline{b} \downarrow_B M$. Now, if $\overline{b}'$ realizes $t(\overline{b}; B \cup \overline{a})$, Lemma VI.3.12 implies $\overline{b}' \downarrow_B M$ and thus $\overline{b}'$ realizes q. But, certainly $t(\overline{b}; (B \cup \overline{a})$ is realized in $M[\overline{a}]$. The other implication is immediate from Theorem 2.3.

2.6 Exercise. Show, using the same example as before, that Corollary 2.5 does not hold for $\mathbf{AT}$.

2.7 Historical Notes. The notion of a powerful isolation relation arose from a desire to unify some proofs from [Buechler 1984b] (only in the preprint) and [Prest 1985]. Although Prest proves Theorem 2.3 for a notion of isolation in modules, I have not succeeded in reducing his argument to this rubric. Nevertheless, the notion seems to isolate one of the essential distinctions between $\mathbf{AT}$ and $\mathbf{S}$. The example in Exercise 2.4 was first suggested to me by Shelah; it was also discovered by Pillay and Steinhorn. Pillay corrected an error in the original proof of Theorem 2.3.

3. Uniqueness of Prime Models

In this section we prove that if T is a countable stable theory which has prime models over every set A then these models are unique up to A-isomorphism. More generally, we show that for any reasonable notion of isolation **I**, any subset of an **I**-constructible model is **I**-constructible. Intuitively, one could justify such a claim by asserting that if the order of a construction was really essential it would be reflected by some ordering in the theory; thus, the theory would be unstable. This intuition is made precise in [Shelah 1975] where Shelah proves the uniqueness of prime models over a set of power $\aleph_1$ using the techniques of stationary sets and relying on stability theory only for the fact that no formula in a stable theory can have the order property. Here, we adopt a more model theoretic course and invoke the machinery of nonforking we have developed earlier.

3.1 Theorem. *Let* **I** *be an isolation relation satisfying the axioms listed in Sections IX.2, IX.3, and X.1. Suppose* $\lambda(\mathbf{I})$ *is regular and* $\kappa(T) \leq \lambda(\mathbf{I})^+$. *If E is strictly* **I**-*prime over A and $A \subseteq M \subseteq E$ for some $M \models T$ then M is strictly* **I**-*prime over A.*

Proof. Fix A, B which is **I**-prime over A, and E which is strictly **I**-prime over A. Without loss of generality $A \subseteq B \subseteq E$. By Theorem IX.4.12, it suffices to show that B is **I**-constructible. The proof proceeds by induction on $|E - A|$. If $|E - A| \leq \lambda(\mathbf{I})$ the result follows from Lemma IX.4.2. Suppose $|E - A| = \mu$ and E is given by the construction $\langle e_i : i < \mu \rangle$. The idea of the induction is to partition the construction of E so that the conclusion of the theorem can be applied to a sequence of smaller models C_{i+1} for $i < |E|$. The main theorem is illustrated by the first diagram (Fig. 2) which concerns E, B, and A. To perform the induction we partition E as in

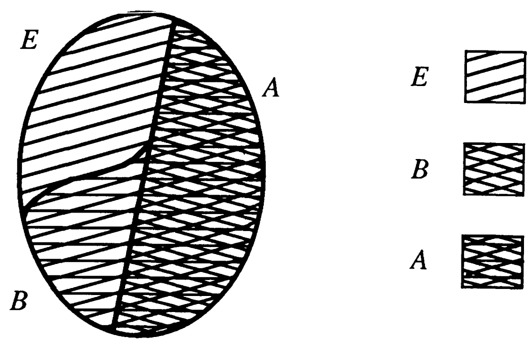

Fig. 2. First Diagram for Theorem X.3.1

the second diagram and apply the hypothesis of induction to $C_{i+1} \cup A$, $C_i \cup A \cup (C_{i+1} \cap B))$, $C_i \cup A$. (Fig. 3). Formally we build $\langle C_i : i < \mu \rangle$ to satisfy the following five conditions.

i) $\langle C_i : i < \mu\rangle$ is an increasing continuous sequence of subsets of $A \cup E$ and $\bigcup C_i = E \cup A$.
ii) $|C_i| \leq |i| + \lambda(\mathbf{I})$, so $|C_{i+1} - C_i| < |E - A|$.
iii) For each $\bar{c} \in C_i$, $\bar{c} \downarrow_{B \cap C_i} B$ and $\bar{c} \downarrow_{A \cap C_i} A$.
iv) C_i is closed. (Recall Definition IX.4.8)
v) C_{i+1} is **I**-constructible over $A \cup C_i$.

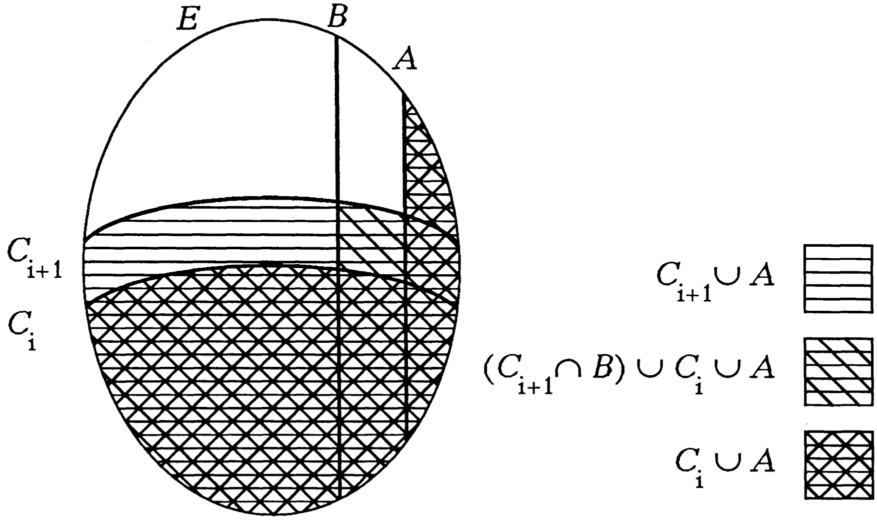

Fig. 3. Second diagram for Theorem X.3.1

The C_i are constructed by induction. Let C_0 be empty. If we have C_i, we define C_i^n for $n < \omega$ by induction and let $C_{i+1} = \bigcup_n C_i^n$. Let C_i^0 contain C_i and a point from $E - C_i$. Suppose we have constructed C_i^{2n}. For each $\bar{c}$ in C_i^{2n}, there exists a $D_{\bar{c}} \subseteq B$ with $|D_{\bar{c}}| < \kappa(T)$ such that $\bar{c} \downarrow_{D_{\bar{c}}} B$. Similarly, there exists an $E_{\bar{c}} \subseteq A$ with $\bar{c} \downarrow_{E_{\bar{c}}} A$. Let $C_i^{2n+1} = \bigcup_{\bar{c} \in C_i^{2n}} D_{\bar{c}} \cup \bigcup_{\bar{c} \in C_i^{2n}} E_{\bar{c}}$. Now for each $e_j \in C_i^{2n+1}$, there is an $F_j \subseteq E_j \cup A$ with $|F_j| < \lambda(\mathbf{I})$ such that $t(e_j; A \cup E_j)$ is isolated by F_j. Let $C_i^{2n+2} = \bigcup_{e_j \in C_i^{2n+1}} F_j$. Now C_{i+1} is a closed subset of E when a) E is regarded as constructed over A and when b) E is regarded as constructed over $A \cup C_i$ (cf. Theorem IX.4.11). Then a) guarantees iv) and b) guarantees v). The choice of C_{i+1} guarantees iii) and i); it is easy to verify ii).

Applying iv) and the induction hypothesis to $A \cup C_i$, $A \cup C_i \cup (C_{i+1} \cap B)$, and $A \cup C_{i+1}$, we conclude $A \cup C_i \cup (C_{i+1} \cap B)$ is **I**-constructible over $A \cup C_i$. Thus for some α, $(B \cap C_{i+1}) - (A \cup C_i) = \langle b_j : j < \alpha\rangle$. Let $B_j = \{b_k : k < j\}$. For each j, $q_j = t(b_j; A \cup C_i \cup B_j)$ is **I**-isolated. If we can show q_j does not fork over $(A \cup (B \cap C_i) \cup B_j)$ for each j, we can conclude from the Open Mapping Theorem that $t(b_j; A \cup (B \cap C_i) \cup B_j)$ is **I**-isolated. This yields that $B \cap C_{i+1}$ is constructible over $A \cup (B \cap C_i)$. But if this holds for each $i < \mu$, B is constructible over A and we finish.

Consider any $\bar{c} \in C_i$. By iii) $\bar{c} \downarrow_{B \cap C_i} B$ so by the monotonicity of non-forking

$$\bar{c} \underset{B \cap C_i}{\downarrow} A \cup (B \cap C_{i+1}).$$

By monotonicity again,

$$\bar{c} \underset{A \cup (B \cap C_i) \cup B_j}{\downarrow} A \cup (B \cap C_i) \cup B_{j+1}.$$

Then by symmetry

$$b_j \underset{A \cup (B \cap C_i) \cup B_j}{\downarrow} A \cup (B \cap C_i) \cup B_j \cup \bar{c}.$$

Since this holds for each $\bar{c} \in C_i$, we have by the finite character of forking that

$$b_j \underset{A \cup (B \cap C_i) \cup B_j}{\downarrow} A \cup C_i \cup B_j$$

as required.

Note that this proof does not require the Existence Axiom IX.2.7. We can state a less technical result if we assume it as well.

3.2 Corollary. *Let* $\mathbf{I}$ *be one of the isolation relations defined in IX.2.1. Suppose* $\lambda(\mathbf{I})$ *is regular,* $\kappa(T) \leq \lambda(\mathbf{I})^+$*, and that for every set* $A \subseteq \mathcal{M}$ *there is a strictly* $\mathbf{I}$*-prime model over* A*. For any* $A \subseteq \mathcal{M}$*, any two* $\mathbf{I}$*-prime models over* A *are isomorphic.*

3.3 Historical Notes. Shelah first proved the uniqueness of prime models for a countable ω-stable theory by induction on rank in [Shelah 1972]. He gave an alternative proof using the machinery of stationary sets rather than the technology of stability theory in [Shelah 1975]. The proof given here is taken from [Shelah 1979]. That proof was inspired by the (unpublished) proof by Ressayre, which we discussed in Section IX.4, that strictly prime models are unique. In [Shelah 1979] Shelah showed that both the hypotheses of countability and stability are necessary for the theorem. First he shows that the theory $\mathrm{REF}_{\aleph_1}$ of $\aleph_1$ refining equivalence relations has prime models over all sets but does not have unique prime models. Then he notes that replacing the equivalence relations by ternary relations such that a linear order on one set indexes a collection of refining equivalence relations on another produces a countable unstable example of the same phenomenon.

This line of argument produced the first proof of the uniqueness of "differential closure". Blum [Blum 1968] showed the theory of differentially closed fields of characteristic zero is ω-stable and deduced the uniqueness from [Shelah 1972]. This result is extended to differentially closed fields of characteristic p by the proof that the theory is stable ([Wood 1973], [Wood 1976], [Wood 1979]) and the application of Theorem X.3.1. These proofs were later given more algebraic form in [Kolchin 1973].

4. Indiscernible Sets in Prime Models

Axiom FI_1 clearly implies that if X is an independent set over M, then X is a maximal independent set in $M[X]$. We would like to improve this result to: if X is an independent set of indiscernibles then $\dim(X, M, M[X]) = |X|$. (Recall Definition V.2.3.) In general, however, this assertion is false. In this section we show that it is true if the cardinality of X is sufficiently large. In Chapter XII we show that for regular types it is always true.

Let A be a set and suppose that M is $\mathbf{I}$-constructible over A. We will show that $\lambda(\mathbf{I})$ provides an upper bound for the length of independent sets over A in M. This upper bound will be a key to the construction of many non-isomorphic models of T in Part D. It is easy to see from Exercise X.1.21 that this property holds if we start with a model.

For simplicity we assume in this section that $\lambda(\mathbf{I})$ is regular. We could strengthen the results to deal with singular $\lambda(\mathbf{I})$ by replacing such hypotheses as $\lambda(\mathbf{I}) \geq \kappa(T)$ by $\mathrm{cf}(\lambda(\mathbf{I})) \geq \kappa(T)$.

4.1 Exercise. Show that if $\lambda(\mathbf{I}) \geq \kappa(T)$, $|A| < \kappa(T)$ and M is $\mathbf{I}$-saturated then every subset of $M[A]$ which is indiscernible over $M \cup A$ has power at most $\kappa(T)$.

It is not so easy to see this when we do not start over a model. We begin by describing a further property of the isolation relations introduced in Chapter IX. It is unclear whether this property follows from the axioms in Chapter IX but it is trivial to verify it for the isolation relations defined in Section IX.2.

4.2 Lemma. *If q is an $\mathbf{I}$-formula with* $\mathrm{dom}^* q \subseteq A$, *$E$ is a set of indiscernible over A realizations of q and $\bar{e}$ realizes* $\mathrm{Av}(E, A)$ *then* $\models q(\bar{e})$.

Proof. If $\mathbf{I}$ is $\mathbf{SET}_\lambda$ or $\mathbf{AT}_\lambda$, the result follows immediately from the definition of average type. For $\mathbf{S}_\lambda$, apply Lemma V.1.7.

4.3 Lemma. *Suppose $E \subseteq A$ is a set of indiscernibles over $A_0 \subseteq A$ with $|E| \geq \kappa(T)$ and $\bar{e}$ realizes* $\mathrm{Av}(E; A)$. *Suppose also that for some $\mathbf{I}$ with $\lambda(\mathbf{I}) \geq \kappa(T)$ and some $\mathbf{I}$-formula q with* $\mathrm{dom}^* q = A_0$, *$q \vdash t(\bar{c}; A)$. Then $q \vdash t(\bar{c}; A \cup \bar{e})$.*

Proof. For any formula $\psi(\bar{x}, \bar{y}; \bar{a})$ such that $\models \psi(\bar{c}, \bar{e}; \bar{a})$ there is a set $E_{\bar{a}} \subset E$ with $|E_{\bar{a}}| < \kappa(T)$ such that $E - E_{\bar{a}}$ is a set of indiscernibles over $E_{\bar{a}} \cup \bar{a} \cup \bar{c} \cup A_0$. Now $\models \psi(\bar{c}, \bar{e}; \bar{a})$ if and only if for one (and thus every) $\bar{e}' \in E - E_{\bar{a}}$, $\models \psi(\bar{c}, \bar{e}'; \bar{a})$. Since $q \vdash t(\bar{c}; A)$ there is a formula $\theta \in q$, with $\models \theta(\bar{x}) \rightarrow \psi(\bar{x}, \bar{e}'; \bar{a})$. Then the $\mathbf{I}$-formula $(\forall \bar{x})\theta(\bar{x}) \rightarrow \psi(\bar{x}, \bar{e}'; \bar{a})$ holds for all $\bar{e}'$. By Lemma 4.2, we conclude $(\forall \bar{x})\theta(\bar{x}) \rightarrow \psi(\bar{x}, \bar{e}; \bar{a})$. Since this holds for any formula ψ we have the lemma.

It is now fairly easy to show

4.4 Corollary. *If M is* **I**-*constructible over A and $E \subset A$ is a set of indiscernibles with $|E| \geq \lambda(\mathbf{I})$ then* $\mathrm{Av}(E; A) \vdash \mathrm{Av}(E; M)$.

Proof. The proof is a straightforward induction on the length of the construction. For the crucial successor stage we must show that if $E \subset A'$, $t(c; A')$ is **I**-isolated by some q with $\mathrm{dom}^* q \subseteq A'$, $\bar{e}$ realizes $\mathrm{Av}(E; A)$ and $t(\bar{e}; A) \vdash t(\bar{e}; A')$ then $t(\bar{e}; A) \vdash t(\bar{e}; A' \cup \bar{c})$. Since $|E| \geq \lambda(\mathbf{I})$ we may assume E is indiscernible over $\mathrm{dom}^* q$. By Lemma 4.3 we know $q \vdash t(\bar{c}; A' \cup \bar{e})$. By the symmetry axiom for **I**-isolation (Axiom IX.3.3) we deduce $t(\bar{e}; A' \cup \bar{c})$ is **I**-isolated over A. If **I** is $\mathbf{SET}_\lambda$ or $\mathbf{AT}_\lambda$ the result is immediate. For $\mathbf{S}_\lambda$ it follows from Lemma IV.3.12.

Now we can show one of the two components of the characterization of **I**-prime models. We show here that if M is **I**-prime over A then M is **I**-atomic over A and every set of indiscernibles in M has cardinality at most $\lambda(\mathbf{I})$. The necessity of the first condition is almost obvious and now we deduce the necessity of the second. The converse to this theorem is proved by induction on rank in Chapter IV of [Shelah 1978].

4.5 Theorem. *Suppose $\lambda(\mathbf{I}) \geq \kappa(T)$ is regular. Assume T admits* **I**-*prime models. If M is strictly* **I**-*prime over A and $E \subseteq M$ is a set of indiscernibles over A then $|E| \leq \lambda(\mathbf{I})$.*

Proof. Suppose $|E| > \lambda(\mathbf{I}) \geq \kappa(T)$. By Theorem V.1.23 choose $E_0 \subseteq E$ with $|E_0| < \kappa(T)$ such that $\mathrm{Av}(E; M)$ is strongly based on $A \cup E_0$. Let p denote the restriction of $\mathrm{Av}(E; M)$ to $A \cup E_0$. Let F be an independent sequence of $\lambda(\mathbf{I})$ realizations of p and let M' be **I**-prime over $B = A \cup E_0 \cup F$. Corollary 4.4 implies $\mathrm{Av}(F; B) \vdash \mathrm{Av}(F; M)$. Thus, $\dim(F, A \cup E_0, M')$ is $\lambda(\mathbf{I})$. By Lemma IX.4.1, M is **I**-prime over $A \cup E_0$. Let f imbed M into M' fixing $A \cup E_0$. Now $f(E)$ and F are equivalent indiscernible sets with at least $\kappa(T)$ elements. Thus, by Theorem V.2.4 they should have the same dimension in M'. But, clearly $\dim(f(E), A \cup E_0, M') > \lambda(\mathbf{I})$ and we finish.

The next result will play an important role in the construction of many non-isomorphic models. It will be used in conjunction with Theorem 4.5 but is proved solely on the basis of the methods of Section 1.

4.6 Theorem. *Let M be* **I**-*saturated and $p, q \in S(M)$ with $p \perp q$. If E is an independent set of realizations of q and N is* **I**-*prime over $M \cup E$ then* $\dim(p, N) = 0$.

Proof. Clearly, if $\bar{a} \in N - M$ realizes p, $\bar{a} \not\downarrow_M E$. But then $p^\omega \not\perp^w q^\omega$, whence by Corollary VI.2.18 $p \not\perp q$.

4.7 Definition. The model M is **I**-minimal over A if M is **I**-saturated and there is no **I**-saturated model N containing A and properly contained in M.

The following lemma allows us to characterize minimal models by the cardinalities of maximal sets of indiscernibles. To apply it we must know every **I**-saturated model is good. We showed in Corollary IV.2.9 and Theorem IV.3.22 that **AT** has that property if T is a countable ω-stable theory and **S** has that property for every superstable theory.

4.8 Lemma. *Fix* $\mathbf{I}$ *with* $\lambda(\mathbf{I}) \geq \kappa(T)$ *and such that every* $\mathbf{I}$*-saturated model is good. Let* M *be* $\mathbf{I}$*-atomic over* A *and suppose* $N \subseteq M$ *is* $\mathbf{I}$*-saturated. Then for any* $\overline{a} \in M - N$*, there is an infinite set* $E \subseteq N$ *of indiscernibles over* A *such that* $t(\overline{a}; N) = \mathrm{Av}(E, N)$.

Proof. Suppose there exists $c \in M - N$. Choose $C \subseteq N$ such that $t(c; N)$ is strongly based on $A \cup C$. Note that $stp(c; A \cup C)$ is **I**-isolated. Now choose $c_i \in N$ for $i < \omega$ so that c_n realizes $stp(c; A \cup C_n)$. The c_i are a strongly independent sequence and therefore indiscernible by Lemma V.1.8 so we have the lemma.

The next result characterizes minimal models.

4.9 Theorem. *Let* T *be superstable and countable.* M *is* $\mathbf{I}$*-minimal over* A *if and only if every set* E *of indiscernibles over* A *in* M *is finite.*

Proof. Suppose E is an infinite set of indiscernibles in M. Choose $N \subseteq M$ to be **I**-prime over $A \cup (E - \{e\}$, for some $e \in E$. By Corollary 4.4, $e \notin N$ so M is not minimal.

The converse is immediate from the lemma.

4.8 Historical Notes. These results are scattered in [Shelah 1978] but Section IV.4 especially Theorem IV.4.9 is most relevant.