Chapter VI
Orthogonality

In this chapter we investigate two extensions of the nonforking notion in an attempt to find the right notion of freeness. In the first section we discuss triples (A, B, C) such that A and B are not only independent over C but are persistently so. We say $t(A; C)$ is orthogonal to $t(B; C)$. In the second section we discuss a way in which A can be even more independent from B over C, namely $t(A; C)$ is orthogonal to every type over B. Our goal is this chapter is to describe the principal properties of these relations so we can use them as tools later. We first consider orthogonality of two types as a freeness relation in the sense of Chapter II. It turns out that, while regarding nonorthogonality of two types as a dependence relation is very natural; viewing nonorthogonality of a type and a set as a dependence relation is somewhat forced. A more natural approach to this latter relation is considered in Section 3 where we introduce an important partial (pre)order on the types over a set: the dominance order.

1. Orthogonality Of Types

In this section we discuss an important extension of the notion of independence: orthogonality. This concept has both a local and global form. The local form describes the orthogonality of two types over subsets of the monster model; the global form describes the orthogonality of two global types (i.e. types over the monster model). The notion of parallelism bridges the gap between these two forms of orthogonality. We begin by defining the local form.

1.1 Definition. Let p and q be complete types over C. We say p is *orthogonal* to q (over C) and write $p \perp q$ if the following holds. For every E containing C and every $\bar{a}$ realizing p, $\bar{b}$ realizing q, if $\bar{a} \downarrow_C E$ and $\bar{b} \downarrow_C E$ then $\bar{a} \downarrow_E \bar{b}$.

To simplify notation, we may write $(\bar{a} \perp \bar{b}; C)$. The role of C turns out to be unimportant; we usually just write $p \perp q$.

We can immediately extend this definition to types with different domains by the following device which we will use for other concepts later.

1.2 Definition. Let p,q be types over C, D, respectively. Then $p \perp q$ if for any nonforking extension p' of p to $S(C \cup D)$ and any nonforking extension q' of q to $S(C \cup D)$, $p' \perp q'$.

Definition 1.2 justifies speaking of the orthogonality of strong types. That is, $stp(c; B) \perp q$ if and only if the unique nonforking extension of $stp(c; B)$ to a complete type over $B \cup c$ is orthogonal to q.

We will investigate orthogonality as a notion of freeness in the sense of Chapter II. There we defined a relation $\mathcal{F}$ between a type p and a set C. This was in fact a relation among a realization $\overline{a}$ of p, the domain B of p and the set C. Much of our discussion focused on the triple $(\overline{a}, B, C)$ but the relation was actually a property of $t(\overline{a}^\frown B; C)$. Here our relation is among a similar triple $(\overline{a}, \overline{b}, C)$ but (and this is the essence of orthogonality) the relation depends not on the entire type $t(\overline{a}^\frown \overline{b}; C)$ but only on its two projections $t(\overline{a}; C)$ and $t(\overline{b}; C)$.

We observed after Axiom II.1.15 that although we defined the notion of freeness in terms of finite sequences it really was a property of the sets which are the ranges of the relevant sequences. Accordingly, this same observation applies to the derived relation, orthogonality, and we can write $(A \perp B; C)$. Similarly, as after Definition II.1.17 we can remove the restriction that A and B be finite because nonorthogonality inherits the finite character of nonforking.

1.3 Proposition. *If $(A \not\perp B; C)$ then for some finite A_0 and B_0 contained in A, B respectively, $(A_0 \not\perp B_0; C)$.*

Exercises 1.4 and 1.6 just describe the translation of properties of orthogonality from set notation to type notation.

1.4 Exercise. If for each finite $\overline{n} \in N$, $p \perp t(\overline{n}; N_0)$ show $p \perp t(N; N_0)$.

This differs from the finite character discussed in Chapter II because we have only the algebraic, as opposed to the logical, form. We have identified finite sets but not specific formulas as a cause of nonorthogonality . We give a more syntactic characterization in Exercise 2.19.

It is easy to see that orthogonality is strictly stronger that nonforking since we can use the base set as the test in the definition of orthogonality. Formally, we have

1.5 Lemma. $(A \perp B; C)$ *implies* $(A \downarrow B; C)$.

1.6 Exercise. Let $p = t(\overline{a}; C)$ and $q = t(\overline{b}; D)$. Then p is orthogonal to q iff for every $\overline{a}'$ realizing p, every $\overline{b}'$ realizing q and every E containing $C \cup D$, if $\overline{a}' \downarrow_C E$ and $\overline{b}' \downarrow_D E$ then $\overline{a}' \downarrow_E \overline{b}'$.

1.7 Exercise. Show that if $p \in S(A)$ is algebraic then no consistent extension of p forks over A.

1.8 Exercise. Conclude that if $p \in S(A)$ is algebraic then every q is orthogonal to p.

There is an important simplification available when checking whether two types are orthogonal. We can replace the quantifiers 'for all $\overline{a}'$, $\overline{b}'$, $\overline{e}'$ by 'there exists an $\overline{a}'$ such that for all $\overline{b}'$ and $\overline{e}'$. This is easily established by an automorphism argument.

1.9 Exercise. Show that $t(A;C) \perp t(B;C)$ iff there exists an A' realizing $t(A;C)$ such that for every E and every B' realizing $t(B;C)$, if $A' \downarrow_C E$ and $B' \downarrow_C E$ then $A' \downarrow_C B'$.

This illustrates an important principle which we will invoke repeatedly in trying to parse our definitions. Because of the homogeneity of the monster model, 'One "for all" is "for free".' That is, in many definitions the first occurrence of an assertion, 'for every element realizing a type p such that ...' can be replaced by the assertion,'there is an element realizing a type p such that ...'.

1.10 Examples. i) The simplest example of two orthogonal types occurs in the theory with one unary predicate U such that both U and $\neg U$ are infinite. Then for any model M, the type p(x), which asserts $x \notin M$ and $U(x)$, and the type $q(x)$, which asserts $\neg U(x)$ and $x \notin M$, are orthogonal over M. Only slightly more complicated is the similar situation of an equivalence relation with two infinite classes.

ii) The next level of complexity arises by considering an equivalence relation with infinitely many infinite classes. Then the type, p, over a model M which fixes an equivalence class but is not realized in M is orthogonal over M to the type, q, which is realized by an element in a 'new' equivalence class.

iii) A much more interesting example arises by considering the theory of $Z_2^{\aleph_0} \oplus Z_4^{\aleph_0}$. Let M be a model of T and consider the types p, q, and r defined as follows. Let p be the type of an element of order four satisfying $2x \neq m$ for each $m \in M$. Let q, r be types which are not realized in M but are satisfied by elements of order two. Let any realization of q be divisible by two but $(2m \neq x) \in q$ for each $m \in M$. Let no realization of r be divisible by two. Then, $p \perp r$, $p \not\perp q$ (since twice some realization of p may realize q), and $q \perp r$. Thus, orthogonality need not be transitive.

We now consider which of the other axioms for freeness hold of orthogonality. It is easy to see that orthogonality is symmetric. But the monotonicity axioms are more problematical.

1.11 Lemma. i) *If* $(A \perp B;C)$ *then* $(B \perp A;C)$.

ii) *Orthogonality satisfies the first monotonicity axiom. If* $(A \perp B;C)$ *and* $A_0 \subseteq A$ *then* $(A_0 \perp B;C)$.

iii) *The following stronger version of ii) is more useful. Suppose that* $t(A;B) \perp t(C;D)$ *and* $A_0 \subseteq A$. *Then* $t(A_0;B) \perp t(C;D)$.

Proof. Check i) immediately from the definition. For ii), fix A'_0, B' and E so that A'_0 realizes $t(A_0;C)$, B' realizes $t(B;C)$, $A'_0 \downarrow_C E$, and $B' \downarrow_C E$. Let A'' be the image of A under an automorphism which takes A_0 to A'_0 while fixing C. Apply the strong extension property (Exercise II.2.8) to choose A' so that $(A' \downarrow E; C)$ and $t(A'; A'_0 \cup C) = t(A''; A'_0 \cup C)$. By symmetry, applying $A'_0 \downarrow_C E$, we have $(A \perp B; C)$ implies $(A' \downarrow B; E)$ so by monotonicity of nonforking $(A'_0 \downarrow B; E)$ as required.

To derive iii) from ii) let A'_0, C' realize $t(A_0;B)$, $t(C,D)$, respectively, with $A'_0 \downarrow_B B \cup D$ and $C' \downarrow_D B \cup D$. We must show $t(A'_0; B \cup D) \perp t(C'; B \cup D)$. Apply the strong extension principle to choose $A' \supseteq A'_0$ with $A' \downarrow_B B \cup D$. By the definition of orthogonality for types over different sets, $t(A; B \cup D) \perp t(C'; B \cup D)$. By part ii) $t(A'_0; B \cup D) \perp t(C'; B \cup D)$. Since this argument holds for any choice of C' which does not depend on $B \cup D$ over D, we conclude $t(A_0; B) \perp t(C; D)$.

In this proof we have carefully traced the argument for each nonforking extension of $t(A;B)$. The second two sentences of the proof of ii) justify the assertion 'without loss of generality $A_0 = A'_0$.' We will not hesitate to make such assumptions later.

The following example shows that upward monotonicity in the third coordinate fails, i.e. $(A \perp B; C)$ does not imply $(A \perp B; C')$ for arbitrary C' containing C. In fact the remedy from Section III.3 (requiring C' to be contained in B) does not work either since in the example $t(C' - C; C) \subseteq t(B; C)$.

1.12 Example. Let T be the theory of $Z_2^{\aleph_0} \oplus Z_4^{\aleph_0}$. Let r be the type over the empty set asserting x has order four. Let $q(y)$ be the type over the empty set asserting y has order two and two does not divide y. Let M be a countable model of T and let N be $M \oplus Z_2 \oplus Z_4$. Consider the following three elements of N

	M	Z_2	Z_4
a	$\bar{0}$	0	1
b	$\bar{0}$	1	0
b'	$\bar{0}$	1	2

Now a realizes the nonforking extension of r to M while B and b$'$ realize the nonforking extension of q to M. We have $(a \perp b; M)$ but $(a \not\perp b; M \cup b')$. For, $t(a; M \cup b, b')$ forks over $M \cup b'$ since $2a = b + b'$.

Using the transitivity of nonforking, it is easy to show the 'correct' form of monotonicity in the third coordinate. Namely, orthogonality is preserved by nonforking extensions. Hrushovski has made important applications ([Hrushovski 1986], [Hrushovski 198?]) of the concept of ***hereditary orthogonality*** (p is hereditarily orthogonal to q if *every* extension of p is orthogonal to q.).

1.13 Lemma. *If $p \perp q$ and p', q' are nonforking extensions of p and q then $p' \perp q'$.*

The following exercise translates this result into the set notation.

1.14 Exercise. If $(A \perp B; C)$ and $C \subseteq C'$ with $A \downarrow_C C'$ and $B \downarrow_C C'$ then $(A \perp B; C')$.

We show now that if we view nonorthogonality as a dependence relation then a suitable modification of the transitivity of independence axiom holds for orthogonality. The clearest argument for the next proposition relies on Theorem 1.37 which asserts that if the nonforking extensions p', q' of the stationary types p and q are orthogonal then so are p and q. Lemma 1.15 is not used in the proof of Theorem 1.37.

1.15 Lemma. *If $B \subseteq C \subseteq D$ and $(A \perp D; C)$ and $(A \perp C; D)$ then $(A \perp D; B)$.*

Proof. Let A' realize $t(A; B)$, D' realize $t(D; B)$, and suppose $(A' \downarrow E; B)$ and $(D' \downarrow E; B)$ for some E containing B. By Exercise 1.9, we may choose A' to realize $stp(A; B)$. We must show $(A' \downarrow D'; E)$. Choose $C' \subseteq D'$ with $t(C'; B) = t(C; B)$. By monotonicity, $(C' \downarrow E; B)$. Now applying the orthogonality of A and C over B we have $(A' \downarrow C'; B \cup E)$. By monotonicity and the independence of A' from E over B, this implies $(A' \downarrow C' \cup E; B)$. ¿From $D' \downarrow_B E$ we deduce by monotonicity and symmetry that $D' \downarrow_{C'} E \cup C'$. Suppose for the moment that $stp(D'; C') \perp stp(A'; C')$. Then $D' \downarrow_{C' \cup E} A'$. Since $A' \downarrow_B C' \cup E$, we deduce $D' \downarrow_B A'$ as required.

The argument that $stp(D'; C') \perp stp(A'; C')$ is somewhat circuitous. Since $stp(D; C) \perp stp(A; C)$ and $A \downarrow_B C$, Theorem 1.39 implies $stp(D; C) \perp stp(A; B)$. There is an automorphism fixing B and mapping D to D' and C to C'. Thus, since orthogonality is preserved under automorphisms, $stp(D'; C') \perp stp(A; C)$. But $stp(A; B) = stp(A'; B)$ and $A' \downarrow_B C'$. So by Lemma 1.13 $stp(D'; C') \perp stp(A'; C')$.

We can drop the assumption that $C \subseteq D$.

1.16 Exercise. Show that if B is a subset of C and $(A \perp C; B)$, and $(A \perp D; C)$ then $(A \perp D; B)$. (Hint: Note that $t(D; C) \perp t(D \cup C; C)$ and then apply Lemma 1.15.

1.17 Example (Stationarity Fails). Let T be the theory of an equivalence relation with infinitely many infinite classes. Then the unique 1-type over the empty set has an unbounded number of pairwise orthogonal non-algebraic extensions.

We will discuss later theories in which each type has a bounded number of mutually orthogonal non-algebraic extensions (i.e. which satisfy for orthogonality the intent of Axiom II.1.23) and will see that they form a particuarly well behaved class. (Such theories are said to have bounded width or be non-multidimensional).

There is one way in which orthogonality is a much stronger relation than independence.

1.18 Definition. We call a notion of freeness $\mathcal{F}$ *trivial* when $(A \mathcal{F} B_0; C)$, $(A \mathcal{F} B_1; C)$, and $(B_0 \mathcal{F} B_1; C)$ together imply $(A \mathcal{F} B_0 \cup B_1; C)$.

For example, if T is any completion of the theory of a single 1-1 unary function then nonforking is a trivial dependence relation. Intuitively, orthogonality should be trivial. For, if $t(A;C) \perp t(B;C)$ then A and C are persistently independent. That is, they remain independent over any set which is independent from each of them; in particular, they should be independent over any realization of a type which is orthogonal to each of $t(A;C)$ and $t(B;C)$. More formally, we show that orthogonality is trivial by showing an even stronger result which we call *strong triviality.*

1.19 Theorem. *Orthogonality is a trivial dependence relation. In fact, let E be an independent set over A. Suppose that for any $\bar{b}$ and each $\bar{e} \in E$, $(\bar{b} \perp \bar{e}; A)$ then $(\bar{b} \perp E; A)$.*

Proof. Without loss of generality, by Proposition 1.3, we assume E is finite. The proof is by induction on $n = |E|$ and the result is obvious if $n = 0$. So assume the result for $n = m$ and we will establish it for $n = m + 1$. Fix any D with $A \subseteq D$ such that $\bar{b} \downarrow_A D$ and $E_{m+1} \downarrow_A D$ Since nonforking preserves independence and since E_{m+1} is independent over A, $\bar{e}_m \downarrow_D E_m$. By induction $(\bar{b} \perp E_m; A)$ so by the definition of orthogonality $\bar{b} \downarrow_D E_m$. Now since $(\bar{b} \perp \bar{e}_m; A)$, we have $(\bar{b} \downarrow \bar{e}_m; D \cup E_m)$. Thus $\bar{b} \downarrow_D E_{m+1}$ as required.

The following exercise provides a useful rephrasing of Theorem 1.19.

1.20 Exercise. Show that if $E = \langle \bar{e}_i : i < \alpha \rangle$ is independent over A and for each i, $p \perp t(\bar{e}_i; A)$ then $p \perp t(E; A)$.

1.21 Exercise. Find a counterexample to the following assertion. If for each $e \in E$, $p \perp t(e; A)$ then $p \perp t(E; A)$.

We now will discuss orthogonality from a more syntactic viewpoint, one which relies heavily on model theory as opposed to the abstract theory of dependence relations.

1.22 Definition. i) Let $p(\bar{x}), q(\bar{y}) \in S(A)$. p is *weakly orthogonal* to q, $p \perp^w q$, if $p(\bar{x}) \cup q(\bar{y})$ is a complete type.

ii) Let p and q be types (almost) over A. p and q are *almost orthogonal*, $p \perp^a q$, if for any a, b realizing p, q, respectively, $a \downarrow_A b$.

The following exercises will be applied repeatedly without reference. They are all proved by permuting Definition 1.22.

1.23 Exercise. Show $t(\bar{a}_1; A) \perp^w t(\bar{a}_2; A)$ iff $t(\bar{a}_1; A) \vdash t(\bar{a}_1; A \cup \{\bar{a}_2\})$ iff $t(\bar{a}_2; A) \vdash t(\bar{a}_2; A \cup \{\bar{a}_1\})$. Hence if $t(\bar{a}_1; A) \perp^w t(\bar{a}_2; A)$ we can conclude $\bar{a}_1 \downarrow_A \bar{a}_2$.

1.24 Exercise. Show that $p \perp^w q$ implies $p \perp^a q$.

1.25 Exercise. Show that $p \perp q$ implies $p \perp^a q$.

1.26 Exercise. Show that if p is weakly orthogonal to $q(\bar{x})^\frown r(\bar{y})$ then p is weakly orthogonal to each of q and r.

1.27 Exercise. If one of p and q is stationary, $p \perp^a q$ implies $p \perp^w q$.

1.28 Exercise. $p \perp^a q$ iff for some a realizing p and any $\bar{b}$ realizing q, $t(\bar{b}; A \cup \{\bar{a}\})$ does not fork over A.

1.29 Exercise. If for some model $M, p, q \in S(M)$ and $p \perp^a q$ then $p \perp^w q$.

We extend by convention the definition of weak orthogonality and almost orthogonality to include types with different domains. Recall the notation $N(A \cup B, A)$ from Section III.3.

1.30 Definition. Let $p \in S(A)$ and $q \in S(B)$ and suppose p', q' extend p and q respectively.

i) $p \perp^w q$ iff for each $p' \in N(A \cup B, A)$ and $q' \in N(A \cup B, B)$, $p' \perp^w q'$.
ii) $p \perp^a q$ iff for each $p' \in N(A \cup B, A)$ and $q' \in N(A \cup B, B)$, $p' \perp^a q'$.

By the preceding exercises, for stationary p and q we have $p \perp^w q$ iff $p \perp^a q$. Thus $\perp^a$ is the weak version of orthogonality where instead of extending p and q freely to an arbitrary extension of their domains, we extend only to $\operatorname{dom} p \cup \operatorname{dom} q$.

The subtle difference between the following two exercises plays an important role in problems relating to the axiomatizability of totally categorical theories [Ahlbrandt 1984].

1.31 Exercise. Show that if $t(a; A)$ is algebraic then it is stationary if and only if a is in the definable closure of A. Conclude that if $t(\bar{a}; C)$ and $t(\bar{b}; C)$ are both algebraic and stationary then $t(\bar{a}; C) \perp^w t(\bar{b}; C)$.

1.32 Exercise. Show the previous result fails if the hypothesis that the types are stationary is dropped.

The following two examples show that, strictly speaking, orthogonality and weak orthogonality are incomparable concepts. Of course, we have already noted that if one of p, q is stationary then $p \perp q$ implies $p \perp^w q$.

1.33 Example (Weak orthogonality need not imply orthogonality). i) We first show that two types over a set A can be weakly orthogonal but not orthogonal.

Let T be the theory of the $Z/(4)$ module $Z_4^{\aleph_0}$. Denote by p the type over the empty set of an element of order 2. Fix a set A of realizations of p, $a \in A$ and q the type of an element of order 4 which satisfies $2x = a$. Letting p' be the nonforking extension of p to A, we will show $p' \perp^w q$ but $p' \not\perp q$.

To see that $p' \perp^w q$, note that if $b \neq c \in \mathcal{M}$ (the monster model) satisfy q there is an automorphism of $\mathcal{M}$ which fixes $p(\mathcal{M})$ pointwise and maps b to c. Exercise 1.23 implies this is more than enough. (For this, regard $\mathcal{M}$ as a free $Z/(4)$-module with basis $\langle b_i : i < \kappa \rangle$. The basis can be chosen so that $b = b_0$ and $c = b_0 + 2b_1$. Then, $\{b, c\} \cup \{b_i : 1 < i < \kappa\}$ is another basis for $\mathcal{M}$. Interchanging b and c and fixing the rest of the basis yields an automorphism of $\mathcal{M}$ which fixes $p(\mathcal{M})$ pointwise.)

To see that $p' \not\perp q$, fix a realization c of q and let q' (p'') be the nonforking extensions q (p') to $A \cup c$. Now, $p' \not\perp q$ since if c' realizes q' and a' realizes p'' we may or may not have $c' = c + a'$.

This counterexample is intrinsic to the situation. That is, $Z_4^{\aleph_0}$ is the prototype of an $\aleph_1$-categorical theory which is not almost strongly minimal. Tsuboi showed that if T is such a theory, A is an infinite dimensional subset of a strongly minimal set D, and p is the unique nonalgebraic type over A which contains $D(x)$ then for some nonalgebraic $q \in S(A)$, $q \perp^w p$. The $\aleph_1$-categoricity of T guarantees that $q \not\perp p$.

ii) We now show that there are examples of types over models which are weakly orthogonal but not orthogonal. Lascar has proved (cf. [Lascar 1982a] and XIII.3.7) that this is impossible in an ω-stable theory.

Let T be the theory CEF_ω^+ (III.4.9) of the structure $(2^\omega, E_i, +)$. Let M be the model whose universe is the collection of eventually constant functions, and let σ, τ be two sequences which do not differ by an eventually constant function. We claim that $p = t(\sigma; M)$ and $q = t(\tau; M)$ are weakly orthogonal but not orthogonal. To see $p \perp^w q$, we need only show that all formulas about σ, τ, and $\sigma + \tau$ with parameters from M are implied by $p \cup q$. This is straightforward using the quantifier eliminability of T. To see that $p \not\perp q$, note that $\tau \downarrow_M \tau + \sigma$ and $\sigma \downarrow_M \tau + \sigma$ but $(\sigma \not\downarrow \tau; M \cup \sigma + \tau)$.

Note that in this example forking reduces to algebraic dependence.

1.34 Exercise. Why are the equivalence relations E_i necessary for the preceding example?

Example 1.33 ii) arose from the following more complicated, but less contrived, example due to Mike Prest.

1.35 Exercise. Let $Z_{(2)}$ denote the additive group of those rational numbers whose denominators are not divisible by two (the integers localized at two). Show there are two types over $Z_{(2)}$ which are weakly orthogonal but not orthogonal.

A more abstract, and so perhaps more believable explanation of the phenomenon in Paragraphs 1.33 through 1.35 is given on page 504 of [Poizat 1985].

1.36 Example (For nonstationary types orthogonality does not imply weak orthogonality). Let T be the theory of two unary predicates P and Q which partition the universe and an equivalence relation with two classes each of which intersects both P and Q infinitely. Then if $p \in S(\emptyset)$ is generated by $P(x)$ and $q \in S(\emptyset)$ is generated by $Q(y)$ it is easy to see that $p \perp q$ but $p \not\perp^w q$.

The following theorem gives the relationship between orthogonality and weak orthogonality.

1.37 Theorem. *Let $p \in S(A)$ and $q \in S(B)$, then $p \perp q$ if and only if for any nonforking extensions of p and q to global types $\hat{p}$ and $\hat{q}$, $\hat{p} \perp^w \hat{q}$.*

Proof. If $p \perp q$ then, by the definition of orthogonality, for any $\overline{a}'$ realizing $\hat{p}$ and any $\overline{b}'$ realizing $\hat{q}$, $\overline{a}' \downarrow_M \overline{b}'$. Since $\hat{p}$ and $\hat{q}$ are stationary, Exercise 1.27 shows $\hat{p} \perp^w \hat{q}$. For the converse, suppose there exist $\overline{a}$,$\overline{b}$ realizing p,q respectively and a set D such that $\overline{a} \downarrow_A D$ and $\overline{b} \downarrow_B D$ but $\overline{a} \not\downarrow_D \overline{b}$. Let $\overline{a}'{}^\frown\overline{b}'$ realize an extension of $t(\overline{a}{}^\frown\overline{b}; D)$ to the monster model which does not fork over D. Then by Corollary II.2.10, $\overline{a}' \not\downarrow_{\mathcal{M}} \overline{b}'$. Since types over models are stationary this contradicts the weak orthogonality of $t(\overline{a}'; \mathcal{M})$ and $t(\overline{b}'; \mathcal{M})$. (Clearly, these types do not fork over A, B, respectively.)

This argument violates at least the spirit of the monster model idea by appealing to $\overline{a}'$ and $\overline{b}'$. This appeal can be avoided by a somewhat more complicated formulation.

1.38 Theorem. *Let $p \in S(A)$ and $q \in S(B)$ with $|A \cup B| < \lambda$, then $p \perp q$ if and only if for any nonforking extensions of p and q to types p' and q' in $S(M)$, where M is a λ-saturated model containing $A \cup B$, $p' \perp^w q'$.*

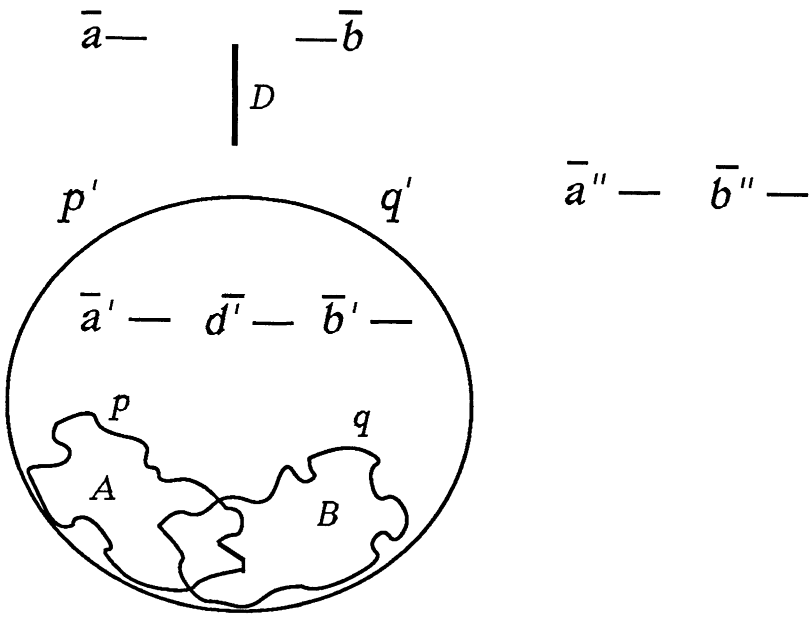

Fig. 1. Theorem VI.1.38

Proof. (Fig. 1). If $p \not\perp q$ then there exists D containing $A \cup B$ with $\overline{a} \downarrow_A D$ and $\overline{b} \downarrow_B D$ but $\overline{a} \not\downarrow_D \overline{b}$ for some $\overline{a}$ and $\overline{b}$ realizing p, q respectively. Without loss of generality, we may assume $D - (A \cup B)$ is a sequence $\overline{d}$ with $|\overline{d}| < \omega$. Now, by the saturation of M, choose $\overline{a}'{}^\frown\overline{b}'{}^\frown\overline{d}' \in M$ realizing $t(\overline{a}{}^\frown\overline{b}{}^\frown\overline{d}; A \cup B)$. Then $\overline{a}' \downarrow_A \overline{d}'$ and $\overline{b}' \downarrow_B \overline{d}'$. Choose $\overline{a}''{}^\frown\overline{b}''$ realizing $t(\overline{a}'{}^\frown\overline{b}'; A \cup B \cup \overline{d}')$ so that $t(\overline{a}''{}^\frown\overline{b}''; M)$ does not fork over $A \cup B \cup \overline{d}'$. Now $(\overline{a}' \not\downarrow \overline{b}'; A \cup B \cup \overline{d}')$ so by Corollary II.2.10 $\bar{a}'' \not\downarrow_M \bar{b}''$. But $\bar{a}'' \downarrow_A M$ and $\bar{b}'' \downarrow_B M$ so this contradicts the hypothesis.

We next prove that parallelism repects orthogonality. Using this observation we are able to improve the formulation of Theorem 1.38. We noted in Lemma 1.13 the routine observation that orthogonality is preserved by nonforking extensions (orthogonality goes up). The fact that orthogonality also goes down is the content of the current lemma.

1.39 Lemma. *Let p,q, and r be arbitrary stationary types. If $p \parallel r$ then $p \perp q$ iff $r \perp q$.*

Proof. It suffices to show $p \perp q$ implies $q \perp r$. Since $p \parallel r$ there exists a global type $\hat{p}$ which is a common nonforking extension of p and r. If $\hat{q}$ is any nonforking extension of q to a global type then $p \perp q$ and Theorem 1.37 imply $\hat{p} \perp^w \hat{q}$. Applying Theorem 1.37 in the other direction, we conclude $r \perp q$.

1.40 Theorem. *If $p, q \in S(M)$ and M is strongly $\kappa(T)$-saturated then $p \perp^w q$ implies $p \perp q$.*

Proof. Since M is strongly $\kappa(T)$-saturated, by Corollary IV.2.8 there are subsets A and B of M with cardinality less than $\kappa(T)$ such that p is strongly based on A and q is strongly based on B. By Theorem 1.38, $p|A \perp q|B$. But $p|A \parallel p$ and $q|B \parallel q$ so $p \perp q$ by Lemma 1.39.

Some conditions on the domain of p and q are necessary to have weak orthogonality imply orthogonality. Example 1.33 showed that we must at least assume the domain is a model. Lascar [Lascar 1982a] has shown that for ω-stable theories this suffices.

There are two parts to the proof of Theorem 1.40. We need to know that the model M is good in the sense of Section IV.2. This part of course transfers immediately to any model of an ω-stable theory. Secondly, we need to know that the nonorthogonality of p and q can be passed to the nonorthogonality of $p|A$ and $q|B$. The above proof accomplishes this by appealing to the $\kappa(T)$-saturation of M. Lascar's argument [Lascar 1982a] substitutes an analysis of regular types.

Although $p \perp q$ was defined without assuming p or q was stationary many of the results in this section depended on the hypothesis that a type was stationary. The natural way to convert a type $p = t(\bar{c}; A)$ to a stationary type is to consider $stp(\bar{c}; A)$. The relation between orthogonality to p and the various strong types extending p is stated in the following corollary which follows easily from Theorem 1.39.

1.41 Corollary. *If $p \in S(A)$ and $q \in S(B)$ then $p \perp q$ if and only if for each $\bar{c}$ realizing p, $stp(\bar{c}; A) \perp q$.*

The following two exercises emphasize, first, the importance of assuming that p and q are stationary in Theorem 1.39 and second the necessity of assuming $stp(\bar{c}; A) \perp q$ for each $\bar{c}$ realizing p in Corollary 1.41.

1.42 Exercise. Let T be the theory of an equivalence relation with two infinite classes. Show that the unique type over the empty set has a pair of nonforking extensions which are orthogonal. Conclude that the hypothesis that p and q are stationary is necessary for Theorem 1.39.

1.43 Exercise. Find an example of types $p, q \in S(\emptyset)$ so that $p \not\perp q$ but for some $\bar{c}$ realizing p, $stp(\bar{c}; \emptyset) \perp q$.

1.44 Historical Notes. The notion of orthogonality was introduced in Chapter V.1 of [Shelah 1978]. Shelah begins with the model theoretic notion of weak orthogonality. Then he defines orthogonality in terms of our Theorem 1.37. Spurred by some observations of Saffe, we noted that orthogonality can be derived in purely algebraic terms from any dependence relation satisfying the axioms of Chapter II. Thus we introduce weak orthogonality as a useful variant which arises in the context of first order theories. The third variant, almost orthogonality, was suggested by Makkai [Makkai 1984].

2. *Orthogonality of a Type and a Set*

The next two sections are devoted to the discussion of several closely related but subtly different notions: orthogonality of a type to a set, dominance, eventual dominance, and several minor variants of these. All of these notions can be derived from an arbitrary freeness relation. That is, just as in Definition 1.1, the extended property is defined from the nonforking relation without the intervention of any other stability theoretic concept. We will explore their full significance in Chapter X when we consider their relations with various notions of isolation.

We first consider the notion 'p is orthogonal to B' as meaning p is orthogonal to every type over B. If $p = t(A; C)$ we can think of this relation as meaning 'A is free from B over C' and so we are studying the kind of relation discussed in Chapter II.

2.1 Definition. i) Let $p \in S(C)$ and B be a set. Then p is *orthogonal* to B (written $p \dashv B$) iff $p \perp q$ for each $q \in S(B)$.

ii) If $p \dashv \emptyset$ and p is not algebraic, p is said to be *unbounded.*

Note that $t(A; C) \dashv B$ is actually an assertion about $t(A; C)$ and $t(B; C)$.

2.2 Examples. i) Let T be the theory of an equivalence relation with infinitely many infinite classes and let $M \models T$. Suppose $M \prec N$ and $a \in N$ is in an equivalence class which is not represented in M. Then, a type p over N which asserts $E(x, a)$ is orthogonal to M. The only difficulty in seeing this is to observe that as well as being orthogonal to any type which fixes an equivalence class different from $[a]$, p is orthogonal to the type which asserts of an element that it is not in any class which is represented in M. (Fig. 2).

ii) A similar situation arises if we consider the third example from Example 1.10. Any type which specifies a new coset of a subgroup with infinite index will be orthogonal to the ground model.

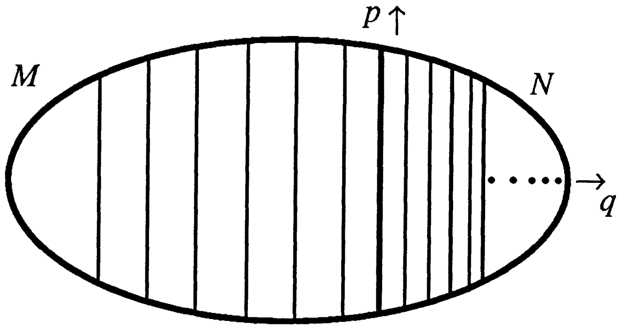

Fig. 2. $p \dashv M$

The following easy exercises will provide some familiarity with the notion.

2.3 Exercise. Show that $p \dashv A$ implies $p \perp stp(c; A)$ for every c. But, if $p \in S(B)$, $p \dashv A$ if and only if for each $\bar{c}$ realizing p, $stp(\bar{c}; B) \dashv A$.

2.4 Exercise. Show that if $p \dashv A$ then $p \dashv \mathrm{cl}(A)$.

It is an easy exercise to show that if $t(A; C) \dashv B$ this relation continues to hold if we shrink A or B. The situation is more complicated if we increase C. A third monotonicity result, replacing B by a B' with $B' \downarrow_B C$, requires much more work and appears as Theorem 2.21.

2.5 Proposition. i) If $t(A; C) \dashv B$, $A' \subseteq A$ and $B' \subseteq B$, it follows that $t(A'; C) \dashv B'$.

ii) If $t(A; C) \dashv B$ and $C \subseteq C'$ with $C' \downarrow_C A$ then $t(A; C') \dashv B$.

iii) If $t(A; C) \not\dashv B$ then there is a $B' \subseteq B$ with $|B'| < \kappa(T)$ such that $t(A; C) \not\dashv B'$.

Part ii) of Proposition 2.5 just says that $p \dashv A$ is preserved by taking a nonforking extension of p.

2.6 Exercise. Show that for any A and B if there is a C with $t(A; B) \dashv C$ then $t(A; B) \dashv \emptyset$.

The following three exercises illustrate some arguments involving trees of types orthogonal to sets. These arguments play an important role in Chapter XVI. The second exercise will usually be applied to $C \subseteq A_0 \subseteq B_0$ and $C \subseteq A_1 \subseteq B_1$.

2.7 Exercise. Show that if $A \downarrow_C B$ and $t(D; A) \dashv C$ then $D \downarrow_C B$.

2.8 Exercise. Let $A_0 \downarrow_C A_1$ and suppose that, for $i = 1, 2$, $t(B_i; A_i) \dashv C$ and $t(D_i; B_i) \dashv A_i$. Conclude $B_0 \downarrow_C B_1$ and $t(D_0; B_0) \perp t(D_1; B_1)$.

2.9 Exercise. Let $D \subseteq A \subseteq (B \cap C)$ and $B \downarrow_A C$. Show that if $t(B; A) \dashv D$ and $t(C; A) \dashv D$ then $t(B \cup C; A) \dashv D$.

At this point the analysis in terms of a freeness relation stops making much sense. The relation does not seem to satisfy the symmetry axiom in general. A literal translation of the transitivity axiom produces nonsense. We will prove a stronger version of monotonicity and then introduce another viewpoint on orthogonality which is more fruitful.

We want to prove that if $t(A; C) \dashv B$ and $C \downarrow_B D$ then $t(A; C) \dashv D$. In some sense we ought to be able to derive this result on the basis of the properties of orthogonality and freeness which we already have in hand. However, such a proof is unknown. Rather, we must introduce here several more complicated notions in order to obtain the result.

The following concept specializes the idea of a product of types which is described in Chapter XIII.2.

2.10 Notation. If $p \in S(A)$ is a stationary type and α is an ordinal, we denote by p^α the type $t(I; A)$ where I is a strongly independent sequence of length α based on p.

2.11 Exercise. Since p is stationary, p^α is well defined.

2.12 Exercise. Find a model N, a submodel M and a $q \in S(N)$ such that $q^\omega | M \neq (q|M)^\omega$. (Hint: consider Example 2.2 i)).

2.13 Exercise. Show that since p is stationary, p^α is stationary for every α.

2.14 Exercise. Show that $p \perp q$ if and only if $p^\omega \perp q$.

The following technical result is notable in two ways. First, it reduces the problem of deciding whether two types p and q are orthogonal to the question of weak orthogonality of associated types. Second, these associated types ($p^\omega|A$ and $q^\omega|A$) need not even have the same domain as p and q but are over an arbitrary subset of their common domain.

2.15 Lemma. *Let $p,q \in S(B)$ be stationary types and suppose $A \subseteq B$. If $(p^\omega|A) \perp^w (q^\omega|A)$ then $p \perp q$.*

Proof. If $p \not\perp q$, there exist $\overline{a}$, $\overline{a}'$, $\overline{b}$, $\overline{b}'$, and $\overline{c}$ such that $\overline{a}$ and $\overline{a}'$ realize p, $\overline{b}$ and $\overline{b}'$ realize q, and each of $\overline{a}$, $\overline{a}'$, $\overline{b}$, and $\overline{b}'$ is independent from $B \cup \overline{c}$ over B but $t(\overline{a}{}^\frown\overline{b}; B \cup \overline{c}) \neq t(\overline{a}'{}^\frown\overline{b}'; B \cup \overline{c})$. Let $\alpha = \sup(\overline{\kappa}(T), |B|^+)$. Now choose a sequence $\overline{e}_i = \overline{a}_i{}^\frown\overline{b}_i$ for $i < \alpha$ such that $\overline{a}_i \downarrow_B A_i$ and $\overline{b}_i \downarrow_B B_i$ but $t(\overline{e}_i; E_i \cup B \cup \overline{c})$ extends $t(\overline{a}{}^\frown\overline{b}; E_i \cup B \cup \overline{c})$ if i is even and $t(\overline{a}'{}^\frown\overline{b}'; E_i \cup B \cup \overline{c})$ if i is odd. Now E realizes the type $p^\alpha|A^\frown q^\alpha|A$. Since $(p^\omega|A) \perp^w (q^\omega|A)$, there is only one type over A of a sequence $\langle \overline{e}'_i : i < \alpha\rangle$ with $\overline{e}'_i = \overline{a}'_i{}^\frown\overline{b}'_i$ which projects to a realization of $p^\alpha|A$ and to a realization of $q^\alpha|A$. Moreover, such a sequence can be chosen with $\overline{e}'_i \downarrow_B E'_i$ and all the $\overline{e}'_i$ realizing the same strong type over B. The resulting sequence, E', is a sequence of indiscernibles over B. Hence, E is a sequence of indiscernibles over B. Since $|E| > |B| + \kappa(T)$, by Corollary V.1.19 there is a final subsequence of E which is indiscernible over $B \cup \overline{c}$. But this contradicts the choice of E, so $p \perp q$.

2.16 Exercise. Show the hypothesis of the last lemma can be replaced by: for each $n < \omega$, $p^n \perp^w q^n$.

2.17 Exercise. Show that if $p, q \in S(A)$, $p \not\perp q$, and $A \subseteq B$ then

i) there exists $\overline{a}$ and $\overline{b}$ realizing p and q with $\overline{a} \downarrow_A B$, $\overline{b} \downarrow_A B$ and $\overline{a} \downarrow_B \overline{b}$.
ii) For some $n < \omega$ there exists $\overline{a}', \overline{b}'$ realizing p^n and q^n with $\overline{a}' \downarrow_A B$, $\overline{b}' \downarrow_A B$ and $\overline{a}' \not\downarrow_B \overline{b}'$.

It is easy to deduce the following summary of the last few lemmas and exercises.

2.18 Corollary. *Let p, $q \in S(B)$ be stationary. The following are equivalent.*

i) $p \perp q$.
ii) $p^\omega \perp^w q^\omega$.
iii) $p^\omega \perp^a q^\omega$.
iv) *For each natural number n, $p^n \perp^w q^n$.*

We noted, after Proposition 1.3, that the finite character we had established for nonorthogonality lacked the syntactic character that the finite character of forking has (Axiom II.1.15, Corollary III.3.15). Of course, weak orthogonality has this syntactic character. Thus, we can now give a syntactic expression for $p(\overline{x}) \not\perp q(\overline{y})$, but only by introducing more variables.

2.19 Exercise. Assume p and q are stationary. Show that if $p, q \in S(A)$ and $p \not\perp q$ then for some integer n and formula $\phi(\overline{x}_0, \ldots, \overline{x}_{n-1}; \overline{y}_0, \ldots, \overline{y}_{n-1})$ and for any $r \in S(A)$, if $\langle \overline{a}_0, \ldots, \overline{a}_{n-1} \rangle$ satisfies p^n, $\langle \overline{b}_0, \ldots, \overline{b}_{n-1} \rangle$ realizes r^n, and $\models \phi(\overline{a}_0, \ldots, \overline{a}_{n-1}; \overline{b}_0, \ldots, \overline{b}_{n-1})$ then $r \not\perp p$.

2.20 Exercise. Show that a fifth equivalent to the conditions in Corollary 2.18 is: For some (any) $\alpha \geq \omega$ $p^\alpha \perp^w q^\alpha$.

Now we can obtain the promised strong monotonicity result.

2.21 Theorem. *Let $p \in S(C)$. Suppose $p \dashv B$ and $C \downarrow_B D$. Then $p \dashv D$.*

Proof. First, we can assume that p is stationary. For, $p \dashv B$ implies that for each $\overline{d}$ realizing p, $stp(\overline{d}; C) \dashv B$. Thus it suffices to prove the theorem for $stp(\overline{d}; C)$ for an arbitrary $\overline{d}$ realizing p. Without loss of generality, invoking Proposition 2.5, $B \subseteq D$ and $B \subseteq C$. Since orthogonality is respected by parallelism, we may also assume without loss of generality that C and D (though not B) are universes of models. Thus if $q \in S(D)$, q is stationary. Let $E = C \cup D$ and let p', q' be nonforking extensions of p, q respectively to $S(E)$. Then $p^\omega = (p'|C)^\omega = (p')^\omega|C$. We establish $p \perp q$ by showing

$p^\omega \perp^w (q')^\omega|C$ and invoking Lemma 2.15. For this, let I realize $(q')^\omega|C$. Since 'one for all is for free' we may assume I realizes $(q')^\omega$. As for stationary types $\perp^a$ implies $\perp^w$ (Exercises 1.9 and 1.27), it suffices to show that for any J realizing p^ω, $J \downarrow_C I$. But $I \downarrow_D E$ implies by monotonicity that $I \downarrow_D C$. By hypothesis, $C \downarrow_B D$ so transitivity yields $I \downarrow_B C$. Now, $p \dashv B$ implies $p^\omega \dashv B$ so applying the definition of orthogonality to p^ω and $t(I; B)$ we have $I \downarrow_C J$ as required.

The following exercise is an easy application of Theorems 1.19, II.2.18, and 1.40.

2.22 Exercise. Show that for any p there does not exist a sequence of types $\langle p_i : i < \kappa(T)\rangle$ with $p \not\perp p_i$ for each i.

As a corollary to Theorem 2.21 and Exercise 2.22 we obtain a convenient test for $p \dashv B$.

2.23 Corollary. *Suppose $B \subseteq C$, $\operatorname{dom} p = C$ and f is any elementary map with domain C which fixes B pointwise. Suppose further that $stp(C; B) = stp(f(C); B)$ and $C \downarrow_B f(C)$. Then, $p \dashv B$ if and only if $p \perp f(p)$.*

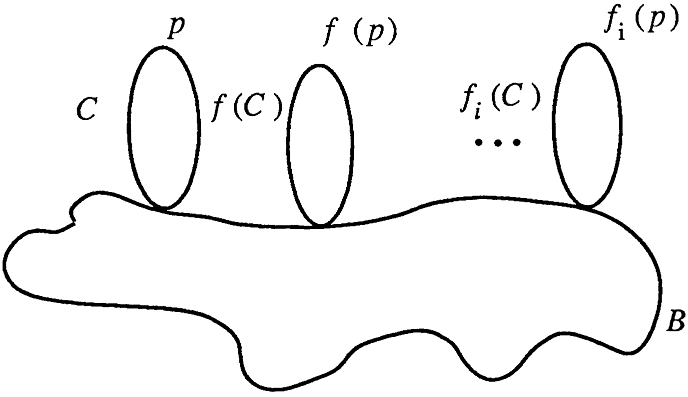

Fig. 3. A test for $p \dashv B$

Proof. (Fig. 3). If $p \dashv B$ then by Theorem 2.21, $p \dashv f(C)$ and, a fortiori, $p \perp f(p)$. Conversely, suppose that $p \not\dashv B$. For $i < |(S(B)| + |T|)^+$, define elementary mappings f_i such that $stp(f_i(C), B) = stp(C, B)$ and $t(f_i(C); \bigcup_{j<i} f_j(C))$ does not fork over B. Without loss, we may assume $f_0 = id|C$ and $f_1 = f$. If $p \perp f(p)$, then the $f_i(p)$ are a sequence of greater than $\kappa(T)$ pairwise orthogonal types. Since $p \not\dashv B$, each $p_i \not\dashv B$. Thus, for some d realizing p, $stp(d; B) \not\perp f_i(p)$ for each i. This contradicts Exercise 2.22 and yields the theorem.

It is essential for this argument that the strong types and not merely the types of B and $f(B)$ are equal. A simple counterexample can be found by considering the theory of an equivalence relation with two infinite classes.

2.24 Exercise. Let E be set of independent sequences and suppose that for each $\overline{e} \in E$ there is a $p_{\overline{e}} \in S(\overline{e})$ with $p_{\overline{e}} \dashv \emptyset$. Let $A_{\overline{e}}$ be a set of independent realizations of $p_{\overline{e}}$. Show $A = \{A_{\overline{e}} : \overline{e} \in E\}$ is an independent set. Show that if $p'_{\overline{e}}$ is the nonforking extension of $p_{\overline{e}}$ to $A_{\overline{e}} \cup \overline{e}$ then i) $p_{\overline{e}} \dashv A - A_{\overline{e}}$ and ii) $p_{\overline{e}} \perp t(A - A_{\overline{e}}; A_{\overline{e}})$.

2.25 Historical Notes. The notion of a type being orthogonal to a set arises in [Shelah 1978]. Our treatment was influenced by that of Lascar and Makkai. The important Corollary 2.18 first appears in [Shelah 1982]. We have tried to avoid arguments involving the construction of an infinite set of independent realizations of a type in favor of arguments directly about the type. They represent a technique which Shelah uses much more often in developing this material and which seems essential for Theorem 2.21. Hrushovski ([Hrushovski 1986], [Hrushovski 198?d]) has proved the remarkable result that in Corollary 2.18 iv) 'each natural number n' can be replaced by '$n \leq 3$'. More precisely, if m is minimal such that for some n $p^n \not\perp^a q^m$ then there is a $k \leq 3$ such that $p^k \not\perp^a q^m$.

3. The Dominance Order

In this section we consider a variant on the notion $t(A; B) \dashv C$. Rather than consider this variant as an independence relation, it is more useful to regard it in Definition 3.2 as a partial ordering relation on types over C.

3.1 Definition. Let $C \subseteq B$ and $p \in S(B)$. Then p is *almost orthogonal* to C if for each $q \in S(C)$, $p \perp^a q$. We write $p \dashv^a C$.

The restriction that $C \subseteq B$ can easily be seen to be superfluous. In fact replacing B by $B - C$ leads to another form of this definition which will be used more frequently.

3.2 Definition. i) For any $\overline{a}$, $\overline{b}$, and C we say $\overline{a}$ *dominates* $\overline{b}$ over C and write $(\overline{a} \rhd \overline{b}; C)$ or $\overline{a} \rhd_C \overline{b}$ if for every $\overline{d}$, $\overline{d} \downarrow_C \overline{a}$ implies $\overline{d} \downarrow_C \overline{b}$.

ii) For any p, $q \in S(C)$, we say p *dominates* q over C and we write $(p \rhd q; C)$ or $p \rhd_C q$ or simply $p \rhd q$ if there exists an $\overline{a}$ realizing p and $\overline{b}$ realizing q such that $\overline{a} \rhd_C \overline{b}$.

iii) If $p \in S(A)$ and $q \in S(B)$, we make the standard extension and say p *dominates* q and write $p \rhd q$ if for any nonforking extensions p' and q' to $A \cup B$, $p' \rhd_{A \cup B} q'$.

iv) We say B and C are *bidominant* or *domination equivalent* and write $B \sqsupseteq\!\!\!\sqsubseteq_A C$ if $B \rhd_A C$ and $C \rhd_A B$. Similarly if $p \rhd q$ and $q \rhd p$ we write $p \sqsupseteq\!\!\!\sqsubseteq q$.

bidominance

While $p \perp^a q$ means for *each* a realizing p and b realizing q $a \downarrow b$, $p \rhd q$ means for *some* a realizing p and b realizing q $a \rhd b$.

The following exercise shows that dominance is a strong negation to orthogonality. In particular, if $t(a; B \cup c) \dashv B$ and $t(a; B \cup c)$ is not algebraic then $c \rhd_B a$.

3.3 Exercise. If $\overline{a} \downarrow_A \overline{b}$ and $t(\overline{b}; A)$ is not algebraic show $(\overline{a} \not\rhd \overline{b}; A)$.

The following is proved by checking through the definitions.

3.4 Lemma. *Let B, $\overline{a}$, and $\overline{c}$ be arbitrary: $t(\overline{a}; B \cup \overline{c}) \dashv^a B$ if and only if $t(\overline{c}; B) \rhd t(\overline{a}{}^\frown\overline{c}; B)$.*

3.5 Exercise. Show using the properties of independence developed in Section II.2 that we can extend the notion $\overline{a} \rhd_C \overline{b}$ of domination of sequences to a notion of domination of sets $A \rhd_C B$.

3.6 Exercise. Show that replacing 'there exists an $\overline{a}$ and a $\overline{b}$' by 'for every $\overline{b}$ there exists an $\overline{a}$' or by 'for every $\overline{a}$ there exists a $\overline{b}$' in Definition 3.2 iii) yields an equivalent definition of p dominates q.

3.7 Exercise. Show that the attempt to strengthen the notion of domination by replacing 'there exists an $\overline{a}$ and a $\overline{b}$' by 'for every $\overline{a}$ and for every $\overline{b}$' yields a relation which is never satisfied, except possibly by algebraic types.

3.8 Exercise. Let T be the theory of an equivalence relation with two infinite classes and let M be a model of T. Show that if a and c are equivalent elements which are not in M then $t(c; M \cup a) \dashv^a M$

3.9 Exercise. Show $p \rhd p$.

The next lemma shows that dominance determines a preorder on the types over a set B. Moreover, we get a useful strong form of transitivity of dominance with varying base sets. The proofs are entirely routine.

3.10 Lemma. i) *The relation $p \rhd_B q$ on types in $S(B)$ is reflexive and transitive.*
ii) *If $A \subseteq B \subseteq C$, $B \rhd_A C$, and $C \rhd_B D$ then $B \rhd_A D$. In particular, if $B \rhd_A C$ and $C \rhd_A D$ then $B \rhd_A D$.*

The following exercise provides a useful reformulation of Lemma 3.10 ii).

3.11 Exercise. Show that if $A \subseteq B \subseteq C$, $t(C; B) \dashv^a A$, and $t(D; C) \dashv^a B$ then $t(D; B) \dashv^a A$.

So far the relation of domination is entirely local. We next explore how it behaves under parallelism so we can extend to a relation on global types. The next lemma is a routine consequence of Corollary II.2.11. It provides the independence conditions necessary to show dominance is preserved when 'going up' and when 'going down'.

3.12 Lemma. *Suppose $A_0 \subseteq A$ and $B \downarrow_{A_0} A$.*

i) $(B \rhd C; A_0)$ *implies* $(B \rhd C; A)$.
ii) $(B{}^\frown C \downarrow A; A_0)$ *implies if* $(B \rhd C; A)$ *then* $(B \rhd C; A_0)$.

Proof. i) Let $D \downarrow_A B$. By monotonicity, $D^\frown A \downarrow_A B$ and thence by transitivity of independence, $D^\frown A \downarrow_{A_0} B$. The hypothesis yields $D^\frown A \downarrow_{A_0} C$ and monotonicity of independence again gives $D \downarrow_A C$.

ii) Let $D \downarrow_{A_0} B$. We must show $D \downarrow_{A_0} C$. Choose A' realizing $stp(A; A_0)$ and with $(A' \downarrow A \cup B \cup C \cup D; A_0)$. Note $t(A'; A_0 \cup B \cup C) = t(A; A_0 \cup B \cup C)$. Thus $(B \rhd C; A')$. Now by Corollary II.2.10 $D \downarrow_{A'} B$; whence by hypothesis $D \downarrow_{A'} C$ and so by Corollary II.2.10 again $D \downarrow_{A0} C$.

We easily deduce the following corollary.

3.13 Corollary. *Let p', $q' \in S(B)$ be nonforking extensions of types p, q which are in $S(A)$. If $p \rhd q$ then $p' \rhd q'$.*

However, we can not conclude from $p' \rhd q'$ that $p \rhd q$. Consider Example 1.12. Using the notation from that example but renaming the types, let $p = t(a; M)$, $q = t(b; M)$, $p' = t(a; M \cup b')$, and $q' = t(b; M \cup b')$. Check that p' (q') is a nonforking extension of p (q). But $(p \not\rhd q; M)$ while $(p' \rhd q'; M \cup b')$ since b is in the prime model over $M \cup \{a, b'\}$. The last assertion follows from X.1.21.

Thus, in order to obtain a parallelism invariant definition we must modify the definition of domination slightly. Since domination is a strong negation to orthogonality and the orthogonality of two types in determined by considering all (appropriate) extensions, it suffices here to find one extension witnessing domination.

3.14 Definition. For $p \in S(A)$ and $q \in S(B)$ we say p *eventually dominates* q and write $p \rhd^e q$ if for some $C \supseteq A \cup B$ and some p', q', nonforking extensions of p and q to $S(C)$, $p' \rhd q'$.

In the natural way we obtain the notion of *eventual domination equivalence* of p and q, written $p \trianglerighteq^e q$.

Clearly $\rhd^e$ is a property of global types and is preserved by parallelism of stationary types. Thus we have established a preorder on the global types. This notion will be exploited extensively later in the book.

The following exercise will be a small but important step in some proofs in Part D.

3.15 Exercise. Let for $i = 1, 2$, $X_i \subseteq M_i$, and suppose $X_i \rhd_{M_0} M_i$ and $M_1 \downarrow_{M_0} M_2$. Show $X_1 \cup X_2 \rhd_{M_0} M_1 \cup M_2$.

3.16 Historical Notes. The treatment of domination presented here follows that of [Makkai 1984]. His account was largely based on [Lascar 1984] and [Lascar 1982a].